Math. Ann. 281, 157–162 (1988)

Mathematische Annalen

Some Perturbation Results for Analytic Semigroups

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The objective of this note is devoted to two particular questions arising in the theory of analytic semigroups. Let $(X, \|\cdot\|)$ be a (real or complex) Banach space and let A be the infinitesimal generator of a linear C_0-semigroup on X. Then it is well-known ([1–2] for instance) that A is closed and hence its domain $D(A)$ equipped with the graph norm

$$\|x\|_A = \|x\| + \|Ax\|$$

becomes a Banach space, which we shall denote by X_A. If A generates an analytic semigroup, then a theorem, due to Hille [5], asserts that this property remains valid under specific perturbations:

Theorem. *Let A be the infinitesimal generator of an analytic semigroup on X and let F be a linear operator $X_A \to X$ such that*

there exist constants α and β so that

$$\|Fx\| \leqq \alpha \|Ax\| + \beta \|x\| \quad \textit{for all} \quad x \in X_A \, . \tag{1}$$

If α can be chosen sufficiently small then $(A+F)$ generates an analytic semigroup on X.

This result fits well to perturbation problems for parabolic partial differential equations. The restriction on the size of α is not as severe at it looks at first glance. For instance, if F is compact from X_A into X and X is reflexive, the estimate (1) can be achieved for arbitrarily small $\alpha > 0$ by choosing β sufficiently large [3].

It is a folk-theorem that reflexivity of X is not needed to prove that $(A+F)$ is a generator if A is one and F is compact from X_A into X. In this general case, however, the argument above does not work as there are compact linear operators X_A into X such that the α in (1) cannot be chosen arbitrarily small (see [4]). As we are not aware of any reference for this perturbation result, we state it with a proof:

Theorem 1. *Let A be the infinitesimal generator of an analytic semigroup on X and let F be a compact linear operator from X_A into X. Then $(A+F)$ generates an analytic semigroup too.*

Proof. We choose a sector $\Sigma=\{\lambda\in\mathbb{C}\mid -\theta\leqq\arg\lambda\leqq\theta\}$, $\theta>\pi/2$, and a constant M so that Σ is contained in the resolvent set of A and for all $\lambda\in\Sigma$

$$\|(\lambda-A)^{-1}\|\leqq\frac{M}{|\lambda|}\ .$$

[If the semigroup generated by A is unbounded, we replace A by $(A-\omega I)$ with some $\omega>0$ to get the above assertion.]

Now

$$\|A(\lambda-A)^{-1}x\|=\|\lambda(\lambda-A)^{-1}x-x\|\leqq(M+2)\|x\|$$

and since for all $x\in X_A$,

$$\|\lambda(\lambda-A)^{-1}x-x\|=\left\|\lambda(\lambda-A)^{-1}\frac{1}{\lambda}Ax\right\| \quad \text{converges to} \quad 0 \quad \text{as} \quad \lambda\to\infty\ ,$$

we conclude by the uniform boundedness of $\|(\lambda-A)^{-1}\|_{X\to X_A}$ that for each $x\in X$

$$\|(\lambda-A)^{-1}x\|_A\to 0 \quad \text{as} \quad |\lambda|\to\infty\ , \qquad \lambda\in\Sigma\ .$$

As F is compact from X_A into X, we infer that $\|(\lambda-A)^{-1}Fx\|_A\to 0$ uniformly for all $x\in X_A$ with $\|x\|_A\leqq 1$. Select some $v>0$ such that for $\lambda\in\Sigma$, $|\lambda|\geqq v$ the norm of $(\lambda-A)^{-1}F$ regarded as an operator from X_A into X_A satisfies $\|(\lambda-A)^{-1}F\|_{X_A,X_A}\leqq\frac{1}{2}$. For these λ, we can set up the Neumann series

$$(\lambda-A-F)^{-1}=(\lambda-A)^{-1}+(\lambda-A)^{-1}F\sum_{j=0}^{\infty}((\lambda-A)^{-1}F)^j(\lambda-A)^{-1}$$

and conclude that

$$\begin{aligned}\|(\lambda-A-F)^{-1}\|&\leqq\frac{M}{|\lambda|}+M\frac{1}{|\lambda|}\|F\|_{X_A,X}\sum_{j=0}^{\infty}2^{-j}\|(\lambda-A)^{-1}\|_{X,X_A}\\&\leqq\frac{1}{|\lambda|}(M+2M\|F\|_{X_A,X}(M+2))=:\frac{\hat{M}}{|\lambda|}\ .\end{aligned}$$

Choosing $\omega>0$ sufficiently large, so that for $\lambda\in\Sigma$ we have $|\lambda+\omega|\geqq v$, we infer that clearly $\lambda+\omega\in\Sigma$, and for $\lambda\in\Sigma$ we have

$$\|(\lambda-(A+F-\omega I))^{-1}\|\leqq\frac{\hat{M}}{|\lambda+\omega|}\leqq\hat{M}\cdot(\sin\theta)^{-1}\frac{1}{|\lambda|}\ .$$

Hence $(A+F-\omega I)$ – and thus also $A+F$ – is the infinitesimal generator of an analytic semigroup on X. □

Remark. Note that is essential that A is a generator. In fact, [6] gives an example of a strongly elliptic operator which is not densely defined and a relatively compact perturbation such that the spectrum of the perturbed operator is the whole complex plane.

Our second goal is to verify that if $(A+F)$ is the generator of a semigroup for all linear operators F satisfying (1) with α sufficiently small and having finite dimensional range then A generates an analytic semigroup.

Theorem 2. *Let X be a (complex) Banach space. Suppose that A is a linear operator in X so that there exists some $\varepsilon > 0$ such that for each $a \in X$, $b^* \in X^*$ with $\|a\| \leqq \varepsilon$, $\|b^*\| \leqq \varepsilon$, $A + ab^*A$ is the infinitesimal generator of a C_0-semigroup in X. Then A generates an analytic semigroup.*

(Here ab^* is the operator defined by $ab^*(x) = b^*(x)a$).

Proof. We set $R(\lambda) = (\lambda I - A)^{-1}$ whenever it exists. Suppose that for some complex λ both $R(\lambda)$ and $(\lambda I - A - ab^*A)^{-1}$ exist. Then we obtain

$$(\lambda I - A - ab^*A)R(\lambda)a = (1 - b^*AR(\lambda)a)a$$

and thus

$$R(\lambda)a = (1 - b^*AR(\lambda)a)(\lambda I - A - ab^*A)^{-1}a \ .$$

Consequently

$$|1 - b^*AR(\lambda)a| \geqq \frac{\|R(\lambda)a\|}{\|(\lambda I - A - ab^*A)^{-1}\| \, \|a\|}$$

and therefore $b^*AR(\lambda)a \neq 1$.

Next, we set for any integer n

$$K_n = \{(a, b^*) \in X \times X^* \,|\, \text{for} \quad \mathrm{Re}\,\lambda \geqq n \ (\lambda I - A - ab^*A)^{-1} \text{ exists and } \|(\lambda I - A - ab^*A)^{-1}\| \leqq n\} \ .$$

By hypothesis $A + ab^*A$ is the infinitesimal generator of a C_0-semigroup for sufficiently small $\|a\|$ and $\|b^*\|$ and hence the ε-ball centered in $(0,0)$ in $X \times X^*$ is covered by $\bigcup_{n \in \mathbb{N}} K_n$. Let n be chosen sufficiently large, so that $(0,0) \in K_n$. To begin with, we shall verify that K_n is closed:

Assume that (a_m, b_m^*) is a sequence in K_n such that $a_m \to a$ and $b_m^* \to b^*$ as $m \to \infty$. By the above consideration, we get the estimate

$$|1 - b_m^*AR(\lambda)a_m| \geqq \frac{\|R(\lambda)a_m\|}{n\|a_m\|}$$

and the right hand side converges to $\dfrac{\|R(\lambda)a\|}{n\|a\|} > 0$ as $m \to \infty$.

Now, fix some $y \in X$ and set $x_m = (\lambda I - A - a_m b_m^* A)^{-1} y$, i.e.

$$(\lambda I - A)x_m - a_m b_m^* A x_m = y \ .$$

Hence

$$b_m^* A x_m = (1 - b_m^* AR(\lambda)a_m)^{-1} b_m^* AR(\lambda)y$$

converges to $(1 - b^*AR(\lambda)a)^{-1} b^*AR(\lambda)y$. On the other hand, $x_m = R(\lambda)a_m b_m^* A x_m + R(\lambda)y$ converges to $x := R(\lambda)y + (1 - b^*AR(\lambda)a)^{-1} R(\lambda)ab^*AR(\lambda)y$. From this equation, we infer that x is a solution of $(\lambda I - A - ab^*A)x = y$ and moreover $\|x\| = \lim_{m \to \infty} \|x_m\| \leqq n\|y\|$. If $(\lambda I - A - ab^*A)$ is not one-to-one then there exists some

$z \neq 0$ so that $(\lambda I - A)z = ab^* Az$, i.e.

$$b^* Az = b^* AR(\lambda) ab^* Az \ .$$

As $b^* AR(\lambda)a \neq 1$ this implies that $b^* Az = 0$, and hence, in turn, $(\lambda I - A)z = 0$ which clearly contradicts the assumption that $(\lambda I - A)^{-1}$ exists. Therefore, $(\lambda I - A - ab^* A)$ is one-to-one and $\|(\lambda I - A - ab^* A)^{-1}\| \leqq n$. As this holds for all λ with $\mathrm{Re}\,\lambda \geqq n$, we deduce that $(a, b^*) \in K_n$ as claimed.

Let X^0 be the closure of $D(A^*)$ in X^*. By Baire's Theorem, we infer that for some sufficiently large n $K_n \cap X \times X^0$ contains an open ball in $X \times X^0$. We choose $\tilde{p} \in X$, $\tilde{q}^* \in X^0$ and $\eta > 0$ such that $\|a - \tilde{p}\| \leqq 2\eta$, $\|b^* - q^*\| \leqq 2\eta$ and $b^* \in X^0$ implies $(a, b^*) \in K_n$. Since $D(A)$ is dense in X and $D(A^*)$ is dense in X^0, we may select $p \in D(A)$ with $\|p - \tilde{p}\| \leqq \eta$ and $q^* \in D(A^*)$ with $\|q^* - \tilde{q}^*\| \leqq \eta$ so that $\|a - p\| \leqq \eta$, $\|b^* - q^*\| \leqq \eta$, $b^* \in X^0$ implies $(a, b^*) \in K_n$. Our goal is to derive an estimate for $\|AR(\lambda)\|$:

Suppose that for some $x \in X$ with $\|x\| = 1$ and some λ with $\mathrm{Re}\,\lambda \geqq n$, we have

$$\|AR(\lambda)x\| > \sup_{\mathrm{Re}\,\mu \geqq n} \frac{1}{\eta} \|AR(\mu)p\| + \sup_{\mathrm{Re}\,\mu \geqq n} \left(\frac{1}{\eta} \|R(\mu)^* A^* q^*\| + \frac{1}{\eta^2} |q^* R(\mu) A p| \right) + \frac{1}{\eta^2}$$

and denote the right hand side of this inequality by M.

Note that each supremum is finite since we chose $p \in D(A)$, $q^* \in D(A^*)$ and $R(\mu)$ is bounded for $\mathrm{Re}\,\mu \geqq n$.

Then

$$\begin{aligned} \|AR(\lambda)(p + \eta x)\| &\geqq \eta \|AR(\lambda)x\| - \|AR(\lambda)p\| \\ &> \sup_{\mathrm{Re}\,\mu \geqq n} \left(\frac{1}{\eta} |q^* R(\mu) A p| + \|R(\mu)^* A^* q^*\| \right) + \frac{1}{\eta} \\ &\geqq \frac{1}{\eta} |q^* R(\lambda) A p| + \frac{1}{\eta} |q^* AR(\lambda)(\eta x)| + \frac{1}{\eta} \\ &\geqq \frac{1}{\eta} |q^* AR(\lambda)(p + \eta x) - 1| \ , \end{aligned}$$

i.e.

$$\|AR(\lambda)(p + \eta x)\| > |q^* AR(\lambda)(p + \eta x) - 1| \cdot \frac{1}{\eta} \ .$$

Since

$$\begin{aligned} \|AR(\lambda)(p + \eta x)\| &= \sup_{\tilde{y}^* \in X^*,\ \|\tilde{y}^*\| \leqq 1} \tilde{y}^* AR(\lambda)(p + \eta x) \\ &= \sup_{\tilde{y} \in X^0,\ \|\tilde{y}^*\| \leqq 1} \tilde{y}^* AR(\lambda)(p + \eta x) \ (\text{as } X^0 \text{ is } w^*\text{-dense in } X^*) \end{aligned}$$

there exists a $\tilde{y}^* \in X^0$ with $\|\tilde{y}^*\| \leqq 1$ so that

$$\eta \tilde{y}^* AR(\lambda)(p + \eta x) > |q^* AR(\lambda)(p + \eta x) - 1| \ .$$

Setting

$$y^* := \frac{1 - q^* AR(\lambda)(p+\eta x)}{\tilde{y}^* AR(\lambda)(p+\eta x)} \tilde{y}^*$$

we obtain $\|y^*\| \leqq 1$ and $\eta y^* AR(\lambda)(p+\eta x) = 1 - q^* AR(\lambda)(p+\eta x)$.

Consequently

$$(q^* + \eta y^*) AR(\lambda)(p+\eta x) = 1 \ . \tag{2}$$

On the other hand

$$\|\eta y^*\| \leqq \eta \ , \quad \|\eta x\| \leqq \eta \quad \text{and} \quad y^* \in X^0 \ ,$$

imply that $(p+\eta x, q^* + \eta y^*)$ lies in K_n which, in turn, implies that

$$(q^* + \eta y^*) AR(\lambda)(p+\eta x) \neq 1$$

contradicting (2).

Thus we have proven that $\|AR(\lambda)\| \leqq M$ for $\operatorname{Re}\lambda \geqq n$. For $\lambda = n + \varrho + i\sigma$ with $\varrho \geqq 0$, we have

$$|\lambda - n| \, \|R(\lambda)\| \leqq \|\lambda R(\lambda)\| = \|AR(\lambda) - I\| \leqq M + 1 \ .$$

We want to extend this estimate to the sector

$$\Sigma := \left\{ n + \varrho + i\sigma \,|\, \varrho \geqq -\frac{1}{2(M+1)} |\sigma| \right\} \ .$$

To this end, we rewrite $(n + \varrho + i\sigma - A)x = y$ as $(n + i\sigma - A)x + \varrho x = y$, i.e.

$$x + \varrho R(n+i\sigma)x = R(n+i\sigma)y$$

and assume that $\varrho \leqq 0$.

As $\|\varrho R(n+i\sigma)\| \leqq \frac{|\sigma|}{2(M+1)} \cdot \frac{M+1}{|\sigma|} = \frac{1}{2}$, it follows from the Neumann series that $1 + \varrho R(n+i\sigma)$ is invertible and that

$$\|x\| \leqq 2 \|R(n+i\sigma)y\| \leqq \frac{2(M+1)}{|\sigma|} \|y\|$$

$$\leqq \frac{(1+4(M+1)^2)^{1/2}}{|\lambda - n|} \|y\|$$

since

$$|\lambda - n|^2 = \varrho^2 + \sigma^2 \leqq \left(\frac{1}{4(M+1)^2} + 1 \right) \sigma^2 \ .$$

Therefore, $|\lambda - n| \, \|R(\lambda)\| \leqq (1 + 4(M+1)^2)^{1/2}$ on the whole sector Σ, and hence A is the infinitesimal generator of an analytic semigroup as claimed. □

Acknowledgements. This work was partially supported by "Fonds zur Förderung der wissenschaftlichen Forschung", Austria, S3206. Moreover, the authors thank Profs. H. Amann and P. Hess for stimulating discussions.

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Received September 22, 1987; in revised form November 11, 1987