Appl Math Optim 12:271–282 (1984)

Applied Mathematics and Optimization

Semilinear Equations in $\mathbb{R}^N$ Without Condition at Infinity

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Communicated by D. Kinderlehrer

Abstract. In this paper we establish that some nonlinear elliptic (and parabolic) problems are well posed in all of $\mathbb{R}^N$ without prescribing the behavior at infinity. A typical example is the following: Let $1 < p < \infty$. For every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ there is a unique $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$-\Delta u + |u|^{p-1}u = f(x) \quad \text{on } \mathbb{R}^N.$$

1. Introduction

The purpose of this paper is to point out that some nonlinear elliptic (and parabolic) problems are well-posed in all of $\mathbb{R}^N$ without conditions at infinity. A typical example is the following:

Theorem 1. *Let* $1 < p < \infty$. *For every* $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ *there exists a unique* $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ *satisfying*

$$-\Delta u + |u|^{p-1}u = f(x) \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{1}$$

Moreover, if $f \geq 0$ a.e. *then* $u \geq 0$ a.e.

Remark 1. It was previously known that for every $f \in L^1(\mathbb{R}^N)$ there exists a unique $u \in L^p(\mathbb{R}^N)$ satisfying (1) (see [3], Theorem 5.11). However, we emphasize that in Theorem 1 there is *no limitation on the growth at infinity of the data* f and the solution u is *unique without prescribing its behavior at infinity.*

2. Proof of Theorem 1

Existence

Let $B_R = \{x \in \mathbb{R}^N;\ |x| < R\}$. We start with some local estimate:

Lemma 1. *Let $R < R'$ and assume $u \in L^p_{\mathrm{loc}}(B_{R'})$ satisfies*

$$-\Delta u + |u|^{p-1}u = f(x) \quad \text{in } \mathscr{D}'(B_{R'}) \tag{2}$$

with $f \in L^1(B_{R'})$. Then

$$\int_{B_R} |u|^p \leqslant C\left(1 + \int_{B_{R'}} |f|\right) \tag{3}$$

where C depends only on p, R and R'.

Remark 2. The conclusion of Lemma 1 is a rather unusual *localization property*. Indeed, let Ω and Ω' be bounded open sets in $\mathbb{R}^N$ such that $\overline{\Omega} \cap \overline{\Omega}' = \varnothing$ and let u be the solution of (1). On the one hand the values of f in Ω' "affect" the solution u in Ω: for example, if $f > 0$ in Ω' and $f \equiv 0$ outside Ω' it follows from the strong maximum principle that $u > 0$ in Ω. On the other hand the values of f in Ω' affect only "mildly" u in Ω: in view of (3) $u|_\Omega$ may be estimated independently of $f|_{\Omega'}$; even if $f \to \infty$ on Ω', $\int_\Omega |u|^p$ still remains bounded.

Proof of Lemma 1. We use a device introduced by P. Baras and M. Pierre [2]. By Kato's inequality (see [10]) and (2) we have

$$-\Delta|u| + |u|^p \leqslant |f| \quad \text{in } \mathscr{D}'(B_{R'}). \tag{4}$$

Let $\zeta \in \mathscr{D}(B_{R'})$ be such that $0 \leqslant \zeta \leqslant 1$ and $\zeta \equiv 1$ on B_R. Multiplying (4) through by ζ^α where α is an integer, and integrating we find

$$\int |u|^p \zeta^\alpha \leqslant \int |f| + C\int |u| \zeta^{\alpha-2} \leqslant \int |f| + C\int |u| \zeta^{\alpha/p}, \tag{5}$$

provided $\alpha - 2 \geqslant \alpha/p$, i.e., $\alpha \geqslant 2p/(p-1)$ and we fix any such α. The conclusion of Lemma 1 follows easily from (5).

Proof of Theorem 1. Existence

Let

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| < n \\ 0 & \text{if } |x| \geqslant n. \end{cases}$$

Let $u_n \in L^p(\mathbb{R}^N)$ be the unique solution of

$$- \Delta u_n + |u_n|^{p-1}u_n = f_n \quad \text{in } \mathscr{D}'(\mathbb{R}^N) \tag{6}$$

(see [3], Theorem 5.11).
We deduce from Lemma 1 that there is a constant C such that

$$\|u_n\|_{L^p(B_R)} \leqslant C$$

where C depends only on p, R and f, and thus we also have

$$\|\Delta u_n\|_{L^1(B_R)} \leqslant C.$$

It follows that (for some subsequence still denoted by u_n) we have

$$u_n \to u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N)$$
$$u_n \to u \quad \text{a.e. on } \mathbb{R}^N.$$

We claim that

$$|u_n|^{p-1}u_n \to |u|^{p-1}u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N).$$

It suffices to verify that $|u_n|^{p-1}u_n$ is a Cauchy sequence in $L^1(B_R)$ for any R. By Kato's inequality and (6) we have

$$- \Delta|u_n - u_m| + \big||u_n|^{p-1}u_n - |u_m|^{p-1}u_m\big| \leqslant |f_n - f_m|.$$

Let $\zeta \in \mathscr{D}(\mathbb{R}^N)$ be such that $0 \leqslant \zeta \leqslant 1$ and $\zeta = 1$ on B_R. We have

$$\int \big||u_n|^{p-1}u_n - |u_m|^{p-1}u_m\big|\zeta \leqslant \int |f_n - f_m|\zeta + \int |u_n - u_m|\,\Delta\zeta$$

and the *RHS* tend to zero as $m, n \to \infty$. Passing to the limit in (6) we obtain (1).

Uniqueness

We shall need the following:

Lemma 2. *Assume $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ satisfies*

$$- \Delta u + |u|^{p-1}u \leqslant 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{7}$$

Then $u \leqslant 0$ a.e. on $\mathbb{R}^N$.,

Remark 3. Lemma 2 is closely related to the results of J. Keller [11] and R. Osserman [13] (see also the earlier works quoted in these papers).

Proof. We use a comparison function of the same type as in Osserman [13] (see also C. Loewner and L. Nirenberg [12]). Set

$$U(x) = \frac{CR^{\alpha}}{(R^2 - |x|^2)^{\alpha}} \quad \text{in } B_R$$

where $R > 0$, $\alpha = 2/(p-1)$ and $C^{p-1} = 2\alpha \max\{N, \alpha + 1\}$. A direct computation shows that

$$-\Delta U + U^p \geqslant 0 \quad \text{in } B_R \tag{8}$$

and thus

$$-\Delta(u - U) + |u|^{p-1}u - U^p \leqslant 0 \quad \text{in } \mathscr{D}'(B_R). \tag{9}$$

Using a variant of Kato's inequality (see Lemma A.1 in the Appendix) we deduce from (9) that

$$-\Delta(u-U)^+ + (|u|^{p-1}u - U^p)\operatorname{sign}^+(u-U) \leqslant 0 \quad \text{in } \mathscr{D}'(B_R) \tag{10}$$

and therefore

$$-\Delta(u-U)^+ \leqslant 0 \quad \text{in } \mathscr{D}'(B_R). \tag{11}$$

From Lemma A.1 and (7) we deduce that

$$-\Delta u^+ + (u^+)^p \leqslant 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N)$$

and therefore

$$-\Delta u^+ \leqslant 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N),$$

i.e., u^+ is subharmonic and in particular $u^+ \in L^{\infty}_{\text{loc}}(\mathbb{R}^N)$. It follows that for some $\delta > 0$ we have

$$(u-U)^+ = 0 \quad \text{for } R - \delta < |x| < R \tag{12}$$

(since $U(x) \to +\infty$ as $|x| \to R$, $x \in B_R$). Combining (11) and (12) we obtain that $(u-U)^+ = 0$ a.e. on B_R, i.e., $u \leqslant U$ a.e. on B_R.

Keeping x fixed and letting $R \to \infty$ we see that $u \leqslant 0$ a.e. on $\mathbb{R}^N$.

Proof of Theorem 1. Uniqueness

Let u_1 and u_2 be two solutions of (1) and let $u = u_1 - u_2$. By Kato's inequality we have

$$-\Delta|u| + \big||u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\big| \leqslant 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{13}$$

On the other hand, there is a constant $\delta > 0$—depending only on p— such that

$$\big||a|^{p-1}a - |b|^{p-1}b\big| \geqslant \delta|a - b|^p \quad \forall a, b \in \mathbb{R}. \tag{14}$$

From (13) and (14) we deduce that

$$-\Delta|u| + \delta|u|^p \leqslant 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

Using Lemma 2 we conclude that $u = 0$.

3. Miscellaneous Remarks and Generalizations

A. Monotone Nonlinearities

The proof of Theorem 1 extends easily to the case where $|u|^{p-1}u$ is replaced by a more general function $g(u)$. Assume $g: \mathbb{R} \to \mathbb{R}$ is a C^1 function such that

$$g'(u) \geqslant a|u|^{p-1} \quad \forall u \in \mathbb{R},$$

for some constants $a > 0$ and $1 < p < \infty$ (for example, $g(u) = \sinh u$, etc. ...).

Theorem 1′. *For every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ there exists a unique $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ with $g(u) \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfying*

$$-\Delta u + g(u) = f(x) \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{16}$$

B. Nonmonotone g's

Let $g(x, u): \mathbb{R}^N \times \mathbb{R}$ be measurable in x and continuous in u. We assume that:

$$g(x, u)\,\text{sign}\,u \geqslant a|u|^p - \omega(x) \quad \text{for a.e. } x \in \mathbb{R}^N, \quad \text{for all } u \in \mathbb{R} \tag{17}$$

where $\omega \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $a > 0$, $1 < p < \infty$ and also

$$h_M(x) = \sup_{|u| \leqslant M} |g(x, u)| \in L^1_{\text{loc}}(\mathbb{R}^N) \quad \text{for all } M > 0. \tag{18}$$

Theorem 2. *There exists $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ such that $g(\cdot, u) \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfying*

$$-\Delta u + g(x, u) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{19}$$

Sketch of the Proof. First we consider the case of a smooth *bounded* domain $\Omega \subset \mathbb{R}^N$.

Claim. There exists $u \in W_0^{1,1}(\Omega)$ such that $g(\cdot, u) \in L^1(\Omega)$, satisfying

$$-\Delta u + g(x, u) = 0 \quad \text{on } \Omega. \tag{20}$$

This type of result is closely related—but not quite contained in [6]. Here it suffices to assume (17) with $a = 0$.

For $r \in \mathbb{R}$ and $n \in \mathbb{N}$ we set

$$\tau_n r = \begin{cases} r & \text{if } |r| \leqslant n \\ n & \text{if } r \geqslant n. \\ -n & \text{if } r \leqslant -n. \end{cases}$$

By the Schauder fixed point theorem there exists $u_n \in W_0^{1,1}(\Omega)$ satisfying

$$-\Delta u_n + g(x, \tau_n u_n) = 0 \quad \text{on} \quad \Omega. \tag{21}$$

Using the fact that $-\int_\Omega \Delta u_n \operatorname{sign} u_n \geqslant 0$ we find

$$\int_\Omega |g(x, \tau_n u_n)| \leqslant 2 \int_\Omega |\omega|.$$

Therefore

$$\int_\Omega |\Delta u_n| \leqslant 2 \int_\Omega |\omega|. \tag{22}$$

After extracting a subsequence we may assume that

$$\begin{aligned} u_n &\to u \quad \text{in } W_0^{1,1}(\Omega) \\ u_n &\to u \quad \text{a.e.} \\ g(x, \tau_n u_n) &\to g(x, u) \quad \text{a.e.} \end{aligned}$$

To show that $g(x, \tau_n u_n) \to g(x, u)$ in $L^1(\Omega)$ we use a new device introduced in [8] by Th. Gallouet and J. M. Morel (with an observation of L. Boccardo).

Set

$$p_M(r) = \begin{cases} 1 & \text{if } r > M \\ 0 & \text{if } -M \leqslant r \leqslant M \\ -1 & \text{if } r < -M \end{cases}$$

where $r \in \mathbb{R}$ and $M > 0$. It is well known that

$$-\int_\Omega \Delta u \cdot p_M(u) \geqslant 0 \quad \forall u \in W_0^{1,1}(\Omega),\ \Delta u \in L^1(\Omega).$$

Therefore we have

$$\int_\Omega g(x,\tau_n u_n)\,p_M(u_n) \leqslant 0,$$

That is,

$$\int_{|u_n|\geqslant M} g(x,\tau_n u_n)\,\mathrm{sign}(u_n) \leqslant 0$$

and hence

$$\int_{|u_n|>M} |g(x,\tau_n u_n)| \leqslant 2\int_{|u_n|>M} |\omega|. \tag{23}$$

From (22) we see that $\|u_n\|_{L^1} \leqslant C$ and thus M meas $[|u_n|>M] \leqslant C$.

Given $\varepsilon>0$ we may *fix* M large enough so that $2\int_{|u_n|>M}|\omega|<\varepsilon$. Next, for any measurable $A\subset\Omega$, we have

$$\begin{aligned}\int_A |g(x,\tau_n u_n)| &\leqslant \int_{\substack{A\\ |u_n|\leqslant M}} |g(x,\tau_n u_n)| + \int_{|u_n|>M} |g(x,\tau_n u_n)| \\ &\leqslant \int_A h_M(x) + \varepsilon \leqslant 2\varepsilon\end{aligned}$$

provided meas $A<\delta$ and δ is small enough. In other words, we have established that

$$\forall\varepsilon>0 \qquad \exists\delta>0 \qquad \text{s.t.} \int_A |g(x,\tau_n u_n)| < 2\varepsilon \quad \text{when meas } A<\delta.$$

We conclude that $g(x,\tau_n u_n)\to g(x,u)$ in $L^1(\Omega)$.

We turn now to problem (19). For each n let

$$\Omega_n = \{x\in\mathbb{R}^N;\ |x|<n\}.$$

By the previous step there exists $u_n\in W_0^{1,1}(\Omega_n)$ such that $g(\cdot,u_n)\in L^1(\Omega_n)$ satisfying

$$-\Delta u_n + g(x,u_n) = 0 \quad \text{on } \Omega_n. \tag{24}$$

From Kato's inequality and (24) we obtain

$$-\Delta|u_n| + g(x,u_n)\mathrm{sign}\,u_n \leqslant 0 \quad \text{in } \mathscr{D}'(\Omega_n).$$

And therefore we also have

$$-\Delta|u_n| + a|u_n|^p \leqslant \omega \quad \text{in } \mathscr{D}'(\Omega_n) \tag{25}$$

$$-\Delta|u_n| + |g(x,u_n)| \leqslant 2|\omega| \quad \text{in } \mathscr{D}'(\Omega_n). \tag{26}$$

Using the same device as in the proof of Theorem 1 we deduce from (25) that

$$u_n \text{ is bounded in } L^p_{\text{loc}}(\mathbb{R}^N).$$

It follows from (26) that

$$g(\cdot, u_n) \text{ is bounded in } L^1_{\text{loc}}(\mathbb{R}^N)$$

and thus

$$\Delta u_n \text{ is bounded in } L^1_{\text{loc}}(\mathbb{R}^N).$$

Hence we may assume that

$$\begin{aligned} u_n &\to u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N) \\ u_n &\to u \quad \text{a.e. on } \mathbb{R}^N \\ g(x,u_n) &\to g(x,u) \quad \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

Finally we prove that $g(x,u_n) \to g(x,u)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$. By a variant of Kato's inequality (see Lemma A.2) we have

$$-\Delta P_M(u) + g(x,u_n)p_M(u_n) \leqslant 0 \quad \text{in } \mathscr{D}'(\Omega_n)$$

where

$$P_M(t) = \int_0^t p_M(s)\,ds.$$

Therefore we have

$$\int_{|u_n|>M} |g(x,u_n)|\zeta \leqslant 2\int_{|u_n|>M} |\omega|\zeta + \int_{|u_n|>M} |u_n|\,\Delta\zeta \quad \forall \zeta \in \mathscr{D}_+(\Omega_n).$$

It follows easily that $g(x,u_n)$ is equi-integrable on bounded sets of $\mathbb{R}^N$ and thus $g(x,u_N) \to g(x,u)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$.

C. Measures or More General Distributions as Right Hand Side Data

Let T be a distribution of the form $T = f + \Delta\phi$ where $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\phi \in L^p_{\text{loc}}(\mathbb{R}^N)$. Then the problem

$$-\Delta u + |u|^{p-1}u = T \quad \text{in } \mathscr{D}'(\mathbb{R}^N) \tag{27}$$

has a unique solution $u \in L^p_{\text{loc}}(\mathbb{R}^N)$.

Indeed it suffices to consider the new unknown $v = u + \phi$ and to apply the result of Section B to v (see also [8] for similar questions on bounded domains).

Suppose now that T is a *measure* on $\mathbb{R}^N$ (not necessarily a bounded measure). Suppose $1 < p < N/(N-2)$ (no restriction when $N = 1, 2$). Then there exists a unique solution $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for (27). Related questions for bounded measures are considered in [5] and [8].

D. *Nonlinearities with Growth Close to Linear*

Suppose $g: \mathbb{R} \to \mathbb{R}$ is continuous and $g(u)$ behaves like $u|\log u|^k$ as $|u| \to \infty$ with $k > 2$. Then for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ there exists $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ with $g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfying

$$-\Delta u + g(u) = f \quad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

As before, we use Kato's inequality to find

$$-\Delta|u| + g(u)\operatorname{sign} u \leqslant |f|.$$

We multiply (28) through by $\eta = e^{-1/\zeta^\beta}$ where $\beta > 2/(k-2)$. Then we estimate $\int |u|\,|\Delta\eta|$ with the help of Young's inequality. Recently Gallouet and Morel have proved the following result. Suppose $g: \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing, odd, convex, and $\int_1^\infty [G(x)]^{-1/2} dx < \infty$ where G is a primitive of g. Then for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ there exists a unique function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ with $g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfying $-\Delta u + g(u) = f$ in $\mathscr{D}'(\mathbb{R}^N)$.

E. *Unbounded Domains*

Let $\Omega \subset \mathbb{R}^N$ be any domain (bounded or unbounded) with smooth boundary. Using the same principles as in the proof of Theorem 1 one can show that for every $f \in L^1_{\mathrm{loc}}(\bar{\Omega})$ and $\phi \in L^1_{\mathrm{loc}}(\partial\Omega)$ there exists a unique $u \in L^p_{\mathrm{loc}}(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta u + |u|^{p-1}u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega. \end{cases}$$

where $1 < p < \infty$ and the boundary condition is understood in some appropriate sense.

F. *Local Regularity*

Let $\Omega \subset \mathbb{R}^N$ be any domain. Let $g: \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing function.

Theorem 3. *Suppose $u \in L^1_{\mathrm{loc}}(\Omega)$ is such that $g(u) \in L^1_{\mathrm{loc}}(\Omega)$ and satisfies*

$$-\Delta u + g(u) = f(x) \quad \text{in } \mathscr{D}'(\Omega) \tag{29}$$

where $f \in L^q_{\mathrm{loc}}(\Omega)$ and $1 < q < \infty$.

Then $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$.

Proof. We may assume that $g(0) = 0$. We have

$$-\Delta|u| + g(u)\operatorname{sign} u \leqslant |f| \quad \text{in } \mathscr{D}'(\Omega)$$

and thus

$$-\Delta|u| \leqslant |f| \quad \text{in } \mathscr{D}'(\Omega).$$

It follows that $u \in L^q_{\mathrm{loc}}(\Omega)$. Set

$$g_n(r) = g(\tau_n r) \quad \text{and} \quad P_n(r) = \operatorname{sign} r \int_0^r |g_n(s)|^{q-1}\, ds$$

so that

$$|P_n(r)| \leqslant |r|\, |g_n(r)|^{q-1} \quad \forall r \in \mathbb{R}.$$

By Lemma A.2 and (29) we have

$$\Delta P_n(u) \geqslant |g_n(u)|^{q-1} \operatorname{sign} u [g(u) - f] \geqslant |g_n(u)|^q - |f|\, |g_n(u)|^{q-1}. \tag{30}$$

Let $\zeta \in \mathscr{D}(\Omega)$ with $0 \leqslant \zeta \leqslant 1$; from (30) we see that

$$\begin{aligned} \int |g_n(u)|^q \zeta^\alpha &\leqslant C \int |P_n(u)| \zeta^{\alpha-2} + \int |f|\, |g_n(u)|^{q-1} \zeta^\alpha \\ &\leqslant C \int |u|\, |g_n(u)|^{q-1} \zeta^{\alpha-2} + \int |f|\, |g_n(u)|^{q-1} \zeta^\alpha \end{aligned}$$

where C is independent of u.

Fix any integer $\alpha \geqslant 2q$; by Hölder's inequality we have

$$\int |g_n(u)|^q \zeta^\alpha \leqslant C \int_{\operatorname{supp} \zeta} (|u|^q + |f|^q).$$

As $n \to \infty$ we see that $g(u) \in L^q_{\mathrm{loc}}(\Omega)$.

G. *Parabolic Equations*

Consider the problem

$$\begin{cases} u_t - \Delta u + |u|^{p-1} u = 0 & \text{on } \mathbb{R}^N \times (0, +\infty) \text{ with } 1 < p < \infty \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^N. \end{cases} \tag{31}$$

Using the same principles as in the proof of Theorem 1 one can show that for every $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ there is a unique function $u \in C^2(\mathbb{R}^N \times (0, +\infty)) \cap C([0, +\infty); L^1_{\mathrm{loc}}(\mathbb{R}^N))$ satisfying (31).

Results of the same nature for the problem

$$\begin{cases} u_t - \Delta\left(|u|^{m-1}u\right) = 0 & \text{on } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^N \end{cases}$$

have been obtained by M. Herrero–M. Pierre [9] when $0 < m < 1$. When $m > 1$ the situation is totally different; see [1, 4, 7].

Appendix: Some Variants of Kato's Inequality

Let $\Omega \subset \mathbb{R}^N$ be any open set.

Lemma A.1. *Let $u \in L^1_{\text{loc}}(\Omega)$ and $f \in L^1_{\text{loc}}(\Omega)$ be such that*

$$\Delta u \geqslant f \quad \text{in } \mathscr{D}'(\Omega).$$

Then

$$\Delta u^+ \geqslant f \operatorname{sign}^+ u \quad \text{in } \mathscr{D}'(\Omega).$$

Lemma A.2. *Let $p : \mathbb{R} \to \mathbb{R}$ be a monotone, nondecreasing function such that p is continuous except at a finite number of jumps and $p(\mathbb{R})$ is bounded.*

Let $P(r) = \int_0^r p(s)\,ds$ and let $u \in L^1_{loc}(\Omega)$ with $\Delta u \in L^1_{\text{loc}}(\Omega)$.

Then

$$\Delta P(u) \geqslant (\Delta u) p(u) \quad \text{in } \mathscr{D}'(\Omega).$$

The proofs are easy modifications of Kato's original argument in [10], and we shall omit them.

Acknowledgments

This paper was written during a visit at the University of Minnesota. I thank the Mathematics Department and the Institute for Mathematics and its Applications for their invitation and hospitality. I thank D. Aronson, C. Kenig, and H. Weinberger for valuable conversations.

References

1. Aronson D, Caffarelli L (in press) The initial trace of a solution of the porous medium equation. Trans. Am Math Soc
2. Baras P, Pierre M (1982) Singularités éliminables d'équations elliptiques semilinéaires, C.R.A.S. Paris 295:519–522; see also Ann Inst Fourier to appear
3. Bénilan Ph, Brezis H, and Crandall M (1975) A semilinear equation in $L^1(\mathbb{R}^N)$, Ann Sc Norm Sup Pisa 2:523–555
4. Bénilan Ph, Crandall M, and Pierre M Solutions of the porous medium equation under optimal conditions on initial values. Indiana Univ Math J 33:51–87

5. Brezis H (1980) Some variational problems of the Thomas-Fermi type. In: Cottle, Giannessi, and Lions (eds) Variational Inequalities and Complementarity Problems. Wiley, New York: 53–73
6. Brezis H, Strauss W (1973) Semilinear second-order elliptic equations in L^1. J Math Soc Japan 25:565–590
7. Dahlberg B, Kenig C (in press) Non-negative solutions of the porous medium equation
8. Gallouet Th, Morel JM (in press) Resolution of a semilinear equation in L^1
9. Herrero M, Pierre M (in press) Some results for the Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$
10. Kato T (1972) Schrödinger operators with singular potentials. Israel J Math 13:135–148
11. Keller JB (1957) On solutions of $\Delta u = f(u)$. Comm Pure Appl Math 10:503–510
12. Loewner C, Nirenberg L (1974) Partial differential equations invariant under conformal or projective transformations. In: Contributions to Analysis. Acad. Press, New York: 245–272
13. Osserman R (1957) On the inequality $\Delta u \geqslant f(u)$. Pacific J Math 7:1641–1647

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