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Elastic Plastic Deformation*

Robert Hardt and David Kinderlehrer

School of Mathematics, University of Minnesota, 206 Church St. S.E., Minneapolis, MN 55455

Abstract. The equilibrium configuration of an elastic perfectly plastic body may be described by its stress or its strain. By use of a first variation formula, a description of the strain tensor, not necessarily unique, is obtained from the stress, which is unique.

Most aspects of this work extend to more general elastic plastic models, in particular ones which lack convexity.

1. Introduction

The equilibrium configuration of an elastic plastic body may be found by minimizing potential energy among appropriate displacements or by resorting to a generalized principle of complementary energy, which provides the stress tensor of the deformed body. The displacement found as the solution of the minimization problem need not be unique and its strain tensor may only be a measure. The stress tensor, obtained by the duality method, is unique, it is pertinent to note, and determines the regions where the body is elastic and plastic. In fact, it depends only on the absolutely continuous part of the strain measure of a corresponding displacement.

This situation suggests inquiring of properties of the stress tensor which lead to information about the displacement, the scope of the present work. It is divided into four parts:

a useful first variation formula;
duality and complementary energy;
some regularity properties of the solution; and
existence of elastic and plastic states and examples illustrating transition to fracture.

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As elastic plastic body occupying a reference configuration $\Omega \subset \mathbb{R}^n$ (with Lipschitz boundary $\partial\Omega$) undergoes deformation, a material point x is transformed to $y = x + u(x)$, with respect to the application of a system of forces. The behavior of the displacement $u(x)$, $x \in \Omega$, is governed by the Von Mises yield criterion and Hencky's Law, (1.1) below, which define a static theory of perfect plasticity [10, 21].

Let

$$\varphi(t) = \begin{cases} \frac{1}{2}t^2 & \text{for } |t| < 1 \\ |t| - \frac{1}{2} & \text{for } |t| \geq 1 \end{cases}, \qquad t \in \mathbb{R}.$$

The potential energy of a smooth virtual displacement $v = (v^1, \ldots, v^n)$ of $\Omega \subset \mathbb{R}^n$ is

$$I(v) = \int_\Omega \varphi(|\varepsilon^D(v)|)\, dx + \frac{\varkappa}{2} \int_\Omega (\operatorname{div} v)^2\, dx - \langle T, v \rangle \tag{1.1}$$

where the distribution of assigned body and surface forces is given by

$$\langle T, \zeta \rangle = \int_\Omega f_i \zeta^i\, dx + \int_{\partial\Omega} g_i \zeta^i\, dS.$$

Above we have used the notations

$$\varepsilon(v) = \tfrac{1}{2}(Dv + Dv^T), \qquad Dv = \left(\frac{\partial v^i}{\partial x_j} \right),$$

for the strain tensor of v,

$$\varepsilon^D(v) = \varepsilon(v) - \frac{1}{n} \operatorname{tr} \varepsilon(v) \mathbb{1} = \varepsilon(v) - \frac{1}{n} \operatorname{div} v \mathbb{1}, \tag{1.2}$$

$$\mathbb{1} = n \times n \text{ identity matrix.}$$

The constant $\varkappa > 0$ is a normalized modulus of compression. For a tensor ψ,

$$\psi^D = \psi - \frac{1}{n} \operatorname{tr} \psi \mathbb{1}$$

is called its "deviator."

Since φ grows only linearly as $|t| \to \infty$, the definition of I must be extended to account for displacements whose strains are merely measures in the same way that the nonparametric area functional must be extended to functions of bounded variation. This has led to the introduction of the spaces ([17, 23, 27, 28])

$$\begin{aligned} BD(\Omega) &= \{v=(v^1,\ldots,v^n)\in L^1(\Omega): \varepsilon_{ij}(v) \text{ is a bounded measure, } 1\le i,\, j\le n\} \\ P(\Omega) &= \{v\in BD(\Omega): \operatorname{div} v\in L^2(\Omega)\} \\ P_0(\Omega) &= \{v\in P(\Omega): v=0 \text{ on } \partial\Omega\} \end{aligned}$$

whose properties are discussed in the works cited above. Some useful ones will be recalled in the next section, but note that $BD(\Omega)\subset L^{n/n-1}(\Omega)$, [27].

For $v\in P(\Omega)$ the matrix-valued measures (tensor-valued measures) $\varepsilon(v)$ and $\varepsilon^D(v)$ may be written

$$\begin{aligned} \varepsilon(v) &= \varepsilon(v)^a\,dx + \varepsilon(v)^s, \qquad \varepsilon(v)^s \perp dx, \\ \varepsilon^D(v) &= \varepsilon^D(v)^a\,dx + \varepsilon^D(v)^s, \qquad \varepsilon^D(v)^s \perp dx. \end{aligned}$$

Since $\operatorname{div} v\in L^2(\Omega)$ gives rise to an absolutely continuous measure, $\varepsilon^D(v)^s=\varepsilon(v)^s$ and

$$\varepsilon^D(v) = \varepsilon^D(v)^a\,dx + \varepsilon(v)^s. \tag{1.3}$$

As a shorthand we define, for $v\in P(\Omega)$,

$$\int_\Omega \varphi(\varepsilon^D(v)) = \int_\Omega \varphi(|\varepsilon^D(v)^a|)\,dx + \int_\Omega |\varepsilon^D(v)^s| \tag{1.4}$$

and

$$I(v) = \int_\Omega \varphi(\varepsilon^D(v)) + \frac{\varkappa}{2}\int_\Omega (\operatorname{div} v)^2\,dx - \langle T, v\rangle, \qquad v\in P(\Omega). \tag{1.5}$$

As mentioned earlier, solutions of an elastic plastic problem may be sought by minimizing (1.5) in a subspace of $P(\Omega)$ satisfying desired side conditions.

Concern here is limited primarily to local solutions. Assume that

$$\langle T,\zeta\rangle = \int_\Omega f\cdot\zeta\,dx, \qquad \zeta\in L^1(\Omega),$$

where

$$f = (f_1,\ldots,f_n), \qquad f_i\in L^\infty(\Omega) \tag{1.6}$$

and I is given by (1.5). An element $u\in P(\Omega)$ is a *local minimum* (in Ω) if

$$I(u)\le I(u+\zeta) \qquad \textit{for } \zeta\in P_0(\Omega). \tag{1.7}$$

The hypothesis (1.6) is not essential and is imposed for technical simplicity.

The stress tensor associated to $v \in P(\Omega)$ is defined by

$$\sigma(v) = \sigma^D(v) + \varkappa \operatorname{div} v \mathbb{1}$$

$$\sigma^D(v) = \varphi_p(|\varepsilon^D(v)^a|) = \begin{cases} \varepsilon^D(v)^a & \text{for } |\varepsilon^D(v)^a| < 1 \\ \dfrac{\varepsilon^D(v)^a}{|\varepsilon^D(v)^a|} & \text{for } |\varepsilon^D(v)^a| \geq 1. \end{cases} \tag{1.8}$$

For a local minimum u the notation (1.8) is abbreviated by writing

$$\sigma = \sigma(u) \quad \text{and} \quad \sigma^D = \sigma^D(u).$$

If u is a smooth local minimum, then σ is formally the solution of the equilibrium equations

$$-\operatorname{div} \sigma = f \qquad \text{in } \Omega, \tag{1.9}$$

where, for a tensor τ with rows $\tau_1, \ldots, \tau_n$, $\operatorname{div} \tau = (\operatorname{div} \tau_1, \ldots, \operatorname{div} \tau_n)$ is an $\mathbb{R}^n$ valued function or distribution, as the case may be.

The elastic set

$$E = \{x \in \Omega : |\sigma^D| < 1\}$$

and the plastic set

$$\Omega - E = \{x \in \Omega : |\sigma^D| \geq 1\}$$

are defined to within an n-dimensional Lebesgue null set. They depend only on the state of stress σ and not on the particular displacement u.

The traction problem for elastic plastic deformation will play a useful role in the duality theory and the various examples. Here the admissible displacements constitute the closed subspace $P_e(\Omega)$ of functions $v \in P(\Omega)$ satisfying

$$\int_\Omega v^i \, dx = \int_\Omega \begin{vmatrix} v^i & v^j \\ x_i & x_j \end{vmatrix} dx = 0, \qquad 1 \leq i, j \leq n. \tag{1.10}$$

Let T be an equilibrated force distribution, namely,

$$\langle T, \zeta \rangle = \int_\Omega f \cdot \zeta \, dx + \int_{\partial\Omega} g \cdot \zeta \, dS, \qquad \zeta \in P(\Omega),$$

obeys

$$\langle T, \gamma \rangle = 0 \qquad \text{whenever } \gamma(x) = c + Bx, \qquad c \in \mathbb{R}^n, \quad B + B^T = 0.$$

One seeks the solution of

Problem 1.1. Find $u \in P_e(\Omega): I(u) = \inf_{P_e(\Omega)} I(v)$.

The discussion so far has concerned only isotropic homogeneous bodies, but the yield criterion and Hencky's Law may be adapted to other materials as well. For example, the functional

$$I(v) = \int_\Omega \varphi(|\varepsilon^D(v)|, x)\, dx + \frac{\varkappa}{2}\int_\Omega (\operatorname{div} v)^2\, dx - \langle T, v\rangle$$

with

$$\varphi(p, x) = \alpha(x)^2 \varphi\left(\frac{1}{\alpha(x)} p\right), \qquad \alpha(x) \geq 1,$$

corresponds to an inhomogeneous isotropic body.

1.1. Variety of Solutions

The variety of solutions displayed by a one dimensional problem offers some indication of the difficulties which lie ahead. Let $\Omega = (0,1) \subset \mathbb{R}$. For $v \in BV(\Omega)$ write the gradient of v, a measure, as

$$Dv = v' dt + D^s v \qquad \text{where } dt \perp D^s v. \tag{1.11}$$

Set

$$g_\delta = \begin{cases} -\delta & t = 0 \\ \delta & t = 1 \end{cases}$$

and consider the functional

$$\begin{aligned} F(v) &= \int_\Omega \varphi(Dv) - \int_{\partial\Omega} gv\, dS \\ &= \int_\Omega \varphi(v')\, dt + \int_\Omega |D^s v| - \delta(v(1) - v(0)) \end{aligned} \tag{1.12}$$

defined for $v \in BV_e(\Omega)$,

$$BV_e(\Omega) = \left\{ v \in BV(\Omega) : \int_\Omega v\, dt = 0 \right\}.$$

Consider

Problem 1.2. To find $u \in BV_e(\Omega): F(u) = \inf_{BV_e(\Omega)} F(v)$.

Clearly for $|\delta| < 1$,

$$u_\delta(t) = \delta(t - \tfrac{1}{2}) \qquad t \in \Omega$$

is the unique solution of Prob. 1.2. In this case the "stress tensor"

$$\sigma = \varphi_p(u'_\delta) = \delta \quad \text{and} \quad F(u_\delta) = -\tfrac{1}{2}\delta^2.$$

As $\delta \to 1$, $u_\delta(t) \to u_1(t) = t - \frac{1}{2}$, witness that the thin rod experiences a transition from a pure elastic state to a pure plastic state as the loading exceeds certain acceptably safe limits. It turns out that $F(u_1) = -\frac{1}{2}$ is the minimum value of $F(v)$ when $\delta = 1$, a property to be clarified later. Since

$$F(w) = -\tfrac{1}{2}$$

for

$$w(t) = \begin{cases} t - \frac{1}{2} - c & 0 \le t < a \\ t - \frac{1}{2} + c & a \le t < 1 \end{cases}, \qquad c \neq 0,$$

w is also a solution. This w exhibits "fracture" or "slippage" at $t = a$.

More generally, let $w \in BV_e(\Omega)$ be monotone and satisfy $w'(t) \ge 1$ a.e. in Ω, cf. (1.11). Then

$$\begin{aligned} F(w) &= \int_\Omega \varphi(w')\,dt + \int_\Omega |D^s w| - (w(1) - w(0)) \\ &= \int_\Omega Dw - (w(1) - w(0)) - \tfrac{1}{2} \\ &= -\tfrac{1}{2}. \end{aligned}$$

In other words, *any* monotone $w(t)$, with average zero, whose absolutely continuous part satisfies $w' \ge 1$ is a solution of Prob. 1.2.

Likewise note that any $w \in BV(\Omega)$ with $w' = \delta$ if $|\delta| < 1$ or $w' \ge 1$ if $\delta = 1$ is a virtual displacement with associated stress $\sigma = \delta$ without necessarily minimizing the functional F. Thus a "weak solution," that is, a w which has the property

$$-\frac{d}{dt}\sigma(w) = 0 \quad \text{in } \Omega$$

$$\sigma = g_\delta \quad \text{on } \partial\Omega,$$

or

$$\int_\Omega \sigma(w)\zeta'\,dt = \delta(\zeta(1) - \zeta(0)), \qquad \zeta \in C^\infty(\Omega),$$

need not be a minimum. This is in marked contrast with the situation usually encountered in the theory of differential equations.

We conclude these introductory remarks by discussing a simple inhomogeneous rod initially occupying $\Omega = (-1,1)$. Let

$$\Phi(p,t) = \begin{cases} k^2\varphi\left(\frac{p}{k}\right) & -1 < t < 0 \\ \varphi(p) & 0 \le t < 1 \end{cases} \tag{1.13}$$

for a fixed $k > 1$. So transition to plastic will occur at $|p| = k$ in the region $(-1,0) \subset \Omega$. Consider the functional

$$\begin{aligned} F(v) &= \int_{-1}^{1} \Phi(Dv,t) - \lambda(v(1) - v(-1)) \\ &= \int_{-1}^{1} \Phi(v',t)\,dt + k\int_{-1}^{0} |D^s v| + \int_{0}^{1} |D^s v| - \lambda(v(1) - v(-1)), \end{aligned}$$

an energy functional for a "traction problem."

For $\lambda = 1$ it is not difficult to verify that F achieves its minimum at $u_0(t) = t$, $F(u_0) = -1$. From this other solutions may be constructed, as before, by setting

$$u(t) = \begin{cases} t & -1 < t < 0 \\ w(t) & 0 \le t < 1 \end{cases} \tag{1.14}$$

where $w \in BV(\Omega)$ is monotone, $w(0) = 0$, and $w' \ge 1$ for $t \ge 0$, cf. (1.11), and

$$\int_0^1 w\,dt = 1.$$

The "stress" associated to u of (1.17) in $(-1,0)$ is

$$\begin{aligned} \sigma(u) &= \begin{cases} u' & |u'| < k \\ u\dfrac{u'}{|u'|} & |u'| \ge k \end{cases}, \qquad -1 < t < 0 \\ &= 1, \qquad\qquad\qquad\qquad -1 < t < 0 \end{aligned}$$

so the region $(-1,0)$ is elastic whereas $[0,1]$ is plastic.

Thus both transitions, from elastic to plastic and plastic to fracture, are present in the equilibrium configuration.

Most aspects of this effort may be extended to more general problems, including, in particular, elastic plastic laws which are neither "perfect" nor convex. In particular, solutions to such problems may display instabilities which suggest a challenging analysis especially from the quasistatic viewpoint. One exception to these extensions is the duality theory of Section 3, which requires a convex functional.

2. First Variation

Two features of the problem complicate the derivation of a useful first variation formula. Since I is not differentiable in the ordinary sense, variations will have to be restricted. One certainly expects ηu, for u a local minimum and η smooth, to be admissible, but unfortunately $P(\Omega)$ is not a local space. In particular, $\operatorname{div}(\eta u)$ lies in $L^{n/n-1}(\Omega)$, but not necessarily in $L^2(\Omega)$. Here we make use of Anzellotti and Giaquinta [4] and a result of Kohn and Temam [18].

Given $u \in P(\Omega)$, denote by S a carrier of $|\varepsilon(u)^s|$ so that $\Omega - S$ and S is a Lebesgue decomposition of Ω with respect to the mutually singular measures dx and $|\varepsilon(u)^s|$. Set

$$\hat{\sigma}^D = \begin{cases} \sigma^D & \text{in } \Omega - S \\ \xi & \text{in } S \end{cases} \tag{2.1}$$

where

$$\xi = \frac{\varepsilon(u)^s}{|\varepsilon(u)^s|},$$

the derivative of $\varepsilon(u)^s$ with respect to its total variation, satisfies

$$\xi = \xi^T, \operatorname{tr}\xi = 0, \quad \text{and} \quad |\xi| = 1 \quad |\varepsilon(u)^s| - \text{a.e.}$$

More formally, with χ_A the characteristic function of the set A,

$$\hat{\sigma}^D = \sigma^D \chi_{\Omega - S} + \xi \chi_S. \tag{2.2}$$

Since σ^D is Lebesgue measurable and ξ is $|\varepsilon(u)^s|$-measurable, $\hat{\sigma}^D$ is $dx + |\varepsilon(u)^s|$-measurable. It is convenient to write

$$\begin{aligned} \hat{\sigma} &= \hat{\sigma}^D + \varkappa \operatorname{div} u \mathbb{1} \\ &= \sigma \chi_{\Omega - S} + \xi \chi_S. \end{aligned} \tag{2.3}$$

Consider a $\zeta \in P(\Omega)$ satisfying $\varepsilon(\zeta)^s \ll |\varepsilon(u)^s|$ and let $\psi = (\psi_{ij})$ be the symmetric traceless tensor such that

$$\varepsilon(\zeta)^s = \psi |\varepsilon(u)^s|, \; |\varepsilon(u)^s| - \text{a.e.}$$

Now $\psi \in L^1(\Omega, |\varepsilon(u)^s|)$ so $\psi \cdot \xi \in L^1(\Omega, |\varepsilon(u)^s|)$,

$$\int_\Omega \hat{\sigma}^D \cdot \varepsilon^D(\zeta) = \int_\Omega \sigma^D \cdot \varepsilon^D(\zeta)^a \, dx + \int_\Omega \psi \cdot \xi |\varepsilon(u)^s| \tag{2.4a}$$

and

$$\int_{\Omega} \hat{\sigma} \cdot \varepsilon(\zeta) = \int_{\Omega} \sigma^D \cdot \varepsilon^D(\zeta)^a \, dx + \int_{\Omega} \psi \cdot \xi |\varepsilon(u)^s|$$

$$= \int_{\Omega} \left(\sigma^D \cdot \varepsilon^D(\zeta)^a + \varkappa \operatorname{div} u \operatorname{div} \zeta \right) dx + \int_{\Omega} \psi \cdot \xi |\varepsilon(u)^s|. \tag{2.4b}$$

Recall that for a local minimum the inhomogeneous terms $f \in L^{\infty}(\Omega)$.

Theorem 2.1. *Let u be a local minimum with stress $\sigma = \sigma(u)$. Then*
(i) $\sigma \in L^s_{\mathrm{loc}}(\Omega)$, $1 \leq s < \infty$ *and*
(ii) *if* $\eta \in H^{1,\infty}(\Omega)$, $\operatorname{supp} \eta \subset \subset \Omega$, *and* $v \in P(\Omega)$ *satisfies*

$$\varepsilon(v)^s \ll |\varepsilon(u)^s|,$$

then

$$\int_{\Omega} \eta \hat{\sigma} \cdot \varepsilon(v) = -\int_{\Omega} \sigma \cdot \eta_x \otimes v \, dx + \int_{\Omega} \eta f \cdot v \, dx \tag{2.5}$$

We begin with a restricted version of (2.5), cf. [3].

Lemma 2.1. *Let u be a local minimum in Ω with stress $\sigma = \sigma(u)$. If $\zeta \in P_0(\Omega)$ satisfies*

$$\varepsilon(\zeta)^s \ll |\varepsilon(u)^s|$$

then

$$\int_{\Omega} \sigma \cdot \varepsilon(\zeta) = \int_{\Omega} f \cdot \zeta \, dx. \tag{2.6}$$

In particular,

$$-\operatorname{div} \sigma = f \tag{2.7}$$

in the sense of distributions.

Proof. The demonstration rests on calculating

$$\frac{d}{dt} I(u + t\zeta)|_{t=0}.$$

For this note that

$$\varepsilon(u + t\zeta) = \left(\varepsilon(u)^a + t\varepsilon(\zeta)^a \right) dx + (\xi + t\psi)|\varepsilon(u)^s|,$$

$$\xi = \frac{\varepsilon(u)^s}{|\varepsilon(u)^s|}, \qquad \psi = \frac{\varepsilon(\zeta)^s}{|\varepsilon(u)^s|}.$$

Thus

$$\int_\Omega \varphi(\varepsilon^D(u+t\zeta)) = \int_\Omega \varphi(\varepsilon^D(u)^a + t\varepsilon^D(\zeta)^a)\,dx + \int_\Omega |\xi + t\psi|\,|\varepsilon(u)^s|.$$

Differentiating this expression keeping in mind that

$$|\xi + t\psi| = \left(1 + 2t\xi\cdot\psi + t^2|\psi|^2\right)^{1/2}, \quad |\varepsilon(u)^s| - \text{a.e.},$$

leads immediately to (2.6).

To show (2.7) it suffices to choose $\zeta \in C_0^\infty(\Omega)$ in (2.6) since $\varepsilon(\zeta)^s = 0$ for such ζ. □

Next we prove a variant of [18].

Lemma 2.2. *If u is a local minimum in Ω, then*

$$\operatorname{div} u \in L^s_{\mathrm{loc}}(\Omega), \qquad 1 \le s < \infty$$

and

$$\sigma = \sigma(u) \in L^s_{\mathrm{loc}}(\Omega), \qquad 1 \le s < \infty.$$

In other words, (i) of the theorem is satisfied.

Proof. Express σ by

$$\sigma = \sigma^D + \varkappa \operatorname{div} u \mathbb{1}.$$

By (2.7), for each i, $1 \le i \le n$,

$$\operatorname{div}\sigma_i^D + \varkappa \frac{\partial}{\partial x_i} \operatorname{div} u = -f_i,$$

$$\sigma_i^D = i\text{th/row of } \sigma^D.$$

Now $\sigma^D \in L^\infty(\Omega)$ and $f_i \in L^\infty(\Omega)$ so

$$\varkappa \frac{\partial}{\partial x_i} \operatorname{div} u = -f_i - \operatorname{div}\sigma_i^D \in H^{-1,\infty}(\Omega) = H^{1,1}(\Omega)',$$

or the gradient of $\operatorname{div} u$ is locally expressible as the distributional derivative of bounded functions. This means that $\operatorname{div} u \in L^s_{\mathrm{loc}}(\Omega)$. (Cf. Morrey [20], p. 70.) □

Proof of the Theorem. Step 1. Extension of (2.6) to $w \in H^{1,p}(\Omega)$, $\operatorname{supp} w \subset\subset \Omega$. Let $\zeta_k \in C_0^\infty(\Omega)$, $\zeta_k \to w$ in $H^{1,p}(\Omega)$ for a given p, $1 \le p < \infty$. Then by the previous lemma

$$\operatorname{div}\zeta_k \operatorname{div} u \to \operatorname{div} w \operatorname{div} u \qquad \text{in } L^p(\Omega)$$

from which it follows immediately that

$$\int_\Omega \sigma \cdot \varepsilon(w)\, dx = \int_\Omega f \cdot w\, dx, \qquad w \in H^{1,p}(\Omega), \quad \operatorname{supp} w \subset\subset \Omega. \tag{2.8}$$

Step 2. Adjustment of ηv. Since it is not known that $I(u + t\eta v)$ is finite, Lemma 2.1, or its argument, cannot be applied directly. With $K = \operatorname{supp}\eta \subset\subset \Omega$, let $K \subset \Omega_0 \subset\subset \Omega$, Ω_0 open with $\partial\Omega_0$ smooth, and $\eta_0 \in C_0^\infty(\Omega_0)$ vanish near $\partial\Omega_0$ satisfy

$$\eta_0 = 1 \qquad \text{in } K.$$

Now calculate that

$$\begin{aligned}
\varepsilon(\eta v) &= \eta\varepsilon(v) + \tfrac{1}{2}(\eta_x \otimes v + v \otimes \eta_x)\, dx,\\
\varepsilon(\eta v)^s &\ll |\varepsilon(u)^s|, \text{ and}\\
\operatorname{div}(\eta v) &= \eta \operatorname{div} v + \eta_x \cdot v, \eta_x \cdot v \in L^{n/n-1}(\Omega).
\end{aligned}$$

By the symmetry of σ and ε,

$$\hat\sigma \cdot \varepsilon(\eta v) = \eta\sigma \cdot \varepsilon(v) + \sigma \cdot \eta_x \otimes v.$$

Now let $h \in H^{2,n/n-1}(\Omega_0)$ be the solution of

$$\begin{aligned}
-\Delta h &= \eta_x \cdot v \quad \text{in } \Omega_0\\
h &= 0 \quad \text{on } \partial\Omega_0,
\end{aligned}$$

(thus one need not assume anything about $\partial\Omega$ itself,) and set

$$w = \eta_0 h_x \in H_0^{1,n/n-1}(\Omega_0)$$

With $\zeta = \eta v + w$ observe that

$$\varepsilon(\zeta)^s = \eta\varepsilon(v)^s \ll |\varepsilon(u)^s|. \tag{2.9}$$

Furthermore,

$$\operatorname{div}\zeta = \eta \operatorname{div} v + \eta_{0x} \cdot h_x.$$

Since the support of η_{0x} is contained in the set where h is harmonic, $\eta_{0x} \cdot h_x$ is smooth on Ω_0. Consequently $\zeta \in P_0(\Omega)$ and fulfills (2.9). Applying (2.6),

$$\int_\Omega \hat\sigma \cdot \varepsilon(\zeta) = \int_\Omega \hat\sigma \cdot \varepsilon(\eta v + w) = \int_\Omega f \cdot (\eta v + w)\, dx.$$

Writing this out in detail and noting that all the integrals are well defined gives that

$$\int_\Omega \eta\hat{\sigma}\cdot\varepsilon(v) = -\int_\Omega \sigma\cdot\eta_x\otimes v\,dx + \int_\Omega \eta f\cdot v\,dx$$
$$+\left\{-\int_\Omega \sigma\cdot\varepsilon(w)\,dx + \int_\Omega f\cdot w\,dx\right\}.$$

Finally, the expression in brackets vanishes by (2.8). □

The next lemma will be an aid in estimating various quantities.

Lemma 2.3. *Let $v \in BD(\Omega)$ and $\eta \in C(\Omega)$, $0 \le \eta \le 1$. For $U \subset \Omega$,*

$$\int_U \eta|\varepsilon^D(v)| \le \int_U \eta\varphi(\varepsilon^D(v)) + \tfrac{1}{2}|U|,$$
$$|U| = \text{measure of } U. \tag{2.10}$$

Proof. For any $t \ge 0$,

$$t \le \tfrac{1}{2}t^2 + \tfrac{1}{2} \text{ and}$$
$$t \le t - \tfrac{1}{2} + \tfrac{1}{2}, \text{ so}$$
$$t \le \varphi(t) + \tfrac{1}{2}.$$

Consequently,

$$\int_U \eta|\varepsilon^D(v)^a|\,dx \le \int_U \eta\varphi(\varepsilon^D(v)^a)\,dx + \tfrac{1}{2}|U|$$

and

$$\int_U \eta|\varepsilon^D(v)| = \int_U \eta|\varepsilon^D(v)^a|\,dx + \int_U \eta|\varepsilon^D(v)^s|$$
$$\le \int_U \eta\varphi(\varepsilon^D(v)^a)\,dx + \tfrac{1}{2}|U| + \int_U \eta|\varepsilon^D(v)^s|$$
$$= \int_U \eta\varphi(\varepsilon^D(v)) + \tfrac{1}{2}|U|.$$

□

The conclusion of the lemma resembles Jensen's inequality,

$$\varphi\left(\frac{1}{|U|}\int_U |\varepsilon^D(v)|\right) \le \frac{1}{|U|}\int_U \varphi(\varepsilon^D(v))$$

which is valid for $v \in BD(\Omega)$. The proof is omitted because (2.10) suffices for our purposes.

Combining the first variation formula with (2.10) provides a useful inequality. Given a local minimum u, we may take $v = u$ in (2.5) with $\eta \in C_0^\infty(\Omega)$, $0 \leq \eta \leq 1$, $\operatorname{supp} \eta = \bar{U}$, say $|U| = |\bar{U}|$. Observe first that since φ is convex, $\varphi(0) = 0$,

$$\varphi(p) \leq \varphi_p(p) \cdot p$$

so that

$$\varphi(\varepsilon^D(u)^a) \leq \sigma^D \cdot \varepsilon^D(u)^a$$

whence

$$\int_U \eta\varphi(\varepsilon^D(u)) \leq \int_U \eta\hat{\sigma}^D \cdot \varepsilon^D(u).$$

Adding the divergence terms and using (2.10),

$$\begin{aligned}\int_U \eta|\varepsilon(u)| &\leq \int_U \eta|\varepsilon^D(u)| + \int_U \eta|\operatorname{div} u|\, dx \\ &\leq \int_U \eta|\varepsilon^D(u)| + \varkappa\int_U \eta(\operatorname{div} u)^2\, dx + \frac{1}{4\varkappa}\int_U \eta\, dx \\ &\leq \int_U \eta\varphi(\varepsilon^D(u)) + \varkappa\int_U \eta(\operatorname{div} u)^2\, dx + \left(\frac{1}{4\varkappa} + \frac{1}{2}\right)|U| \\ &\leq \int_U \eta\hat{\sigma}^D \cdot \varepsilon^D(u) + \varkappa\int_U \eta(\operatorname{div} u)^2\, dx + \left(\frac{1}{4\varkappa} + \frac{1}{2}\right)|U|\end{aligned}$$

Now applying (2.5), we obtain for a local minimum u

$$\begin{aligned}&\int_U \eta|\varepsilon(u)| \leq -\int_U \sigma \cdot \eta_x \otimes u\, dx + \int_U \eta f \cdot u\, dx + C_0|U|, \\ &C_0 = \frac{1}{4\varkappa} + \frac{1}{2}.\end{aligned} \tag{2.11}$$

In the study of duality, a version of Lemma 2.1 for the traction problem is useful.

Lemma 2.4. *Let $f_i, g_i \in L^\infty(\Omega)$, $1 \leq i \leq n$, and set*

$$\langle T, \zeta \rangle = \int_U f \cdot \zeta\, dx + \int_{\partial\Omega} g \cdot \zeta\, ds, \qquad \zeta \in P(\Omega). \tag{2.12}$$

Let u be a solution of Prob. 1.1 with stress $\sigma = \sigma(u)$. If $\zeta \in P(\Omega)$ satisfies

$$\varepsilon^s(\zeta) \ll |\varepsilon(u)^s|$$

then

$$\int_\Omega \hat{\sigma} \cdot \varepsilon(\zeta) = \langle T, \zeta \rangle \tag{2.13}$$

Also,

$$-\operatorname{div} \sigma = T$$

in the sense of distributions, namely,

$$\int_\Omega \sigma \cdot \varepsilon(\zeta)\, dx = \langle T, \zeta \rangle \quad \textit{for } \zeta \in C^\infty(\overline{\Omega}). \tag{2.14}$$

The proof is omitted.

3. Duality or Complementary Energy

In equilibrium, the elastic plastic body has a potential energy which may be characterized in two ways. It may be the minimum of potential energy among admissible virtual displacements, as in Prob. 1.1, or it may be realized by Castigliano's Principle, as the minimum of the stress energy among suitably equilibrated tensor fields. In duality theory the two viewpoints are reconciled by showing that the extrema are connected and, moreover, when a minimizing displacement exists, its stress tensor resolves the problem of stress energy. An alternative way of expressing this is to say that a saddle point condition is satisfied and the solution and its stress tensor are in duality.

Duality theory has been considered in Duvaut and Lions [10], Kohn and Temam [18], Strang [22], Suquet [23], Temam [24, 25], and Temam and Strang [28], and is described here for the orientation and convenience of the reader. The uniqueness of the stress tensor and the saddle point/duality condition are verified. These latter properties have only recently been clarified [18]. A different proof is given here based on the first variation formula and the approximation theorem found in the appendix.

Duality theory is treated more generally in Ekeland and Temam [11], cf. also Courant and Hilbert [9], p. 210. Of special interest here are the works of Brezis [5] and Goffman and Serrin [15].

Our so-called primal problem is Prob. 1.1 for a distribution of forces

$$\langle T, \zeta \rangle = \int_\Omega f \cdot \zeta\, dx + \int_{\partial\Omega} g \cdot \zeta\, dS \qquad \zeta \in L^1(\Omega)$$

satisfying

$f \in L^\infty(\Omega), \qquad g \in L^\infty(\Omega)$ and

T is equilibrated.

Its dual problem is defined in terms of virtual stress tensors $\tau = (\tau_{ij}) \in L^\infty(\Omega; \mathbb{R}^{n^2})$ obeying

$$\left.\begin{aligned} \tau &= \tau^T \\ |\tau^D| &\leq 1 \end{aligned}\right. \quad \text{in } \Omega \tag{3.1}$$

and

$$\begin{aligned} -\operatorname{div}\tau &= f \quad && \text{in } \Omega \\ \tau\nu &= g \quad && \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Equations (3.2) mean

$$\int_\Omega \tau \cdot \varepsilon(\zeta)\, dx = \langle T, \zeta \rangle, \qquad \zeta \in C^\infty(\bar{\Omega}).$$

Denote by $\mathbb{K}$ the tensor fields satisfying (3.1) and (3.2), a closed convex subset of $L^2(\Omega; \mathbb{R}^{n^2})$. The energy of τ is

$$I^*(\tau) = \frac{1}{2}\int_\Omega |\tau^D|^2\, dx + \frac{1}{2\varkappa n^2}\int_\Omega (\operatorname{tr}\tau)^2\, dx \tag{3.3}$$

The dual problem is [24, 28].

Problem 3.1. To find $\sigma \in \mathbb{K}$: $I^*(\sigma) = \min_{\mathbb{K}} I^*(\tau)$.

Thus the solution of the dual problem resolves the variational inequality

$$\sigma \in \mathbb{K}: \int_\Omega \left\{ \sigma^D \cdot (\tau^D - \sigma^D) + \frac{1}{\varkappa n^2}\operatorname{tr}\sigma \operatorname{tr}(\tau - \sigma) \right\} dx \geq 0 \qquad \text{for } \tau \in \mathbb{K} \tag{3.4}$$

Now it is easily checked that the bilinear form of (3.4) is coercive (on all symmetric tensors in $L^2(\Omega; \mathbb{R}^{n^2})$); consequently, if $\mathbb{K} \neq \varnothing$, the solution of the dual problem is unique, [16], p. 24.

Proposition 3.1. *If* $v \in P_e(\Omega)$ *and* $\tau \in \mathbb{K}$, *then*

$$I(v) \geq -I^*(\tau). \tag{3.5}$$

Proof. Assume first that $v \in C^\infty(\Omega) \cap P_e(\Omega)$. Then by (3.2),

$$\begin{aligned} I(v) &= \int_\Omega \varphi(|\varepsilon^D(v)|)\, dx + \frac{\varkappa}{2}\int_\Omega (\operatorname{div} v)^2\, dx - \langle T, v \rangle \\ &= \int_\Omega \varphi(|\varepsilon^D(v)|)\, dx + \frac{\varkappa}{2}\int_\Omega (\operatorname{div} v)^2\, dx - \int_\Omega \tau \cdot \varepsilon(v)\, dx. \end{aligned}$$

Let $E = \{x \in \Omega\colon |\varepsilon^D(v)| < 1\}$ and $P = \Omega - E$ and recall that

$$\tau \cdot \varepsilon(v) = \tau^D \cdot \varepsilon^D(v) + \frac{1}{n}\operatorname{tr}\tau \operatorname{div} v.$$

Using these notations we compute that

$$
\begin{aligned}
I(v) &= \frac{1}{2}\int_E |\varepsilon^D(v)|^2\,dx + \int_P \left(|\varepsilon^D(v)| - \tfrac{1}{2}\right)dx + \frac{\varkappa}{2}\int_\Omega (\operatorname{div} v)^2\,dx \\
&\quad - \int_\Omega \tau^D\cdot\varepsilon^D(v)\,dx - \frac{1}{n}\int_\Omega \operatorname{tr}\tau \operatorname{div} v\,dx \\
&= -I^*(\tau) + \frac{1}{2}\int_E \left(|\varepsilon^D(v)|^2 - 2\tau^D\cdot\varepsilon^D(v) + |\tau^D|^2\right)dx \\
&\quad + \int_P \left(\sqrt{\frac{\varkappa}{2}}\operatorname{div} v - \frac{1}{\sqrt{2\varkappa}\,n}\operatorname{tr}\tau\right)^2 dx \\
&\geq -I^*(\tau) + \int_P \left(|\varepsilon^D(v)| - \tfrac{1}{2} - \tau^D\cdot\varepsilon^D(v) + \tfrac{1}{2}|\tau^D|^2\right)dx \\
&\geq -I^*(\tau) + \int_P \left(|\varepsilon^D(v)| - \tfrac{1}{2} - |\tau^D|\,|\varepsilon^D(v)| + \tfrac{1}{2}|\tau^D|^2\right)dx.
\end{aligned}
$$

Finally, note that the polynomial

$$p(t) = a - \tfrac{1}{2} - at + \tfrac{1}{2}t^2, \qquad \text{with } a \geq 1,$$

satisfies $p(t) \geq 0$ for $0 \leq t \leq 1$. Thus

$$|\varepsilon^D(v)| - \tfrac{1}{2} - |\tau^D|\,|\varepsilon^D(v)| + \tfrac{1}{2}|\tau^D|^2 \geq 0 \qquad \text{in } P$$

whence

$$I(v) \geq -I^*(\tau).$$

Given $v \in P_e(\Omega)$, choose a sequence $v_\delta \in C^\infty(\bar{\Omega}) \cap P_e(\Omega)$ such that $I(v_\delta) \to I(v)$ according to Thm. A.2 (of the Appendix), subtracting a rigid motion if necessary to satisfy the normalization requirement (1.10). □

Theorem 3.1. *Let u be a solution of Prob.* 1.1 *and $\bar{\sigma}$ a solution of Prob.* 3.1. *Then*

$$\inf_{P_e(\Omega)} I(v) = I(u) = -I^*(\bar{\sigma}) = -\min_{\mathbb{K}} I^*(\tau) \tag{3.6}$$

and

$$\bar{\sigma} = \sigma(u). \tag{3.7}$$

Proof. According to (2.14), $\sigma = \sigma(u) \in \mathbb{K}$. In view of the uniqueness of the solution of Prob. 3.1 and Proposition 3.1, it suffices to show that

$$I(u) = -I^*(\sigma) \qquad \text{for } \sigma = \sigma(u).$$

By the first variation formula (2.13),

$$\int_\Omega \hat{\sigma}\cdot\varepsilon(u) = \langle T, u\rangle$$

or, with $E=\{|\sigma^D|<1\}$ and $P=\Omega - E$,

$$\begin{aligned}
I(u) &= \int_\Omega \varphi(\varepsilon^D(u)) + \frac{\varkappa}{2}\int_\Omega (\operatorname{div} u)^2\,dx - \int_\Omega \hat{\sigma}\cdot\varepsilon(u)\\
&= \int_\Omega \varphi(|\varepsilon^D(u)^a|)\,dx + \int_\Omega |\varepsilon(u)^s| + \frac{\varkappa}{2}\int_\Omega (\operatorname{div} u)^2\,dx\\
&\quad - \int_\Omega \sigma\cdot\varepsilon(u)^a\,dx - \int_\Omega |\varepsilon(u)^s|\\
&= -\frac{1}{2}\int_E |\varepsilon^D(u)^a|^2\,dx + \int_P \left(|\varepsilon^D(u)^a| - |\varepsilon^D(u)^a| - \frac{1}{2}\right)dx\\
&\quad - \frac{\varkappa}{2}\int_\Omega (\operatorname{div} u)^2\,dx\\
&= -\frac{1}{2}\int_E |\varepsilon^D(u)^a|^2\,dx - \frac{1}{2}\int_P dx - \frac{\varkappa}{2}\int_\Omega (\operatorname{div} u)^2\,dx\\
&= -\frac{1}{2}\int_\Omega |\sigma^D|^2\,dx - \frac{1}{2\varkappa n^2}\int_\Omega (\operatorname{tr}\sigma)^2\,dx
\end{aligned}$$

since $\operatorname{tr}\sigma = \varkappa n \operatorname{div} u$. □

4. Some Properties of Local Minima

Recall that for a local minimum u it is assumed that the inhomogeneous term $f\in L^\infty(\Omega)$, (1.6). Also recall that if $v\in BD(U)$, $U\subset\mathbb{R}^n$ and $v=0$ on ∂U, then $v\in L^{n/n-1}(U)$ and

$$\|v\|_{L^{n/n-1}(U)} \le C_s\int_U |\varepsilon(v)| \tag{4.1}$$

where C_s depends only on n, [27].

Theorem 4.1. *Let u be a local minimum and let $\Omega'\subset\subset\Omega$. Then there is a $p>\dfrac{n}{n-1}$, $p=p(\Omega')$, such that*

$$u\in L^p(\Omega')$$

Proof. Given $a\in\Omega'$ and $\rho>0$, $4\rho<\operatorname{dist}(\Omega',\partial\Omega)$, choose $\eta\in C_0^\infty(B_{2\rho}(a))$, $\eta=1$ on $B_\rho(a)$, $0\le\eta\le 1$. Thus $\zeta=\eta u$ is admissible in the first variation formula and by (2.11),

$$\int_{B_{2\rho}(a)} \eta |\varepsilon(u)| \leq -\int_{B_{2\rho}(a)} \sigma \cdot (\eta_x \otimes u)\, dx + \int_{B_{2\rho}(a)} \eta f \cdot u\, dx + C_0 |B_{2\rho}| \tag{4.2}$$

Note that by (4.1),

$$\begin{aligned} \|\eta u\|_{L^{n/n-1}(\Omega)} &\leq C_s \int_\Omega |\varepsilon(\eta u)| \\ &\leq C_s \left\{ \int_\Omega \eta |\varepsilon(u)| + \int_\Omega |u \otimes \eta_x|\, dx \right\} \end{aligned} \tag{4.3}$$

Setting (4.2) in (4.3) and recalling that $\sigma \in L^\infty_{\mathrm{loc}}(\Omega)$,

$$\begin{aligned} \|\eta u\|_{L^{n/n-1}(\Omega)} &\leq C_s \bigg\{ -\int_{B_{2\rho}(a)} \sigma \cdot (\eta_x \otimes u)\, dx + \int_{B_{2\rho}(a)} \eta f \cdot u\, dx \\ &\qquad + C_0 |B_{2\rho}| + \int_{B_{2\rho}} |u \otimes \eta_x|\, dx \bigg\} \\ &\leq C\left(\|\sigma\|_{L^\infty(B_{2\rho}(a))} + 1 \right) \frac{1}{\rho} \int_{B_{2\rho}(a)} |u|\, dx + C \|f\|_{L^\infty(\Omega)} \int_{B_{2\rho}(a)} |u|\, dx + C\rho^n \end{aligned}$$

Since $\eta = 1$ on $B_\rho(a)$, after dividing both sides by ρ^{n-1} and restricting on the left to $B_\rho(a)$, we obtain the inequality, valid for some $M > 0$,

$$\left(\frac{1}{\rho^n} \int_{B_\rho(a)} |u|^{n/n-1} dx \right)^{1-(1/n)} \leq M \left\{ \frac{1}{(2\rho)^n} \int_{B_{2\rho}(a)} |u|\, dx + 1 \right\},$$

$$\rho \leq \tfrac{1}{4} \operatorname{dist}(\Omega', \partial\Omega).$$

The conclusion now follows from the reverse Hölder inequality (Gehring [13], Giaquinta and Modica [14].) □

An interesting characterization of $\hat{\sigma}$ is provided by a result of Anzellotti [2], a different proof of which is given here. To begin, let u be a local minimum with stress σ and set

$$\tau_\rho(a) = \frac{1}{|B_\rho|} \int_{B_\rho(a)} \sigma\, dx \tag{4.4}$$

Thus $\tau_\rho \in C(\Omega)$, $\tau_\rho \to \sigma$ in $L^p_{\text{loc}}(\Omega)$, $1 \le p < \infty$, and

$$-\operatorname{div}\tau_\rho = f_\rho = \frac{1}{|B_\rho|}\int_{B_\rho(a)} f\,dx,$$

$$f_\rho \to f \quad \text{in } L^p_{\text{loc}}(\Omega), \qquad 1 \le p < \infty. \tag{4.5}$$

Theorem 4.2. *Let u be a local minimum and define τ_ρ by* (4.4). *Then for each p, $1 \le p < \infty$,*

$$\tau_\rho \to \hat{\sigma} \qquad \text{in } L^p_{\text{loc}}\big(\Omega, dx + |\varepsilon(u)^s|\big) \quad \text{as } \rho \to 0.$$

Prior to giving the demonstration we remark that the conclusion holds as well for any suitable mollifier α_h $(h \to 0)$ with

$$\tau_h = \alpha_h * \sigma.$$

Proof. Note that it suffices to prove the theorem for the case $p=1$ since $\hat{\sigma} \in L^\infty(\Omega, dx + |\varepsilon(u)^s|)$.

Let $\Omega' \subset\subset \Omega$ be a fixed subdomain and $\eta \in C_0^\infty(\Omega)$ a fixed function, $0 \le \eta \le 1$, with $\eta = 1$ on Ω'. Thus $\zeta = \eta u$ is admissible in (2.5) so

$$\int_\Omega \eta\hat{\sigma}\cdot\varepsilon(u) = -\int_\Omega \sigma\cdot u\otimes\eta_x\,dx + \int_\Omega \eta f\cdot u\,dx \tag{4.6}$$

Since τ_ρ is a locally Lipschitz tensor,

$$\int_\Omega \tau_\rho\cdot\varepsilon(\zeta) = \int_\Omega f_\rho\cdot\zeta\,dx \tag{4.7}$$

for any $\zeta \in BD(\Omega)$, $\operatorname{supp}\zeta \subset\subset \Omega$. In particular, for $\zeta = \eta u$,

$$\int_\Omega \eta\tau_\rho\cdot\varepsilon(u) = -\int_\Omega \tau_\rho\cdot u\otimes\eta_x\,dx + \int_\Omega \eta f_\rho\cdot u\,dx. \tag{4.8}$$

Subtracting (4.8) from (4.6) gives

$$\int_\Omega \eta(\sigma - \tau_\rho)\cdot\varepsilon(u) = -\int_\Omega (\sigma - \tau_\rho)\cdot u\otimes\eta_x\,dx + \int_\Omega \eta(f - f_\rho)\cdot u\,dx. \tag{4.9}$$

Let S denote the carrier of $|\varepsilon(u)^s|$ and recall that

$$\hat{\sigma}^D\cdot\varepsilon(u) = \xi\cdot\xi|\varepsilon(u)^s| = |\varepsilon(u)^s| \text{ on } S,$$

c.f. (2.1). Also $|\tau_\rho^D| \le 1$ in Ω and, of course,

$$\tau_\rho^D\cdot\xi \le |\tau_\rho^D|.$$

Consequently,

$$
\begin{aligned}
0 &\leq \int_{\Omega'} \left(1 - |\tau_\rho^D|\right)|\varepsilon(u)^s| \\
&\leq \int_{\Omega'} \left(\xi - \tau_\rho^D\right) \cdot \xi |\varepsilon(u)^s| \\
&\leq \int_\Omega \eta \left(\hat{\sigma} - \tau_\rho^D\right) \cdot \varepsilon(u)^s \\
&= -\int_\Omega (\sigma - \tau_\rho) \cdot u \otimes \eta_x \, dx + \int_\Omega \eta (f - f_\rho) \cdot u \, dx - \int_\Omega \eta (\sigma - \tau_\rho) \cdot \varepsilon(u)^a \, dx
\end{aligned}
$$

by (4.9). It follows from (4.5) that each term in the last line tends to zero as $\rho \to 0$; thus

$$\lim_{\rho \to 0} \int_{\Omega'} \left(1 - |\tau_\rho^D|\right)|\varepsilon(u)^s| = 0$$

and

$$\lim_{\rho \to 0} \int_{\Omega'} \left(1 - \xi \cdot \tau_\rho^D\right)|\varepsilon(u)^s| = \lim_{\rho \to 0} \int_{\Omega'} \left(\xi - \tau^D\right) \cdot \xi |\varepsilon(u)^s| = 0.$$

Indeed, since $|\tau_\rho^D| \leq 1$,

$$\lim_{\rho \to 0} \int_{\Omega'} \left(1 - |\tau_\rho^D|^2\right)|\varepsilon(u)^s| = 0.$$

So

$$
\begin{aligned}
\int_{\Omega'} |\xi - \tau_\rho^D|^2 |\varepsilon(u)^s| &= \int_{\Omega'} \left(1 + |\tau_\rho^D|^2 - 2\xi \cdot \tau_\rho^D\right)|\varepsilon(u)^s| \\
&= 2\int_{\Omega'} \left(1 - \xi \cdot \tau_\rho^D\right)|\varepsilon(u)^s| + \int_{\Omega'} \left(|\tau_\rho^D|^2 - 1\right)|\varepsilon(u)^s| \\
&\to 0 \text{ as } \rho \to 0,
\end{aligned}
$$

or $\tau_\rho^D \to \zeta$ in $L^2(\Omega'; |\varepsilon(u)^s|)$, hence in $L^1(\Omega'; |\varepsilon(u)^s|)$. The conclusion is now immediate since $\tau_\rho \to \sigma$ in $L^1(\Omega) \equiv L^1(\Omega; dx)$ (and $\operatorname{div} v = 0$ $|\varepsilon(u)^s| -$ a.e.) □

5. The Strain Measure in the Elastic Set

For a given local minimum $u \in P(\Omega)$ with stress

$$\sigma = \sigma^D + \varkappa \operatorname{div} u \mathbb{1}$$

we have regarded the set

$$\{a \in \Omega\colon |\sigma^D(a)| < 1\}, \qquad \sigma^D(a) \equiv \sigma^D(u(a)),$$

as the elastic set. As such, it is determined within a Lebesgue null set, for example as the set of $a \in \Omega$ for which the family of averages

$$\tau_\rho(a) = \frac{1}{|B_\rho|}\int_{B_\rho(a)} \sigma\, dx$$

satisfies

$$\lim_{\rho \to 0} \tau_\rho(a) = \sigma(a) \text{ exists and } |\sigma^D(a)| < 1.$$

This notion may be slightly refined by choosing a sequence ρ_k for which

$$\tau_{\rho_k} \to \hat{\sigma},\, dx + |\varepsilon(u)^s| \text{ a.e.,} \qquad \text{as } k \to \infty,$$

which exists in view of Theorem 4.2, and defining

$$E = \left\{a \in \Omega\colon \lim_{k \to \infty} |\tau_{\rho_k}^D(a)| < 1\right\} \tag{5.1}$$

A different choice of sequence yields a set $\tilde{E}$ with

$$\int_{E \Delta \tilde{E}} dx + \int_{E \Delta \tilde{E}} |\varepsilon(u)^s| = 0$$

Theorem 5.1. *If u is a local minimum and E is defined by* (6.1), *then*

$$\int_E |\varepsilon(u)^s| = 0.$$

Proof. Let $\Omega' \subset \Omega$ be open and $E' = E \cap \Omega'$. The object is to show that $S \cap E'$ has $|\varepsilon(u)^s|$ measure zero. But

$$\lim_{k \to \infty} \tau_{\rho_k}^D = \hat{\sigma}^D = \xi, \quad \text{and} \quad |\xi| = 1 \quad |\varepsilon(u)^s| - \text{a.e.},$$

or

$$\left\{a \in \Omega'\colon \lim_{\rho_k \to 0} |\tau_{\rho_k}^D(a)| < 1\right\}$$

has $|\varepsilon(u)^s|$-measure zero. □

Note that the Heaviside function

$$v(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

is not a local solution of the one dimensional problem since $\sigma(v) \equiv 0$ but

$$\int_{-1}^{1} |\varepsilon(v)^s| \equiv \int_{-1}^{1} |D^s v| = 1$$

A point $a \in \Omega$ is an elastic density point if $a \in E$ (for some choice of sequence ρ_k) and it is a Lebesgue point of σ. In particular, there are a symmetric traceless matrix $\sigma^D(a)$, $|\sigma^D(a)| < 1$, and a $\delta(a) \in \mathbb{R}$ such that

$$\frac{1}{|B_\rho|} \int_{B_\rho(a)} \left(|\sigma^D - \sigma^D(a)|^p + |\operatorname{div} u - \delta(a)|^p \right) dx \to 0 \quad \text{as } \rho \to 0. \tag{5.2}$$

for all p, $1 \leq p < \infty$. Note in particular that at an elastic density point a, for small ρ, $\int_{B_\rho(a)} |\operatorname{div} u| \, dx \leq C\rho^n$.

Theorem 5.2. *If u is a local minimum and a is an elastic density point, then there are $M > 0$ and $r_0 > 0$ such that*

$$\int_{B_\rho(a)} |\varepsilon(u)| \leq M\rho^n \quad \text{for } \rho \leq r_0. \tag{5.3}$$

It will be clear that M depends on the data of the problem for r_0 sufficiently small. The condition above is not a general property of measures, but relies on the local minimum property. For example, let

$$v(t) = v_k(t) = \begin{cases} 0 & t < 0 \\ kt & t \geq 0 \end{cases}, \qquad k > 1$$

so to be consistent with (5.1)

$$\sigma(v)(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & t = 0 \\ 1 & t > 0. \end{cases}$$

Thus $t = 0$ is an elastic density point according to the definition although

$$\frac{1}{2\rho} \int_{-\rho}^{\rho} |Dv| = k.$$

Proof. We may choose $u = 0$ and $\rho > 0$ small. After subtraction of a rigid motion from u we may suppose that

$$\int_{B_{2\rho}} u^i \, dx = \int_{B_{2\rho}} \begin{vmatrix} u^i & u^j \\ x_i & x_j \end{vmatrix} dx = 0, \qquad 1 \leq i, j \leq n. \tag{5.4}$$

Set

$$\eta(x) = \begin{cases} 1 & |x| \le \rho \\ 2 - \dfrac{|x|}{\rho} & \rho < |x| < 2\rho \\ 0 & |x| \ge 2\rho \end{cases}$$

so that (2.5) holds. Indeed, by (2.11),

$$\int_{B_{2\rho}} \eta |\varepsilon(u)| \le -\int_{B_{2\rho}} \sigma \cdot \eta_x \otimes u \, dx + \int_{B_{2\rho}} \eta f \cdot u \, dx + C\rho^n$$

so

$$\int_{B_{2\rho}} \eta |\varepsilon^D(u)| \le -\int_{B_{2\rho}} \sigma \cdot \eta_x \otimes u \, dx + \int_{B_{2\rho}} \eta f \cdot u \, dx + C\rho^n, \tag{5.5}$$

keeping in mind that $\int_{B_\rho} |\operatorname{div} u| \, dx \le C\rho^n$.

By the divergence theorem, noting that $\sigma^D(0)$ and $\delta(0)$ are constants,

$$\int_\Omega \sigma_0 \cdot \varepsilon(\eta u) = 0, \sigma_0 = \sigma^D(0) + \varkappa \delta(0) \mathbb{1}.$$

Thus

$$\int_{B_{2\rho}} \eta \sigma^D(0) \cdot \varepsilon^D(u) = -\int_{B_{2\rho}} \sigma_0 \cdot \eta_x \otimes u \, dx - \varkappa \delta(0) \int_{B_{2\rho}} \eta \operatorname{div} u \, dx. \tag{5.6}$$

Observe that since $|\sigma^D(0)| < 1$, there is an $\alpha > 0$ such that

$$\begin{aligned} &0 < \alpha < |\psi - \sigma^D(0)| && |\varepsilon^D(u)| - \text{a.e.}, \\ &\psi = \frac{\varepsilon^D(u)}{|\varepsilon^D(u)|}, |\psi| = 1 && |\varepsilon^D(u)| - \text{a.e.} \end{aligned} \tag{5.7}$$

Thus, subtracting (5.6) from (5.5) and using (5.7),

$$\begin{aligned} \alpha \int_{B_{2\rho}} \eta |\varepsilon^D(u)| &\le \int_{B_{2\rho}} \eta \left(|\varepsilon^D(u)| - \sigma^D(0) \cdot \varepsilon^D(u) \right) \\ &\le -\int_{B_{2\rho}} (\sigma - \sigma_0) \cdot \eta_x \otimes u \, dx + \int_{B_{2\rho}} \eta f \cdot u \, dx + C\rho^n \\ &\le \frac{1}{\rho} \int_{B_{2\rho}} |\sigma - \sigma_0| \, |u| \, dx + \int_{B_{2\rho}} |f| \, |u| \, dx + C\rho^n \\ &\le \left(\frac{1}{\rho} \|\sigma - \sigma_0\|_{L^n(B_{2\rho})} + c_n \|f\|_{L^\infty(\Omega)} \rho \right) \|u\|_{L^{n/n-1}(B_{2\rho})} + C\rho^n. \end{aligned}$$

Now employing the Sobolev-Korn inequality, note (5.4),

$$\alpha\int_{B_{2\rho}}\eta|\varepsilon^D(u)| \le C_1\left(\frac{1}{\rho}\|\sigma-\sigma_0\|_{L^n(B_{2\rho})}+\|f\|_{L^\infty}\rho\right)\int_{B_{2\rho}}|\varepsilon(u)| + C\rho^n.$$

Adding to this the inequality

$$\alpha\int_{B_{2\rho}}\eta|\operatorname{div} u|\,dx \le C\rho^n$$

we obtain, for some constants $C_2, C_3 > 0$,

$$\begin{aligned}\int_{B_\rho}|\varepsilon(u)| &\le \int_{B_{2\rho}}\eta|\varepsilon(u)|\\ &\le C_2\left(\frac{1}{\rho}\|\sigma-\sigma_0\|_{L^n(B_{2\rho})}+\|f\|_{L^\infty}\rho\right)\int_{B_{2\rho}}|\varepsilon(u)| + C_3\rho^n.\end{aligned}$$

Whence

$$\begin{aligned}\rho^{-n}\int_{B_\rho}|\varepsilon(u)| &\le 2^nC\left(\frac{1}{\rho}\|\sigma-\sigma_0\|_{L^n(B_{2\rho})}+\|f\|_{L^\infty}\rho\right)(2\rho)^{-n}\int_{B_{2\rho}}|\varepsilon(u)| + C_3\\ &= \lambda(\rho)(2\rho)^{-n}\int_{B_{2\rho}}|\varepsilon(u)| + C_3.\end{aligned}$$

By (5.2) it is possible to find an $r_0 > 0$ such that

$$\lambda(\rho) \le \mu < 1 \quad \text{for } \rho \le 2r_0.$$

Thus

$$\rho^{-n}\int_{B_\rho}|\varepsilon(u)| \le r_0^{-n}\int_{B_{r_0}}|\varepsilon(u)| + \frac{C_3}{1-\mu}, \qquad \rho \le r_0. \qquad \square$$

A somewhat more precise result holds, one which makes more evident the remarks preceding the proof of the last theorem.

Theorem 5.3. *Let u be a local minimum and let $a \in \Omega$ be an elastic density point. With $\sigma_a = \sigma^D(a) + \varkappa\delta(a)\mathbb{1}$,*

$$\lim_{\rho\to 0}\rho^{-n}\int_{B_\rho(a)}|\varepsilon(u)-\sigma_a\,dx| = 0.$$

First note

Lemma 5.1. *Let $\sigma_a^D \in \mathbb{R}^N$. If $|\sigma_a^D| \le \alpha < 1$, there is a $\beta = \beta(\alpha) > 0$ such that*

$$\beta|t\sigma^D - \sigma_a^D| \le \left(\sigma^D - \sigma_a^D\right)\cdot\left(t\sigma^D - \sigma_a^D\right) \quad \textit{for } |\sigma^D| = 1, \quad \sigma^D \in \mathbb{R}^N, \quad t \ge 1.$$

The proof, which is elementary geometry, is omitted.

Proof of the Theorem. Choose $a = 0$. Let $w(x) = \sigma_0 x$. With η as in the previous theorem, we may apply (2.5) to $u - w$ and η, from which we obtain

$$\int_{B_{2r}} \eta\hat{\sigma} \cdot (\varepsilon(u) - \sigma_0\, dx) = \int_{B_{2r}} \sigma \cdot (u - w) \otimes \eta_x\, dx + \int_{B_{2r}} \eta f(u - w)\, dx.$$

Similarly, since σ_0 is a constant matrix,

$$\int_{B_{2r}} \eta\sigma_0 \cdot (\varepsilon(u) - \sigma_0\, dx) = \int_{B_{2r}} \sigma_0 \cdot (u - w) \otimes \eta_x\, dx.$$

Subtracting gives that

$$\int_{B_{2r}} \eta(\hat{\sigma} - \sigma_0) \cdot (\varepsilon(u) - \sigma_0\, dx) = \int_{B_{2r}} (\sigma - \sigma_0) \cdot (u - w) \otimes \eta_x\, dx$$

$$+ \int_{B_{2r}} \eta f(u - w)\, dx.$$

The left-hand side may be divided into a number of pieces, to wit,

$$\int_{B_{2r}} \eta(\hat{\sigma} - \sigma_0) \cdot (\varepsilon(u) - \sigma_0\, dx) = \int_{E_{2r}} \eta|\sigma^D - \sigma^D(0)|^2\, dx$$

$$+ \int_{P_{2r}} \eta(\sigma^D - \sigma^D(0)) \cdot (\varepsilon^D(u)^a - \sigma^D(0))\, dx + \int_{B_{2r}} \eta(\hat{\sigma} - \sigma_0) \cdot \varepsilon(u)^s$$

$$+ \varkappa \int_{B_{2r}} \eta(\operatorname{div} u - \delta(0))^2\, dx$$

where $E_{2r} = E \cap B_{2r}$ and $P_{2r} = B_{2r} - E_{2r}$. By Lemma 5.1,

$$\beta \int_{P_{2r}} \eta|\varepsilon^D(u)^a - \sigma^D(0)|\, dx \le \int_{P_{2r}} \eta(\sigma^D - \sigma^D(0)) \cdot (\varepsilon^D(u)^a - \sigma^D(0))\, dx$$

assuming that $|\sigma^D(0)| \le \alpha < 1$ for some α. Since

$$\hat{\sigma} \cdot \varepsilon(u)^s = |\varepsilon(u)^s| \text{ and}$$
$$\sigma_0 \cdot \varepsilon(u)^s = \sigma^D(0) \cdot \varepsilon(u)^s \quad |\varepsilon(u)^s| - \text{a.e.},$$

$$(1 - \alpha) \int_{B_{2r}} \eta|\varepsilon(u)^s| \le \int_\Omega \eta(\hat{\sigma} - \sigma_0) \cdot \varepsilon(u)^s$$

Thus for a $C > 0$,

$$\int_{P_{2r}} \eta|\varepsilon^D(u) - \sigma^D(0)\,dx| \le C\int_{B_{2r}} (\sigma - \sigma_0)\cdot(u - w)\otimes\eta_x\,dx$$

$$+ C\int_{B_{2r}} \eta f\cdot(u - w)\,dx$$

$$\le \frac{C'}{r}\|\sigma - \sigma_0\|_{L^n(B_{2r})}\|u - w\|_{L^{n/n-1}(B_{2r})}$$

$$+ C''r\|u - w\|_{L^{n/n-1}(B_{2r})}$$

$$\le C_1\left(\frac{1}{r}\|\sigma - \sigma_0\|_{L^n(B_{2r})} + r\right)\int_{B_{2r}} |\varepsilon(u) - \sigma_0\,dx|,$$

assuming, as before, that u satisfies the normalization conditions (5.4) in B_{2r}. Consequently, for a constant $M > 0$,

$$r^{-n}\int_{P_r} |\varepsilon^D(u) - \sigma^D(0)\,dx| \le M\left(\frac{1}{r}\|\sigma - \sigma_0\|_{L^n(B_{2r})} + r\right)\left(r^{-n}\int_{B_{2r}} |\varepsilon(u)| + 1\right) \tag{5.8}$$

The first factor on the right-hand side tends to zero with r and the second remains bounded by the previous theorem; thus,

$$\lim_{r\to 0} r^{-n}\int_{P_r} |\varepsilon^D(u) - \sigma^D(0)|\,dx = 0.$$

Since it is part of the hypothesis that

$$\lim_{r\to 0} r^{-n}\int_{E_r} |\sigma^D - \sigma^D(0)|\,dx = \lim_{r\to 0} r^{-n}\int_{B_r} |\operatorname{div} u - \delta(0)|\,dx = 0,$$

the theorem is proved. □

The expression (5.8) suggests that the strain in the plastic set P_{2r} decreases more rapidly than the strain in E_{2r}.

6. Elastic States

The discussion is directed toward establishing the presence of an elastic state in a body Ω subjected to a given force distribution T. The question of plastic states will be considered in the next section. Taken together, the two sections illustrate that elastic sets arise from global restrictions on the data T whereas the transition to plastic behavior is a local question which depends in some sense on the smoothness of T. To fix the ideas we shall discuss the traction problem, Prob. 1.1.

If T and its derivatives are suitably small, then there is a smooth solution of Prob. 1.1 which satisfies $|\varepsilon^D| < 1$ in Ω, [1]. In this case the solution is also unique (in $P_e(\Omega)$.) To check the uniqueness in this class, one employs Thms. 5.1 and 5.2 which imply that any solution has a bounded (locally) strain matrix.

Let $\mathbb{S} > 0$ denote the Korn-Sobolev constant such that

$$\|v\|_{L^{n/n-1}(\Omega)} + \|v\|_{L^1(\partial\Omega)} \leq \mathbb{S}\int_\Omega |\varepsilon(v)|, \qquad v \in P_e(\Omega). \tag{6.1}$$

Suppose now that Ω is entirely plastic so

$$\sigma^D = \frac{\varepsilon^D(u)^a}{|\varepsilon^D(u)^a|}$$

and a.e. in Ω.

$$|\varepsilon^D(u)^a| \geq 1$$

Setting $\zeta = u$ in the variational formula (2.13) yields that

$$\int_\Omega |\varepsilon^D(u)| + \varkappa\int_\Omega (\operatorname{div} u)^2\,dx = \int_\Omega \hat{\sigma}^D \cdot \varepsilon^D(u) + \varkappa\int_\Omega (\operatorname{div} u)^2\,dx$$

$$= \langle T, u\rangle \leq \|T\|\{\|u\|_{L^{n/n-1}(\Omega)} + \|u\|_{L^1(\partial\Omega)}\} \leq \mathbb{S}\|T\|\int_\Omega |\varepsilon(u)|, \tag{6.2}$$

$$\|T\| = \|f\|_{L^n(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}.$$

Recall that

$$\begin{aligned} |\varepsilon(u)^a| &= \left|\varepsilon^D(u)^a + \frac{1}{n}\mathbb{1}\operatorname{div} u\right| \\ &\leq \left(|\varepsilon^D(u)^a|^2 + (\operatorname{div} u)^2\right)^{1/2} \\ &\leq |\varepsilon^D(u)^a| + \tfrac{1}{2}(\operatorname{div} u)^2 \qquad \text{a.e. in } \Omega \end{aligned}$$

since

$$\sqrt{\alpha^2 + \beta^2} \leq \alpha + \tfrac{1}{2}\beta^2 \quad \text{if } \alpha \geq 1.$$

This informs us that

$$\int_\Omega |\varepsilon(u)| \leq \int_\Omega |\varepsilon^D(u)| + \frac{1}{2}\int_\Omega (\operatorname{div} u)^2\,dx.$$

Using this in (6.2),

$$(1-\mathbb{S}\|T\|)\int_\Omega|\varepsilon^D(u)| + \left(\varkappa - \frac{1}{2}\mathbb{S}\|T\|\right)\int_\Omega(\operatorname{div} u)^2\,dx \le 0.$$

Thus an elastic state must exist in Ω if

$$\|T\| < \frac{1}{\mathbb{S}}\min\{1,2\varkappa\}. \tag{6.3}$$

Furthermore, the weakened condition

$$\|T\| \le \frac{1}{\mathbb{S}} \tag{6.4}$$

may be used to show that $I(v)$ is bounded below in $P_e(\Omega)$ and thus forms the basis of an existence theory. A limit analysis would provide a more careful estimate of the best constant $\mathbb{S}$ [28].

7. Prisms

Left unresolved so far is whether or not both elastic and plastic states may be present in the same body, or, more generally, how plastic states arise. The convenient form assumed by special solutions of antiplanar shear and torsion problems exhibit elastic/plastic behavior and even plastic slippage, or fracture.

Given an infinite prism

$$\Omega = \Sigma \times \mathbb{R}, \qquad \Sigma \subset \mathbb{R}^{n-1} \text{ Lipschitz region},$$

and an equilibrated force distribution independent of the axial coordinate x_n, an evident definition of a solution arises by consideration of the restricted functional, for each interval $J=(a,b)$,

$$I_J(v) = \int_{\Sigma\times J}\varphi(\varepsilon^D(v)) + \frac{\varkappa}{2}\int_{\Sigma\times J}(\operatorname{div} v)^2\,dx - \int_{\Sigma\times J} g\cdot v\,dS\,dx_n \tag{7.1}$$

A displacement u is a solution (in Ω) provided that for any interval J,

$$\begin{aligned} &u \in P(\Sigma\times J): I_J(u) \le I_J(u+\zeta)\\ &\zeta \in P(\Omega), \quad \operatorname{supp}\zeta \subset \bar{\Sigma}\times J. \end{aligned} \tag{7.2}$$

Note that ζ need not vanish on $\partial\Sigma\times J$. The functions f_i and g_i depend only on $x'=(x_1,\ldots,x_{n-1})$.

The easiest form to use in actual calculation is not (7.2) but the saddle point condition (3.6). To express this, introduce the stress energy

$$I_J^*(\tau) = \frac{1}{2}\int_{\Sigma\times J}|\tau^D|^2\,dx + \frac{1}{2\varkappa n^2}\int_{\Sigma\times J}(\operatorname{tr}\tau)^2\,dx \tag{7.3}$$

and the distribution

$$\langle T_J, \zeta\rangle = \int_{\Sigma\times J} f\cdot\zeta\,dx + \int_{\partial\Sigma\times J} g\cdot\zeta\,dS\,dx_n + \int_{\Sigma\times b}(\sigma\cdot e_n)\zeta\,dx'$$
$$-\int_{\Sigma\times a}(\sigma\cdot e_n)\cdot\zeta\,dx' \tag{7.4}$$

where

$$\sigma = \sigma(u) = \sigma^D(u) + \varkappa\,\mathrm{div}\,u\mathbf{1},$$

is the stress tensor of the given solution of (7.2). According to the first variation formula (2.14)

$$\begin{aligned} -\mathrm{div}\,\sigma &= f \quad \text{in } \Omega \\ \sigma\nu &= g \quad \text{on } \partial\Sigma\times\mathbb{R} \end{aligned} \tag{7.5}$$

in the sense of distributions, where ν is the outward normal to $\partial\Sigma$. That f_i, g_i define an equilibrated system of forces means that for any affine rigid motion γ

$$\int_{\Sigma\times J} f\cdot\gamma\,dx + \int_{\partial\Sigma\times J} g\cdot\gamma\,dS\,dx_n = 0.$$

Hence

$$\begin{aligned} \langle T_J, \gamma\rangle &= \int_{\Sigma\times b}(\sigma e_n)\gamma\,dx' - \int_{\Sigma\times a}(\sigma e_n)\gamma\,dx' \\ &= \int_{\Sigma\times J}\mathrm{div}\,\sigma\cdot\gamma\,dx - \int_{\partial\Sigma\times J}(\sigma\nu)\cdot\gamma\,dS\,dx_n \\ &= -\int_{\Sigma\times J} f\cdot\gamma\,dx - \int_{\partial\Sigma\times J} g\cdot\gamma\,dS\,dx_n \\ &= 0 \end{aligned}$$

by (7.5). Thus T_J is also equilibrated so it makes sense to pose Prob. 1.1 for the functional

$$\tilde{I}_J(v) = \int_{\Sigma\times J}\varphi(\varepsilon^D(v)) + \frac{\varkappa}{2}\int_{\Sigma\times J}(\mathrm{div}\,v)^2\,dx - \langle T_J, v\rangle. \tag{7.6}$$

Note that if γ is a rigid motion then

$$\tilde{I}_J(v+\gamma) = \tilde{I}_J(v).$$

According to Thm. 3.3,

$$\inf_{P_e(\Sigma\times J)}\tilde{I}_J(v) = -\inf_{\mathbb{K}} I_J^*(\tau)$$

and

$$\tilde{I}_J(u) = -I_J^*(\bar{\sigma})$$

if and only if u is a solution of Prob. 1.1 and $\bar{\sigma} = \sigma(u)$. Given an interval J, assume that u has been adjusted by a rigid motion so that $u \in P_e(\Sigma \times J)$. Choose $\eta \in H^{1,\infty}(\mathbb{R})$,

$$\begin{aligned} \eta &= 1 \quad \text{in } J \\ \eta &= 0 \quad \text{outside } (a-h, b+h) \end{aligned}$$

and η linear in $(a-h, a)$ and in $(b, b+h)$. The first variation formula, or a trivial variant of it, may be applied to $\eta u = \zeta$ so

$$\int_\Omega \sigma \cdot \varepsilon(\zeta) = \int_\Omega f \cdot \zeta\, dx + \int_{\partial\Sigma \times \mathbb{R}} g \cdot \zeta\, dS\, dx_n.$$

Expanding this and letting $h \to 0$ yields precisely that

$$\int_{\Sigma \times J} \hat{\sigma} \cdot \varepsilon(u) = \langle T_J, u \rangle. \tag{7.7}$$

Separating the left- and right-hand sides into elastic and plastic regions as in the proof of Thm. 3.2, one obtains that

$$\tilde{I}_J(u) = -\tilde{I}_J^*(\sigma) \tag{7.8}$$

or, explicitly

$$\begin{aligned} &\int_{\Sigma \times J} \varphi(\varepsilon^D(u)) + \frac{\varkappa}{2} \int_{\Sigma \times J} (\operatorname{div} u)^2\, dx - \left\{ \int_{\Sigma \times J} f \cdot u\, dx + \int_{\partial\Sigma \times J} g \cdot u\, dS\, dx_n \right. \\ &\qquad \left. + \int_{\Sigma \times b} (\sigma e_n) u\, dx' - \int_{\Sigma \times a} (\sigma e_n) u\, dx' \right\} \\ &\quad = -\left\{ \frac{1}{2} \int_{\Sigma \times J} |\sigma^D|^2\, dx + \frac{1}{2\varkappa n^2} \int_{\Sigma \times J} (\operatorname{tr} \sigma)^2\, dx \right\}. \end{aligned} \tag{7.9}$$

Observe that (7.7) and (7.8) are equivalent. Thus u is a solution of Prob. 1.1 for $\Sigma \times J$, for every J, if and only if it is a solution of (7.2). More exactly

Proposition 7.1. *Let $u \in P(\Sigma \times J)$ for each (bounded) interval J. The displacement u is a solution of the prism problem* (7.2) *if and only if u satisfies the saddle point conditions* (7.7) *or* (7.8) *for each J.*

7.1. Antiplanar Shear

Antiplanar shear of Ω consists in finding solutions of (7.2) which have the form

$$u(x) = (0,\ldots,0,u^n(x')), \qquad x' \in \Sigma. \tag{7.10}$$

For motivation, a few remarks about the nature of solutions of (2.2) not subject to (7.10) might be pertinent. Since T is independent of the axial coordinate x_n and the boundary conditions involve only functions on $\partial\Sigma \times \mathbb{R}$, whenever u is a solution, the displacements

$$\begin{aligned} v(x) &= u(x', x_n + c), \qquad c \in \mathbb{R}, \text{ and} \\ v(x) &= u(x) + \gamma(x), \qquad \gamma \text{ an affine rigid motion,} \end{aligned}$$

are again solutions, which furthermore have the same stress tensor. Uniqueness may fail in a more significant manner, for example, if Ω is all elastic with respect to u, the addition of a small St. Venant's torsion solution (e.g., [19] p. 310) provides another solution with a different stress tensor.

Solutions in antiplanar shear may be found by means of an obvious variational principle. If $v = (0,\ldots,0,v^n(x'))$, $x' \in \Sigma$, then

$$\begin{aligned} \varepsilon(v) &= \varepsilon^D(v) = \tfrac{1}{2}(e_n \otimes Dv^n + Dv^n \otimes e_n), \\ |\varepsilon(v)| &= \frac{1}{\sqrt{2}}|Dv^n|, \\ Dv^n &= \text{gradient of } v^n, \end{aligned}$$

and $v \in BD(\Sigma \times J)$ if and only if $v^n \in BV(\Sigma)$, the functions of bounded variation in Σ. Consequently with L the length of the interval J,

$$\begin{aligned} I_J(v) &= LF(v^n), \\ F(v^n) &= \int_\Sigma \varphi\left(\frac{1}{\sqrt{2}} Dv^n\right) - \int_\Sigma f_n v^n \, dx' - \int_{\partial\Sigma} g_n v^n \, dS. \end{aligned} \tag{7.11}$$

This leads to the discussion of

Problem 7.1. To find $\psi \in BV(\Sigma)$: $F(\psi) = \inf_{BV_e(\Sigma)} F(w)$, where

$$BV_e(\Sigma) = \left\{ w \in BV(\Sigma) : \int_\Sigma w \, dx' = 0 \right\}.$$

Considerations identical to those of Sect. 6 show that a solution ψ exists whenever

$$\|f_n\|_{L^n(\Sigma)} + \|g_n\|_{L^\infty(\Sigma)} \le 1/\mathcal{S}_0$$

where $\mathbb{S}_0$ is the isoperimetric constant in the estimation

$$\left(\int_\Sigma |w|^{n/n-1}dx'\right)^{1-1/n} + \int_{\partial\Sigma}|w|dS \leq \mathbb{S}_0\int_\Sigma |Dw| \quad \text{for } w \in BV_e(\Sigma).$$

So the sizes of f_n and g_n, not their smoothness, govern the existence of a solution.

For a function $w \in BV(\Sigma)$ its gradient will be written

$$Dw = w_{x'}dx' + D^s w, \quad \text{where } dx' \perp D^s w. \tag{7.12}$$

Set

$$q = \begin{cases} (w_{x'}, 0) & |w_{x'}| < \sqrt{2} \\ \left(\dfrac{w_{x'}}{|w_{x'}|}, 0\right) & |w_{x'}| \geq \sqrt{2}. \end{cases}$$

With these notations, the displacement $v = (0, \ldots, 0, w)$ has the stress tensor

$$\sigma(v) = \frac{1}{2}(q \otimes e_n + e_n \otimes q) = \begin{pmatrix} & & q_1 \\ & 0 & \vdots \\ & & q_{n-1} \\ q_1 & \cdots\, q_{n-1} & 0 \end{pmatrix}$$

which has the special property

$$(\sigma(v)e_n)\cdot v = (q \cdot e_n)\cdot v = 0.$$

This facilitates checking (7.7) or (7.9) for the $u = (0, \ldots, 0, \psi)$ obtained from the solutions ψ of Problem 7.1, whose details are omitted in the interest of brevity.

7.2. *Existence of a Plastic State*

Examination of the behavior of f and g leads to prediction of a plastic state. To fix the ideas, assume that $\Sigma \subset \mathbb{R}^2$ contains the semidisc $G = \{|x| < 1, x_2 > 0\}$, with the segment $(-1, 1)$ of the real axis in $\partial\Sigma$, and that $u = (0, 0, \psi)$ is a solution of the problem of antiplanar shear in which

$$f_2 = 0 \quad \text{in } G$$

$$g_2 = \begin{cases} 0 & x_1 > 0, x_2 = 0 \\ \delta & x_1 < 0, x_2 = 0 \end{cases}, \qquad \delta \neq 0, \quad \delta \in \mathbb{R}.$$

For a solution elastic in all of G, ψ would satisfy

$$\begin{aligned} \Delta\psi &= 0 \quad \text{on } G \\ \psi_{x_2} &= g_2 \quad \text{on } (-1, 1) \end{aligned} \tag{7.13}$$

and

$$|D\psi| < \sqrt{2} \text{ in } G. \tag{7.14}$$

Such a ψ satisfying (7.13) is given by

$$\psi(x') = \frac{\delta}{\pi}\operatorname{Im}(z\log z) + h(x'), \qquad z = x_1 + ix_2, \quad x_2 > 0 \tag{7.15}$$

when $h(x')$ is harmonic in a neighborhood of $x'=0$. The gradient of ψ fails to obey (7.14). Thus any solution with f_2, g_2 as above must have some plastic set.

Since f, g may be chosen so that

$$\|f_2\|_{L^2(\Sigma)} + \|g_2\|_{L^1(\partial\Sigma)} = \delta < 1/\mathbb{S}_0,$$

the argument of Sect. 6 may be employed to show the presence of an elastic set. Even if f and g are smooth with large gradients the same notions lead to states of stress where both elastic and plastic regions are present. The example of antiplanar shear was adopted for ease of presentation; a similar analysis holds for solutions of Prob. 1.1.

7.3. *Examples in Antiplanar Shear*

In the sequel we use the notation

$$Dw = w'(t)\,dt + D^s w, \; D^s w \perp dt$$

for a real valued function of the real variable t of bounded variation.

A first example of what might be termed fracture in antiplanar shear may be obtained by scaling the example Prob. 1.2. and studying the transition as the forces are increased. Let $\Sigma = (0,1)\times(0,1) \subset \mathbb{R}^2$ and $\Omega = \Sigma \times \mathbb{R}$. Set $f = 0$ and

$$g = \begin{cases} -\delta e_3 & x_1 = 0, \qquad 0 < x_2 < 1 \\ \delta e_3 & x_1 = 1, \qquad 0 < x_2 < 1 \\ 0 & \text{elsewhere on } \partial\Sigma \end{cases}$$

for $|\delta| < \dfrac{1}{\sqrt{2}}$, a solution of (7.2) as given by

$$u(x) = (0,0,2\delta x_1), \qquad 0 < x_1 < 1,$$

which corresponds to simple shear of a linearly elastic body. For $\delta = 1/\sqrt{2}$, solutions of (7.2) are given by

$$u(x) = (0,0,\psi(x_1)) \qquad 0 < x_1 < 1 \tag{7.16}$$

where ψ is any monotone function on $(0,1)$ with $\psi' \geq \sqrt{2}$.

The stress tensor associated to u is

$$\sigma = \frac{1}{\sqrt{2}}(e_1 \otimes e_3 + e_3 \otimes e_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{7.17}$$

If ψ is smooth, the body is merely stretched in some fashion. Discontinuities of ψ represent planes of slippage in the material occupying Ω.

The slippage or fracture exhibits other forms, even with the same stress (7.17). Let $\eta(x_3)$, $x_3 \in \mathbb{R}$, be a monotone function with $d\eta \perp dx_3$, and let ψ be as in (7.16). Then

$$u(x) = (\eta(x_3), 0, \psi(x_1)), \qquad 0 < x_1 < 1, \quad x_3 \in \mathbb{R} \tag{7.18}$$

also satisfies the conditions of Prop. 7.1.

The example illustrates that discontinuities in u can arise in more than one way. But these cannot be arbitrary, for if u has a discontinuity along a surface with normal ν, then this discontinuity is in the direction $\sigma^D \nu$ and $|\sigma^D \nu| = 1/\sqrt{2}$. Other restrictions must also be obeyed; these will be studied in a subsequent note.

7.4. Examples in Torsion

Torsion of the prism Ω is a solution u of (7.2) with $f = 0 = g$. The trivial solution need not be the only one, as mention of the classical St. Venant solution illustrates. Frequently solutions are sought in an analogous form, $(n = 3)$,

$$u(x) = \tau(-x_2 x_3, x_1 x_3, \psi(x_1, x_2)) \qquad x \in \Omega \tag{7.19}$$

where $\tau \in \mathbb{R}$ is a parameter. One class of examples in this category is given by merely extending the domains of the solution of elastic plastic torsion of a finite cylindrical column. This problem has been studied extensively [4, 6, 7, 8, 12, 29, 30].

One illustration concerns the right circular cylinder

$$\Omega = B \times \mathbb{R}, \quad B = \{x' = (x_1, x_2) : |x'| < 1\}.$$

Here a solution is

$$u(x) = \tau(-x_2 x_3, x_1 x_2, 0), \qquad x \in \Omega,$$

the same as the St. Venant's solution for the elastic cylinder.

It would be of interest to classify all torsion solutions of (7.2), or for example, those which satisfy

$$\int_{\Sigma \times (k, k+1)} \left(|\varepsilon^D(u)| + (\operatorname{div} u)^2 \, dx\right) \leq M$$

for each $k \in \mathbb{R}$. They needn't have the representation (7.19).

To illustrate the last remark, consider the hollow cylinder

$$\Omega = A \times \mathbb{R}, \qquad A = \{1 < |x'| < 2\}.$$

As in the case of the circular cylinder, one solution is

$$u(x) = \tau(-x_2x_3, x_1x_3, 0) \qquad x \in \Omega$$

with

$$\varepsilon(u) = \frac{\tau}{2}\begin{pmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

$$|\varepsilon(u)| = \frac{|\tau|}{\sqrt{2}}\rho, \qquad \rho = |x'|,$$

and stress tensor

$$\sigma = \begin{cases} \varepsilon(u) & \dfrac{|\tau|}{\sqrt{2}}\rho < 1 \\ \bar{\sigma} & \dfrac{|\tau|}{\sqrt{2}}\rho \geq 1 \end{cases},$$

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}.$$

In particular, if $\tau \geq \sqrt{2}$, the configuration is entirely plastic. Now new solutions may be found by setting

$$u(x) = (-\tau(x_3)x_2, \ \tau(x_3)x_1, 0), \qquad x \in \Omega,$$

where τ is any monotone function of x_3 satisfying

$$\tau' \geq \sqrt{2} \text{ a.e.}$$

8. Appendix—Approximation

The approximation theorem given here, valid for Lipschitz domains, is used in the discussion of duality. Unfortunately its proof is technical. A similar theorem is due to Anzellotti and Giaquinta [4], with a different proof. We state Thm. A.1 below since (A.1) is the basis of (A.4).

Theorem A.1. *Let $v \in BD(\Omega)$, $\operatorname{supp} v \subset\subset \Omega$. Then there are $v_\delta \in C_0^\infty(\Omega)$ such that*

$$\lim_{\delta \to 0} \int_\Omega \psi \cdot \varepsilon(v_\delta)\, dx = \int_\Omega \psi \cdot \varepsilon(v) \tag{A.1}$$

for any continuous symmetric tensor ψ on Ω.

Since the proof of this statement is routine, and anyway may be seen as a special case of Thm. A.2 below, it is omitted.

In $I(v)$ given by (1.5), let T be defined by

$$\langle T, \zeta \rangle = \int_\Omega f \cdot \zeta\, dx + \int_{\partial\Omega} g \cdot \zeta\, dS, \qquad \zeta \in P(\Omega), \tag{A.2}$$

where

$$f \in L^n(\Omega) \quad \text{and} \quad g \in L^\infty(\partial\Omega).$$

Theorem A.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Given $v \in P(\Omega)$, there exist $v_\delta \in C^\infty(\overline{\Omega})$, $0 < \delta < 1$, with the properties*

$$\begin{gathered} v_\delta \to v \text{ in } L^{n/n-1}(\Omega) \cap L^1(\partial\Omega) \text{ and} \\ I(v_\delta) \to I(v) \text{ as } \delta \to 0 \end{gathered} \tag{A.3}$$

if T satisfies (A.2);

$$\lim_{\delta \to 0} \int_\Omega \psi \cdot \varepsilon(v_\delta)\, dx = \int_\Omega \psi \cdot \varepsilon(v) \tag{A.4}$$

for any continuous symmetric tensor ψ on $\overline{\Omega}$,

$$\lim_{\delta \to 0} \int_E |\varepsilon^D(v_\delta)|\, dx = \int_E |\varepsilon^D(v)|, \tag{A.5}$$

$$\lim_{\delta \to 0} \int_E \varphi(|\varepsilon^D(v_\delta)|)\, dx = \int_E \varphi(\varepsilon^D(v)), \text{ and} \tag{A.6}$$

$$\lim_{\delta \to 0} \int_E (\operatorname{div} v_\delta)^2\, dx = \int_E (\operatorname{div} v)^2\, dx \tag{A.7}$$

whenever $E \subset \Omega$ is $dx + |\varepsilon(v)^s|$-measurable and

$$\int_{\partial E} dx + \int_{\partial E} |\varepsilon(v)^s| = 0.$$

Proof. The demonstration is separated into three parts.

Step A.1. Convolution. Let $U \subset \mathbb{R}^n$ be open and $E \subset U$ be open. It is obvious that for $w \in BD(U)$,

$$\int_E |\varepsilon(w)| = \sup_{\Psi}\left\{-\int_E w \operatorname{div}\psi\, dx\right\}$$

$$\Psi = \Psi_E = \{\psi = (\psi_{ij}) \in C_0^1(E) : |\psi| \le 1, \psi = \psi^T\}.$$

This formula renders evident that $\int_E |\varepsilon(w)|$ is lower semicontinuous with respect to convergence in $L^1(E)$. To find an analogous expression for the functional in (A.6), supposing momentarily that w is smooth, one checks that

$$\int_E \varphi(|\varepsilon^D(w)|)\, dx = \frac{1}{2}\int_E \{|\varepsilon^D(w)|^2 - |\varepsilon^D(w) - \sigma^D(w)|^2\}\, dx$$

$$= \sup_{\tilde{\Psi}} \frac{1}{2}\int_E (|\varepsilon^D(w)|^2 - |\varepsilon^D(w) - \psi|^2)\, dx$$

where

$$\tilde{\Psi} = \tilde{\Psi}_E = \{\psi \in \Psi_E : \operatorname{tr}\psi = 0\}.$$

After an integration by parts, one is led to the formula

$$\int_E \varphi(|\varepsilon^D(w)|)\, dx = \sup_{\tilde{\Psi}} -\int_E \left(\frac{1}{2}|\psi|^2 + w\cdot\operatorname{div}\psi\right) dx$$

an expression which has meaning for any $w \in BD(U)$. In fact, it is easily verified that ([5, 24])

$$\int_E \varphi(\varepsilon^D(w)) = \sup_{\tilde{\Psi}} -\int_E \left(\frac{1}{2}|\psi|^2 + w\cdot\operatorname{div}\psi\right) dx, \qquad w \in BD(E). \tag{A.8}$$

As in the previous case, lower semicontinuity of the functional with respect to convergence in $L^1(E)$ is answered by (A.8).

Let $\bar{\Omega} \subset U$, where U is open, and let $E \subset \bar{\Omega}$ satisfy the hypotheses of the theorem. Let α_k, $k > 0$, be a family of mollifiers. Set

$$E_\lambda = \{x : \operatorname{dist}(x, E) < \lambda\}, \qquad \lambda > 0,$$

small enough that $E_\lambda \subset U$.

Given $w \in BD(U)$,

$$w_h = w * \alpha_h \to w \quad \text{in } L^{n/n-1}_{\mathrm{loc}}(U)$$

so in particular,

$$\int_E \varphi\big(\varepsilon^D(w)\big) \le \liminf_{h\to 0} \int_E \varphi\big(|\varepsilon^D(w_h)|\big)\,dx. \tag{A.9}$$

Let $\eta \in C_0^\infty(E_{2\lambda})$ satisfy $\eta = 1$ on E_λ, $0 \le \eta \le 1$. From (A.8), with $\theta_h = \sigma^D(w_h)$,

$$\begin{aligned}
\int_E \varphi\big(|\varepsilon^D(w_h)|\big)\,dx &\le \int_{E_\lambda} \varphi\big(|\varepsilon^D(w_h)|\big)\,dx \\
&= \frac{1}{2}\int_{E_\lambda} \big(|\varepsilon^D(w_h)|^2 - |\varepsilon^D(w_h) - \theta_h|^2\big)\,dx \\
&\le \frac{1}{2}\int_{E_{2\lambda}} \big(|\varepsilon^D(w_h)|^2 - |\varepsilon^D(w_h) - \eta\theta_h|^2\big)\,dx \\
&= -\int_{E_{2\lambda}} \Big(\frac{1}{2}|\eta\theta_h|^2 + w_h\cdot \operatorname{div}(\eta\theta_h)\Big)\,dx \\
&= -\frac{1}{2}\int_{E_{2\lambda}} \Big(\frac{1}{2}|\eta\theta_h^*\alpha_h|^2 + w\cdot \operatorname{div}(\eta\theta_h^*\alpha_h)\Big)\,dy \\
&\quad + \frac{1}{2}\int_{E_{2\lambda+h}} \big(|\eta\theta_h^*\alpha_h|^2 - |\eta\theta_h|^2\big)\,dy \\
&= -A + B,
\end{aligned}$$

after interchanging the order of integration. Now $\eta\theta_h \in \tilde{\Psi}_{E_{2\lambda}}$, from which it follows that $\eta\theta_h^*\alpha_h \in \tilde{\Psi}_{E_{2\lambda+h}}$. Thus

$$-A \le \int_{E_{2\lambda+h}} \varphi\big(\varepsilon^D(w)\big).$$

It is a standard fact about convolutions that

$$\int_{E_{2\lambda+h}} |\eta\theta_h^*\alpha_h|^2\,dx \le \int_{E_{2\lambda+h}} |\eta\theta_h|^2,$$

so $B \le 0$. Consequently

$$\int_E \varphi\big(|\varepsilon^D(w_h)|\big)\,dx \le \int_{E_{2\lambda+h}} \varphi\big(\varepsilon^D(w)\big), \qquad \lambda > 0,\quad h > 0$$

so small that $E_{2\lambda+h} \subset U$. It follows from this that

$$\limsup_{h\to 0} \int_E \varphi\big(|\varepsilon^D(w_h)|\big)\,dx \le \int_E \varphi\big(\varepsilon^D(w)\big). \tag{A.10}$$

From (A.9) and (A.10),

$$\lim_{h \to 0} \int_E \varphi(|\varepsilon^D(w_h)|)\, dx = \int_E \varphi(\varepsilon^D(w)) \tag{A.11}$$

whenever

$$\int_{\partial E} dx + \int_{\partial E} |\varepsilon^D(w)^s| = 0. \tag{A.12}$$

Furthermore,

$$\lim_{h \to 0} \int_E |\varepsilon(w_h)|\, dx = \int_E |\varepsilon(w)| \tag{A.13}$$

when (A.12) holds. If in addition,

$$\operatorname{div} w \in L^{n/n-1}(U),$$

then

$$\lim_{h \to 0} \|\operatorname{div} w_h - \operatorname{div} w\|_{L^{n/n-1}(\Omega)} = 0. \tag{A.14}$$

Step A.2. Approximation of $v \in P(\Omega)$ by $w \in BD(U)$ having $\operatorname{div} w \in L^{n/n-1}(U)$, $\Omega \subset U$.

Here we follow Morrey [20, p. 76]. Since $\partial\Omega$ is Lipschitz, for each $x_0 \in \partial\Omega$, there is a cylinder $Q_{r,L}(x_0, a_0)$ with axis in the direction a_0, $|a_0| = 1$, such that after a suitable rotation and translation of axes, say $y = C(x - x_0)$,

$$B_r' \times (-L, L) = C(Q_{r,L}(x_0, a_0) - x_0), \qquad B_{B'} = \{|y'| < r, y' \in \mathbb{R}^{n-1}\}$$

and

$$\{y : g(y') < y_n < L\} = C(Q_{r,L}(x_0, a_0) \cap \Omega - x_0)$$

where g is a Lipschitz function. The Lipschitz constant of g, r, and L may be taken independent of $x_0 \in \partial\Omega$.

Let $\Omega_1, \dots, \Omega_N$ be a finite covering of $\partial\Omega$ by such cylinders with the property that

$$\sum_1^N \int_{\Omega \cap \Omega_j} |\varepsilon(v)| < \frac{1}{4}\delta \text{ and}$$

$$\sum_1^N \|\operatorname{div} v\|_{L^{n/n-1}(\Omega \cap \Omega_j)} < \frac{1}{4}\delta. \tag{A.15}$$

Let Ω_0 be a neighborhood of $\Omega - \bigcup_1^N \Omega_j$ in Ω satisfying $\operatorname{dist}(\Omega_0, \partial\Omega) \le L/2$ and choose $\eta_0 \in C_0^\infty(\Omega)$, $\eta_0 = 1$ on Ω_0, $0 \le \eta_0 \le 1$. Let $\eta_j \in C_0^\infty(\Omega_j)$, $1 \le j \le N$, be such that $0 \le \eta_j \le 1$ and

$$\sum_0^N \eta_j = 1 \text{ in a neighborhood of } \Omega.$$

Note that whenever $\psi(x)$ is defined in $\Omega \cap \Omega_j$, $\tilde{\psi}(x) = \psi(x + \lambda a_j)\eta_j(x)$, a_j the axis direction of Ω_j, is defined whenever $x + \lambda a_j \in \Omega_j$. Also, $\eta_j(x) = 0$ whenever $(x - x_j)\cdot a_j > L/2$ where $x_j \in \partial\Omega$ is the "center" of Ω_j, by the choice of Ω_0.

Now set

$$\begin{aligned} w(x) = F_\lambda v(x) &= \sum_0^N \eta_j(x) v(x + \lambda a_j) \\ &= \eta_0(x)v(x) + \sum_1^N \eta_j(x) v(x + \lambda a_j) \\ &= \eta_0(x)v(x) + \sum_1^N \eta_j(x) v_{\lambda,j}(x) \end{aligned} \tag{A.16}$$

which is well defined in a neighborhood $U = U_\lambda \subset \bigcup_0^N \Omega_j$, $\sum \eta_j = 1$ in U.

The claim is that for $\lambda > 0$ sufficiently small, $w = F_\lambda v$ provides a suitable approximation to v. First observe that

$$\begin{aligned} \|w - v\|_{L^{n/n-1}(\Omega)} &= \left\| \eta_0 v + \sum_1^N \eta_j v_{\lambda,j} - \sum_0^N \eta_j v \right\|_{L^{n/n-1}(\Omega)} \\ &\le \sum_1^N \|\eta_j(v_{\lambda,j} - v)\|_{L^{n/n-1}(\Omega\cap\Omega_j)}. \end{aligned}$$

Since $v_{\lambda,j} \to v$ in $L^{n/n-1}(\Omega \cap \Omega_j)$, for λ sufficiently small

$$\|w - v\|_{L^{n/n-1}(\Omega)} < \delta. \tag{A.17}$$

Let us give the proof of (A.6). For $\delta > 0$, $t > 0$,

$$|\varphi(s + t) - \varphi(t)| \le s \tag{A.18}$$

Also, since $\sum_0^N \eta_{jx} = 0$ in Ω,

$$\begin{aligned}\varepsilon^D(w) &= \sum_0^N \eta_j \varepsilon^D(v_{\lambda,j}) + \frac{1}{2}\sum_0^N \left(\eta_{jx} \otimes v_{\lambda j} + v_{\lambda,j} \otimes \eta_{jx} - \frac{2}{n}\eta_{jx} \cdot v_{\lambda,j} \mathbb{1} \right) \\ &= \sum_0^N \eta_j \varepsilon^D(v_{\lambda,j}) \\ &\quad + \frac{1}{2}\sum_0^N \left(\eta_{jx} \otimes (v_{\lambda,j} - v) + (v_{\lambda,j} - v) \otimes \eta_{jx} - \frac{2}{n}\eta_{jx} \cdot (v_{\lambda,j} - v) \mathbb{1} \right)\end{aligned}$$

and

$$\varepsilon^D(v) = \sum_0^N \varepsilon^D(\eta_j v) = \sum_0^N \eta_j \varepsilon^D(v).$$

Writing $E_j = E \cap \Omega_j$ for $E \subset \Omega$, and using (A.18),

$$\begin{aligned}&\left| \int_E \varphi(\varepsilon^D(w)) - \int_E \varphi(\varepsilon^D(v)) \right| \\ &\quad \le \left| \int_{E_0} \varphi(\eta_0 \varepsilon^D(v)) - \int_{E_0} \varphi(\eta_0 \varepsilon^D(v)) \right| + \sum_1^N \int_E \eta_j |\varepsilon^D(v_{\lambda,j})| \\ &\quad + \frac{1}{2}\sum_0^N \int_E \left| \eta_{jx} \otimes (v_{\lambda,j} - v) + (v_{\lambda,j} - v) \otimes \eta_{jx} - \frac{2}{n}\eta_{jx} \cdot (v_{\lambda,j} - v) \mathbb{1} \right| dx \\ &\quad + \sum_1^N \int_{E_j} \eta_j |\varepsilon^D(v)|.\end{aligned}$$

Since for λ sufficiently small

$$\int_\Omega \eta_j |\varepsilon^D(v_{\lambda,j})| \le \int_{\Omega \cap \Omega_j} |\varepsilon^D(v)|$$

and $v_{\lambda,j} \to v$ in $L^1(\Omega \cap \Omega_j)$ as $\lambda \to 0$, it is possible to choose λ so small that (cf. (A.15))

$$\left| \int_E \varphi(\varepsilon^D(w)) - \int_E \varphi(\varepsilon^D(v)) \right| < \frac{\delta}{2}$$

A similar but easier calculation may be used to prove (A.5).

Now observe that

$$\begin{aligned}\operatorname{div} w &= \sum_0^N \eta_j \operatorname{div} v_{\lambda,j} + \sum_0^N \eta_{jx}\cdot v_{\lambda,j} \\ &= \sum_0^N \eta_j \operatorname{div} v_{\lambda,j} + \sum_0^N \eta_{jx}\cdot (v_{\lambda j} - v)\end{aligned} \qquad \text{in } \Omega.$$

Although $\operatorname{div} v_{\lambda,j} \to \operatorname{div} v$ in $L^2(\Omega\cap\Omega_j)$, one only knows that $v_{\lambda,j}\to v$ in $L^{n/n-1}(\Omega\cap\Omega_j)$. This leads to (A.14).

Step A.3. Approximation of $v\in P(\Omega)$ by $w_\delta \in C^\infty(\overline{\Omega})$.

Combining the previous steps, given $\delta>0$ choose $w=F_\lambda v$, cf. (A.16), and $w_\delta = w*\alpha_h$, h so small that

$$\|w_\delta - v\|_{L^{n/n-1}(\Omega)} < \delta/2$$

$$\left|\int_\Omega |\varepsilon(w_\delta)|\,dx - \int_\Omega |\varepsilon(v)\right| < \delta/2$$

$$\left|\int_\Omega \varphi\big(|\varepsilon^D(w_\delta)|\big)\,dx - \int_\Omega \varphi\big(\varepsilon^D(v)\big)\right| < \delta/2, \text{ and}$$

$$\|\operatorname{div} w_\delta - \operatorname{div} v\|_{L^{n/n-1}(\Omega)} < \delta/2.$$

Step A.4. Correction for divergence term.

The approximation w_δ of the last step must be adjusted slightly to account for the divergence. Suppose that R is large enough that $\Omega\subset\subset B_R$ and choose $z_\delta \in C_0^\infty(B_R)$ satisfying

$$\|z_\delta - \chi_\Omega \operatorname{div} v\|_{L^2(B_R)} < \delta,$$

$\chi_\Omega =$ the characteristic function of Ω. Let ζ_δ denote the solution of

$$\begin{aligned}\Delta\zeta_\delta &= z_\delta - \chi_\Omega \operatorname{div} w_\delta && \text{in } B_R\\ \zeta_\delta &= 0 && \text{on } \partial B_R\end{aligned}$$

and set $q_\delta = \zeta_{\delta x}$. Then by the well known estimates of Calderon and Zygmund (cf. [20])

$$\begin{aligned}\|q_\delta\|_{H^{1,n/n-1}(B_R)} &\le \|\zeta_\delta\|_{H^{2,n/n-1}(B_R)}\\ &\le C\|z_\delta - \operatorname{div} w_\delta \chi_\Omega\|_{L^{n/n-1}(B_R)}\\ &\le C\Big(\|z_\delta - \chi_\Omega \operatorname{div} v\|_{L^{n/n-1}(B_R)}\\ &\qquad + \|\operatorname{div} v - \operatorname{div} w_\delta\|_{L^{n/n-1}(\Omega)}\Big)\\ &\le C_1\delta. \end{aligned} \tag{A.19}$$

Choosing δ/C_1 instead of δ, we may assume that $C_1 = 1$. Finally, we set

$$v_\delta = w_\delta + q_\delta$$

so

$$\|\operatorname{div} v_\delta - \operatorname{div} v\|_{L^2(\Omega)} < \delta.$$

Using (A.19) and the estimates for w_δ it is an easy matter to check that the sequence v_δ has the required properties. For this, note especially (A.18). □

Note Added in Proof (4/20/83)

Our treatment of duality has elements in common with a new paper of F. Demengel and R. Temam, Convex function of a measure and applications (to appear).

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