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# Bounds for the First Eigenvalue of a Spherical Cap

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**Abstract.** We study upper and lower bounds for the lowest eigenvalue $\lambda$ of the Laplace operator on a spherical cap $C_\theta$ in $m$-dimensional space ($m \geq 3$). We prove that these bounds are sharp by finding asymptotic expressions for $\lambda$ as $\theta \to \pi$ and as $\theta \to 0$.

## 1. Introduction

The purpose of this paper is to study upper and lower bounds as well as asymptotic behavior of the lowest eigenvalue $\lambda$ of the Laplacian on a spherical cap in $m$-dimensional space, $m \geq 3$.

In Section 2 we find the bounds mentioned above by using probabilistic methods. For the case $m = 3$ these are compared to the ones given by Pinsky [5].

Sect. 3 is devoted to the asymptotic expressions of $\lambda$ as the cap tends to the whole sphere, and also as the radius of the cap tends to zero. In order to obtain these expressions we use a result of Hobson [4] concerning zeroes of Legendre functions as well as the bounds for $\lambda$ given by Friedland and Hayman [3].

Finally, in Sect. 4 we examine the sharpness of our results.

## 2. Upper and Lower Bounds

Consider the eigenvalue problem

$$\left[(\sin\theta)^{2-m}\frac{d}{d\theta}\left((\sin\theta)^{m-2}\frac{dF}{d\theta}\right)\right] + \lambda F = 0 \tag{2.1}$$

$$F(\theta_0) = 0, \qquad 0 < \theta < \theta_0 < \pi,$$

and let $(X_t)$ be the diffusion on $(0, \pi)$ with generator

$$L = \frac{1}{2}\left[(\sin\theta)^{2-m}\frac{d}{d\theta}\left((\sin\theta)^{m-2}\frac{d}{d\theta}\right)\right]$$

killed when it reaches $\theta_0$. Define

$$T_{\theta_0} = \inf\{t > 0 : X_t = \theta_0\}$$

and let

$$Vf(\theta) = E^\theta \int_0^{T_{\theta_0}} f(X_t)\, dt$$

be the associated potential operator.

If $\mathfrak{D}_L$ denotes the domain of $L$, one has for $f \in \mathfrak{D}_L$

$$-VLf = f \quad \text{in } (0, \theta_0)$$

and hence

$$Lf = -\frac{\lambda}{2} f \quad \text{if and only if } f = \frac{\lambda}{2} Vf,$$

from which it follows that

$$\frac{\lambda}{2} \geq \frac{1}{\|V\|}. \tag{2.2}$$

Now

$$|Vf(\theta)| \leq \|f\| \sup_\theta E^\theta(T_\alpha)$$

and moreover

$$\|V\| = \sup_\theta E^\theta(T_\alpha).$$

**Theorem 2.1.**

$$\lambda \geq \frac{1}{\int_0^{\theta_0} \frac{1}{(\sin x)^{m-2}} \int_0^x (\sin t)^{m-2}\, dt}.$$

*Proof.* We shall find an upper bound for $\|V\|$ which, together with (2.2), will give the desired lower bound for $\lambda$.

Let $g$ be such that $Lg = 1$, $g(0) = 0$, $g'(0) = 0$ and apply Dynkin's formula to obtain

$$g(\theta_0) - g(\theta) = E^{\theta}(T_\alpha).$$

Now clearly

$$g(\theta) = 2\int_0^{\theta} \frac{dx}{(\sin x)^{m-2}} \int_0^x (\sin t)^{m-2}\,dt.$$

So $g(\theta) \geq 0$ for $0 < \theta < \pi$ and therefore

$$\|V\| = \sup_{\theta} E^{\theta}(T_\alpha) \leq g(\theta_0). \tag{2.3}$$

From (2.2) and (2.3) we obtain

$$\lambda \geq \frac{2}{g(\theta_0)},$$

where

$$g(\theta) = 2\int_0^{\theta_0} \frac{dx}{(\sin x)^{m-2}} \int_0^x (\sin t)^{m-2}\,dt. \qquad \square \tag{2.4}$$

Observe that in the case $m = 3$ we have

$$g(\theta_0) = 4\left|\log\cos\frac{\theta_0}{2}\right|$$

and so we obtain a better lower bound than the one given by Pinsky [5]. We shall see in Sect. 3 that this bound is in accordance with the asymptotic behavior of $\lambda$.

**Theorem 2.2.** *If $j_{(m-3)/2}$ is the first zero of the Bessel function $J_{(m-3)/2}$ then*

$$\lambda \leq j^2_{(m-3)/2}/\theta_0^2.$$

*Proof.* In order to find an upper bound for $\lambda$, we follow Pinsky [5] and compare the equation

$$\frac{d^2F}{d\theta^2} + (m-2)\cot\theta\frac{dF}{d\theta} + \lambda F = 0, \qquad 0 < \theta < \theta_0 < \pi$$

which is clearly equivalent to (2.1) with the equation

$$\frac{d^2F}{d\theta^2} + \frac{(m-2)}{\theta}\frac{dF}{d\theta} + \tilde{\lambda}F = 0, \qquad 0 < \theta < \theta_0 < \pi, \tag{2.5}$$

whose solution is

$$F(\theta) = \theta^{(3-m)/2} J_{(3-m)/2}\left(\sqrt{\tilde{\lambda}}\,\theta\right),$$

where $J_{(3-m)/2}$ is the Bessel function of order $(3-m)/2$.

Thus the smallest eigenvalue $\tilde{\lambda}$, of (2.5) is $\tilde{\lambda} = (j_{(m-3)/2})^2/\theta_0^2$, $j_{(m-3)/2}$ being the first zero of the Bessel function $J_{(m-3)/2}$.

Using now the comparison theorem in [1] we get

$$\lambda \leq j^2_{(m-3)/2}/\theta_0^2. \qquad \square$$

## 3. The Characteristic Constant of a Spherical Cap and Asymptotic Results

In this section we would like to obtain asymptotic expressions for the first eigenvalue of a cap of the $(m-1)$-dimensional unit sphere $(m \geq 3)$.

The result that we obtain turns out to be different in the cases $m = 3$ and $m > 3$.

Following Hobson [4] we consider the Legendre's associated function of the first kind $P_n^l(\mu)$ defined for unrestricted values of the degree $n$ and the order $l$. This function is a particular integral of the ordinary linear differential equation of the second order

$$(1-\mu^2)\frac{d^2u}{d\mu^2} - 2\mu\frac{du}{d\mu} + \left\{n(n+1) - \frac{l^2}{1-\mu^2}\right\}u = 0,$$

which is known as Legendre's associated equation of degree $n$ and order $l$. Next, we quote a result concerning the zeros of $P_n^{-l}(\cos\theta)$ considered as a function of $n$ and when $\theta$ is near $\pi$. The proof of the following lemma can be found in Hobson [4].

**Lemma 3.1.** *The smallest value of $n$ for which $P_n^{-l}(\cos\theta)$ vanishes satisfies the following asymptotic relations as $\theta$ tends to $\pi$: if $l = 0$, then*

$$n \sim \frac{1}{2\log\dfrac{2}{\pi-\theta}};$$

*if $l > 0$, then*

$$n - l \sim \frac{\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l)}\tan^{2l}\left\{\frac{\pi-\theta}{2}\right\}.$$

In what follows we define the characteristic constant of a spherical cap of the $(m-1)$-dimensional unit sphere and see its relationship with both the above lemma and the first eigenvalue.

Let $C_{\theta_0}$ be the cap of the $(m-1)$-dimensional unit sphere $S_m(0,1)$ defined by $\cos\theta_0 < x_1 \leq 1$. The characteristic constant $\alpha(\theta_0)$ for such a cap is given by the positive root of the equation

$$\alpha(\theta_0)\{\alpha(\theta_0)+(m-2)\} = \lambda(\theta_0),$$

where

$$\lambda(\theta_0) = \inf_{f \in F_{\theta_0}} J(f) = \inf \frac{\int_{C_{\theta_0}} |\mathrm{grad}\, f|^2 \, d\sigma_\xi}{\int_{C_{\theta_0}} |f|^2 \, d\sigma_\xi},$$

and

$F_{\theta_0} =$ {$f$, functions depending only on $x_1$, nonnegative, Lipschitzian, nonidentically zero on $S_m(0,1)$ and vanishing outside the cap $C_{\theta_0}$}.

This infimum is attained at the solution of the *Laplace-Beltrami* equation

$$\Delta f + \lambda f = 0$$

on $C_{\theta_0}$, where $\lambda = \lambda(\theta_0)$ is the lowest eigenvalue of this equation.

The fact that we may take axisymmetric functions is due to Sperner [6]. If we write $x_1 = \cos\theta$, then all functions $f$ on the cap $C_{\theta_0}$ can be considered as functions of the variable $\theta$ ($\theta \in [0,\pi]$). Therefore the elements of the class $F_{\theta_0}$ are functions $f(\theta)$ defined in $[0,\pi]$, Lipschitzian, nonnegative, nonidentically zero on $[0,\pi]$ and which vanish in $[\theta_0,\pi]$. Furthermore,

$$\lambda(\theta_0) = \inf_{f \in F_{\theta_0}} \frac{\int_0^{\theta_0} f'(\theta)^2 \sin^{m-2}\theta \, d\theta}{\int_0^{\theta_0} f(\theta)^2 \sin^{m-2}\theta \, d\theta}.$$

Regarding the minimum value of $J(f)$ we have the following lemma due to Friedland and Hayman [3].

**Lemma 3.2.** *Let $f(\theta)$ be a Lipschitzian function in $[0,\pi]$, not identically zero and such that $f(\theta)=0$, $\theta_0 \leq \theta \leq \pi$. Then $J(f) \geq J(F) = \lambda(\theta_0)$, where $u = (\sin\theta)^{(1/2)(m-2)}F(\theta)$ is a solution of the differential equation*

$$\frac{d^2u}{d\theta^2} + \left\{\lambda + \tfrac{1}{4}(m-2)^2 + \frac{(m-2)(4-m)}{4\sin^2\theta}\right\}u = 0 \tag{3.1}$$

*and the positive number* $\lambda$ *is so chosen that* $F$ *remains analytic at* $\theta = 0$

$$F'(0) = 0, \qquad F(\theta_0) = 0 \quad \textit{and} \quad F(\theta) > 0 \quad \textit{for}\ 0 < \theta < \theta_0.$$

The smallest zero of the function $F(\theta)$ is $\theta_0$. The differential equation (3.1) can also be written in the following way

$$\frac{d}{d\theta}\left\{(\sin\theta)^{m-2}\frac{dF}{d\theta}\right\} = -\lambda \sin^{m-2}\theta F(\theta)$$

or

$$\frac{d^2F}{d\theta^2} + (m-2)\operatorname{cot g}\theta\frac{dF}{d\theta} + \lambda F = 0. \tag{3.2}$$

**Theorem 3.1.** *The following asymptotic relations hold as* $\theta \to \pi$

$$\alpha(\theta) \sim \frac{1}{2\log\dfrac{2}{\pi-\theta}}, \quad \textit{if}\ m = 3;$$

$$\alpha(\theta) \sim \frac{\Gamma(m-2)}{\Gamma\left(\dfrac{m-1}{2}\right)\Gamma\left(\dfrac{m-3}{2}\right)}\left\{\frac{\pi-\theta}{2}\right\}^{m-3} \quad \textit{if}\ m > 3.$$

*Proof.* The differential equation (3.2) is equivalent to

$$(1-z^2)w'' - (2\nu+1)zw' + \alpha(\alpha+2\nu)w = 0$$

for

$$z = \cos\theta, \nu = \frac{m-2}{2} \quad \textit{and} \quad \alpha(\alpha+m-2) = \lambda.$$

This means that the function $F$ in (3.2) is a Gegenbauer function of degree $\alpha$ and order $\nu$. We also observe that $\alpha$ is the characteristic constant of $C_{\theta_0}$. In the standard nomenclature it is denoted by $C_\alpha^\nu(z)$. The Gegenbauer functions can be represented in terms of the Legendre's associated functions of the first kind in the following way [2]

$$C_\alpha^\nu(z) = \frac{2^{\nu-(1/2)}\Gamma(\alpha+2\nu)\Gamma(\nu+(1/2))(z^2-1)^{(1/4)-(1/2)\nu}}{\Gamma(2\nu)\Gamma(\alpha+1)}P_{\alpha+\nu-(1/2)}^{(1/2)-\nu}. \tag{3.3}$$

We are interested in the zeros of $C_\alpha^\nu(\cos\theta)$ as a function of $\alpha$. More precisely, we are interested in an asymptotic expression for $\alpha$ (as a zero of $C_\alpha^\nu(\cos\theta)$), in terms of $\theta$, as $\theta$ approaches the value $\pi$. This asymptotic expression will lead us to the corresponding asymptotic expression for the characteristic constant of a spherical cap $C_\theta$ in terms of $\theta$ (as $\theta \to \pi$). Lemma 3.1 states these relations for the functions $P_n^{-l}(\cos\theta)$.

Therefore, by means of (3.3) we obtain the corresponding relations for the functions $C_\alpha^\nu(z)$. All we have to do is to write in Lemma 3.1 the values

$$-l = \frac{1}{2} - \nu = \frac{3-m}{2},$$

$$n = \alpha + \nu - \frac{1}{2} = \alpha + \frac{m-3}{2}.$$

Finally, the results that we obtain are

$$\alpha(\theta) \sim \frac{1}{2\log\dfrac{2}{\pi-\theta}} \quad \text{if } m = 3,$$

and

$$\alpha(\theta) \sim \frac{\Gamma(m-2)}{\Gamma\left(\dfrac{m-1}{2}\right)\Gamma\left(\dfrac{m-3}{2}\right)}\left\{\frac{\pi-\theta}{2}\right\}^{m-3} \quad \text{if } m > 3.$$

**Remark 3.1.** In the case $m = 4$ equation (3.1) is

$$\frac{d^2u}{d\theta^2} + (\lambda+1)u = 0$$

which is a Sturm-Liouville equation. The eigenfunctions of this equation are

$$u(\theta) = \sin\sqrt{\lambda+1}\,\theta.$$

The condition $u(\theta_0) = 0$ implies that

$$\lambda = \frac{n^2\pi^2}{\theta_0^2} - 1, \qquad n = 1,2,\ldots$$

so the least eigenvalue is

$$\lambda(\theta_0) = \frac{\pi^2}{\theta_0^2} - 1.$$

## 4. Sharpness of Results

*The lower bound near $\theta = \pi$ for $m = 3$:* In Thm. 2.1 we obtained

$$\lambda(\theta) \geq \frac{1}{2\left|\log\cos\frac{1}{2}\theta\right|}. \tag{4.1}$$

Since $\lambda(\theta) \sim \alpha(\theta)$ for $m = 3$ then the asymptotic behavior obtained in Sect. 2 for $\alpha(\theta)$ is the same for $\lambda(\theta)$. If we compare (4.1) with this asymptotic expression we see that this lower bound for $\lambda(\theta)$ is sharp.

*The lower bound near $\theta = \pi$ for $m \geq 4$:* In Sect. 2 we obtained the global lower bound

$$\lambda(\theta) \geq \frac{1}{\displaystyle\int_0^\theta \frac{dx}{\sin^{m-2}x}\int_0^x \sin^{m-2}t\,dt}.$$

We shall prove now that this lower bound is also sharp for $m \geq 4$. What we shall do is to compare this lower bound with the asymptotic expression that we obtained for $\alpha(\theta)$ in the case $m \geq 4$.

Since $\lambda/\alpha \to m-2$ as $\theta \to \pi$, the sharpness of our lower bound comes as a consequence of the following

**Lemma 4.1.**

$$\lim_{\theta\to\pi}(\pi-\theta)^{m-3}\int_0^\theta \frac{dx}{\sin^{m-2}x}\int_0^x \sin^{m-2}t\,dt = \frac{2^{m-3}\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{m-3}{2}\right)}{\Gamma(m-1)}.$$

*Proof.* We have as $\theta$ tends to $\pi$

$$(\pi-\theta)^{m-3}\int_0^\theta \frac{dx}{\sin^{m-2}x}\int_0^x \sin^{m-2}t\,dt \sim \frac{(-1)^m}{(m-3)\cos^{m-2}\theta}\int_0^\theta \sin^{m-2}t\,dt.$$

Therefore,

$$\lim_{\theta\to\pi}(\pi-\theta)^{m-3}\int_0^\theta \frac{dx}{\sin^{m-2}x}\int_0^x \sin^{m-2}t\,dt$$

$$= \frac{1}{m-3}\int_0^\pi \sin^{m-2}t\,dt = \frac{\sigma_m}{(m-3)\sigma_{m-1}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{m-1}{2}\right)}{(m-3)\Gamma\left(\frac{m}{2}\right)}.$$

On the other hand, from the duplication formula for the Gamma function we have that

$$\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{m}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(m-1)}{2^{m-2}}.$$

Thus

$$\frac{\sqrt{\pi}}{\Gamma\left(\frac{m}{2}\right)} = \frac{2^{m-2}\Gamma\left(\frac{m-1}{2}\right)}{\Gamma(m-1)} = \frac{2^{m-3}(m-3)\Gamma\left(\frac{m-3}{2}\right)}{\Gamma(m-1)}.$$

Then

$$\frac{\sqrt{\pi}\,\Gamma\left(\frac{m-1}{2}\right)}{(m-3)\Gamma\left(\frac{m}{2}\right)} = \frac{2^{m-3}\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{m-3}{2}\right)}{\Gamma(m-1)}.$$

Thus the lemma is proved. □

*The upper bound near $\theta = 0$ for $m = 3$:* We shall show now that for $m = 3$ the upper bound obtained by Pinsky [5] may be combined with the lower bound obtained by Friedland and Hayman [3] to show that

$$\lim_{\theta \to 0} \theta^2 \lambda(\theta) = j_0^2,$$

where $j_0 \cong 2.4$ is the first zero of the Bessel function $J_0$. In fact, Hayman and Friedland have shown that

$$\alpha(\theta) \geq \frac{1}{2} j_0 \left( \frac{1}{\sin^2 \frac{\theta}{2}} - \frac{1}{2} \right)^{1/2} - \frac{1}{2}$$

for $0 < \theta < \pi/2$. Since $\lambda(\theta) = \alpha(\theta)(\alpha(\theta) + 1)$ then

$$\lambda(\theta) \geq \left[ \frac{1}{2} j_0 \left( \frac{1}{\sin^2 \frac{\theta}{2}} - \frac{1}{2} \right)^{1/2} - \frac{1}{2} \right] \left[ \frac{1}{2} j_0 \left( \frac{1}{\sin^2 \frac{\theta}{2}} - \frac{1}{2} \right)^{1/2} + \frac{1}{2} \right].$$

Therefore,

$$\lambda(\theta) \geq \frac{1}{4} j_0^2 \left( \frac{1}{\sin^2 \frac{\theta}{2}} - \frac{1}{2} \right) - \frac{1}{4}, \qquad 0 < \theta < \frac{\pi}{2}.$$

On the other hand, Pinsky obtains

$$\lambda(\theta) \leq \frac{j_0^2}{\theta^2}.$$

The last two inequalities yield

$$\lim_{\theta \to 0} \theta^2 \lambda(\theta) = j_0^2.$$

*The upper bound near $\theta = 0$ for $m \geq 4$:* We have obtained the upper bound

$$\lambda(\theta) \leq j_{(m-3)/2}^2 / \theta^2$$

where $j_{(m-3)/2}$ is the first zero of the Bessel function $J_{(m-3)/2}$. Friedland and Hayman [3] have obtained the following lower bound for the characteristic constant $\alpha(\theta)$ of $C_\theta$, $0 < \theta < \pi/2$, and $m \geq 4$,

$$\alpha(\theta) \geq j_{(m-3)/2} \left( \frac{1}{(m-1)\int_0^\theta (\sin t)^{m-2}\, dt} \right)^{1/m-1} - \tfrac{2}{5} j_{(m-3)/2} - \tfrac{1}{2}(m-2).$$

Since $\lambda(\theta) = \alpha(\theta)(\alpha(\theta) + m - 2)$ one obtains

$$j_{(m-3)/2}^2 \left[ \left( \frac{1}{(m-1)\int_0^\theta (\sin t)^{m-2}\, dt} \right)^{1/m-1} - \frac{2}{5} \right]^2 - \frac{(m-2)^2}{4} \leq \lambda(\theta) \leq \frac{j_{(m-3)/2}^2}{\theta^2}$$

From these inequalities one can deduce that also for $m \geq 4$

$$\lim_{\theta \to 0} \theta^2 \lambda(\theta) = j_{(m-3)/2}^2.$$

## References


1. Debiard A, Gaveau B, Mazet E (1975) Théoremes de comparison en géometrie Riemannienne. Comptes Rendus Acad Sciences Paris, Ser A, 281:455–458
2. Erdelyi A, Magnus W, Oberhettinger F, Tricomi F (1953) Higher transcendental functions, vol 1. McGraw-Hill, New York
3. Friedland S, Hayman WK (1976) "Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions. Comment Math Helvetici 51:133–161
4. Hobson EW (1931) The theory of spherical and ellipsoidal harmonics. Cambridge Univ. Press
5. Pinsky MA (1981) The first eigenvalue of a spherical cap. Appl Math Optim 7:137–139
6. Sperner E (1973) Zur Symmetrisierung von Funktionen auf spharen. Math Z 134:317–327


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