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Some L^1 Existence and Dependence Results for Semilinear Elliptic Equations under Nonlinear Boundary Conditions*

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Abstract. In this paper we study questions of existence, uniqueness, and continuous dependence for semilinear elliptic equations with nonlinear boundary conditions. In particular, we obtain results concerning the continuous dependence of the solutions on the nonlinearities in the problem, which in turn implies analogous results for a related parabolic problem. Such questions arise naturally in the study of potential theory, flow through porous media, and obstacle problems.

1. Introduction

In this article we establish some results concerning the existence, uniqueness, and continuous dependence on the data of solutions of boundary-value problems for semilinear elliptic equations of the special form

$$\text{(BVP)}\qquad \begin{aligned} &\text{(i)} \quad \beta(u)-\Delta u \ni f \quad \text{on } \Omega,\\ &\text{(ii)} \quad u_\nu+\gamma(u)\ni 0 \quad \text{on } \Gamma, \end{aligned}$$

where $\Omega\subset\mathbf{R}^N$ is open, connected, bounded, and locally lies on one side of its C^2 boundary Γ, Δ denotes the Laplace operator in $\mathbf{R}^N$, $f\in L^1(\Omega)$, and u_ν denotes

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the exterior normal derivative of u on Γ. The nonlinearities β and γ are maximal monotone graphs in $\mathbf{R}$ (see, e.g., [8]). In particular, they may be multivalued and this allows (ii) to include the Dirichlet condition $u=0$ (taking γ to be the monotone graph defined by $\gamma(0)=\mathbf{R}$) and the Neumann condition $u_\nu=0$ (taking $\gamma(r)=0$ for all r) as well as many other possibilities.

In order to discuss the notion of solution of (BVP) we will use, we require a little notation. The Sobolev space consisting of those functions whose derivatives up to order k lie in $L^p(\Omega)$ will be denoted by $W^{k,p}(\Omega)$. We use $d\Gamma$ to denote area measure on Γ and $L^p(\Gamma)$ will mean the L^p space defined by this measure. The statement "a.e. on Ω" means with respect to Lebesgue N-measure and the statement "a.e. on Γ" means with respect to $d\Gamma$. A solution of (BVP) will mean a triple $[u, v, w] \in W^{1,1}(\Omega)\times L^1(\Omega)\times L^1(\Gamma)$ such that

$$v(x)\in\beta(u(x)) \quad \text{a.e. on } \Omega, \qquad w(x)\in\gamma(u(x)) \quad \text{a.e. on } \Gamma\,[1] \tag{1.1}$$

and

$$\int_\Omega v\rho+\int_\Omega \nabla u\cdot\nabla\rho+\int_\Gamma w\rho=\int_\Omega f\rho \qquad \text{for every} \quad \rho\in W^{1,\infty}(\Omega). \tag{1.2}$$

Here and below the integrals over Ω are with respect to Lebesgue measure and the integrals over Γ are with respect to $d\Gamma$. Of course, if we write $\beta(u)$ in place of v and $\gamma(u)$ in place of w, (1.2) is just the expression which would result from assuming everything is smooth and single-valued (so $v=\beta(u)$ and $w=\gamma(u)$), multiplying (BVP)(i) by ρ, integrating once by parts, and using the boundary condition (BVP)(ii). The definition above uses the fact that the trace of $u\in W^{1,1}(\Omega)$ on Γ is well defined in $L^1(\Gamma)$ (Theorem 4.2 of [18]). Observe that we use the same notation u for u and its trace when convenient.

Some hypotheses on the graphs β and γ are necessary in order for (BVP) to have a solution. To begin, we clearly need that the closure of the domains of β and γ are not disjoint and so it is natural to assume that

$$D(\beta)\cap D(\gamma)\neq\varnothing; \tag{1.3}$$

however, we will assume more. For example, if $a\in\mathbf{R}$, $\beta(a)=\mathbf{R}$, and (u, v, w) is a solution of (BVP), then $u\equiv a$, $v=f$, and $0=w(x)\equiv\gamma(u(x))=\gamma(a)$ a.e. In particular, for this β we must have

$$D(\beta)\cap\gamma^{-1}(0)\neq\varnothing \tag{1.4}$$

in order for (BVP) to have any solutions at all. We will assume that (1.4) holds, which in turn amounts to assuming that there is at least one constant function f for which (BVP) has a solution (u, v, w) which is a constant.

Another necessary condition for the existence of a solution of (BVP) is obtained by choosing $\rho\equiv 1$ in (1.2) to find

$$\int_\Omega f=\int_\Omega v+\int_\Gamma w. \tag{1.5}$$

[1] Hereafter when we write the expression $v\in\eta(u)$ where u and v are functions on a measure space and η is a monotone graph, we will mean that v and u are integrable with respect to the measure defined by the context and $v(x)\in\eta(u(x))$ a.e.

Using the notation $\varphi_+ = \sup R(\varphi)$ and $\varphi_- = \inf R(\varphi)$ for a graph φ in $\mathbf{R}$ with range $R(\varphi)$, (1.5) immediately implies that

$$\beta_-|\Omega| + \gamma_-|\Gamma| \leq \int_\Omega f \leq \beta_+|\Omega| + \gamma_+|\Gamma|, \tag{1.6}$$

where $|\Omega|$ is the Lebesgue N-measure of Ω and $|\Gamma|$ is the area of Γ.

Since (1.6) arises from an approximation of the condition imposed by a single choice of test function ρ, it is not quite sufficient for the existence of a solution of (BVP). It will be convenient to use the graph

$$B(r) = |\Gamma|\gamma(r) + |\Omega|\beta(r) \tag{1.7}$$

for the statement of necessary and sufficient conditions in the following theorem. Observe that (1.4) implies $B(r)$ is maximal monotone (since $D(\beta) \cap D(\gamma) \neq \varnothing$) and $B_+ = |\Gamma|\gamma_+ + |\Omega|\beta_+$, etc. Then (1.5) in fact leads at once to the necessary condition stated in the following theorem. This theorem also completely characterizes the cases in which (BVP) is solvable—in particular, it follows from the theorem that (BVP) is solvable if (1.6) holds with strict inequalities.

Theorem A. *Let* (1.4) *hold and* $f \in L^1(\Omega)$. *If* (BVP) *has a solution* $[u, v, w]$, *then*

$$\int_\Omega f \in R(B). \tag{1.8}$$

Conversely, (BVP) *has a solution if* $\int_\Omega f \in$ *interior* $(R(B))$ *or, equivalently,*

$$B_- < \int_\Omega f < B_+. \tag{1.9}$$

Lastly, if $\int_\Omega f = B_+ \in R(B)$ (*respectively,* $\int_\Omega f = B_- \in R(B)$), *then the solvability of* (BVP) *is determined as follows: if* $\beta_+ > \beta_-$, *then* (BVP) *has a solution exactly when* (*the unique within a constant*) *solutions* z *of the linear problem*

$$-\Delta z = f - \beta_+ \quad \text{in } \Omega, \qquad z_\nu = -\gamma_+ \quad \text{on } \Gamma \tag{1.10}$$

(*respectively,* $-\Delta z = f - \beta_-$ *in* Ω, $z_\nu = -\gamma_-$ *on* Γ) *are bounded below* (*respectively, above*) *on* Ω. *If* $\beta_+ = \beta_-$ *and* $\gamma_+ > \gamma_-$, *then* (BVP) *has a solution exactly when solutions of* (1.10) (*respectively, etc.*) *are bounded below* (*respectively, above*) *on* Γ. *If* $\beta_+ = \beta_-$ *and* $\gamma_+ = \gamma_-$, *then* (BVP) *has a solution.*

We remark that a sufficient condition for the solutions of (1.10) to be bounded is that $f \in L^p(\Omega)$ for some $p > \max(1, N/2)$. This can be proved by the method of Murthy and Stampacchia [17].

Theorem A will be proved in the next section together with the following result concerning the continuous dependence of v on β and γ where $[u, v, w]$ is the solution of (BVP) provided by Theorem A. Using nonlinear semigroup theory, this result implies, as is recalled in Section 3, the continuous dependence of solutions of an associated nonlinear parabolic problem on the nonlinearities in the problem. The resulting continuous dependence result for this parabolic problem was a principal motivation for this investigation.

Theorem B. *Let β^k, γ^k be maximal monotone graphs with $0 \in \beta^k(0) \cap \gamma^k(0)$ and $f_k \in L^1(\Omega)$ for $k = 1, 2, \ldots, \infty$. Assume that $\beta^k \to \beta^\infty$ and $\gamma^k \to \gamma^\infty$ in the sense of maximal monotone graphs*[2] *and $f_k \to f_\infty$ in $L^1(\Omega)$ as $k \to \infty$. Let $B^k = \beta^k|\Omega| + \gamma^k|\Gamma|$,*

$$B_-^\infty < \int_\Omega f_\infty < B_+^\infty \tag{1.11}$$

and $[u_k, v_k, w_k]$ be a solution of

$$(\mathrm{BVP})_k \quad \begin{aligned} &\beta^k(u_k) - \Delta u_k \ni f \quad \text{on } \Omega, \\ &(u_k)_\nu + \gamma^k(u_k) \ni 0 \quad \text{on } \Gamma \end{aligned}$$

for large k (the existence being guaranteed by Theorem A in view of (1.11) and the other assumptions). Then $v_k \to v_\infty$ in $L^1(\Omega)$ and $w_k \to w_\infty$ in $L^1(\Gamma)$ as $k \to \infty$.

Problems related to (BVP) have received a great deal of attention. If β, γ are Lipschitz continuous on $\mathbf{R}$ and β', $\gamma' \geq \varepsilon > 0$ and $f \in L^2(\Omega)$, the existence of a solution follows from standard variational arguments. Brezis in [6] and [7] handles the case in which β is the identity, γ is a maximal monotone graph, and $f \in L^2(\Omega)$. The case in which $f \in L^1(\Omega)$ and β, γ are continuous nondecreasing functions from $\mathbf{R}$ to $\mathbf{R}$ with $\beta' \geq \varepsilon > 0$ is studied in Brezis and Strauss [10]. Other aspects of (BVP) are generalized in [10]—for example, Δ can be replaced by more general elliptic operators, etc. We could consider this generality as well, but do not do so here since they will be developed in [5]. In [2] Benilan has proven existence results for $f \in L^1(\Omega)$, γ a maximal monotone graph, and β continuous and strictly increasing. The analog of Theorem A for f in $L^2(\Omega)$ was obtained by Schatzman [19] by variational methods—however, the gap between results for $f \in L^1(\Omega)$ and for $f \in L^2(\Omega)$ is substantial (indeed, think of the linear Dirichlet problem in this regard). Brezis has pointed out that formally the problem can be regarded as a "range of the sum" question, although the L^1 setting falls outside the scope of Brezis and Nirenberg [9]. Finally, see Magenes *et al.* [16], Alikakos and Rostamian [1], and Diaz [13] for some other special cases, references, and applications to some corresponding parabolic problems. The problem addressed in Theorem B in the case where Ω is $\mathbf{R}^N$ (and so there is no boundary) is rather different in character and was studied by Benilan and Crandall [4].

Section 2 of the text contains the proofs of Theorems A and B and several auxiliary considerations concerning (BVP). The "simplicity" of these proofs is one of the contributions of this paper. Section 3 is devoted to explaining the relationship of the results to the associated parabolic problem. In Section 4 we discuss the case $f \in L^p$ for $p > 1$ and an associated obstacle problem.

2. Proofs of Theorems A and B

Before proceeding, we will simplify the writing by using the assumption (1.4) to

[2] This convergence of (e.g.) the β^k can be expressed in terms of the inverse $(I + \beta^k)^{-1}$ of $r \to r + \beta^k(r)$; it means that $(I + \beta^k)^{-1}(r) \to (I + \beta^\infty)^{-1}(r)$ for $r \in \mathbf{R}$.

reduce to the situation in which

$$\beta(0) \cap \gamma(0) \ni 0; \tag{2.1}$$

this is done by choosing $a \in D(\beta) \cap \gamma^{-1}(0)$, $b \in \beta(a)$, and putting $\tilde{u} = u - a$. In terms of $\tilde{u}$ the problem becomes $\tilde{\beta}(\tilde{u}) - \Delta\tilde{u} = f - b$ in Ω and $\tilde{u}_\nu + \tilde{\gamma}(\tilde{u}) = 0$ on Γ where $\tilde{\beta}(r) = \beta(r+a) - b$ and $\tilde{\gamma}(r) = \gamma(r+a)$. Since (2.1) holds for $\tilde{\beta}$, $\tilde{\gamma}$ in place of β, γ we hereafter assume (2.1).

In fact, the heart of the proofs of both Theorems A and B is the following sharpened form of Theorem B:

Theorem B′. *Let β^k, γ^k be maximal monotone graphs with $0 \in \beta^k(0) \cap \gamma^k(0)$ and $f_k \in L^1(\Omega)$ for $k = 1, 2, \ldots, \infty$. Assume that $\beta^k \to \beta^\infty$ and $\gamma^k \to \gamma^\infty$ in the sense of maximal monotone graphs and $f_k \to f_\infty$ in $L^1(\Omega)$ as $k \to \infty$. Let $[u_k, v_k, w_k]$ be a solution of*

$$(\mathrm{BVP})_k \quad \begin{aligned} &\beta^k(u_k) - \Delta u_k \ni f_k \quad \text{on } \Omega,\\ &u_{k\nu} + \gamma^k(u_k) \ni 0 \quad \text{on } \Gamma \end{aligned}$$

for finite $k = 1, 2, \ldots$. Let $B^k(r) = |\Omega|\beta^k(r) + |\Gamma|\gamma^k(r)$ for all k and assume that

$$B_-^\infty \le \int_\Omega f_\infty \le B_+^\infty. \tag{2.2}$$

Then we have:

(i) *The sequences $\{v_k\}$, $\{w_k\}$ converge to limits v_∞ in $L^1(\Omega)$ and w_∞ in $L^1(\Gamma)$ as $k \to \infty$.*

(ii) *If $\{u_k\}$ has a limit point u_∞ in $L^1(\Omega)$, then $[u_\infty, v_\infty, w_\infty]$ is a solution of $(\mathrm{BVP})_\infty$.*

(iii) *If $\{\int_\Omega u_k\}$ is bounded, then $\{u_k\}$ is bounded in $W^{1,q}(\Omega)$ for $1 \le q < N/(N-1)$ (and hence is precompact in $L^1(\Omega)$).*

(iv) *If*

$$B_+^\infty < \int_\Omega f_\infty < B_-^\infty \tag{2.3}$$

then $\{\int_\Omega u_k\}$ is bounded.

Clearly, Theorem B′ implies Theorem B if we know that whenever $[u, v, w]$ and $[\hat{u}, \hat{v}, \hat{w}]$ are solutions of (BVP) then $v = \hat{v}$ and $w = \hat{w}$. This follows at once from the first assertion of Proposition E below. We proceed below by deducing Theorem A from Theorem B′ and then we prove Theorem B′.

Proof of Theorem A. In order to prove Theorem A we use the following standard approximation scheme: let $\lambda > 0$ and β_λ, γ_λ be the Yosida approximations of β, γ; that is

$$B_\lambda = (I - (I + \lambda\beta)^{-1})/\lambda, \qquad \gamma_\lambda = (I - (I + \lambda\gamma)^{-1})/\lambda. \tag{2.4}$$

Let

$$\hat{\beta}_\lambda(r) = \lambda r + \beta_\lambda(r). \tag{2.5}$$

Then $\hat{\beta}_\lambda$ and its inverse are both Lipschitz continuous homeomorphisms of **R**. The problem

$$\hat{\beta}_\lambda(u_\lambda)-\Delta u_\lambda=f \quad \text{in } \Omega, \qquad u_{\lambda\nu}+\gamma_\lambda(u_\lambda)=0 \quad \text{on } \Gamma \tag{2.6}$$

has a solution $[u_\lambda, v_\lambda, w_\lambda]=[u_\lambda, \hat{\beta}_\lambda(u_\lambda), \gamma_\lambda(u_\lambda)]$ for $\lambda>0$ by the results of [10]. Alternatively, if the reader prefers a self-contained presentation, for $f\in L^2(\Omega)$ it is a simple matter to exhibit the solution of (2.6) as a minimum of the functional

$$\Phi(u)=(\tfrac{1}{2})\int_\Omega (|\nabla u|^2+j_1(u)-fu)+\int_\Gamma j_2(u),$$

where $j_1'=\hat{\beta}_\lambda, j_2'=\gamma_\lambda$. Now replace f by an approximation $f_\lambda\in L^2(\Omega)$ where $f_\lambda\to f$ in $L^1(\Omega)$ as $\lambda\downarrow 0$. Since $\beta_\lambda\to\beta$ and $\gamma_\lambda\to\gamma$ as maximal monotone graphs as $\lambda\downarrow 0$, we may use Theorem B′ to conclude that if f satisfies (1.10) there is a solution of (BVP).

Assume now that $\int_\Omega f=B_+$ and (BVP) has a solution $[u, v, w]$. Then the identity (1.5) implies $w\equiv\gamma_+$ and $v\equiv\beta_+$ so $z=u$ satisfies (in the obvious sense—we assume some simple facts here concerning the Neumann problem which are reviewed again in Proposition C below)

$$-\Delta z=f-\beta_+, \qquad z_\nu=-\gamma_+. \tag{2.7}$$

Moreover, the relation $v(x)=\beta_+\in\beta(u(x))$ a.e. shows that either $\beta\equiv\beta_+$ or u is bounded below (by $\inf\{r: \beta_+\in\beta(r)\}$) and in the former case we must have (by (2.1)) $\beta\equiv 0$. Similarly, either u is bounded below on Γ or $\gamma\equiv 0$. If $\beta_+>\beta_-$ then u is bounded below so (2.7) has a solution z which is bounded below. Conversely, if (2.7) has a solution z which is bounded below on Ω, then for a large enough constant c $[z+c, \beta_+, \gamma_+]$ solves (BVP). Assume therefore that $\beta\equiv\beta_+=0$ and $\gamma_+>\gamma_-$. As above, a solution of (BVP) is then a solution of (2.7) which is bounded below on Γ and if z is a solution of (2.7) which is bounded below on Γ, then $[z+c, 0, \gamma_+]$ is a solution of (BVP) for large enough c. In the remaining case, $\beta=\gamma=0$, $\int_\Omega f=0$ and if z solves $-\Delta z=f$, $z_\nu=0$ (see Proposition C), then $[z, 0, 0]$ is a solution of (BVP).

We turn now to the proof of Theorem B′. The theorem itself is a fairly direct consequence of several ingredients. First we recall some basic facts concerning the linear Neumann problem

$$\text{(NP)} \quad -\Delta u=f \quad \text{in } \Omega, \qquad u_\nu=g \quad \text{on } \Gamma,$$

where $f\in L^1(\Omega)$ and $g\in L^1(\Gamma)$; a solution of (NP) is a $u\in W^{1,1}(\Omega)$ such that

$$\int_\Omega \nabla u\cdot\nabla\rho=\int_\Omega f\rho+\int_\Gamma g\rho$$

for $\rho\in W^{1,\infty}(\Omega)$. Set

$$I(f, g)=\int_\Omega f+\int_\Gamma g.$$

We will use the following proposition:

Proposition C. *Let $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$.*

(i) (NP) *has a solution if and only if $I(f, g) = 0$.*
(ii) *Solutions of* (NP) *are unique within a constant so if $I(f, g) = 0$ there is a unique solution u with the property $\int_\Omega u = 0$; this solution will be denoted by $u = G(f, g)$.*
(iii) *For $1 \le q < N/(N-1)$ there is a constant C depending only on q and Ω such that if $I(f, g) = 0$, then*

$$\|G(f, g)\|_{W^{1,q}(\Omega)} \le C(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}). \tag{2.8}$$

(iv) *There is a bounded linear mapping T: $L^1(\Omega) \to L^1(\Gamma)$ such that the solution $u \in W_0^{1,1}(\Omega)$ of the Dirichlet problem $-\Delta u = f$ in Ω, $u = 0$ on Γ is also a solution of* (NP) *with $g = Tf$. Moreover, if $1 \le p < \infty$, then the restriction of T to $L^p(\Omega)$ is a bounded linear operator from $L^p(\Omega)$ into $L^{(N-1)p/(N-p)}(\Gamma)$ if $N > p$ and from $L^p(\Omega)$ into $L^q(\Gamma)$ for any finite q if $p \ge N$.*
(v) *If $I(f, g) = 0$ and $u = G(f, g)$, then*

$$I(f\chi_{\{u>0\}}, g\chi_{\{u>0\}}) \ge I(f^-\chi_{\{u=0\}}, g^-\chi_{\{u=0\}}),$$

where $r^+ = \max(r, 0)$, $r^- = r^+ - r$, χ_A is the characteristic function of the set A, and $\{u > 0\}$ denotes the set of $x \in \Omega$ (or Γ) on which $u(x) > 0$, etc.

Sketch of Proof of Proposition C. For the moment, let τu denote the trace of $u \in W^{1,1}(\Omega)$ in $L^1(\Gamma)$. Let M be the operator in $L^1(\Omega) \times L^1(\Gamma)$ with the graph

$$\{[(u, w), (f, g)] : [u, f, g] \in W^{1,1}(\Omega) \times L^1(\Omega) \times L^1(\Gamma),\ w = \tau u \text{ and } u \text{ is a solution of (NP)}\}. \tag{2.9}$$

Clearly, M is a linear, single-valued, and densely defined operator in $L^1(\Omega) \times L^1(\Gamma)$. It follows from [10] that M is an m-accretive linear operator (i.e., $(I + \lambda M)^{-1}$ is an everywhere defined nonexpansive self-mapping of $L^1(\Omega) \times L^1(\Gamma)$ for $\lambda > 0$). Moreover, if P is the projection of $L^1(\Omega) \times L^1(\Gamma)$ onto $L^1(\Omega)$, Lemma 23 of [10] implies that

$$P(I + M)^{-1}\colon L^1(\Omega) \times L^1(\Gamma) \to W^{1,q}(\Omega) \tag{2.10}$$

boundedly for $1 \le q < N/(N-1)$. Thus M has a compact resolvent and $(f, g) \in R(M)$ if and only if (f, g) is orthogonal to the null space of the adjoint M^* of M. Now if (u, z) is in the null-space of M we have $z = \tau u$ and

$$-\Delta u = 0 \text{ in } \Omega \quad \text{and} \quad u_\nu = 0 \text{ on } \Gamma.$$

We claim that then u is a constant (and therefore z is the same constant). To see this, we rely on the classical fact that the problem

$$-\Delta \rho = (f_i)_{x_i} \text{ in } \Omega \quad \text{and} \quad \rho_\nu = 0 \text{ on } \Gamma$$

has a classical solution whenever each f_i is smooth and compactly supported in Ω and then use ρ as a test function in the relation satisfied by u. The null-space of M is thus one-dimensional and (because of the compact resolvent) so is the null-space of M^*. But constants (as elements of $L^\infty(\Omega)\times L^\infty(\Omega)$) are in the null-space of M^*, whence the sufficiency of the condition $I(f,g)=0$ for the solvability of (NP). We have argued that (i) and (ii) hold. The estimate (iii) follows from similar considerations. Since a solution u of (NP) also solves

$$u-\Delta u=f+u\in L^1(\Omega), \qquad u+u_\nu=g+u\in L^1(\Gamma)$$

we have $u=P(I+M)^{-1}(f+u, g+\tau u)\in W^{1,q}(\Omega)$. Thus G is a mapping from the subspace of $L^1(\Omega)\times L^1(\Gamma)$ on which $I(f,g)=0$ into $W^{1,q}(\Omega)$. It clearly has a closed graph and is therefore bounded, whence (iii).

We turn our attention to (iv). A solution u of the Dirichlet problem also satisfies

$$-\int_\Omega u\,\Delta\rho+\int_\Gamma u_\nu\rho=\int_\Omega f\rho \tag{2.11}$$

for $\rho\in C^2(\bar\Omega)$ if f is smooth. If we choose a regular function g on Γ and solve $-\Delta\rho=0$ in Ω, $\rho=g$ on Γ, the maximum principle implies that

$$\|\rho\|_{L^\infty(\Omega)}\le\|g\|_{L^\infty(\Gamma)}$$

so (2.11) yields

$$\int_\Gamma u_\nu g\le\|f\|_{L^1(\Omega)}\|g\|_{L^\infty(\Gamma)}.$$

We conclude that the linear mapping $f\to u_\nu=Tf$ where u is the solution of the Dirichlet problem (which is well defined for smooth f) is bounded from $L^1(\Omega)$ into $L^1(\Gamma)$ whence the result for $p=1$. For $1<p<\infty$ the result follows from the $W^{2,p}$ regularity estimates and trace theorems for $W^{1,p}(\Omega)$.

To establish (v) we use the following variant of Lemma 2 of [10]:

Lemma D. *Let $1\le p\le\infty$, $f\in L^p(\Omega)$, $g\in L^p(\Gamma)$, and u be a solution of* (NP). *Let η be a maximal monotone graph in* $\mathbf{R}$ *and $0\in\eta(0)$. Let p' be the Holder conjugate of p, $a\in L^{p'}(\Omega)$, $b\in L^{p'}(\Gamma)$, $a(x)\in\eta(u(x))$ a.e. on Ω, and $b(x)\in\eta(u(x))$ a.e. on Γ. Then*

$$\int_\Omega af+\int_\Gamma bg\ge 0.$$

In order to deduce (v) from the lemma we choose

$$\eta(r)=\begin{cases}\{1\} & \text{for } r>0,\\ [0,1] & \text{for } r=0,\\ \{0\} & \text{for } r<0,\end{cases}$$

and $a(x)=1$ on $\{u>0\}$, $a(x)=0$ on $\{u=0$ and $f\ge 0\}$, and $a(x)=1$ on $\{u=0$ and $f<0\}$ (respectively, define b on Γ by $b(x)=1$ on $\{u>0\}$, $b(x)=0$ on $\{u=0$ and $g\ge 0\}$, and $b(x)=1$ on $\{u=0$ and $g<0\}$).

Sketch of Proof of Lemma D. The lemma follows from the fact that the m-accretive operator M in $L^1(\Omega)\times L^1(\Gamma)$ satisfies the conditions (I) and (II) formulated below. We will use the notation $J_\lambda=(I+\lambda M)^{-1}=(J_{\lambda 1}, J_{\lambda 2})$ to indicate the projections of J_λ on $L^1(\Omega)$ and $L^1(\Gamma)$. The first condition can be verified by the arguments of [10].

$$\int_\Omega (J_{\lambda 1}(f,g)-J_{\lambda 1}(\hat{f},\hat{g}))^+ + \int_\Gamma (J_{\lambda 2}(f,g)-J_{\lambda 2}(\hat{f},\hat{g}))^+$$
$$\leq \int_\Omega (f-\hat{f})^+ + \int_\Gamma (g-\hat{g})^+ \tag{I}$$

for (f,g), $(\hat{f},\hat{g})\in L^1(\Omega)\times L^1(\Gamma)$. The second condition is trivial—we have

$$J_{\lambda 1}(b,b)=J_{\lambda 2}(b,b)=b \tag{II}$$

for all constants b. Now one shows that the (analog of) Lemma D holds for any linear densely defined m-accretive operator M in a product space $L^1(\mu)\times L^1(\nu)$ (μ and ν are measures) which satisfies the analogs of (I) and (II). This can be done by the method of proof of Lemma 3 of [10]. In particular, if (J_1, J_2): $L^1(\mu)\times L^1(\nu)\to L^1(\mu)\times L^1(\nu)$ has the properties (I) and (II) then one checks that for any convex lower-semicontinuous function j: $\mathbf{R}\to[0,\infty]$ one has

$$\int j(T_1(f,g))\,d\mu+\int j(T_2(f,g))\,d\nu\leq\int j(f)\,d\mu+\int j(g)\,d\nu.$$

The next proposition is formulated for the reader's convenience for the generalization

$$(\mathrm{BVP})_{f,g}\quad \begin{array}{ll}\text{(i)} & \beta(u)-\Delta u\ni f \quad \text{on } \Omega,\\ \text{(ii)} & u_\nu+\gamma(u)\ni g \quad \text{on } \Gamma,\end{array}$$

where $f\in L^1(\Omega)$ and $g\in L^1(\Gamma)$ of (BVP). A solution of $(\mathrm{BVP})_{f,g}$ is a triple $[u, v, w]$ as before, but (1.2) is replaced by

$$\int_\Omega v\rho+\int_\Omega \nabla u\cdot\nabla\rho+\int_\Gamma (w-g)\rho=\int_\Omega f\rho \tag{1.2$'$}$$

if $\rho\in W^{1,\infty}(\Omega)$.

Proposition E. *Let* $f,\hat{f}\in L^1(\Omega)$, $g,\hat{g}\in L^1(\Gamma)$, $[u,v,w]$ *be a solution of* $(\mathrm{BVP})_{f,g}$, *and* $[\hat{u},\hat{v},\hat{w}]$ *be a solution of* $(\mathrm{BVP})_{\hat{f},\hat{g}}$. *Then*

$$\int_\Omega (v-\hat{v})^+ + \int_\Gamma (w-\hat{w})^+\leq\int_\Omega (f-\hat{f})^+ + \int_\Gamma (g-\hat{g})^+. \tag{i}$$

Moreover, if $g=0$ *and* $b\geq 0$ *then*

$$\int_\Omega (|v|-b)^+\leq\int_\Omega (|f|-b)^+ \tag{ii}$$

and

$$\int_\Gamma (|w|-b)^+ \le \int_\Gamma (|T(f-v)|-b)^+, \tag{iii}$$

where the operator T is from Proposition C(iv).

Proof of Proposition E. According to the assumptions $z = u - \hat{u}$ is a solution of

$$-\Delta z = f - \hat{f} + \hat{v} - v, \qquad z_\nu = g - \hat{g} + \hat{w} - w.$$

Proposition C(v) implies that

$$I((f-\hat{f}+\hat{v}-v)\chi_{\{z>0\}}, (g-\hat{g}+\hat{w}-w)\chi_{\{z>0\}})$$
$$\ge I((f-\hat{f}+\hat{v}-v)^-\chi_{\{z=0\}}, (g-\hat{g}+\hat{w}-w)^-\chi_{\{z=0\}}).$$

Now the relations $v(x) \in \beta(u(x))$ a.e., etc., imply that $v-\hat{v}$, $w-\hat{w} \ge 0$ (respectively ≤ 0) on $\{z>0\}$ (respectively, $\{z<0\}$) and then manipulation of the above inequality (using $\int_\Omega v^+ \le \int \chi_{\{z>0\}} v + \int \chi_{\{z=0\}} v^+$, $v^+ \le (f-v)^- + f^+$, etc.) implies (i). To obtain (ii), let $[u, v, w]$ be a solution of $(\mathrm{BVP})_{f,0}$. If $a, b, c \ge 0$, $b \in \beta(a)$, and $c \in \gamma(a)$ then $[a, b, c]$ is a solution of $(\mathrm{BVP})_{b,c}$. By (i) we have

$$\int_\Omega (v-b)^+ + \int_\Gamma (w-c)^+ \le \int_\Omega (f-b)^+$$

and so

$$\int_\Omega (v-b)^+ \le \int_\Omega (f-b)^+. \tag{2.12}$$

So far we have obtained this inequality for $b \in \beta([0,\infty) \cap D(\gamma))$. If $D(\gamma)$ contains $D(\beta)$, then it will hold for all $b>0$, since if $b \ge \beta(\mathbf{R})$ we have $(v-b)^+ = 0$. If $b \in \beta(a)$ and $a > D(\gamma)$, we proceed by letting $\lambda > 0$ and observing that $[u, v, w]$ is a solution of

$$\beta(u) - \Delta u \ni f \quad \text{on } \Omega, \qquad u_\nu + \gamma_\lambda(u) \ni \gamma_\lambda(u) - w \quad \text{on } \Gamma,$$

while $[a, b, \gamma_\lambda(a)]$ is a solution of

$$\beta(a) - \Delta a \ni b \quad \text{on } \Omega, \qquad a_\nu + \gamma_\lambda(a) \ni \gamma_\lambda(a) \quad \text{on } \Gamma.$$

Using the estimate (i) on these problems yields

$$\int_\Omega (v-b)^+ + \int_\Gamma (w-\gamma_\lambda(a))^+ \le \int_\Omega (f-b)^+ + \int_\Gamma (\gamma_\lambda(u) - w - \gamma_\lambda(a))^+.$$

Since $|w| \ge |\gamma_\lambda(u)|$ and $\gamma_\lambda(a) \to \infty$ as $\lambda \downarrow 0$ (by properties of the Yosida approximation) the inequality (2.12) follows upon sending λ to 0. In a similar fashion one proves that

$$\int_\Omega (v+b)^- \le \int_\Omega (f+b)^-$$

for $0 \le b$ and adding this and (2.12) yields (ii).

It remains to prove (iii). To this end let $\hat{u}$ be the solution of

$$-\Delta\hat{u}=f-v \text{ in } \Omega \quad \text{and} \quad \hat{u}=0 \text{ on } \Gamma;$$

according to Proposition C(iv) $\hat{u}$ also solves

$$-\Delta\hat{u}=f-v \quad \text{in } \Omega, \qquad \hat{u}_\nu = T(f-v) \quad \text{on } \Gamma.$$

Thus (using that $\hat{u}=0$ on Γ) we have the solution $[u-\hat{u}, 0, w]$ of

$$-\Delta(u-\hat{u})=0 \quad \text{in } \Omega, \qquad (u-\hat{u})_\nu+\gamma(u-\hat{u})\ni T(v-f) \quad \text{on } \Gamma.$$

If $b\in\gamma(a)$ we have the solution $[a, 0, b]$ of

$$-\Delta a=0 \text{ in } \Omega \quad \text{and} \quad a_\nu+\gamma(a)\ni b \text{ on } \Gamma.$$

Applying (i) again to the latter two problems (with $\beta=0$ now) we find that

$$\int_\Gamma (w-b)^+ \le \int_\Gamma (T(v-f)-b)^+$$

as desired. This completes the proof.

We now complete the proof of Theorem B′.

Proof of Theorem B′. We use the notations and assumptions of the theorem.

Step 1. The sequence $\{v_k\}$ is weakly sequentially compact in $L^1(\Omega)$. This follows from the following criterion for weak sequential compactness of a subset $\mathbf{F}$ of $L^1(\mu)$ where μ is a finite measure:

$$\lim_{b\to\infty} \sup_{\mathbf{F}} \int (|f|-b)^+ \, d\mu = 0.$$

Indeed, this condition is easily seen to be equivalent to the uniform integrability of the family $\mathbf{F}$ (that is, for every $\varepsilon>0$ there is a $\delta>0$ such that $\int_A |f| \, d\mu \le \varepsilon$ if $\mu(A)\le\delta$) and this implies the weak sequential compactness. Since we have

$$\int_\Omega (|v_k|-b)^+ \le \int_\Omega (|f_k|-b)^+$$

by Proposition E(ii) and $\{f_k\}$ is convergent in $L^1(\Omega)$ by assumption, the weak sequential compactness of $\{v_k\}$ follows. Observe in particular (taking $b=0$) that $\{v_k\}$ is bounded in $L^1(\Omega)$.

Step 2. The sequence $\{w_k\}$ is weakly sequentially compact in $L^1(\Gamma)$. First note that the sequence $\{T(f_k-v_k)\}$ is weakly sequentially compact in $L^1(\Gamma)$ because of Step 1, the assumed convergence of the f_k, and the continuity of the linear operator T as a map from $L^1(\Omega)$ into $L^1(\Gamma)$. Since Proposition E(iii) implies that

$$\int_\Gamma (|w_k|-b)^+ \le \int_\Gamma (|T(f_k-v_k)|-b)^+$$

we conclude as above.

Step 3. The sequence $\{v_k\}$ is precompact in $L^1(\Omega)$. In view of the uniform integrability of the sequence, it suffices to prove that for each compact subset Ω' of Ω, $\{v_k\}$ is precompact in $L^1(\Omega')$. For this we use the following local version of Proposition C(v).

Lemma F. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $f \in L^1_{\mathrm{loc}}(\Omega)$, $-\Delta u = f$, and $\zeta \in C^2(\Omega)$ be nonnegative and compactly supported in Ω. Then*

$$\int_{\{u>0\}} (f\zeta + u\,\Delta\zeta) \ge \int_{\{u=0\}} f^-\zeta. \tag{2.13}$$

Proof of Lemma F. We may assume that $u = 0$ holds in a neighborhood of Γ since otherwise we may replace u by ψu where $\psi = 1$ in a neighborhood of the support of ζ and has compact support in Ω. Since

$$-\nabla \cdot ((\zeta+\varepsilon)\nabla u) = (\zeta+\varepsilon)f - \nabla u \cdot \nabla \zeta$$

Lemma 2 of [10] implies that if η is a maximal monotone graph containing the origin, $a \in L^\infty(\Omega)$, $a \in \eta(u)$, and $\varepsilon > 0$, then

$$\int_\Omega ((\zeta+\varepsilon)f - \nabla u \cdot \nabla\zeta)a \ge 0.$$

Now choose η and a as in the proof of Proposition C(v), let $\varepsilon \to 0$ and use the fact that $\nabla u = 0$ a.e. on $\{u = 0\}$ to see that

$$\int_{\{u>0\}} (\zeta f - \nabla u \cdot \nabla\zeta) \ge \int_{\{u=0\}} f^-\zeta.$$

Next we claim that

$$\int_{\{u>0\}} \nabla u \cdot \nabla\zeta = -\int_{\{u>0\}} u\,\Delta\zeta;$$

this may be seen by choosing a suitable sequence of smooth approximations η_k of the Heaviside function and passing to the limit in the relation

$$\int_\Omega \eta_k(u)\nabla u \cdot \nabla\zeta = -\int_\Omega N_k(u)\,\Delta\zeta,$$

where $N_k(r) = \int_0^r \eta_k(s)\,ds$.

To continue, fix $y \in \mathbf{R}^N$, observe that

$$-\Delta(u_k(x) - u_k(x+y)) = f(x) - f(x+y) + (v_k(x+y) - v_k(x)),$$

and apply Lemma F (as we did with Proposition C(v) to get Proposition E(i)) with ζ supported in $\{x \in \Omega\colon \text{distance } (x, \Gamma) < |y|\}$ to conclude that then

$$\int_\Omega \zeta(x)|v_k(x+y) - v(x)|$$

$$\le \int_\Omega |\Delta\zeta(x)|\,|u_k(x) - u_k(x+y)| + \int_\Omega |\zeta(x)|\,|f_k(x) - f_k(x+y)|.$$

The sequence $\{u_k-(1/|\Omega|)\int_\Omega u_k\}$ is bounded in $W^{1,q}(\Omega)$ by the boundedness of $\{v_k\}$ and $\{w_k\}$ (from Steps 1 and 2) and hence is precompact in $L^1(\Omega)$ by Proposition C(iii). Likewise, translations act equicontinuously on $\{f_k\}$ in $L^1_{\text{loc}}(\Omega)$ by the assumptions. We conclude by appropriate choice of ζ that

$$\lim_{|y|\to 0}\sup_k \int_{\Omega'} |v_k(x+y)-v_k(x)|=0$$

and this implies the desired compactness.

Step 4. If $[u_{k_n}, v_{k_n}, w_{k_n}]$ is a subsequence of $[u_k, v_k, w_k]$ and $v_{k_n}\to v_\infty$ weakly in $L^1(\Omega)$, $w_{k_n}\to w_\infty$ weakly in $L^1(\Gamma)$, and $u_{k_n}\to u_\infty$ in $L^1(\Omega)$, then $[u_\infty, v_\infty, w_\infty]$ is a solution of $(\mathrm{BVP})_\infty$. To establish this, first note that $\{u_{k_n}\}$ is bounded in $W^{1,q}(\Omega)$ by Proposition C(iii) applied to $\{u_{k_n}-(1/|\Omega|)\int_\Omega u_{k_n}\}$ so we can assume that $\nabla u_{k_n}\to\nabla u_\infty$ weakly in $L^q(\Omega)$ for $1\le q<N/(N-1)$. Now we must show that

$$\int_\Omega v_\infty\rho+\int_\Omega \nabla u_\infty\cdot\nabla\rho+\int_\Gamma w_\infty\rho=\int_\Omega f_\infty\rho \tag{2.14}$$

for every $\rho\in W^{1,\infty}(\Omega)$ and

$$v_\infty\in\beta^\infty(u_\infty),\qquad w_\infty\in\gamma^\infty(u_\infty). \tag{2.15}$$

Equation (2.14) follows immediately from the assumptions. Condition (2.15) follows from the next simple lemma.

Lemma G. *Let η and η^k, $k=1,2,\ldots$, be maximal monotone graphs in* **R** *and $\eta^k\to\eta$ in the sense of graphs as $k\to\infty$. Let (S,μ) be a σ-finite measure space, u_k, u, v_k, $v\in L^1(\mu)$, $v_k\in\eta^k(u_k)$ for $k=1,2,\ldots$, and $u_k\to u$ strongly in $L^1(\mu)$ and $v_k\to v$ weakly in $L^1(\mu)$ as $k\to\infty$. Then $v\in\eta(u)$.*

Sketch of Proof. It is standard in the theory of accretive operators that if v satisfies

$$\int (v-\hat v)\,\mathrm{sign}_0(u-\hat u)\,d\mu\ge 0\qquad\text{whenever}\quad \hat u,\ \hat v\in L^1(S)\ \text{and}\ \hat v\in\eta(\hat u) \tag{2.16}$$

(where $\mathrm{sign}_0(r)=1, -1$, or 0 according as $r>0$, $r<0$, or $r=0$), then $v\in\eta(u)$. Moreover, owing to the assumptions, if $\hat v\in\eta(\hat u)$ then there are sequences $\hat v_k\in\eta^k(\hat u_k)$ such that $\hat u_k\to\hat u$, $\hat v_k\to\hat v$ in $L^1(\mu)$. Finally, the monotonicity of η^k implies $\int(v_k-\hat v_k)p(u_k-\hat u_k)\,d\mu\ge 0$ where $p(r)=r/m$, 1, -1 according as $|r|\le m$, $r>m$, and $r<-m$. We may pass to the limit as $k\to\infty$ in this relation and then as $m\to\infty$ to find (2.16).

We know from Proposition E that if $[u_\infty, v_\infty, w_\infty]$ and $[\hat u_\infty, \hat v_\infty, \hat w_\infty]$ are two solutions of $(\mathrm{BVP})_\infty$, then $v_\infty=\hat v_\infty$ and $w_\infty=\hat w_\infty$. We therefore further conclude that if $\{u_k\}$ is precompact in $L^1(\Omega)$, then $(\mathrm{BVP})_\infty$ has a solution and if $[u_\infty, v_\infty, w_\infty]$ is a solution

$$v_k\to v_\infty\ \text{in}\ L^1(\Omega)\quad\text{and}\quad w_k\to w_\infty\ \text{weakly in}\ L^1(\Gamma). \tag{2.17}$$

Step 5. If $\{\int_\Omega u_k\}$ is bounded then $\{u_k\}$ is precompact in $L^1(\Omega)$ and $L^1(\Gamma)$ and (2.3) implies that $\{\int_\Omega u_k\}$ is bounded. The first assertion follows from Proposition C(iii). As regards the second, we argue by contradiction. Indeed, assume for example that

$$\int_\Omega u_k \to \infty. \tag{2.18}$$

Proposition C(iii), $-\Delta u_k = f_k - v_k$ in Ω, $u_{k\nu} = -w_k$ on Γ, and the boundedness of v_k and w_k from Steps 1 and 2 imply that $\{u_k - (1/|\Omega|)\int_\Omega u_k\}$ is bounded in $W^{1,q}(\Omega)$ for $1 \le q < N/(N-1)$. The precompactness of this sequence in both $L^1(\Omega)$ and $L^1(\Gamma)$ follows and we can conclude (passing to a subsequence if necessary) that

$$u_k \to \infty \quad \text{a.e. on } \Omega \quad \text{and a.e. on } \Gamma \tag{2.19}$$

and there are $v_\infty \in L^1(\Omega)$ and $w_\infty \in L^1(\Gamma)$ such that (2.17) holds. It follows from the assumptions and (2.19) that $v_\infty \ge \beta_+^\infty$ and $w_\infty \ge \gamma_+^\infty$ a.e. The compatability condition

$$\int_\Omega v_k + \int_\Gamma w_k = \int_\Omega f_k$$

then yields

$$|\Omega|\beta_+^\infty + |\Gamma|\gamma_+^\infty \ge \int_\Omega f_\infty$$

in the limit, and this contradicts (2.3). Hence (2.18) cannot hold (even along a subsequence). In this way we see that $\{\int_\Omega u_k\}$ is bounded and $\{u_k\}$ is precompact in $L^1(\Omega)$ and $L^1(\Gamma)$.

We note that the existence assertion of Theorem A now follows from what has already been proved (recall the argument which showed Theorem A follows from Theorem B′ and the above steps).

Step 6. The sequence $\{w_k\}$ converges in $L^1(\Gamma)$. We begin by assuming that $B_+^\infty > B_-^\infty$ so that $B_+^k > B_+^k$ for large k. By the previous steps we know that the problem $(\mathrm{BVP})_k$ has a solution $[u_k, v_k, w_k]$ if

$$B_+^k < \int_\Omega f_k < B_+^k \tag{2.20$_k$}$$

and we set $S_k f_k = [S_{1k} f_k, S_{2k} f_k] = [v_k, w_k]$. By Proposition E(i) S_k is a nonexpansive mapping of the subset of $L^1(\Omega)$ on which $(2.20)_k$ holds into $L^1(\Omega)\times L^1(\Gamma)$ and we may therefore extend it by continuity to the set

$$D(S_k) = \left\{ f \in L^1(\Omega) \colon B_-^k \le \int_\Omega f \le B_+^k \right\}.$$

Moreover, we know that if $f_k \in D(S_k)$ and $f_k \to f_\infty$ in $L^1(\Omega)$ then $\{S_{1k} f_k\}$ is precompact in $L^1(\Omega)$ and every limit point is $S_{1\infty} f_\infty$. Hence $S_{1k} f_k \to S_{1\infty} f_\infty$ in $L^1(\Omega)$. Let $(2.20)_\infty$ hold and consider $\{S_{2k} f_\infty\}$. If this sequence is precompact

$L^1(\Gamma)$ we conclude as above that $S_{2k}f_\infty \to S_{2\infty}f_\infty$ in $L^1(\Gamma)$ and then (using the nonexpansiveness) that $S_{2k}f_k \to S_{2\infty}f_\infty$ in $L^1(\Gamma)$. Moreover, using the nonexpansiveness, it will suffice to check the precompactness for a dense set of f_∞'s satisfying $(2.20)_\infty$ and once we have this we get the desired conclusion: $S_{2k}f_k \to S_{2\infty}f_\infty$ in $L^1(\Gamma)$.

Lemma H. *If* $f_\infty \in L^2(\Omega)$ *and satisfies* $(2.20)_\infty$, *then* $\{S_{2k}f_\infty\}$ *is precompact in* $L^1(\Gamma)$.

Sketch of Proof of Lemma H. For the moment let $[u_k, v_k, w_k]$ be the solution of $(\text{BVP})_k$ with f_k replaced by f_∞. It will suffice to show that $\{u_k\}$ is bounded in $L^2(\Omega)$. Indeed, we have

$$u_k - \Delta u_k = f_\infty + u_k - v_k \quad \text{in } \Omega, \qquad u_{k\nu} + \gamma^k(u_k) \ni 0 \quad \text{on } \Gamma$$

and we argue below that $\{v_k\}$ is bounded in $L^2(\Omega)$ and $\{w_k\}$ is bounded in $L^2(\Gamma)$. If we show $\{u_k\}$ is also bounded in $L^2(\Omega)$, then $\{f_\infty + u_k - v_k\}$ is as well and we conclude from Brezis [6] that $\{u_k\}$ is bounded in $W^{2,2}(\Omega)$. This implies that $\{u_{k\nu}\}$ and hence $\{w_k\}$ is precompact in $L^2(\Gamma)$ and so also in $L^1(\Gamma)$.

In order to see that $\{u_k\}$ is bounded in $L^2(\Omega)$ we use Proposition E(ii) and (iii) as in the proof of Lemma 3 of [10] to conclude that

$$\int_\Omega j(v_k) \le \int_\Omega j(f_\infty) \tag{2.21}$$

and

$$\int_\Gamma j(w_k) \le \int_\Gamma j(T(f_\infty - v_k)) \tag{2.22}$$

for every even lower-semicontinuous convex function $j\colon \mathbf{R} \to [0, \infty]$. Taking $j(r) = r^2$ we deduce first that $\{v_k\}$ is bounded in $L^2(\Omega)$ and then (using Proposition C(iv)) that $\{w_k\}$ is bounded in $L^2(\Omega)$. Hence

$$-\Delta u_k = \hat{f}_k \ \text{in } \Omega \quad \text{and} \quad u_{k\nu} = \hat{g}_k,$$

where $\hat{f}_k$, $\hat{g}_k$ are bounded in $L^2(\Omega)$ and $L^2(\Gamma)$ and then we conclude by the usual duality argument that $\{-\nabla u_k\}$ is bounded in $L^2(\Omega)$. Moreover, we know that $\{\int_\Omega u_k\}$ is bounded, so by the Poincaré inequality we are done.

There remains the case in which $\beta^\infty = \gamma^\infty \equiv 0$, which is trivial by Proposition E(ii) and (iii) with $b = 0$ (in this case $v_\infty = 0$, $w_\infty = 0$).

We end this section with some simple remarks concerning the uniqueness of the solutions of (BVP). We know that if $[u, v, w]$ and $[\hat{u}, \hat{v}, \hat{w}]$ are two solutions, then $v = \hat{v}$ and $w = \hat{w}$. However, it is not true that u and $\hat{u}$ must coincide. For example, if $\beta = \gamma \equiv 0$, then u is only unique up to an arbitrary constant. In fact, the uniqueness of v and w implies that this is the general case, in the sense that we must have $\hat{u} = u + c$ for some c. Using the arguments of Lemma 3.5 of [3] we see that if u is not unique then v must be a constant. We suspect that then w must be constant on each component of Γ but have not established this. If, for example, $\gamma \equiv 0$ (so w is constant) and $v = b$ is a constant, the compatability relation shows that $b|\Omega| = \int_\Omega f$. Then $b \in \beta(u)$ and $b = (1/|\Omega|) \int_\Omega f$ imply $(1/|\Omega|) \int_\Omega f \in \beta(u)$ and u is unique if $\beta^{-1}((1/|\Omega|) \int_\Omega f)$ is a singleton. In the event we knew w is constant as well, we could conclude that u is unique if $B^{-1}(\int_\Omega f)$ is a singleton. See Theorem 3 of [19] in this regard.

3. The Parabolic Problem

Let $T > 0$ and $Q = \Omega \times (0, T)$. We consider the "parabolic" problem

$$\begin{aligned} &z_t - \Delta u = f \quad \text{on } Q, \\ &u_\nu + \gamma(u) \ni 0 \qquad \text{on} \quad (0, T) \times \Gamma, \\ &z \in \beta(u) \quad \text{on } Q, \\ &z(0, x) = z_0(x) \quad \text{on } \Omega, \end{aligned} \tag{3.1}$$

in which β and γ are maximal monotone graphs with $0 \in \beta(0) \cap \gamma(0)$, $f \in L^1(\Omega)$, and $z_0 \in L^1(\Omega)$. The partial differential equation above could be formally replaced by either the inclusion $\beta(u)_t - \Delta u \ni f$ or by the inclusion $z_t - \Delta\varphi(z) \ni f$ where $\varphi = \beta^{-1}$. Both forms appear in the literature and here we are emphasizing that they are coextensive. By a classical solution of (3.1) we mean a function $z \in C(\bar{Q})$ with $z_t \in C(\bar{Q})$ for which there is a $u \in C(\bar{Q})$ which is twice continuously differentiable with respect to the space variables so that the conditions in (3.1) are satisfied in the pointwise sense. If β is everywhere defined, smooth, β' is bounded away from zero, γ and f are smooth, and z_0 is smooth and compactly supported in Ω, it can be shown by standard methods that (3.1) has a classical solution. We are going to define generalized solutions of (3.1) when these regularity conditions are not satisfied by taking limits of classical solutions of approximating problems in which they are satisfied.

We will say that a sequence of problems

$$\begin{aligned} &z_{kt} - \Delta u_k = f_k \quad \text{on } Q, \\ &u_{k\nu} + \gamma^k(u_k) \ni 0 \qquad \text{on} \quad (0, T) \times \Gamma, \\ &z_k \in \beta^k(u_k) \quad \text{on } Q, \\ &z_k(x, 0) = z_{k0}(x) \quad \text{on } \Omega, \end{aligned} \tag*{$(3.1)_k$}$$

of the same form as (3.1) converges to the problem (3.1) provided we have

$$\left.\begin{aligned} &\text{(i) } \beta^k \to \beta \text{ and } \gamma^k \to \gamma \text{ in the graph sense,} \\ &\text{(ii) } f_k \to f \text{ in } L^1(Q), \\ &\text{(iii) } z_{k0} \to z_0 \text{ in } L^1(\Omega), \end{aligned}\right\} \tag{3.2}$$

as $k \to \infty$.

Definition. A generalized solution z of (3.1) is a limit in $C([0, T); L^1(\Omega))$ of a sequence of classical solutions z_k of a sequence of problems $(3.1)_k$ which converges to (3.1).

For each $t \geq 0$ let B^t be the maximal monotone graph

$$B^t(r) = t|\Gamma|\gamma(r) + |\Omega|\beta(r). \tag{3.3}$$

One sees, in the obvious way, that if (3.1) has a classical solution then the compatability condition

$$B^t_- \leq \int_\Omega z_0 + \int_0^t \left(\int_\Omega f(x, s)\, dx \right) ds \leq B^t_+ \quad \text{for} \quad 0 < t < T \tag{3.4}$$

holds. This condition is not quite strong enough for what follows and we will use the following stronger variant:

$$\begin{aligned}
&\text{Either} \quad \gamma_+ + \beta_+ = \infty \quad \text{or} \quad \int_\Omega f(x, t)\, dx \leq |\Gamma|\gamma_+ \\
&\qquad \text{for} \quad 0 < t < T \text{ and } \int_\Omega z_0 \leq |\Omega|\beta_+ \quad \text{and} \\
&\text{either} \quad \gamma_- + \beta_- = -\infty \quad \text{or} \quad \int_\Omega f(x, t)\, dx \geq |\Gamma|\gamma_- \\
&\qquad \text{for} \quad 0 < t < T \text{ and } \int_\Omega z_0 \geq |\Omega|\beta_-.
\end{aligned} \tag{3.5}$$

To motivate (3.5), consider the case in which $f(x, t) = f(x)$ is independent of t and (3.4) holds for all T. Then (3.4) is obviously equivalent to (3.5). We have the following theorem:

Theorem I. *Let* (3.5) *hold. Then a necessary and sufficient condition that* (3.1) *have a generalized solution is that*

$$\beta_- \leq z_0(x) \leq \beta_+ \quad \text{a.e.} \tag{3.6}$$

Moreover, if (3.5) *and* (3.6) *hold, then the generalized solution of* (3.1) *is unique. Finally, if* (3.5) *and* (3.6) *hold and* $(3.1)_k$ *is any sequence of problems converging to* (3.1) *with generalized solutions* z_k *we have*

$$z_k \to z \quad \textit{in} \ C([0, T]; L^1(\Omega)),$$

where z *is the generalized solution of* (3.1).

Outline of Proof. The proof of this result relies upon nonlinear semigroup theory and the results of Section 2. In order to apply the nonlinear semigroup theory to (3.1) we define a (possibly multivalued) operator A in $L^1(\Omega)$ associated with (3.1) by $f - v \in Av$ if there is a $u \in \beta^{-1}(v)$ and a $w \in \gamma(u)$ such that $[u, v, w]$ is a solution of (BVP). We now exhibit the dependence of A on β, γ and define

$$A_{\beta,\gamma} = \{[v, f - v]: v, f \in L^1(\Omega) \text{ and there is a solution } [u, v, w] \text{ of (BVP)}\}. \tag{3.7}$$

$A = A_{\beta,\gamma}$ is accretive in $L^1(\Omega)$ by Proposition E(i).

Step 1. We determine the closure of $D(A)$ where $A=A_{\beta,\gamma}$ is defined in (3.7). If $[u, v, w]$ solves (BVP), then clearly $v(x)\in\beta(u(x))$ a.e., so $v(x)\in R(\beta)$ a.e. On the other hand, if $v\in L^1(\Omega)$ and satisfies

$$v(x)\in R(\beta) \text{ a.e.} \quad \text{and} \quad \beta_-|\Omega|<\int_\Omega v<\beta_+|\Omega| \tag{3.8}$$

we will show that $v\in\overline{D(A)}$. Once this is established, it follows that

$$\overline{D(A)}=\{v\in L^1(\Omega)\colon \beta_-\le v\le\beta_+ \text{ a.e.}\} \tag{3.9}$$

since the closure of the set of v's satisfying (3.8) is given by (3.9) (except in the trivial case $\beta_-=\beta_+$).

Let $v\in L^1(\Omega)$, (3.8) hold, and $\lambda>0$. We claim there is a solution $[u_\lambda, v_\lambda, w_\lambda]$ of the problem

$$\begin{aligned} &\beta(u_\lambda)-\lambda\Delta u_\lambda\ni v \quad \text{in } \Omega,\\ &u_{\lambda\nu}+\gamma(u_\lambda)\ni 0 \quad \text{on } \Gamma \end{aligned} \tag{3.10}$$

with $v_\lambda\in\beta(u_\lambda)$, etc. In particular, $v_\lambda\in D(A)$. The existence assertion follows from Theorem A (applied with the graph $r\to(1/\lambda)\beta(r)$ in place β) upon dividing the equation by λ. We may rewrite (3.10) as

$$\begin{aligned} &\beta^\lambda(u^\lambda)-\Delta u^\lambda\ni v \quad \text{in } \Omega,\\ &u^\lambda_\nu+\gamma^\lambda(u^\lambda)\ni 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.11}$$

where

$$\beta^\lambda(r)=\beta(r/\lambda), \qquad \gamma^\lambda(r)=\lambda\gamma(r/\lambda), \quad \text{and} \quad u^\lambda=\lambda u_\lambda. \tag{3.12}$$

Of course, a solution of (3.11) is a triple $[u^\lambda, v^\lambda, w^\lambda]\in W^{1,1}(\Omega)\times L^1(\Omega)\times L^1(\Gamma)$ with the obvious properties. In this correspondence $v^\lambda=v_\lambda$ and we claim that $v^\lambda\to v$ as $\lambda\downarrow 0$ (at least along a subsequence) in $L^1(\Omega)$, whence $v\in\overline{D(A)}$. Indeed, there is a maximal monotone graph $\hat\beta$ such that $\beta^\lambda\to\hat\beta$ as $\lambda\downarrow 0$. Concerning $\hat\beta$ we only need to note that

$$\operatorname{int}\hat\beta(0)=(\beta_-,\beta_+). \tag{3.13}$$

Similarly, there is a sequence $\lambda_n\downarrow 0$ and a maximal monotone graph $\hat\gamma$ such that $\gamma_{\lambda_n}\to\hat\gamma$ as $n\to\infty$ and

$$\hat\gamma(0)\ni 0. \tag{3.14}$$

It follows that a solution $[\hat u, \hat v, \hat w]$ of

$$\begin{aligned} &\hat\beta(\hat u)-\Delta\hat u\ni v,\\ &\hat u_\nu+\hat\gamma(\hat u)\ni 0 \end{aligned}$$

is $\hat u=0$, $\hat v=v$, and $\hat w=0$. We conclude from Theorem B that $v_\lambda\to v$ in $L^1(\Omega)$ as $\lambda\downarrow 0$, completing the proof.

Step 2. In order to apply the nonlinear semigroup theory to the abstract problem

$$z' + Az \ni f, \qquad z(0) = z_0 \tag{3.15}$$

in $L^1(\Omega)$ we need to know the closure R_λ of $R(I+\lambda A)$ in $L^1(\Omega)$ for $\lambda > 0$. However, it is immediate from Theorem A that

$$R_\lambda = \{f \in L^1(\Omega)\colon B_-^\lambda \le \int_\Omega f \le B_+^\lambda\}. \tag{3.16}$$

To give the idea of the proof it will suffice to let f be independent of t. Let $\bar{A}$ be the closure of A in $L^1(\Omega)$. It follows from (3.16) that if f, z_0 satisfies (3.5) then, if $t_0 = 0 < t_1 < t_2 < \cdot < \cdot$ is an increasing sequence, the difference scheme

$$\frac{z_i - z_{i-1}}{t_i - t_{i-1}} + \bar{A} z_i \ni f \qquad \text{for} \quad i = 1, 2, \cdot, \cdot \tag{3.17}$$

has a unique solution $z_0, z_1, \ldots$. It follows that if z_0 is in the closure of $D(A)$, that is (3.6) holds, then (3.15) has a unique mild solution (see, e.g., [5] and [12] concerning this notion). In the general case of time-dependent f satisfying (3.5) we use a suitable approximation of f by step functions and proceed in the obvious way to conclude that if (3.5) and (3.6) hold, then (3.15) has a (unique) mild solution.

If we have a sequence of problems $(3.1)_k$ with associated operators $A_k = A_{\beta_k, \gamma_k}$ which together with (3.1) may be solved by this method, then the "graph convergence" inclusion

$$A \subset \liminf A_k \tag{3.18}$$

guarantees that the mild solution z_k of $z_k' + A_k z_k \ni 0$, $z_k(0) = z_{k0}$, of $(3.1)_k$ converges to the solution z of (3.1) in $C([0, T); L^1(\Omega))$ as $k \to \infty$ (see Theorem 6 of [12]). Moreover, the relation (3.18) is an immediate consequence of Theorem B.

Step 3. If (3.5) and (3.6) hold, then a generalized solution exists. Indeed, we can approximate (3.1) by a sequence of problems $(3.1)_k$ which converge to (3.1) and which have classical (and hence mild) solutions z_k. By Step 2 the z_k converge in $C([0, T)\colon L^1(\Omega))$ to the mild solution z of (3.1), which is then a generalized solution according to the definition. In fact, a refinement of this argument shows that if (3.5) and (3.6) hold, the problems $(3.1)_k$ converge to (3.1) and z_k is a generalized solution of $(3.1)_k$, then $z_k \to z$ in $C([0, T)\colon L^1(\Omega))$. This relies upon the fact that the notion of graph convergence is convergence in a metric topology and so the notion of convergence "of problems" we are using is also metrizable.

Step 4. To conclude the proof, we remark that under assumption (3.5) any generalized solution is a mild solution. In particular, (3.6) holds and generalized solutions are unique since mild solutions are unique. To see this, first note that because of (3.18) generalized solutions are integral solutions in the sense of [2] (see also [5] when it appears). Then use the fact that if an integral solution z of (3.15) satisfies $z(0) \in R_\lambda$ for $\lambda > 0$, then $z(t)$ lies in the closure of $D(A)$ and hence is the mild solution.

This completes the outline of the proof.

The notions above are very general. There are interesting questions about generalized solutions which we have not answered—for example, the uniqueness is open in the event we do not know that generalized solutions are mild solutions. We do know, however, that the class of generalized solutions strictly includes the class of mild solutions. An example of this can be given by choosing $\beta = \gamma = 0$, $f = 0$. Using the approximations

$$\beta_k(r) = r/k \quad \text{and} \quad \gamma_k = 0$$

one can see that constants are generalized solutions in this case while only 0 is a mild solution.

We remark that the recent interesting works of Caffarelli and Friedman [11] and Friedman and Huang [14] determine the limit of the solutions of a sequence of problems

$$u_{kt} - \Delta\varphi_k(u_k) = 0 \quad \text{for } t > 0,\ x \in \mathbf{R}^N, \qquad u_k(x, 0) = g(x)$$

in some cases where the φ_k are everywhere defined and have the limit φ_∞ given by $\varphi_\infty(r) = \emptyset$, $[0, \infty)$, $\{0\}$, $[0, -\infty)$ according as $|r| > 1$, $r = 1$, $|r| < 1$, or $r = -1$ and the function g does *not* take values in $D(\varphi_\infty)$ a.e. There are interesting questions about analogous results in our setting.

4. The L^p Case

We consider the analog of Theorem B′ for the case in which $f_\infty \in L^p(\Omega)$ and $f_k \to f_\infty$ in $L^p(\Omega)$ where $1 < p < N$.

Theorem J. *Let β^k, γ^k be maximal monotone graphs with $0 \in \beta^k(0) \cap \gamma^k(0)$, $1 \le p < \infty$, and $f_k \in L^p(\Omega)$ for $k = 1, 2, \ldots, \infty$. Assume that $\beta^k \to \beta^\infty$ and $\gamma^k \to \gamma^\infty$ in the sense of maximal monotone graphs and $f_k \to f_\infty$ in $L^p(\Omega)$ as $k \to \infty$. Let $[u_k, v_k, w_k]$ be a solution of* (BVP)$_k$ *for finite $k = 1, 2, \ldots$ and* (2.2) *hold. Then*

(i) *$\{v_k\}$ is convergent in $L^p(\Omega)$.*

If, moreover, $1 < p < N$, then

(ii) *$\{w_k\}$ is convergent in $L^{(N-1)p/(N-p)}(\Gamma)$ and*

(iii) *$\{\nabla u_k\}$ is bounded in $L^{Np/(N-p)}(\Omega)$.*

Corollary K. *Assume the hypotheses of Theorem J*(ii) *and* (iii) *and also* (2.3). *Then* (BVP)$_\infty$ *has a solution $[u_\infty, v_\infty, w_\infty]$ belonging to $W^{1,Np/(N-p)}(\Omega) \times L^p(\Omega) \times L^{(N-1)p/(N-p)}(\Gamma)$. Furthermore, $v_k \to v_\infty$* in *$L^p(\Omega)$ and $w_k \to w_\infty$ in $L^{(N-1)p/(N-p)}(\Gamma)$.*

Proof of Theorem J. Since $f_k \to f_\infty$ in $L^p(\Omega)$, there is a convex function $j: \mathbf{R} \to [0, \infty)$ with $\lim_{|s| \to \infty} (j(s)/|s|^p) = \infty$ for which $\{\int_\Omega j(f_k)\}$ is bounded. By (2.21), (putting f_k in place of f_∞), $\int_\Omega j(v_k)$ is bounded and this implies that $\{|v_k|^p\}$ is uniformly integrable. Since $\{v_k\}$ is convergent in $L^1(\Omega)$ by Theorem B′, it is then also convergent in $L^p(\Omega)$, and we have proved (i).

To prove (ii), we use Proposition C and (i) above to conclude that $T(f_k - v_k) \to T(f_\infty - v_\infty)$ in $L^{(N-1)p/(N-p)}(\Gamma)$ where v_∞ is the $L^p(\Omega)$ limit of the v_k. Using (2.22), Theorem B′, and arguing as above we conclude that $\{|w_k|^{(N-1)p/(N-p)}\}$ is uniformly integrable and $\{w_k\}$ is convergent in $L^1(\Gamma)$, so $\{w_k\}$ is convergent in $L^{(N-1)p/(N-p)}(\Gamma)$.

To prove (iii) we argue by duality. Let h be the solution of mean zero of

$$-\Delta h = \sum_1^N (\rho_i)_{x_i} \text{ in } \Omega \quad \text{and} \quad h_\nu = 0 \text{ on } \Gamma, \tag{4.1}$$

where $\rho_i \in C_0^\infty(\Omega)$ for $i = 1, \cdot, \cdot, N$. We have

$$\|h\|_{W^{1,q}(\Omega)} \le C \sum_1^N \|\rho_i\|_{L^q(\Gamma)} \qquad \text{for} \quad 1 \le q < \infty, \tag{4.2}$$

where C depends on Ω and q. There seems to be no convenient reference for this estimate, but it is proved by standard methods. Using h as a test function in (1.2) yields

$$\int_\Omega \sum_1^N u_{kx_i}\rho_i = \int_\Omega h(v_k - f_k) + \int_\Gamma h w_k. \tag{4.3}$$

Let $q = Np/(Np + p - N)$, which is the Holder conjugate of $Np/(N-p)$ and satisfies $q < N$. Using Sobolev and imbedding inequalities together with (4.2) we thus have

$$\|h\|_{L^{p/(p-1)}(\Omega)}, \|h\|_{L^{(N-1)p/N(p-1)}(\Gamma)} \le C \sum \|\rho_i\|_{L^q(\Omega)}. \tag{4.4}$$

Equations (4.3) and (4.4) together imply that

$$\left| \int_\Omega \sum_1^N u_{kx_i}\rho_i \right| \le C(\|v_k - f_k\|_{L^p(\Omega)} + \|w_k\|_{L^{(N-1)p/(N-p)}(\Gamma)}) \sum_1^N \|\rho_i\|_{L^q(\Omega)},$$

and we conclude that $\{\nabla u_k\}$ is bounded in $L^{Np/(N-p)}(\Omega)$ as asserted.

The final remarks we make concern the obstacle problem: let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ and put

$$K = \{v \in H_0^1(\Omega) \colon v \ge \Psi \text{ in } \Omega\}.$$

We consider the solution of the variational inequality

$$u \in K \quad \text{and} \quad (Au, v-u) \ge (f, v-u) \qquad \text{for} \quad v \in K, \tag{4.5}$$

where $A = -\Delta$ and $(\cdot,\cdot)$ is the inner-product of $L^2(\Omega)$. A standard way (see Kinderlehrer and Stampacchia [15, Chapter IV]) to approximate the solution u of this problem is to choose $\alpha \in C^\infty(\mathbf{R})$ satisfying $0 \le \alpha'$, $\alpha(r) = 0$ for $r > 0$, $\alpha(r) < 0$ for $r < 0$, and $\alpha(r)$ is linear for large r and then put

$$\beta_\varepsilon(r) = \varepsilon r + \alpha(r)/\varepsilon.$$

Let u_ε be the solution of

$$\beta_\varepsilon(u_\varepsilon - \Psi) - \Delta u_\varepsilon = f \quad \text{on } \Omega, \qquad u_\varepsilon = 0 \quad \text{on } \Gamma.$$

The arguments in Chapter IV of [15] establish that if $\infty > p \ge 2$ and $f, \Delta\Psi \in L^p(\Omega)$, then u_ε converges weakly in $W^{2,p}(\Omega)$ to the solution u of the variational inequality.

The point we wish to make here is that in fact one has strong convergence in $W^{2,p}(\Omega)$.

Indeed, if $\hat{u}_\varepsilon = u_\varepsilon - \Psi$ then $\hat{u}_\varepsilon$ is a solution of

$$\beta_\varepsilon(\hat{u}_\varepsilon) - \Delta\hat{u}_\varepsilon = f + \Delta\Psi \text{ in } \Omega \quad \text{and} \quad \hat{u}_\varepsilon = 0 \text{ on } \Gamma.$$

By Theorem J we conclude that $-\Delta u_\varepsilon$ is compact in $L^p(\Omega)$ and so by the Calderon-Zygmund estimates $\hat{u}_\varepsilon \to u - \psi$ strongly in $W^{2,p}(\Omega)$. Similar considerations may be used to prove the continuous dependence of u in $W^{2,p}(\Omega)$ on f and $\Delta\Psi$ in $L^p(\Omega)$.

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