Appl Math Optim 20:223-235 (1989)

Applied Mathematics and Optimization

Systems Governed by Ordinary Differential Equations with Continuous, Switching and Impulse Controls

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Abstract. Optimal control problem for systems governed by ordinary differential equations with continuous, switching and impulse controls are studied. It is proved that the value function of the problem is the unique viscosity solution of the corresponding Hamilton-Jacobi-Bellman system.

1. Introduction

In this paper we study an infinite horizontal optimal control problem for systems governed by ordinary differential equations with continuous, switching and impulse controls. Formal application of the Bellman Dynamic Programming Principle leads to the following Hamilton-Jacobi-Bellman (HJB for short) system for the value function $u \equiv (u^1, \ldots, u^m)$ of the optimal control problem:

$$\max\{\lambda u^d(x) - H^d(x, Du^d(x)), u^d(x) - M^d[u](x), u^d(x) - Nu^d(x)\} = 0,$$
$$x \in \mathbb{R}^n, \quad d \in \Lambda \equiv \{1, 2, \ldots, m\}, \tag{1.1}$$

where $H^d(x, p)$ is some given function (see Section 2), M^d and N, we call them switching and impulse obstacle operators, respectively, are given as follows:

$$M^d[u](x) = \min_{d \neq \tilde{d}}\{u^{\tilde{d}}(x) + k(d, \tilde{d})\}, \tag{1.2}$$

$$Nu^d(x) = \inf_{\xi \in K}\{u^d(x + \xi) + l(\xi)\}, \tag{1.3}$$

where $k(\cdot, \cdot)$ and $l(\cdot)$ are some given functions and $K \subset \mathbb{R}^n$ (see Section 2).

The major contribution of this paper is to prove that the value function $u(\cdot) \equiv (u^1(\cdot), \ldots, u^m(\cdot))$ of our control problem is the unique viscosity solution of (1.1). The difficulty in the proof is due to the appearance of the impulse obstacle operator N, which is nonlocal. However, the notion of viscosity solutions is of a local nature in some sense. To overcome this difficulty, we combine the ideas of [2], [5], and [9] and repeat the relevant arguments. The key point in the proof is that under some mild and reasonable conditions, the number of iterations of the arguments is finite.

Optimal switching problems were discussed by Capuzzo-Dolcetta and Evans [5] for ODE (ordinary differential equation) systems, by Evans and Friedman [7], Lenhart and Belbas [11], and Lions [12] for stochastic ODE systems, and by Stojanovic and Yong [15], [16] for abstract evolution systems in infinite dimensions. Optimal impulse control problems were also studied by many authors. Among them we mention works of Barles [1], Bensoussan [3], Bensoussan and Lions [4], Menaldi [13], Menaldi and Robin [14], and the extensive references cited therein. Also, we mention the work by Belbas and Lenhart [2] for optimal switching and impulse problems of stochastic ODE systems. Finally, we should point out that our approach is different from that given in [1].

2. Control Problem

In this section we give some preliminaries. Let V be a metric space, $\Lambda = \{1, 2, \dots, m\}$ and K be a closed subset of $\mathbb{R}^n$ satisfying the following:

$$\xi_1, \xi_2 \in K \quad \Rightarrow \quad \xi_1 + \xi_2 \in K. \tag{2.1}$$

Let $g: \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$, $f: \mathbb{R}^n \times \Lambda \to \mathbb{R}$, $k: \Lambda \times \Lambda \to \mathbb{R}^+ \equiv [0, \infty)$, $l: K \to \mathbb{R}^+$ be continuous mappings satisfying the following conditions, which are assumed throughout this paper:

(i) There exist constants L, L_0, $\gamma > 0$, $0 < \delta \leq 1$ such that, for all x, $\hat{x} \in \mathbb{R}^n$, $(v, d) \in V \times \Lambda$,

$$|g(x, v, d) = g(\hat{x}, v, d)| \leq L|x - \hat{x}|, \tag{2.2}$$

$$|g(x, v, d)| \leq L + L_0|x|, \tag{2.3}$$

$$|f(x, v, d) - f(\hat{x}, v, d)| \leq L(1 + |x|^\gamma + |\hat{x}|^\gamma)|x - \hat{x}|^\delta, \tag{2.4}$$

$$|f(x, v, d)| \leq L(1 + |x|^{\gamma+\delta}). \tag{2.5}$$

(ii) For all d, $\hat{d}$, $\tilde{d} \in \Lambda$, with $d \neq \hat{d} \neq \tilde{d}$,

$$k(d, \tilde{d}) < k(d, \hat{d}) + k(\hat{d}, \tilde{d}), \tag{2.6}$$

$$k(d, d) = 0. \tag{2.7}$$

(iii) There exists a constant $l_0 > 0$ such that

$$l(\xi) \geq l_0, \tag{2.8}$$

and

$$l(\xi + \hat{\xi}) \leq l(\xi) + l(\hat{\xi}), \qquad \forall \xi, \hat{\xi} \in K, \tag{2.9}$$

$$\lim_{\xi \in K, |\xi| \to \infty} \frac{|\xi|^{\gamma+\delta}}{l(\xi)} = 0, \tag{2.10}$$

where γ and δ are the same as those in (i).

Next, let us introduce the following control sets: let $\lambda > 0$ and $d \in \Lambda$ be given. We define

$$\mathcal{V} = \{v(\cdot): [0, \infty) \to V \mid v(\cdot) \text{ is measurable}\},$$

$$\mathcal{A}^d = \Big\{ d(\cdot) = \sum_{i \ge 1} d_{i-1} \chi_{[\theta_{i-1}, \theta_i)}(\cdot): [0, +\infty) \to \Lambda \mid d_0 = d,\ \theta_0 = 0,\ \theta_i \in [0, +\infty],$$

$$\forall i \ge 1;\ \theta_i \uparrow +\infty,\ d_{i+1} \ne d_i \text{ if } \theta_{i+1} < \infty;\ \sum_{i \ge 1} k(d_{i-1}, d_i) e^{-\lambda \theta_i} < \infty \Big\}.$$

$$\mathcal{K} = \Big\{ \xi(\cdot) = \sum_{j \ge 1} \xi_j \chi_{[\tau_j, \infty)}(\cdot): [0, \infty) \to K \mid \tau_j \in [0, \infty],\ \forall j \ge 1;$$

$$\tau_j \uparrow +\infty;\ \sup_{t \ge 0} |\xi(t)| e^{-\tilde{\mu} t} \le R;\ \sum_{j \ge 1} l(\xi_j) e^{-\lambda \tau_j} < \infty \Big\},$$

where $\lambda > 0$, R, $\tilde{\mu} \ge 0$ are given constants. We call any $v(\cdot) \in \mathcal{V}$, $d(\cdot) \in \mathcal{A}^d$, and $\xi(\cdot) \in \mathcal{K}$ an admissible continuous, switching and impulse control, respectively.

For $(x, d) \in \mathbb{R}^n \times \Lambda$ and any $(v(\cdot), d(\cdot), \xi(\cdot)) \in \mathcal{V} \times \mathcal{A}^d \times \mathcal{K}$, the response of the dynamic system is the unique solution of the following equation:

$$y_x(t) = x + \int_0^t g(y_x(s), d(s))\, ds + \xi(t), \qquad t \ge 0. \tag{2.11}$$

The following result is obvious.

Proposition 2.1. *Let $d \in \Lambda$, x, $\hat{x} \in \mathbb{R}^n$, $(v(\cdot), d(\cdot), \xi(\cdot)) \in \mathcal{V} \times \mathcal{A}^d \times \mathcal{K}$. Then the corresponding solutions $y_x(\cdot)$ and $y_{\hat{x}}(\cdot)$ of (2.11) satisfy the following*:

$$|y_z(t)| \le e^{L_0 t}(|x| + Lt) + |\xi(t)| + L_0 \int_0^t e^{L_0(t-s)} |\xi(s)|\, ds, \qquad t \ge 0, \tag{2.12}$$

$$|y_x(t) - y_{\hat{x}}(t)| \le e^{Lt} |x - \hat{x}|, \qquad t \ge 0. \tag{2.13}$$

Hereafter, we assume the following:

$$\lambda > (\gamma + \delta) \max\{\tilde{\mu}, L_0\}. \tag{2.14}$$

It is easy to see that under our assumptions, for any $(x, d) \in \mathbb{R}^n \times \Lambda$, $(v(\cdot), d(\cdot), \xi(\cdot)) \in \mathcal{V} \times \mathcal{A}^d \times \mathcal{K}$ and the corresponding trajectory $y_x(\cdot)$, the following cost functional is well defined

$$J_x^d(v(\cdot), d(\cdot), \xi(\cdot)) = \int_0^\infty f(y_x(s), v(s), d(s))\, e^{-\lambda s}\, ds + \sum_{i \ge 1} k(d_{i-1}, d_i)\, e^{-\lambda \theta_i} + \sum_{j \ge 1} l(\xi_j)\, e^{-\lambda \tau_j}. \tag{2.15}$$

The right-hand side of (2.15) represents the sum of running, switching and impulse costs. The constant $\lambda > 0$ is called the discount factor. Now, we defined the value function $u(\cdot) \equiv (u^1(\cdot), \dots, u^m(\cdot))$ of the control problem in the following way:

$$u^d(x) = \inf_{\mathcal{V} \times \mathcal{A}^d \times \mathcal{K}} J_x^d(v(\cdot), d(\cdot), \xi(\cdot)), \qquad (x, d) \in \mathbb{R}^n \times \Lambda. \tag{2.16}$$

Then, our optimal control problem is the following:

Problem (P). For any given $(x, d) \in \mathbb{R}^n \times \Lambda$, find $(v^*(\cdot), d^*(\cdot), \xi^*(\cdot)) \in \mathcal{V} \times \mathcal{A}^d \times \mathcal{K}$, such that

$$J_x^d(v^*(\cdot), d^*(\cdot), \xi^*(\cdot)) = u^d(x). \tag{2.17}$$

Next, we give some basic properties of the value function.

Lemma 2.2. *The value function $u(\cdot)$ satisfies the following: for all $(x, d) \in \mathbb{R}^n \times \Lambda$,*

$$|u^d(x)| \leq C(1+|x|^{\gamma+\delta}), \tag{2.18}$$

$$|u^d(x) - u^d(\hat{x})| \leq C(1+|x|^\gamma + |\hat{x}|^\gamma)|x-\hat{x}|^\delta, \tag{2.19}$$

where $C = C(L, L_0\gamma, \delta, \lambda, \tilde{\mu}, R)$.

The proof is very similar to that given in [5] or [16]. The following result is the Dynamic Programming Principle. The proof is essentially the same as that in [5].

Proposition 2.3. *The value function $u(\cdot)$ satisfies the following: for any $(x, d) \in \mathbb{R}^n \times \Lambda$,*

$$u^d(x) \leq M^d[u](x) \equiv \min_{\tilde{d} \neq d}\{u^{\tilde{d}}(x) + k(d, \tilde{d})\}, \tag{2.20}$$

$$u^d(x) \leq Nu^d(x) \equiv \inf_{\xi \in K}\{u^d(x+\xi) + l(\xi)\}, \tag{2.21}$$

$$u^d(x) \leq \inf_{v(\cdot) \in \mathcal{V}} \left\{ \int_0^t f(y_x(s), v(s), d)\, e^{-\lambda s}\, ds + u^d(y_x(t))\, e^{-\lambda t} \right\}, \qquad \forall t \geq 0, \tag{2.22}$$

where $\dot{y}_x(s) = g(y_x(s), v(s), d), 0 < s \leq t, y_x(0) = x$. Moreover, if the strict inequalities hold at some point x_0 both in (2.20) and (2.21), then there exists a $t_0 > 0$ such that

$$u^d(x_0) = \inf_{v(\cdot) \in \mathcal{V}} \left\{ \int_0^t f(y_x(s), v(s), d)\, e^{-\lambda s}\, ds + u^d(y_x(t))\, e^{-\lambda t} \right\}, \qquad 0 \leq t \leq t_0. \tag{2.23}$$

From Proposition 2.3 we see that if $u(\cdot) \in C^1(\mathbb{R}^n)^m \equiv \{\mathbb{R}^m\text{-valued } C^1 \text{ functions on } \mathbb{R}^n\}$, then it satisfies the following HJB system:

$$\max\{\lambda u^d(x) - H^d(x, Du^d(x)), u^d(x) - M^d[u](x), u^d(x) - Nu^d(x)\} = 0, \quad \forall x \in \mathbb{R}^n, \quad d \in \Lambda, \tag{2.24}$$

where

$$H^d(x, p) = \inf_{v \in V}\{\langle p, g(x, v, d)\rangle + f(x, v, d)\}. \tag{2.25}$$

However, in general we cannot guarantee the value function $u(\cdot)$ to be C^1. Hence, we need the following notion (see [5] and [6]):

Definition 2.4. Function $u(\cdot) \in C(\mathbb{R}^n)^m \equiv \{\mathbb{R}^m\text{-valued continuous functions on } \mathbb{R}^n\}$ is called a *viscosity solution* of (2.24), if for any $d \in \Lambda$ and any $\varphi(\cdot) \in C^1(\mathbb{R}^n)$ with $u^d(\cdot) - \varphi(\cdot)$ attaining a local maximum (minimum) at $x_0 \in \mathbb{R}^n$, then

$$\max\{\lambda u^d(x_0) - H^d(x_0, D\varphi(x_0)), u^d(x_0) - M^d[u](x_0), u^d(x_0) - Nu^d(x_0)\} \le 0 \ (\ge 0).$$

Then, by Proposition 2.3, we can prove the following result (see [5]):

Theorem 2.5. *The value function $u(\cdot)$ of Problem (P) is a viscosity solution of* (2.24).

3. Uniqueness of Viscosity Solutions

This section contains the main contribution of this paper. We will prove the uniqueness of the viscosity solutions of the following system:

$$\max\{\lambda u^d(x) - H^d(x, Du^d(x)), u^d(x) - M^d[u](x), u^d(x) - Nu^d(x)\} = 0, \quad \forall x \in \mathbb{R}^n, \quad d \in \Lambda. \tag{3.1}$$

Here, $H^d(x, p)$ does not have to be in the form of (2.25). However, it is clear that Definition 2.4 is still good for (3.1).

Before proving our uniqueness theorems, let us first give the following:

Lemma 3.1. *For each $d \in \Lambda$, the function $Nu(\cdot)$ is continuous in $\mathbb{R}^n$.*

The proof is simple if we note (2.10) and Lemma 2.2.

Lemma 3.2. *Suppose $u \equiv (u^1, u^2, \ldots, u^m)$ is a viscosity solution of* (3.1). *Then*

$$u^d(x) \le \min\{M^d[u](x), Nu^d(x)\}, \qquad \forall (x, d) \in \mathbb{R}^n \times \Lambda. \tag{3.2}$$

By Lemma 3.1, similar to [5], we can easily prove the above lemma.

Now let us consider sublinear viscosity solutions. To this end, let us define for $\mu \ge 0$ that

$$Q_\mu(\mathbb{R}^n) = \left\{ u \in UC(\mathbb{R}^n)^m \,\middle|\, \sup_{x \in \mathbb{R}^n} \frac{|u(x)|}{1 + |x|^\mu} < \infty \right\},$$

where $UC(\mathbb{R}^n)^m$ is the set of all $\mathbb{R}^m$-valued, uniformly continuous functions on $\mathbb{R}^n$.

Theorem 3.3. *Let $\mu, \nu \in [0, 1)$. Let $k(\cdot, \cdot)$, $l(\cdot)$ be given continuous functions satisfying* (2.6)-(2.9), *and*

$$\max_{d \neq \tilde{d}} k(d, \tilde{d}) < l_0 \leq \inf_{\xi \in K} l(\xi), \tag{3.3}$$

$$\lim_{\xi \in K, |\xi| \to \infty} \frac{1+|\xi|^{\mu}}{l(\xi)} = 0. \tag{3.4}$$

For each $d \in \lambda$, let $H^d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be continuous satisfying

$$|H^d(x, p) - H^d(x, q)| \leq L(1+|x|^{\nu})|p-q|, \qquad \forall x, p, q \in \mathbb{R}^n, \tag{3.5}$$

$$|H^d(x, p) - H^d(x, q)| \leq w_1(|p||x-y|) + w_2(r; |x-y|),$$
$$\forall p \in \mathbb{R}^n, \quad |x|, |y| \leq r, \tag{3.6}$$

where $w_1(\cdot)$ and $w_2(r; \cdot)$ are strictly increasing continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^+$, with $w_1(0) = 0$, $w_2(r; 0) = 0$, for any $r \geq 0$. If $u, \hat{u} \in Q_{\mu}(\mathbb{R}^n)$ are two viscosity solutions of (3.1), *then $u = \hat{u}$.*

Remark 3.4. Condition (3.3) means that switching is "cheaper" than impulse. Also, we should note that in the above theorem, K only needs to satisfy (2.1), in particular, K may be $\mathbb{R}^n$.

Proof of Theorem 3.3. Since u, $\hat{u} \in Q_{\mu}(\mathbb{R}^n)$, we can find a strictly increasing continuous function $w(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ with $w(0) = 0$, such that

$$|u(x) - u(\hat{x})|, |\hat{u}(x) - \hat{u}(\hat{x})| \leq w(|x - \hat{x}|) \leq C_1(1 + |x - \hat{x}|^{\mu}),$$
$$\forall x, \hat{x} \in \mathbb{R}^n. \tag{3.7}$$

Next, we let σ, $\eta > 0$ be such that (see (3.3))

$$l_0 - \max_{d, \tilde{d}} k(d, \tilde{d}) - w(\sigma) \geq \eta. \tag{3.8}$$

Then we have the following lemma which will play a crucial role in our proof.

Lemma 3.5. *Let $\{d_i, \xi_i, y_i\}_{i=0}^{j}$ be a sequence satisfying*

$$\hat{u}^{d_i}(y_i) = \hat{u}^{d_i}(y_i + \xi_i) + l(\xi_i), \qquad 0 \leq i \leq j, \tag{3.9}$$

$$|y_{i+1} - y_i - \xi_i| < \sigma, \qquad 0 \leq i \leq j. \tag{3.10}$$

Then there exists a constant $\hat{C} > 0$, only depending on σ, m, μ, $l(\cdot)$, C_1, and η, such that

$$j \leq \hat{C}. \tag{3.11}$$

Proof. From (3.2) and (3.7)-(3.10) we have

$$\hat{u}^{d_i}(y_i) = \hat{u}^{d_i}(y_i + \xi_i) + l(\xi_i) \geq \hat{u}^{d_{i+1}}(y_{i+1}) + \eta \geq \cdots \geq \hat{u}^{d_{\hat{i}}}(y_{\hat{i}}) + (\hat{i} - i)\eta \tag{3.12}$$

for all $0 \le i \le \hat{i} \le j$. Without loss of generality, we assume that $d_{\hat{i}} = d_0$, $\hat{i} \ge j/m$. Then,

$$\hat{u}^{d_0}(y_{\hat{i}}) \ge \hat{u}^{d_0}(y_0) + \frac{j}{m}\,\eta. \tag{3.13}$$

On the other hand, we have (see (3.7))

$$l(\xi_i) = \hat{u}^{d_i}(y_i) - \hat{u}^{d_i}(y_i + \xi_i) \le C_1(1 + |\xi_i|^{\mu}).$$

Hence, by (3.4), we can find $C_2 > 0$ such that

$$|\xi_i| \le C_2, \qquad 0 \le i \le j. \tag{3.14}$$

Also, from (3.10), we have

$$|y_{\hat{i}} - y_0| \le \sum_{i=0}^{j-1} |y_i - y_{i+1}| \le j(\sigma + C_2).$$

Hence, (3.13) gives

$$\frac{j}{m}\,\eta \le C_1[1 + j^{\mu}(\sigma + C_2)^{\mu}].$$

Since $\mu < 1$, (3.11) follows. □

Now we return the proof of Theorem 3.3.

Let $\varepsilon, \alpha \in (0, 1)$ be fixed. We define

$$\Phi^d(x, y) = u^d(x) - \hat{u}^d(y) - \frac{1}{\varepsilon}|x - y| - \alpha(\langle x\rangle + \langle y\rangle), \qquad x, y \in \mathbb{R}^n, \quad d \in \Lambda,$$

where $\langle x\rangle = (1 + |x|^2)^{1/2}$. Since $\mu < 1$, it is clear that there exists $(\hat{x}_0, \hat{y}_0) \in \mathbb{R}^{2n}$ such that for some $d_0 \in \Lambda$ (see [5], [15], and [16])

$$\Phi^{d_0}(x_0, y_0) = \max_d \Phi^d(\hat{x}_0, \hat{y}_0) = \sup_{x,y} \max_d \Phi^d(x, y) \tag{3.15}$$

and

$$\hat{u}^{d_0}(y_0) < M^{d_0}[\hat{u}](y_0). \tag{3.16}$$

Now, we assume

$$\hat{u}^{d_0}(y_0) = N\hat{u}^{d_0}(y_0) = \hat{u}^{d_0}(y_0 + \xi_0) + l(\xi_0) \tag{3.17}$$

for some $\xi_0 \in K$. Then, by (3.2) and (3.14), we have

$$\Phi^{d_0}(x_0 + \xi_0, y_0 + \xi_0) \ge \Phi^{d_0}(x_0, y_0) - 2C_2\alpha. \tag{3.18}$$

Next, we let $\zeta: \mathbb{R}^{2n} \to [0, 1]$ be C^1 such that

$$\begin{cases} \operatorname{supp} \zeta \subset \overline{\mathcal{O}((0,0), \sigma)}; & |D\zeta| \le \dfrac{2}{\sigma}; \\ \zeta(0, 0) = 1; & \zeta(x, y) < 1, \quad \text{if } (x, y) \ne (0, 0), \end{cases}$$

where $\mathcal{O}((0,0),\sigma)=\{(x,y)\in\mathbb{R}^{2n} \mid |x|^2+|y|^2<1\}$. We set

$$\zeta_1(x,y)=\zeta(x-\hat{x}_0-\xi_0,\, y-\hat{y}_0-\xi_0)$$

and define

$$\Psi_1^d(x,y)=\Phi^d(x,y)+2C_2\alpha\zeta_1(x,y), \qquad x,y\in\mathbb{R}^n, \quad d\in\Lambda.$$

Then we can apply the above argument to $\Psi_1^d(x,y)$. By repeating the above procedure, and by Lemma 3.5, we know that there exists a $j\le\hat{C}$ such that, for some $x_j,\, y_j\in\mathbb{R}^n$, $d_j\in\Lambda$,

$$\begin{cases}\hat{u}^{d_j}(y_j)<\min\{M^{d_j}[\hat{u}](y_j),\, N\hat{u}^{d_j}(y_j)\},\\ \Psi_j^{d_j}(x_j,y_j)=\sup\limits_{x,y}\max\limits_{d}\Psi_j^d(x,y),\\ \Psi_j^d(x,y)=\Phi^d(x,y)+C_2\alpha\sum\limits_{i=1}^{j}2^i\zeta_i(x,y), \qquad x,y\in\mathbb{R}^N, \quad d\in\Lambda,\end{cases} \tag{3.19}$$

where $\zeta_{i+1}(x,y)=\zeta(x-x_i-\xi_i,\, y-y_i-\xi_i)$. Then, from $\Psi_j^{d_j}(0,0)\le\Psi_j^{d_j}(x_j,y_j)$, we have (see [9])

$$\alpha(\langle x_j\rangle+\langle y_j\rangle)\le C_1(2+|x_j|^{\mu}+|y_j|^{\mu})+\tilde{C},$$

for some $\tilde{C}$, independent of ε and α. Hence, noting $\mu<1$, we have $R_\alpha>0$, independent of ε, such that

$$|x_j|,\, |y_j|\le R_\alpha. \tag{3.20}$$

Hence, from $2\Psi_j^{d_j}(x_j,y_j)\ge\Psi_j^{d_j}(x_j,x_j)+\Psi_j^{d_j}(y_j,y_j)$, we have

$$\frac{1}{\varepsilon}|x_j-y_j|^2\le w(|x_j-y_j|)+C_2\alpha\sum_{i=1}^{j}2^i\zeta_i(x_j,y_j)\le w(|x_j-y_j|)+C_2 2^{\hat{C}+1}\alpha. \tag{3.21}$$

Next, by the definition of viscosity solutions, from (3.5), (3.6), (3.19), and (3.20), we have

$$\begin{aligned}\lambda(u^{d_j}(x_j)-\hat{u}^{d_j}(y_j))\le w_1\left(\frac{2}{\varepsilon}|x_j-y_j|^2\right)&+w_2(R_\alpha;\, |x_j-y_j|)\\ &+\alpha L(2+|x_j|^{\nu}+|y_j|^{\nu})(1+C_2 2^{\hat{C}+2}/\sigma).\end{aligned} \tag{3.22}$$

Then we may assume that $(x_j,y_j)\equiv(x_{j(\varepsilon,\alpha)},\, y_{j(\varepsilon,\alpha)})\to(x^\alpha,x^\alpha)$, as $\varepsilon\to0$ (see (3.21)). Hence, (3.22) gives

$$\lambda[u^{d_j}(x^\alpha)-\hat{u}^{d_j}(x^\alpha)]\le w_1(C_2 2^{\hat{C}+1}\alpha)+2\alpha L(1+2^{\hat{C}+2}C_2/\sigma)(1+|x^\alpha|^{\nu}). \tag{3.23}$$

Then, from (3.19), we see that for any fixed $x\in\mathbb{R}^n$, $\alpha>0$,

$$\begin{aligned}u^d(x)-\hat{u}^d(x)-2\alpha\langle x\rangle\le&\frac{1}{\lambda}w_1(C_2 2^{\hat{C}+1}\alpha)+C_2 2^{\hat{C}+1}\alpha\\ &-2\alpha\left[\langle x^\alpha\rangle-\frac{L}{\lambda}(1+2^{\hat{C}+2}C_2/\sigma)(1+|x^\alpha|^{\nu})\right].\end{aligned}$$

Since $\nu < 1$, letting $\alpha \to 0$, we obtain

$$u^d(x) \le \hat{u}^d(x), \qquad d \in \Lambda, \quad x \in \mathbb{R}^n.$$

By symmetry, we complete the proof. □

The idea of the proof essentially comes from [2], [5], and [9]. Because of the nonlocal nature of the impulse obstacle operator N, we have to repeat some procedures. Then Lemma 3.5 plays a crucial role. But for Lemma 3.5 to be true, we need conditions (3.3) and (3.4) with $\mu < 1$ and u, $\hat{u}$ uniformly continuous. Finally, at the end of the proof, we see that $\nu < 1$ is also crucial.

Next we discuss the case where (3.3) does not hold, namely, switching is not necessarily "cheaper" than impulse. For this case we have the following uniqueness result.

Theorem 3.6. *Let* $\nu \in [0, 1)$. *Let* $K \subset \mathbb{R}^n_+ \equiv \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \ge 0, 1 \le i \le n\}$ *satisfy* (2.1). *Let* $k(\cdot, \cdot)$ *and* $l(\cdot)$ *be continuous functions satisfying* (2.6)-(2.10), *and*

$$\lim_{\xi \in K, |\xi| \to \infty} \frac{1 + |\xi|}{l(\xi)} = 0. \tag{3.24}$$

For each $d \in \Lambda$, *let* $H^d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be continuous satisfying*

$$|H^d(x, p) - H^d(x, q)| \le L(1 + |x|^\nu)|p - q|, \qquad \forall x, p, q \in \mathbb{R}^n, \tag{3.25}$$

$$|H^d(x, p) - H^d(x, q)| \le w_0(|p||x - y|, |x - y|), \qquad \forall x, y, p \in \mathbb{R}^n, \tag{3.26}$$

where $w_0(\cdot, \cdot): \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, strictly increasing in each argument and* $w_0(0, 0) = 0$. *Let* $u, \hat{u} \in UC(\mathbb{R}^n)^m$ *be two viscosity solutions of* (3.1). *Then* $u = \hat{u}$.

The basic idea of proving the above theorem is the same as that of Theorem 3.4. However, since u and $\hat{u}$ are not necessarily sublinear and condition (3.3) is not assumed, to get a result similar to Lemma 3.5 we have to impose the condition $K \subset \mathbb{R}^n_+$. But still K is not necessarily $\mathbb{R}^n_+$ (say, K could be the set of all points in $\mathbb{R}^n_+$ with integer coordinates).

Proof of Theorem 3.6. First, from $u, \hat{u} \in UC(\mathbb{R}^n)^m$, we know that (see [5]) there exists a strictly increasing continuous function $w(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ with $w(0) = 0$, such that

$$|u(x) - u(\hat{x})|, |\hat{u}(x) - \hat{u}(\hat{x})| \le w(|x - \hat{x}|) \le C_1(1 + |x - \hat{x}|),$$
$$\forall x, \hat{x} \in \mathbb{R}^n. \tag{3.27}$$

Next we let $\sigma > 0$ be such that

$$w(\sigma) < l_0, \tag{3.28}$$

where l_0 is given in (2.8). Then we have the following lemma.

Lemma 3.7. *For $\sigma>0$ given as in* (3.28), *the result of Lemma* 3.5 *holds.*

Proof. From (3.9)-(3.10) and (3.27)-(3.28) we have

$$w(\sigma)<l_0\le l(\xi_i)\le w(|\xi_i|)\le C_1(1+|\xi_i|). \tag{3.29}$$

Hence, by (3.24) and the monotonicity of $w(\cdot)$, we obtain

$$\sigma\le|\xi_i|\le C_2, \qquad 0\le i\le j, \tag{3.30}$$

for some C_2 independent of i. We let

$$\hat{k}=\max_{d,\tilde{d}} k(d,\tilde{d}).$$

Then, for $0\le i<\hat{i}\le j$, similar to (3.12), we have (note (2.9))

$$\begin{aligned}\hat{u}^{d_i}(y_i)&\ge\hat{u}^{d_{i+1}}(y_{i+1})+l(\xi_i)-w(\sigma)-\hat{k}\ge\cdots\\&\ge\hat{u}^{d_{\hat{i}}}(y_{\hat{i}})+l(\xi_i+\xi_{i+1}+\cdots+\xi_{\hat{i}})-(\hat{i}-i)(w(\sigma)+\hat{k}).\end{aligned} \tag{3.31}$$

By (3.30), we have (note (3.10))

$$|y_{\hat{i}}-y_i|\le(\hat{i}-i)(\sigma+C_2). \tag{3.32}$$

Again we may assume that $d_{\hat{i}}=d_0$, $\hat{i}\ge j/m$. Then, from (3.29)-(3.31),

$$l(\xi_0+\xi_1+\cdots+\xi_{\hat{i}})\le j[C_1(1+2\sigma+C_2)+\hat{k}]+C_1. \tag{3.33}$$

On the other hand, since $\xi_0,\ldots,\xi_j\in\mathbb{R}^n_+$, we have (see (3.30))

$$|\xi_0+\cdots+\xi_{\hat{i}}|\ge\frac{1}{\sqrt{n}}(|\xi_0|+\cdots+|\xi_{\hat{i}}|)\ge\frac{j\sigma}{m\sqrt{n}}. \tag{3.34}$$

Then our conclusion follows from (3.24), (3.33), and (3.34). □

Now we return the proof of Theorem 3.6.

From the proof of Theorem 3.3 we see that it suffices to prove that for each $\varepsilon>0$, $d\in\Lambda$, the function $u^d(x)=\hat{u}^d(y)-(1/\varepsilon)|x-y|^2$ is bounded from above on $\mathbb{R}^n\times\mathbb{R}^n$. To prove this, let ε, $\alpha>0$. We consider the following function:

$$\Phi_0^d(x,y)=u^d(x)-\hat{u}^d(y)-\frac{1}{\varepsilon}|x-y|-\alpha(|x|^2+|y|^2), \qquad x,y\in\mathbb{R}^n, \quad d\in\Lambda.$$

Then, applying the argument used in the proof of Theorem 3.4, we can obtain the following: there exists $j\le\hat{C}$ such that

$$\begin{cases}\hat{u}^{d_j}(y_j)<\min\{M^{d_j}[\hat{u}](y_j),\ N\hat{u}^{d_j}(y_j)\},\\ \Phi_j^{d_j}(x_j,y_j)=\sup\limits_{x,y}\max\limits_{d}\Phi_j^d(x,y),\\ \Phi_j^d(x,y)=\Phi_0^d(x,y)+\alpha\sum\limits_{i=1}^{j}K_i\zeta_i(x,y), \qquad x,y\in\mathbb{R}^n,\quad d\in\Lambda,\\ K_{i+1}=K_i+2C_2(|x_i|+|y_i|+C_2),\end{cases} \tag{3.35}$$

where $(x_{i+1}, y_{i+1}) \in \mathcal{O}((x_i+\xi_i, y_i+\xi_i), \sigma)$, $\zeta_{i+1}(x, y) = \zeta(x - x_i - \xi_i, y - y_i - \xi_i)$. Then, by the definition of viscosity solutions, we have

$$\lambda(u^{d_j}(x_j) - \hat{u}^{d_j}(y_j)) \le w_0\left(\frac{2}{\varepsilon}|x_j - y_j|^2, |x_j - y_j|\right)$$
$$+ 2\alpha L(2 + |x_j|^\nu + |y_j|^\nu)\left(|x_j| + |y_j| + 2\sum_{i=1}^{j} K_i/\sigma\right). \tag{3.36}$$

It is clear that

$$|x_i| \le |x_j| + j(\sigma + C_2), \qquad 0 \le i \le j.$$

Thus, we have some $\tilde{C}$, independent of ε and α, such that

$$\sum_{i=1}^{j} K_i \le \tilde{C}(1 + |x_j| + |y_j|). \tag{3.37}$$

Then, from $\Phi_j^{d_j}(0, 0) \le \Phi_j^{d_j}(x_j, y_j)$, we can obtain (similarly as before)

$$\alpha(|x_j|^2 + |y_j|^2) \le (2C_1 + \alpha\tilde{C})(1 + |x_j| + |y_j|).$$

Hence,

$$\alpha(|x_j| + |y_j|) \le C. \tag{3.38}$$

Similarly, from $2\Phi_j^{d_j}(x_j, y_j) \ge \Phi_j^{d_j}(x_j, x_j) + \Phi_j^{d_j}(y_j, y_j)$, we can derive that

$$\frac{1}{\varepsilon}|x_j - y_j|^2 \le C, \qquad \forall \alpha \in (0, 1). \tag{3.39}$$

Hence, from (3.35)-(3.39), for any $(x, y) \in \mathbb{R}^{2n}$, $d \in \Lambda$, we have

$$u^d(x) - \hat{u}^d(y) - \frac{1}{\varepsilon}|x - y|^2 - \alpha(|x|^2 + |y|^2)$$
$$\le \frac{1}{\lambda} w_0(2C, \sqrt{C\varepsilon}) - \alpha(|x_j|^2 + |y_j|^2) + \alpha\tilde{C}(1 + |x_j| + |y_j|)$$
$$+ \frac{2\alpha L}{\lambda}(2 + |x_j|^\nu + |y_j|^\nu)[|x_j| + |y_j| + 2\tilde{C}(1 + |x_j| + |y_j|)/\sigma].$$

Then, noting $\nu < 1$, by letting $\alpha \to 0$, we obtain

$$u^d(x) - \hat{u}(y) - \frac{1}{\varepsilon}|x - y|^2 \le w_0(2C, \sqrt{C\varepsilon})/\lambda, \qquad \forall x, y \in \mathbb{R}^n.$$

Then the proof of Theorem 3.3 applies. □

It is easy to see that if (2.2), (2.4), and (2.5) hold with $\gamma = 0$ and for some $\nu \in [0, 1)$,

$$|g(x, v, d)| \le L + L_0|x|^\nu, \qquad \forall (x, v, d) \in \mathbb{R}^n \times V \times \Lambda, \tag{3.40}$$

then (3.5)-(3.6) and (3.25)-(3.26) hold. Hence, from Theorems 2.5, 3.3, and 3.6, we obtain a characterization of the value function for Problem (P).

Theorem 3.8. *Suppose g and f satisfy* (2.2), (2.4), (2.5) (*with* $\gamma = 0$), *and* (3.40). *Suppose the conditions of either Theorem* 3.3 *or Theorem*3.6 *concerning* K, $k(\cdot, \cdot)$, *and* $l(\cdot)$ *hold. Then the value function* $u(\cdot)$ *of Problem* (P) *is the unique viscosity solution of* (2.24) *with* $H^d(x, p)$ *given by* (2.25).

Remark 3.9. It is well known that for general quasi-variational inequalities in unbounded domain with unbounded data, the solutions in some (weighted) Sobolev spaces are not necessarily unique (see [3] and [4], for example). In this case, to characterize the value functions for optimal impulse (stochastic) control problems as the maximal solutions of the corresponding Bellman equations (kinds of quasi-variational inequalities), some kinds of coercivity conditions on the data (namely f) were needed (see [13] and [14]). We have seen that in our present situation, we have imposed some coercivity conditions on $l(\cdot)$ (see (3.4) and (3.24)). These conditions ensure Lemma 3.1 which is needed in the proof of later uniqueness results. We refer the reader to [8] and [10] for some classical uniqueness results about quasi-variational inequalities.

We should note that the cases we have discussed in the above include the case where g and f are uniformly bounded, for which Capuzzo-Dolcetta and Evans discussed optimal switching problem [5].

Finally, we should point out that it is still open whether Theorem 3.8 holds without assuming $\gamma = 0$ and $\nu < 1$.

Acknowledgments

The author would like to thank Professor S. Stojanovic of the University of Cincinnati for some helpful discussions. The author also appreciated the comments of the referees on the original version of the paper.

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Accepted 20 September 1988