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Generalized Solution of Some Parabolic Equations with a Random Drift*

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Abstract. Some stochastic partial differential equations arising from a turbulent transport model are studied using Hida's theory of Brownian functionals. For the spatially homogeneous case, the solutions are constructed as a regular or generalized Brownian functional, depending on a small parameter. The regularity property of such solutions is also determined. However, for the spatially nonhomogeneous equations, only generalized solutions in a series form involving iterated singular Wiener integrals are found.

1. Introduction

This paper is mainly concerned with the fundamental solution of a certain stochastic parabolic equation as a generalized Brownian functional in the sense of Hida [3]. The equation arises from turbulent transport theory. Let $u(x, t, \omega)$ be the concentration of a passive substance convected by a turbulent fluid, such as smoke in turbulent air. For an incompressible flow with the turbulent velocity field $v(x, t, \omega)$, the concentration u satisfies the heat equation with a random drift [1, p. 31]:

$$\begin{cases} \dfrac{\partial u}{\partial t} + v \cdot \nabla u = \frac{1}{2}\nu \nabla^2 u, \\ u(x, 0, \omega) = \delta(x), \end{cases} \tag{1.1}$$

where ν denotes the molecular diffusivity and the initial concentration $\delta(x)$ is the Dirac delta function representing a unit point source at the origin. Usually

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the molecular diffusivity is relatively small and often neglected. Then, by some physical arguments, the mean concentration is assumed to satisfy an effective diffusion equation (see, e.g., pp. 606–614 of [10]). In fact, under some technical assumptions, this kind of diffusion approximation can sometimes be justified [7]. The diffusion approximation allows as to approximate the random velocity $v(x, t, \omega)$ locally by a white noise $\dot{\eta}_t(x)$ satisfying

$$\begin{cases} d\eta_t = b(x, t)\, dt + \sigma(x, t)\, dB_t, \\ \eta_0 = y, \end{cases} \tag{1.2}$$

where b, σ are the drift vector and the diffusion matrix, respectively, y is the initial state, and B_t is the standard Brownian motion. Taking the molecular diffusion into account, we are led to the model equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \eta_t \circ \nabla u = \tfrac{1}{2}\nu \nabla^2 u, \\ u(x, 0) = \delta(x). \end{cases} \tag{1.3}$$

Here the symbol "$\circ$" in the random drift term denotes the symmetric (Stratonovich) scalar product. While the diffusion approximation alleviates the computational difficulty considerably, it also creates some serious mathematical problems. For example, consider the simple one-dimensional problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dot{B}_t \circ \dfrac{\partial u}{\partial x} = \tfrac{1}{2}\varepsilon \dfrac{\partial^2 u}{\partial x^2}, \\ u(x, 0) = \delta(x), \end{cases} \tag{1.4}$$

where $\nu = \varepsilon$ is taken as a small parameter. Regarding the solution $u = q^{\varepsilon}_{x,t}(\dot{B})$ as a Brownian functional depending on ε, it is easy to check by the stochastic calculus that

$$q^{\varepsilon}_{x,t}(\dot{B}) = p(x - B_t, \varepsilon t),$$

where

$$p(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

As $\varepsilon \to 0$, we expect, at least formally, that

$$q^{\varepsilon}_{x,t}(\dot{B}) \to q_{x,t}(\dot{B}) = \delta(x - B_t),$$

which, as shown by Kuo [9], is a generalized Brownian functional, known as Donsker's delta function. Clearly, for $\varepsilon > 0$, the solution $q^{\varepsilon}_{x,t}$ is a regular Brownian functional. The remarkable change in the solution behavior as $\varepsilon \to 0$ is due to a singular perturbation (for the deterministic case, see [12]) of the problem (1.4).

Also we note that, in Itô's form, it may be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dot{B}_t \cdot \dfrac{\partial u}{\partial x} = \frac{1}{2}\nu^{\varepsilon} \dfrac{\partial^2 u}{\partial x^2}, \\ u(x, 0) = \delta(x), \end{cases} \tag{1.5}$$

where $\nu^{\varepsilon} = (1+\varepsilon)$. Therefore, for a parabolic Itô equation such as (1.5), the solution is a generalized Brownian functional unless ν^{ε} is sufficiently large ($\nu^{\varepsilon} > 1$ in this case).

In this paper the solution to a stochastic equation is called a "regular" or a "generalized" solution according to whether it is a regular or a generalized Brownian functional. For notational simplicity, we study only one-dimensional stochastic parabolic equations suggested by the model equation (1.3) in the turbulent transport theory. In Section 2 we briefly review Hida's generalized Brownian functionals. A few definitions and technical lemmas on some mixed singular stochastic integrals are also introduced. Then, in Section 3, a spatially homogeneous turbulent transport equation is treated. By the Fourier transform and the Wiener–Hermite expansion, the solution is constructed explicitly. Thereby the singular behavior, as $\varepsilon \to 0$, and the regularity of the solution are also examined. When the coefficients are spatially nonhomogeneous, the situation is more complex. Such an equation is considered in Section 4. Here, for technical convenience, the stochastic parabolic equation is written in Itô's form. By an iteration procedure, the solution is constructed as a series of iterated singular stochastic integrals which is shown to converge to a generalized Brownian functional. In view of the above example (1.5), in general, we cannot expect to find a regular solution. The regularity question for this equation seems difficult and remains open to us (see remark (2) following Theorem 4.2).

We wish to point out that regular or even strong solutions to parabolic Itô equations have been studied by several authors (for references, see [1]). However, most of the abstract results are applicable only when the spatial domain is bounded. The regular solution of a simpler parabolic Ito equation with spatially homogeneous coefficients has been treated by Shimizu [13] using Hida's theory of Brownian functionals. However, no one seems to have studied the generalized solution and the regularity question as presented here. It will also become clear that most techniques used in this paper can be applied to problems in higher dimensions and to other types of stochastic equations.

2. Preliminaries

To fix the notations, let us first briefly review Hida's generalized Brownian functionals (G.B.F.s) (see Chapter 8 of [4]). The white noise distribution μ is a standard Gaussian measure on $\mathcal{S}^*$ with the characteristic functional

$$C(\xi) = \int_{\mathcal{S}^*} e^{i\langle \dot{B}, \xi \rangle}\, d\mu(\dot{B}) = e^{-\|\xi\|^2/2}, \qquad \xi \in \mathcal{S},$$

where $\mathcal{S}^*$ is the dual of the Schwartz space $\mathcal{S}=\mathcal{S}(\mathbb{R})$ with the duality relation $\langle\cdot,\cdot\rangle$, and $\|\cdot\|$ denotes the $L^2(\mathbb{R})$-norm.

Let $(L^2)=L^2(\mathcal{S}^*,\mu)$ stand for the space of regular or L^2-Brownian functionals $\varphi(\dot{B})$. By the Wiener–Itô decomposition theorem, we have

$$(L^2)=\sum_{n=0}^{\infty}\oplus\mathcal{H}_n,$$

where $\mathcal{H}_n$ is the set of nth multiple Wiener integrals

$$I_n(F)=\int_{\mathbb{R}^n}F(t_1,\ldots,t_n)\,dB_{t_1}\cdots dB_{t_n}, \tag{2.1}$$

for $F\in\hat{L}_n^2$, the symmetric $L_n^2=L^2(\mathbb{R}^n)$. Denote by $H_n^k=H^k(\mathbb{R}^n)$ the Sobolev space of order k. Let $\hat{H}_n^k=\hat{L}_n^2\cap H_n^k$ and $\hat{H}_n^{-k}=(\hat{H}_n^k)^*$. For $F\in H_n^{-\alpha_n}$ with $\alpha_n=(n+1)/2$, the corresponding multiple Wiener integral $I_n(F)$ is known to be a generalized Brownian functional of degree n. Now let $\mathcal{H}_n^{(n)}=\{I_n(f):F\in H_n^{\alpha_n}\}$ and $\mathcal{H}_n^{(-n)}=\{I_n(F):F\in H_n^{-\alpha_n}\}$. Then we can define

$$(L^2)^+=\sum_{n=0}^{\infty}\oplus\mathcal{H}_n^{(n)}$$

and

$$(L^2)^-=\sum_{n=0}^{\infty}\oplus\mathcal{H}_n^{(-n)},$$

which are called the space of test functionals and the space of generalized Brownian functionals, respectively.

An interesting application of G.B.F.s is due to Kuo [9], who has defined Donsker's delta function $\delta_{t,x}(\dot{B})=\delta(x-B_t)$ in this framework. The regularity property of such a function was subsequently investigated by Kallianpur and Kuo [6]. Some analytical techniques developed in these references will be found very useful to the problem under study. In the subsequent analysis, it is necessary to introduce multiple Wiener integrals with a mild singular kernel F, which are related to the iterated stochastic integrals

$$I_t^{(n)}(F)=\int_0^t\cdots\int_0^{t_{n-1}}F(t_1,\ldots,t_n)\,dB_{t_1}\cdots dB_{t_n} \tag{2.2}$$

in the usual way: $I_n(F)=n!\,I_t^{(n)}(F)$. To be specific, let the kernel F satisfy the condition

$$|F(t_1,\ldots,t_n)|\le A[(t_1-t_2)\cdots(t_{n-1}-t_n)t_n]^{-1/2} \tag{2.3}$$

for some positive constant A. Then $I_t^{(n)}(F)$ can be defined as a G.B.F. according to the following lemma.

Lemma 2.1. *Let condition* (2.3) *hold. Then the singular iterated stochastic integral* $I_t^{(n)}(F)$ *can be defined as a* G.B.F. *in* $\mathcal{H}_n^{(-n)}$. *Furthermore, for each* $t>0$, *there exists a constant* $A>0$ *such that*

$$\|I_t^{(n)}(F)\|_{\mathcal{H}_n^{(-n)}}\le 4A(\pi^{3/2}t)^{n/2}/n[\Gamma(n/2)]^{3/2}\qquad\text{for}\quad n\ge 2, \tag{2.4}$$

where Γ *denotes the Gamma function.*

Proof. Let

$$F_t(t_1, \ldots, t_n) = F(t_1, \ldots, t_n)\mathcal{I}_t(t_1, \ldots, t_n), \tag{2.5}$$

where $\mathcal{I}_t(t_1, \ldots, t_n) = \chi_{S_t^n}(t_1, \ldots, t_n)$ denotes the indicator function over the simplex $S_t^n = \{0 \le t_n \le \cdots \le t_1 \le t\}$ in $\mathbb{R}^n$. Let $\tilde{F}_t$ be the symmetrization of F_t. Then

$$\|I_t^{(n)}(F)\|_{\mathcal{H}_n^{(-n)}} = \frac{1}{n!}\|I_n(\tilde{F}_t)\|_{\mathcal{H}_n^{(-n)}} \le \|F_t\|_{H_n^{-\alpha_n}}. \tag{2.6}$$

On the other hand, by definition,

$$\|F\|^2_{H_n^{-\alpha_n}} = \int_{\mathbb{R}^n} (1+|\lambda|^2)^{-\alpha_n}|\hat{F}(\lambda)|^2\, d\lambda,$$

where $\hat{F}$ is the Fourier transform of F,

$$\hat{F}(\lambda) = \int_{\mathbb{R}^n} F(\tau)\, e^{i\lambda\cdot\tau}\, d\tau$$

with $\tau = (t_1, \ldots, t_n) \in \mathbb{R}^n$. Note that the sup-norm

$$\|\hat{F}\|_\infty \le \|F\|_{L_n^1} \qquad \text{with} \quad L_n^1 = L^1(\mathbb{R}^n),$$

and

$$\int_{\mathbb{R}^n} (1+|\lambda|^2)^{-\alpha_n}\, d\lambda \le \sigma_n \qquad \text{for} \quad \alpha_n = \frac{n+1}{2},$$

where $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$ denotes the surface area of the unit sphere in $\mathbb{R}^n$ [9, p. 173]. It follows from (2.6) that

$$\|I_t^{(n)}(F)\|_{\mathcal{H}_n^{(-n)}} \le \frac{2\pi^{n/4}}{[\Gamma(n/2)]^{1/2}}\|F_t\|_{L_n^1}. \tag{2.7}$$

In the meantime, from conditions (2.3) and (2.5), we get

$$\|F_t\|_{L_n^1} \le A d_n(t),$$

where

$$d_n(t) = \int_{S_t^n} [(t_1 - t_2)\cdots(t_{n-1} - t_n)t_n]^{-1/2}\, dt_1 \cdots dt_n.$$

By a change of variables $s_1 = (t_1 - t_2), \ldots, s_{n-1} = (t_{n-1} - t_n)$ and $s_n = t_n$, the above integral $d_n(t)$ can be shown to be dominated by the Dirichlet integral [14, p. 258]:

$$\int_{T_t^n} (s_1 \cdots s_n)^{-1/2}\, ds_1 \cdots ds_n = 2(\pi t)^{n/2}/n\Gamma(n/2),$$

where $T_t^n = \{t_1 \ge 0, \ldots, t_n \ge 0\colon t_1 + \cdots + t_n \le t\}$ is a tetrahedra in $\mathbb{R}^n$. Therefore

$$\|F_t\|_{L_n^1} \le 2A(\pi t)^{n/2}/n\Gamma(n/2)$$

or, in view of (2.7), the desired inequality (2.4) now follows. □

As a variant of an iterated Wiener integral, consider the mixed integral of the form

$$I_n(x,t)=\left(\int_0^t\int\right)\cdots\left(\int_0^{t_{n-1}}\int\right)f_1(x,x_1;t,t_1)\cdots f_n(x_{n-1},x_n;t_{n-1},t_n)\times(dx_1\,dB_{t_1})\cdots(dx_n\,dB_{t_n}). \tag{2.8}$$

Let

$$F_{x,t}^{(n)}(t_1,\ldots,t_n)=\int_{\mathbb{R}^n}f_1(x,x_1;t,t_1)\cdots f_n(x_{n-1},x_n;t_{n-1},t_n)\,dx_1\cdots dx_n. \tag{2.9}$$

Suppose the following conditions hold:

(A1) For each $k\geq 1$, the function $f_k(\cdot,\cdot;t,s)$ is bounded and jointly continuous over $\mathbb{R}^2$ for every t, $s\in\mathbb{R}^+$, and uniformly integrable over $\mathbb{R}$ separately.

(A2) $\int_{\mathbb{R}^k}|F_{x,t}^{(k)}(t_1,\ldots,t_k)|^2\,dt_1\cdots dt_k<\infty$, for $x\in\mathbb{R}$, $t\geq 0$, and $k\leq n$.

Then, as usual, the integral $I_n(x,t)$ can be defined in the (L^2)-sense and rewritten as

$$I_n(x,t)=\int_0^t\cdots\int_0^{t_{n-1}}F_{x,t}^{(n)}(t_1,\ldots,t_n)\,dB_{t_1}\cdots dB_{t_n}. \tag{2.10}$$

To do so we have to appeal to a Fubini type of theorem [5, Lemma 4.1] to change the order of mixed integrations. For instance, take the integral $I_2(x,t)$, which can be written as

$$I_2(x,t)=\int_0^t\left[\int_0^s f_1(x,x_1;t,t_1)\tilde{f}_2(x_1,t_1,t_2)\,dB_{t_2}\right]dx_1\,dB_{t_1}$$

with $\tilde{f}_2(x_1,t_1,t_2)=\int f_2(x_1,x_2;t_1,t_2)\,dx_2$. The above integral is of the form (2.10) provided that the order of the last two integrations may be interchanged. Under conditions (A1) and (A2), it is easy to check that the assumptions of Lemma 4.1 in [5] are met so that the interchange is justified. For higher-order integrals, it resolves in a similar fashion.

Now let us turn to the stochastic integral $I_n(x,t)$ with a singular kernel. In particular, in lieu of (A1) and (A2), we assume that:

(B1) For each $k\geq 1$, the function $f_k(x,y;t,s)$ is jointly continuous for x, $y\in\mathbb{R}$ and t, $s\geq 0$.

(B2) There exist positive constants A and ν_1 independent of n, such that

$$|F_{x,t}^{(n)}(t_1,\ldots,t_n)|\leq A^n p(x,\nu_1 t)[(t_1-t_2)\cdots(t_{n-1}-t_n)t_n]^{-1/2},$$

where, as before, $p(x,t)=(1/\sqrt{2\pi t})\,e^{-x^2/2t}$.

(B3) For each $k\geq 1$, there is a regular function f_k^ε which satisfies conditions (A1) and (A2) for each $\varepsilon>0$, such that f_k^ε converges to f_k pointwise for $t>s\geq 0$.

Then the mixed singular integral of the form (2.8) may be defined as follows.

Definition 2.1. Let $F_{x,t}^{(n),\varepsilon}$ be a regularized version of $F_{x,t}^{(n)}$ defined by (2.9) and let

$$I_n^\varepsilon(x, t) = \int_0^t \cdots \int_0^{t_{n-1}} F_{x,t}^{(n),\varepsilon}(t_1, \ldots, t_n)\, dB_{t_1} \cdots dB_{t_n}.$$

Then we define the integral (2.8), if it exists, as the limit

$$I_n(x, t) = \lim_{\varepsilon \downarrow 0} I_n^\varepsilon(x, t) \quad \text{in } (L^2)^-.$$

We note that, by regularization, the integral $I_n(x, t)$ can be written in the form (2.10). The existence of such an integral as a G.B.F. is ensured by the following lemma.

Lemma 2.2. *Under conditions* (B1)-(B3), *the integral $I_n(x, t)$ defined above exists as a* G.B.F. *of order n such that, for $n \geq 2$,*

$$\|I_n(x, t)\|_{\mathscr{H}_n^{(-n)}} \leq 4p(x, \nu t)A^n(\pi^{3/2}t)^{n/2}/n[\Gamma(n/2)]^{3/2}. \tag{2.11}$$

Proof. By regularization, the proof follows closely that of Lemma 2.1 and is not given here. □

For simplicity, the same notations for norms, inner product, and the duality pairing are used for different Gelfand's triplets as long as there is no confusion. Otherwise, appropriate subscripts will be attached to indicate to which spaces they are associated.

3. Spatially Homogeneous Equations

Consider the following stochastic equation with time-dependent coefficients:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + \dot{\eta}_t \circ \dfrac{\partial u^\varepsilon}{\partial x} = \frac{1}{2}\varepsilon\nu(t)\dfrac{\partial^2 u^\varepsilon}{\partial x^2} + c(t)u^\varepsilon, \\ u^\varepsilon(x, 0) = \delta(x), \end{cases} \tag{3.1}$$

where

$$\begin{cases} d\eta_t = a(t)\, dt + \sigma(t)\, dB_t, \\ \eta_0 = x_0. \end{cases} \tag{3.2}$$

Suppose that the coefficients satisfy:

(C1) $a(t)$, $\nu(t)$, $c(t) \in L^1_{\text{loc}}(\mathbb{R}^+)$ and $\sigma(t) \in L^2_{\text{loc}}(\mathbb{R}^+)$, that is, they are locally L^1 and L^2 functions over $\mathbb{R}^+ = [0, \infty)$, respectively.

(C2) There exist constants σ_0, $\nu_0 > 0$ such that $\sigma(t) \geq \sigma_0$ and $\nu(t) \geq \nu_0$ for a.e. $t \geq 0$.

For convenience, define

$$\begin{cases} a_t = x_0 + \displaystyle\int_0^t a(s)\, ds, \\ c_t = \displaystyle\int_0^t c(s)\, ds, \\ \tau_t^\varepsilon = \displaystyle\int_0^t [\sigma^2(s) + \varepsilon\nu(s)]\, ds \qquad \text{with} \quad \tau_t = \tau_t^0, \\ \text{and} \\ \langle \dot{B}, \sigma\rangle_t = \dfrac{1}{\sqrt{\tau_t}} \displaystyle\int_0^t \sigma(s)\, dB_s. \end{cases} \tag{3.3}$$

Then we can construct the solution to equation (3.1) as follows:

Theorem 3.1. *Let the coefficients a, σ, ν, and c satisfy conditions* (C1) *and* (C2) *given above. Then equation* (3.1) *has a unique solution $u^\varepsilon = q_{x,t}^\varepsilon(\dot{B})$ which is a regular Brownian functional having the following representation*:

$$q_{x,t}^\varepsilon(\dot{B}) = p(x - a_t, \tau_t^\varepsilon) e^{c_t} \sum_{n=0}^\infty \frac{1}{n!\, 2^n} \left(\frac{\tau_t}{\tau_t^\varepsilon}\right)^{n/2} H_n\left(\frac{x - a_t}{\sqrt{2\tau_t^\varepsilon}}\right) H_n\left(\frac{\langle \dot{B}, \sigma\rangle_t}{\sqrt{2}}\right), \tag{3.4}$$

where $H_n(x)$ is the Hermite polynomial of order n.

Proof. The theorem will be proved in the case when $a = c \equiv 0$. The general case can be verified by a simple modification. Therefore, it will be shown that, setting $a = c \equiv 0$ in (3.4), the solution is given by

$$q_{x,t}^\varepsilon(\dot{B}) = p(x, \tau_t^\varepsilon) \sum_{n=0}^\infty \frac{1}{n!\, 2^n} \left(\frac{\tau_t}{\tau_t^\varepsilon}\right)^{n/2} H_n\left(\frac{x - a_t}{\sqrt{2\tau_t^\varepsilon}}\right) H_n\left(\frac{\langle \dot{B}, \sigma\rangle_t}{\sqrt{2}}\right). \tag{3.5}$$

By the Fourier transform, $\hat{u}^\varepsilon(\lambda, t) = \int u^\varepsilon(x, t)\, e^{i\lambda x}\, dx$, equation (3.1) with $a = c \equiv 0$ can be easily solved to give

$$\hat{u}^\varepsilon(\lambda, t) = \exp\{i\lambda \langle \dot{B}, \sigma\rangle_t \sqrt{\tau_t} - \tfrac{1}{2}\lambda^2(\tau_t^\varepsilon - \tau_t)\}.$$

Thus, by the inverse Fourier transform, it yields

$$u^\varepsilon = q_{x,t}^\varepsilon(\dot{B}) = \frac{1}{2\pi} \int \exp\{i\lambda(\langle \dot{B}, \sigma\rangle_t \sqrt{\tau_t} - x) - \tfrac{1}{2}\lambda^2(\tau_t^\varepsilon - \tau_t)\}\, d\lambda, \tag{3.6}$$

which is clearly well defined as a regular Brownian functional.

To obtain the series representation, we formally expand the integrand in (3.6) and interchange the order of summation and integration to obtain

$$q_{x,t}^\varepsilon(\dot{B}) = \frac{1}{2\pi} \sum_{n=0}^\infty \int \exp\{-i\lambda x - \tfrac{1}{2}\lambda^2 \tau_t^\varepsilon\} \frac{(i\lambda)^n}{n!} \left(\frac{\tau_t}{2}\right)^{n/2} H_n\left(\frac{\langle \dot{B}, \sigma\rangle_t}{\sqrt{2}}\right) d\lambda, \tag{3.7}$$

where we have made use of a formula for the generating function of the Hermite polynomials [4, p. 311]. By invoking an integration formula [4, p. 312], we get

$$\frac{1}{2\pi}\int (i\lambda)^n \exp\{-i\lambda x - \tfrac{1}{2}\lambda^2\tau_t^\varepsilon\} = (2\tau_t^\varepsilon)^{-n/2} p(x, \tau_t^\varepsilon) H_n\left(\frac{x}{\sqrt{2\tau_t^\varepsilon}}\right). \tag{3.8}$$

In view of (3.7) and (3.8), representation (3.5) follows.

In justifying the above procedure, we only show that the series (3.5) converges in (L^2). Since $H_n(\langle \dot{B}, \sigma\rangle_t/\sqrt{2}) \in \mathscr{H}_n$ and $\|H_n(\langle\cdot, \sigma\rangle_t/\sqrt{2})\| = n!2^n$, the (L^2)-norm of $q_{x,t}^\varepsilon$ given by (3.5) yields

$$\|q_{x,t}^\varepsilon\|^2 = \sum_{n=0}^{\infty} (n!\,2^n)^{-1} \left| p(x, \tau_t^\varepsilon) H_n\left(\frac{x}{\sqrt{2\tau_t^\varepsilon}}\right)\right|^2 \left(\frac{\tau_t}{\tau_t^\varepsilon}\right)^n.$$

In view of the estimate [9, p. 173],

$$\sup_{\substack{t>0\\ x\in\mathbb{R}}} \left| (\pi n!\,2^n)^{-1/2} e^{-x^2/4t} H_n\left(\frac{x}{\sqrt{2t}}\right)\right| = O(n^{-1/12}),$$

it follows that there exists a constant $K>0$ and a positive integer n_0 such that

$$\sum_{n=n_0}^{\infty} (n!\,2^n)^{-1} \left| p(x, \tau_t^\varepsilon) H_n\left(\frac{x}{\sqrt{2\tau_t^\varepsilon}}\right)\right|^2 (r_t^\varepsilon)^n \le K(\tau_t^\varepsilon)^{-1} e^{-x^2/2\tau_t^\varepsilon} \sum_{n=n_0}^{\infty} n^{-1/6}(r_t^\varepsilon)^n$$

which converges for $t>0$ and $x\in\mathbb{R}$, since the ratio

$$r_t^\varepsilon = \frac{\tau_t}{\tau_t^\varepsilon} < 1 \qquad \text{for} \quad \varepsilon > 0.$$

With the aid of the above results, it is straightforward to show that, in fact, the series (3.5) converges in (L^2) to the unique solution u^ε given by (3.6). □

Now, as $\varepsilon\downarrow 0$, equation (3.1) reduces to

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dot{\eta}_t \circ \dfrac{\partial u}{\partial x} = c(t)u, \\ u(x,0) = \delta(x). \end{cases} \tag{3.9}$$

As mentioned before, the above equation does not have a regular solution. The next theorem states that there exists a unique generalized solution u, which is defined as the weak limit, in $(L^2)^-$, of the perturbed solution u^ε as $\varepsilon\downarrow 0$. But let us first define a modified Donsker's delta function.

Lemma 3.1. *Let η_t be given as in equation* (3.2). *Then the Donsker's delta function $\delta(x-\eta_t)$ is a generalized Brownian functional which has the following representation*:

$$\delta(x-\eta_t) = p(x-a_t, \tau_t) \sum_{n=0}^{\infty} (n!\,2^n)^{-1} H_n\left(\frac{x-a_t}{\sqrt{2\tau_t}}\right) H_n\left(\frac{\langle \dot{B}, \sigma\rangle_t}{\sqrt{2}}\right). \tag{3.10}$$

Proof. The proof is similar to that of Theorem 2 in [9], and thus is omitted. □

Theorem 3.2. *For a fixed $t>0$, as $\varepsilon\downarrow 0$, the solution $u^\varepsilon = q^\varepsilon_{x,t}(\dot B)$ of equation* (3.1) *converges to the generalized solution*

$$u = q_{x,t}(\dot B) = e^{c_t}\delta(x-\eta_t) \tag{3.11}$$

in the sense that, for any $\varphi\in(L^2)^+$,

$$\langle q^\varepsilon_{x,t}, \varphi\rangle \to \langle q_{x,t}, \varphi\rangle \qquad \textit{uniformly in} \quad x\in\mathbb{R}. \tag{3.12}$$

Proof. The fact that the G.B.F. (3.11) is the unique generalized solution of (3.9) can be verified by applying a Fourier transform (now, in the distributional sense) to equation (3.9) in a manner similar to the proof of Theorem 3.1. Here we only prove the convergence statement (3.12) with $a=c\equiv 0$.

For $\varphi\in(L^2)^+$, let

$$g^\varepsilon_{t,\varphi}(x) = \langle q^\varepsilon_{x,t}, \varphi\rangle \quad \text{and} \quad g_{t,\varphi}(x) = \langle q_{x,t}, \varphi\rangle.$$

Noting (3.5) and (3.10), we have

$$\begin{aligned} |g^\varepsilon_{t,\varphi}(x) - g_{t,\varphi}(x)|^2 &= \langle q^\varepsilon_{x,t} - q_{x,t}, \varphi\rangle^2 \\ &\le \|\varphi\|^2_{(L^2)^+} \sum_{n=0}^{\infty} (n!\,2^n)^{-2} h^\varepsilon_n(x,t) \left\| \left(\frac{\langle\cdot,\sigma\rangle_t}{\sqrt{2}}\right)\right\|^2_{(L^2)^-}, \end{aligned} \tag{3.13}$$

where

$$h^\varepsilon_n(x,t) = \left\{ \left(\frac{\tau_t}{\tau^\varepsilon_t}\right)^{n/2} p(x,\tau^\varepsilon_t) H_n\left(\frac{x}{\sqrt{2\tau^\varepsilon_t}}\right) - p(x,\tau_t) H_n\left(\frac{x}{\sqrt{2\tau_t}}\right)\right\}^2.$$

By invoking the Fourier integral representation (3.8) and the like, it can be shown that

$$\begin{aligned} h^\varepsilon_n(x,t) &\le (2\tau_t)^n \int \lambda^{2n} e^{-\lambda^2\tau_t}[1 - e^{-(1/2)\lambda^2(\tau^\varepsilon_t - \tau_t)}]^2\, d\lambda \\ &\le \tfrac{1}{4}(\tau^\varepsilon_t - \tau_t)^2 (2\tau_t)^n \int \lambda^{2(n+2)} e^{-\lambda^2\tau_t}\, d\lambda \\ &= (\varepsilon\nu_t)^2 \tau_t^{5/2} \Gamma(n+\tfrac{5}{2})/2^{n+2}, \end{aligned} \tag{3.14}$$

where we have used the fact $(1-e^{-t})\le t$, $\forall t\ge 0$, and $\nu_t = \int_0^t \nu(s)\, ds$.

Let $f_t(s) = (\sigma(s)/\sqrt{\tau_t})\mathcal{I}_t(s)$, where $\mathcal{I}_t$ is the indicator function on $[0,t]$. Then $\langle \dot B, \sigma\rangle_t = \langle \dot B, f_t\rangle$ and the Fourier transform $\hat f_t(\lambda) = \int f_t(s)\, e^{i\lambda s}\, ds$ has the obvious bound $|\hat f_t(\lambda)| \le (1/\sqrt{\tau_t}) \int_0^t |\sigma(s)|\, ds \le \sqrt{t}$.

Now an application of Lemma 1 in [9] gives

$$\left\| H_n\left(\frac{\langle\cdot,\sigma\rangle_t}{\sqrt{2}}\right)\right\|^2_{(L^2)^-} \le 2(2t)^n n!\,\sigma_n, \qquad n\ge 2. \tag{3.15}$$

Since $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$, inequalities (3.14) and (3.15) imply that

$$(n!\,2^n)^{-2} h^\varepsilon_n(x,t) \left\| H_n\left(\frac{\langle \dot B,\sigma\rangle_t}{\sqrt{2}}\right)\right\|^2_{(L^2)^-} \le \varepsilon^2 \nu_t^2 \tau_t^{-5/2} \frac{\Gamma(n+\frac{5}{2})}{n!\,\Gamma(n/2)} \left(\frac{\sqrt{\pi}\, t}{4}\right)^n. \tag{3.16}$$

By the ratio test and Stirling's formula, it can be shown that the series

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+\frac{5}{2})}{n!\,\Gamma(n/2)} t^k$$

converges for $t \geq 0$. Therefore, upon substituting (3.16) and (3.13) into (3.13), we get

$$|g_{t,\varphi}^{\varepsilon}(x) - g_{t,\varphi}(x)| \leq K_t \varepsilon,$$

where, for a fixed $t > 0$, K_t is a constant independent of x. This proves the theorem. □

The following theorem is concerned with some regularity results for the solutions of the stochastic equations.

Theorem 3.3. *Let the assumptions in Theorem* 3.1 *hold and let $q_{x,t}^{\varepsilon}$ and $q_{x,t}$ denote the solutions of the stochastic equations* (3.1) *and* (3.9), *respectively. For each $\varphi \in (L^2)$ and $\psi \in (L^2)^+$ define*

$$g_{t,\varphi}^{\varepsilon}(x) = (q_{x,t}^{\varepsilon}, \varphi) \quad \text{and} \quad g_{t,\psi}(x) = \langle q_{x,t}, \psi \rangle.$$

Then, for each fixed $t > 0$, the functions $g_{t,\varphi}^{\varepsilon}$ and $g_{t,\psi}$ belong to $\mathcal{S}$. Furthermore, for $\psi \in (L^2)^+$, the uniform convergence (3.12) *can be strengthened so that $g_{t,\psi}^{\varepsilon} \to g_{t,\psi}$ in $\mathcal{S}$.*

Proof. To prove the regularity property, introduce the Sobolev space $\mathcal{S}_k = H^k(\mathbb{R})$ of order k with the norm

$$\|g\|_k = \left\{ \int (1+\lambda^2)^k |\hat{g}(\lambda)|^2 \, d\lambda \right\}^{1/2} \quad \text{for} \quad g \in \mathcal{S}_k, \tag{3.17}$$

where $\hat{g}(\lambda)$ is the Fournier transform of g. By definition and (3.5), we have

$$g_{t,\varphi}^{\varepsilon}(x) = p(x, \tau_t^{\varepsilon}) \sum_{n=0}^{\infty} (n!\,2^n)^{-1} \left(\frac{\tau_t}{\tau_t^{\varepsilon}}\right)^{1/2} \left(H_n\left(\frac{\langle \cdot, \sigma \rangle_t}{\sqrt{2}}\right), \varphi \right) H_n\left(\frac{x}{\sqrt{2\tau_t^{\varepsilon}}}\right). \tag{3.18}$$

In view of (3.8) and (3.17), equation (3.18) yields

$$\begin{aligned}
\|g_{t,\varphi}^{\varepsilon}\|_k^2 &\leq \|\varphi\|_{(L^2)}^2 \sum_{n=0}^{\infty} (n!\,2^n)^{-2} \frac{\tau_t}{\tau_t^{\varepsilon}} \left\| H_n\left(\frac{\langle \cdot, \sigma \rangle_t}{\sqrt{2\tau_t^{\varepsilon}}}\right) \right\|_{(L^2)}^2 \left\| p(\tau_t^{\varepsilon}, \cdot) H_n\left(\frac{\cdot}{\sqrt{2\tau_t^{\varepsilon}}}\right) \right\|_k^2 \\
&\leq \|\varphi\|_{(L^2)}^2 \sum_{n=0}^{\infty} (n!\,2^n)^{-2} \left(\frac{\tau_t}{\tau_t^{\varepsilon}}\right) (n!\,2^n)(2\tau_t^{\varepsilon})^n \int (1+\lambda^2)^k \lambda^{2n} e^{-\lambda^2 \tau_t^{\varepsilon}} \, d\lambda \\
&\leq \|\varphi\|_{(L^2)}^2 \sum_{n=0}^{\infty} \frac{\tau_t^n}{n!} \int (1+\lambda^2)^k \lambda^{2n} e^{-\lambda^2 \tau_t^{\varepsilon}} \, d\lambda.
\end{aligned} \tag{3.19}$$

Making use of the fact $(1+\lambda^2)^k \leq 2^{k-1}(1+\lambda^{2k})$, the above integral has the bound

$$\int (1+\lambda^2)^k \lambda^{2n} e^{-\lambda^2 \tau_t^{\varepsilon}} \, d\lambda \leq 2^{k-1} \left\{ \frac{\Gamma(n+1)}{n!\,(\tau_t^{\varepsilon})^n} + \frac{\Gamma(n+k+\frac{1}{2})}{(n+k)!(\tau_t^{\varepsilon})^{n+k}} \right\}.$$

Again, by the Stirling formula, it is easy to check that the last series in (3.19) converges for each $k>0$. This implies that $g^{\varepsilon}_{t,\varphi}\in\mathscr{S}=\bigcap_{k\geq 0}\mathscr{S}_k$.

With an appropriate change of norms, the proof of the fact $g_{t,\varphi}\in\mathscr{S}$ is quite similar, and the convergence $g^{\varepsilon}_{t,\psi}\to g_{t,\psi}$ in $\mathscr{S}$ as $\varepsilon\downarrow 0$ can be verified as in Theorem 3.2. □

In passing, a few remarks are in order.

(1) Since the proofs are based on the method of the Fourier transform, the above theorems may be generalized to stochastic equations in several space-dimensions. This requires the introduction of the spaces $(L^2)^+$, $(L^2)^-$, etc., for functionals of multidimensional Brownian motions.

(2) Note that, if $\eta_t=B_t$ and $c\equiv 0$ in (3.1) and (3.9), the results of this section reduce to that of Kuo's Theorem 2 in [9] and Theorem 1 of Kallianpur and Kuo [6]. In addition, $q^{\varepsilon}_{x,t}(\dot{B})=p(x-B_t,\varepsilon t)$ may be regarded as a regularization of the Donsker's delta function $\delta(x-B_t)$ for $\varepsilon>0$.

(3) By taking $\varphi=\psi=1$ and then $\varphi=q_{y,t}$, Theorem 3.3 implies that the first two moments of $q^{\varepsilon}_{x,t}$ and the first moment of $q_{x,t}$ are C^{∞}-functions in each space variable for $t>0$.

4. Spatially Nonhomogeneous Equations

To avoid undue notational complication, let us first consider a special case:

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dot{\eta}_t(x)\cdot\dfrac{\partial u}{\partial x}=\tfrac{1}{2}\nu\dfrac{\partial^2 u}{\partial x^2}, & t>0,\quad x\in\mathbb{R},\\ u(x,0)=\delta(x),\end{cases}\tag{4.1}$$

where $\eta_t(x)$ is the Wiener integral

$$\eta_t(x)=\int_0^t \sigma(x,s)\,dB_s,\tag{4.2}$$

and $\nu>0$ is a constant. Note that in contrast with (3.1), the product in the random drift term $\dot{\eta}_t\cdot\partial u/\partial x$ is now taken in the Itô sense. Let g be Green's function associated with (4.1):

$$g(x,t)=p(x,t)=\frac{1}{\sqrt{2\pi t}}\exp-\left\{\frac{1}{2t}x^2\right\}.\tag{4.3}$$

Rewriting (4.1) in an integral form, we get

$$u(x,t)=g(x,t)-\int_0^t g(x-y,t-s)\frac{\partial u(y,s)}{\partial y}\sigma(y,s)\,dy\,dB_s.\tag{4.4}$$

Since we are interested in a generalized solution, the above stochastic integral should be interpreted in the sense of Kubo [8]. An obvious procedure for constructing the solution to (4.4) is by an iteration method (see, e.g., [13]). To this end, we set

$$\begin{cases} u_0(x, t) = g(x, t), \\ \text{and} \\ u_{n+1}(x, t) = g(x, t) - \displaystyle\int_0^t g(x-y, t-s) \frac{\partial u_n(y, s)}{\partial y} \sigma(y, s)\, dy\, dB_s \\ \qquad \text{for} \quad n = 1, 2, \ldots. \end{cases} \tag{4.5}$$

Thereby, we obtain

$$u_n(x, t) = g(x, t) + \sum_{k=1}^{\infty} (-1)^k I_k(x, t), \tag{4.6}$$

where I_k is a k-iterated integral of the form

$$\begin{aligned} I_k(x, t) = &\left(\int_0^t \int\right) \cdots \left(\int_0^{t_{k-1}} \int\right) [g(x-x_1, t-t_1)\sigma(x_1, t_1)] \\ &\times [g_x(x_1 - x_2, t_1 - t_2)\sigma(x_2, t_2)] \cdots [g_x(x_{k-1} - x_k, t_{k-1} - t_k)\sigma(x_k, t_k)] \\ &\times g_x(x_k, t_k)(dx_1\, dB_{t_1}) \cdots (dx_k\, dB_{t_k}), \\ &\text{with} \quad g_x(y, t) = \left.\frac{\partial g(x, t)}{\partial x}\right|_{x=y}. \end{aligned} \tag{4.7}$$

Since, referring to (4.3), the integrand in (4.7) is singular in t's, the above iterated stochastic integral $I_k(x, t)$ cannot be defined in the usual sense. In fact, this has motivated us to define such singular integrals (Definition 2.1) in Section 2. Based on this definition, we can show that

Lemma 4.1. *Suppose that the function σ is in $L^\infty(\mathbb{R} \times \mathbb{R}^+)$. Then, for each $t > 0$ and $x \in \mathbb{R}$, the singular stochastic integral $I_n(x, t)$ given by* (4.7) *exists as a* G.B.F. *in $\mathcal{H}_n^{(-n)}$ for $n = 1, 2, \ldots$.*

Proof. This is an easy consequence of Lemma 2.2. To see this, let

$$f_1(x, y; t, s) = g(x-y, t-s)\sigma(y, s),$$

$$f_k(x, y; t, s) = g_x(x-y, t-s)\sigma(y, s) \qquad \text{for} \quad k = 2, \ldots, n-1,$$

and

$$f_n(x, y; t, s) = g_x(x-y, t-s)\sigma(y, s) g_x(y, s).$$

Condition (B1) is clearly met. Then define $F_{x,t}^{(n)}$ as in (2.9), or

$$\begin{aligned} &F_{x,t}^{(n)}(t_1, \ldots, t_n) \\ &\quad = \int_{\mathbb{R}^n} [g(x-x_1, t-t_1)\sigma(x, t_1)] \cdots [g_x(x_{n-1} - x_n, t_{n-1} - t_n)\sigma(x_n, t_n)] \\ &\qquad \times g_x(x_n, t_n)\, dx_1 \cdots dx_n \mathcal{I}_t(t_1, \ldots, t_n). \end{aligned} \tag{4.8}$$

Since, for each $\nu_1 < \nu$, there exists $\alpha > 0$ such that (see Lemma 4.2)

$$|g_x(x, t)| \le \alpha t^{-1/2} p(x, \nu_1 t), \tag{4.9}$$

condition (B2) is satisfied. To verify (B3), we simply take $g^\varepsilon(x, t) = g(x, t+\varepsilon)$ for $\varepsilon > 0$. It is straightforward to show $F^{(n),\varepsilon}_{x,t}$, with g replaced by g^ε in (4.8), is a regularization of $F^{(n)}_{x,t}$. The conclusion now follows from Lemma 2.2. □

In view of this lemma, the sequence $u_n(x, t)$ given by (4.6) is well defined in $(L^2)^-$. It converges to the generalized solution of equation (4.1), or, equivalently, the integral equation (4.4), as asserted by the next theorem.

Theorem 4.1. *If $\sigma \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$, then equation (4.1) has a unique generalized solution $u = q_{x,t}(\dot{B})$, which admits the following representation:*

$$q_{x,t}(\dot{B}) = g(x, t) + \sum_{n=1}^{\infty} (-1)^n I_n(x, t). \tag{4.10}$$

For each $t > 0$, the above series converges in $(L^2)^-$, uniformly in x over $\mathbb{R}$.

Proof. Consider the regularized version of equation (4.4) in which g is replaced by g^ε, as mentioned above. Then its solution $u^\varepsilon(x, t) = q^\varepsilon_{x,t}(\dot{B})$ has, in lieu of (4.10), the representation

$$q^\varepsilon_{x,t}(\dot{B}) = g^\varepsilon(x, t) + \sum_{n=1}^{\infty} (-1)^n I^\varepsilon_n(x, t), \tag{4.11}$$

where

$$I^\varepsilon_n(x, t) = \int_{\mathbb{R}^n} F^{(n),\varepsilon}_{x,t}(t_1, \ldots, t_n)\, dB_{t_1} \cdots dB_{t_n}. \tag{4.12}$$

By invoking (4.9) and Lemma 2.2 for $0 \le \varepsilon \le t$ and $n \ge 2$, we have

$$\|I^\varepsilon_n(x, t)\|^2_{\mathscr{H}^{(-n)}_n} \le 16 g_1^2(0, t)(2A^2\pi^{3/2}t)^n n!/n^2[\Gamma(n/2)]^{3n}, \tag{4.13}$$

where $g_1(x, t) = p(x, \nu_1 t)$ with $\nu_1 < \nu$, and $A = \alpha\|\sigma\|_\infty$. Again, by Stirling's formula, the series with the nth term given by the right-hand side of (4.13) converges for each fixed $t > 0$ (independent of x and ε). Also, if $u^\varepsilon_n(x, t)$ denotes the nth partial sum of the series (4.11), then $\|u^\varepsilon_n(x, t)\|^2_{(L^2)^-} = |g^\varepsilon(x, t)|^2 + \sum_{k=1}^n \|I^\varepsilon_k(x, t)\|^2_{\mathscr{H}^{(-n)}_n}$. Therefore, $q^\varepsilon_{x,t}$ defined by (4.11) converges in $(L^2)^-$ to the generalized solution $q_{x,t}$ as indicated in (4.10). □

Now we consider the stochastic parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dot{\eta}_t(x) \cdot \dfrac{\partial u}{\partial x} = Lu, \qquad t > 0, \quad x \in \mathbb{R}, \\ u(x, 0) = \delta(x), \end{cases} \tag{4.14}$$

where

$$Lu = \tfrac{1}{2} a(x, t) \frac{\partial^2 u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t) u,$$

and, without the loss of generality, $\eta_t(x)$ is taken as the stochastic integral (4.2). For, if it has the general form (1.2), the deterministic drift item can always be incorporated in the definition of L.

Suppose L is a regular, strongly elliptic operator such that its coefficients satisfy the following conditions:

(D1) There is a constant $\nu > 0$ so that $a(x, t) \geq \nu$ for all $t \geq 0$ and $x \in \mathbb{R}$.
(D2) The functions a, b, c are bounded and uniformly Hölder continuous on $\mathbb{R} \times \mathbb{R}^+$.

Then, by appealing to the theory of parabolic equations, the reduced equation of (4.14) ($\eta_t \equiv 0$) has a unique Green's function $h(x, y; t, s)$ with the following properties [2, pp. 23–24]:

Lemma 4.2. *Let assumptions* (D1) *and* (D2) *be satisfied and let ν_1 be any constant with $0 < \nu_1 < \nu$. Then there exists a constant $A > 0$ such that*

(i) $|h(x, y; t, s)| \leq A(t-s)^{-1/2} \exp\{-(x-y)^2/2\nu_1(t-s)\}$ *and*
(ii) $|(\partial/\partial x) h(x, y; t, s)| \leq A(t-s)^{-1} \exp\{-(x-y)^2/2\nu_1(t-s)\}$,

for $t > s > 0$ and $x \in \mathbb{R}$.

To construct the solution to the problem (4.14), we may proceed as before by the method of iteration. Let $u(x, t) = q_{x,t}(\dot{B})$ be such a solution. Then the series representation is given by

$$q_{x,t}(\dot{B}) = J_0(x, t) + \sum_{n=1}^{\infty} (-1)^n J_n(x, t), \tag{4.15}$$

where

$$J_0(x, t) = h(0, x; 0, t),$$

and, for $n \geq 1$,

$$\begin{aligned} J_n(x, t) = \left(\int_0^t \int\right) \cdots \left(\int_0^{t_{n-1}} \int\right) & h(x, x_1; t, t_1)[\sigma(x_1, t_1) h_x(x_1, x_2; t_1, t_2)] \\ & \cdots [\sigma(x_n, t_n) h_x(x_{n-1}, x_n; t_{n-1}, t_n)] h_x(0, x_n; 0, t_n)(dx_1\, dB_{t_1}) \\ & \cdots (dx_n\, dB_{t_n}) \end{aligned} \tag{4.16}$$

with

$$h_x(x_1, x_2; t_1, t_2) = \frac{\partial}{\partial x} h(x_1, x; t_1, t_2)|_{x=x_2}.$$

Define

$$\begin{aligned} & G_{x,t}^{(n)}(t_1, \ldots, t_n) \\ & \quad = \mathcal{I}_t(t_1, \ldots, t_n) \int_{\mathbb{R}^n} h(x, x_1; t, t_1)[\sigma(x_1, t_1) h_x(x_1, x_2; t_1, t_2)] \\ & \quad \cdots [\sigma(x_n, t_n) h_x(x_{n-1}, x_n; t_{n-1}, t_n)] h_x(0, x_n; 0, t_n)\, dx_1 \cdots dx_n. \end{aligned} \tag{4.17}$$

Then, by Lemma 4.2

$$|G_{x,t}^{(n)}(t_1, \ldots, t_n)| \leq \beta_n g_1(x, t)[(t_1 - t_2) \cdots (t_{n-1} - t_n)t_n]^{-1/2} \times \mathcal{I}_t(t_1, \ldots, t_n), \tag{4.18}$$

where

$$\beta_n = [A\sqrt{2\pi\nu_1}]^{n+1} \|\sigma\|_\infty^n.$$

In view of this inequality, it is easy to prove the following theorem.

Theorem 4.2. *Suppose conditions* (D1) *and* (D2) *are satisfied and* $\sigma \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$. *Then there exists a unique generalized solution* $u = q_{x,t}(\dot{B})$ *of equation* (4.14), *which can be expressed in the form* (4.15). *This series solution converges, for each* $t > 0$, *in* $(L^2)^-$ *uniformly in* x *over* $\mathbb{R}$.

Proof. Because of the bound (4.18), it should be clear that this theorem can be proved in a way parallel to Theorem 4.1. Therefore, the proof is omitted. □

Finally, we would like to make a few comments.

(1) By comparison with the spatially homogeneous case, we have been unable to show the solution of (4.1) as being regular for sufficiently large ν. This is due to the necessary use of the estimate for the derivative of Green's function given by (ii) in Lemma 4.2, which gives rise to iterated singular Wiener integrals.

(2) In addition, the regularity results similar to Theorem 3.3 are lacking for the same reason. However, for Zakai's equation, the regularity property of the solution has been shown by applying the Malliavan calculus [11]. But, in this case, it could also be proved more simply by the present approach, since the multiple Wiener integrals involved are nonsingular.

(3) The analysis in this section cannot be extended directly to a multi-dimensional problem. This is so because the associated multiple Wiener integrals would have a nonintegrable singularity. To overcome this difficulty, it is necessary to enlarge Hida's class of G.B.F.s in such a way that the kernel in the integral representation admits a generalized function.

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