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FINITE MINKOWSKI PLANES IN WHICH EVERY CIRCLE-SYMMETRY IS AN AUTOMORPHISM

1. INTRODUCTION

Our main result is as follows: a finite Minkowski plane in which every circle admits an automorphism fixing it pointwise is either a 'Miquelian' plane or a plane associated to a non-projective sharply 3-transitive group.

1.1. DEFINITIONS. Let M be a set of *points* provided with a family $\mathscr{C}$ of subsets called *circles* and two other families $\mathscr{L}_1$, $\mathscr{L}_2$ of subsets called *lines* (or more precisely $\mathscr{L}_1$-*lines* and $\mathscr{L}_2$-*lines*, respectively). We say that two circles are *tangent* if their intersection is a unique point. The quadruple $(M, \mathscr{L}_1, \mathscr{L}_2, \mathscr{C})$ is a *Minkowski plane* (Heise and Karzel [6]) if the following axioms hold:

(M1) *$\mathscr{L}_1$ and $\mathscr{L}_2$ are partitions of M;*
(M2) *every $\mathscr{L}_1$-line meets every $\mathscr{L}_2$-line in a unique point;*
(M3) *there is a circle containing at least three points;*
(M4) *through three distinct points, such that no two of them are on a common line, there is a unique circle;*
(M5) *every circle intersects every line in a unique point;*
(T) *given a circle C, a point $p \in C$ and a point $p' \notin C$, with p and p' not on a line, there is one and only one C' through p', such that $C \cap C' = \{p\}$.*

If a Minkowski plane is finite—i.e. the set of points is finite—all its lines and all its circles have the same number of points $q+1$, where $q \geqslant 2$ is called the *order* of the plane. Actually, *in finite planes, the 'tangency axiom' (T) is a consequence of the preceding conditions* (Heise and Karzel [6], Percsy [13]).

Two Minkowski planes M and M' are called *isomorphic* if there exists an *isomorphism* from M onto M', i.e. a one-to-one mapping preserving lines and circles. An isomorphism from M on M is called an *automorphism* of M.

1.2. *Examples of Minkowski Planes*

A classical example of Minkowski plane (the 'Miquelian' plane) is given by the geometry of a ruled quadric Q in a three-dimensional projective space: M is the set of points of Q, $\mathscr{L}_1$ and $\mathscr{L}_2$ are the two families of lines contained in Q, and the circles are the non-degenerate plane sections of Q.

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In view of this example, we say that a Minkowski plane is *embeddable* in a three-dimensional projective space P if it is isomorphic to a plane $(M, \mathscr{L}_1, \mathscr{L}_2, \mathscr{C})$, where M is a set of points of P, $\mathscr{L}_1$ and $\mathscr{L}_2$ are families of lines of P contained in M, and each element of $\mathscr{C}$ is the intersection of M and of a plane of P.

Given a finite field F_q of order q, there is a unique (up to isomorphism) Minkowski plane that is embeddable in $PG(3, q)$: this plane is isomorphic to the geometry of a ruled quadric in $PG(3, q)$ and will be denoted by $Mn(F_q)$. This property is a consequence of Segre's results on 'reguli' [16] (Percsy [15]).

By using Benz's method ([1] or [2, p. 296]),[1] a non-embeddable Minkowski plane can be obtained from any sharply 3-transitive set of permutations, containing the identity, that is not a projective group. For instance, there is a Minkowski plane associated with every non-projective sharply 3-transitive group. Such a group has degree $q = p^{2m} + 1$, where p is an odd prime and m a positive integer (Passman [11, p. 261]) and the corresponding Minkowski plane will be denoted by $Mn(N_q)$.[2]

According to Pedrini [12], and contrarily to a guess of Heise and Sörensen [7], there exists also finite sharply 3-transitive sets of permutation, containing the identity, that are not groups.[3] Recently, Wilbrink [18] obtained independently a slight generalization of Pedrini's construction. By a short and neat proof, he establishes the existence of (at least) one Minkowski plane of order p^n for each odd prime p and each integer $n \geqslant 3$, which is not isomorphic to $Mn(F_q)$ or $Mn(N_q)$.

1.3. *A Common Description of all known Finite Models*

We give briefly a common description of all known finite Minkowski planes in a manner that differs slightly from that of Wilbrink.

We first introduce the following (right distributive) quasi-field. Let S be the multiplication subgroup of all (non-zero) squares of $F = GF(q)$, for q odd, and let σ be a non-trivial field automorphism of F. A new multiplication is defined in F by

$$x \circ y = \begin{cases} xy & \text{if } y \in S; \\ x^{\sigma} y & \text{if } y \notin S. \end{cases}$$

The set F, provided with its original addition and the new multiplication $\circ$, becomes a quasi-field, which is not a field, and which will be denoted by Q_q^{σ}. The proof follows easily from Dembowski [3, (4), p. 222], for instance, by using the property that S is a subgroup of index 2 and $S^{\sigma} = S$. The particular case $\sigma^2 = 1$ leads to the regular nearfield N_q of rank 2 over its kernel and

[1] Let us recall that in the finite case, Axiom (T) is not needed, so Benz's result applies.

[2] This notation will be explained in 1.3.

[3] Moreover, the geometry called '3-rete'—obtained by Pedrini from such a set—is actually a Minkowski plane (Axiom (T) is not needed).

odd order q (or $N(2, \sqrt{q})$ according to the notation of Dembowski [3, p. 33]).

Following Wilbrink [18], we define a set of permutations on $F \cup \{\infty\}$ (where $\infty \notin F$), as follows:

$PGL(Q_q^\sigma) = G_1 \cup (G_2\bar{\sigma})$, where $G_1 = PSL(2, q)$ and $G_2 = PGL(2, q)\backslash G_1$ in their natural action on $F \cup \{\infty\}$, and $\bar{\sigma}$ is the permutation fixing ∞ that coincides with σ on F.

We thus obtain a sharply 3-transitive set on $F \cup \{\infty\}$ (Pedrini [12] for a particular value of σ and Wilbrink [18] for the general case). If $Q_q^\sigma = N_q$, we obtain the non-projective sharply 3-transitive group of degree $q+1$ (described in Passman [11, p. 261], for instance). Accordingly, we shall write $PGL(F_q)$ instead of $PGL(2, q)$ (here q may be even).

Now let $K, +, \circ$ be either the field F_q of arbitrary order q, or the regular nearfield N_q of rank 2 over its kernel and odd order q, or the (proper) quasi-field Q_q^σ of odd order q, where $\sigma^2 \neq 1$. The *Minkowski plane $Mn(K)$ over K* is defined as follows:

(i) $M = \bar{K} \times \bar{K}$, where $\bar{K} = K \cup \{\infty\}$ and $\infty \notin K$;
(ii) $\mathscr{L}_1 = \{\{(k, x_2) \mid x_2 \in \bar{K}\} \mid k \in \bar{K}\}$
and $\mathscr{L}_2 = \{\{(x_1, k) \mid x_1 \in \bar{K}\} \mid k \in \bar{K}\}$;
(iii) $\mathscr{C} = \{\{(x_1, x_2) \mid x_2 = \phi(x_1)\} \mid \phi \in PGL(K)\}$.

Finally, let us mention a useful description of $PGL(K)$—or, equivalently, of the circles of $Mn(K)$. It is well known that the elements of $PGL(F_q)$ can be written in the form $x \mapsto (ax+b)/(cx+d)$ (for $a, b, c, d \in F_q$, with $ad - bc \neq 0$). We generalize this property by considering a natural involutary automorphism α of the multiplicative loop of Q_q^σ: for a non-zero $a \in Q_q^\sigma$, a^α is the inverse of a with respect to the multiplication of the *associated field* F_q. Now $PGL(Q_q^\sigma)$ is the set of all permutations ϕ of $Q_q^\sigma \cup \{\infty\}$ of one of the following forms:

(i) $$\phi(x) = \begin{cases} x \circ a + b, & \text{if } x \in Q_q^\sigma; \\ \infty, & \text{if } x = \infty; \end{cases}$$

(ii) $$\phi(x) = \begin{cases} (x+b)^\alpha \circ a + c, & \text{if } x \in Q_q^\sigma - \{-b\}; \\ \infty, & \text{if } x = -b; \\ c, & \text{if } x = \infty; \end{cases}$$

where $a, b, c \in Q_q^\sigma$ and $a \neq 0$.

This description explains better the notation $PGL(Q_q^\sigma)$. Also, by using it, one proves immediately that the residual affine plane at (∞, ∞) (see 2.1) of $Mn(K)$ is a translation plane which can be coordinatized over K.

1.4. *Circle-Symmetries*

There is a natural way to define, in a Minkowski plane, a *symmetry with respect to each circle C*; it is a permutation σ_C of M, fixing all points of C, and mapping each $p \notin C$ on the unique point $p' \notin C$ such that the lines through p' meet C in the same points as the lines through p (see Axioms M1, M2, M5). So σ_C is an involutary permutation of M, preserving lines, whose set of fixed points is C; it is not necessarily an automorphism of the plane.

For the finite Minkowski planes $Mn(K)$ introduced in 1.3, we have the following precise result, proved in 2.8:

THEOREM A. (i) *If* $K = F_q$ *or* N_q, *every symmetry with respect to a circle is an automorphism.*

(ii) *If* $K = Q_q^\sigma$, *where* $\sigma^2 \neq 1$, *none of the symmetries with respect to a circle is an automorphism.*

Our main result states that the property involved in Theorem A actually characterizes the planes over F_q or N_q (i.e. the planes associated to a sharply 3-transitive group):

THEOREM B. *A finite Minkowski plane in which every symmetry with respect to a circle is an automorphism is isomorphic to a plane over* F_q *or* N_q.

Let us note that a non-trivial automorphism of a Minkowski plane fixing a circle C pointwise must be the symmetry σ_C (Dienst [4, §2]). Theorem B can thus be stated in the following way:

COROLLARY. *A finite Minkowski plane in which every circle admits a non-trivial automorphism fixing it pointwise is isomorphic to a plane over* F_q *or* N_q.

The proof of Theorem B is based especially on the Hering–Kantor–Seitz–Shult classification of finite groups with a split BN-pair of rank 1 [8]. Let us recall that, for even order Minkowski planes, there is a stronger result than Theorem B (Heise [5], Percsy [13]): an even order plane is isomorphic to $Mn(F_q)$ for some even prime power q. But our proof works also in that case, and we did not omit it *a priori*. Let us also remark that some of our results hold for infinite Minkowski planes; this is briefly discussed in 4.9.

Finally, we list here some information about the group Σ generated by all symmetries with respect to a circle in a Minkowski plane of finite order q, in which all these symmetries are automorphisms.

THEOREM C. *The subset of all elements of* Σ *stabilizing* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *is a normal subgroup* Σ^+ *of index* 2. *It contains a normal subgroup* T, *isomorphic to the direct product* $T_1 \times T_2$, *where* T_i *is the subgroup of* Σ *of all elements stabilizing every* $\mathscr{L}_i$*-line* ($i = 1, 2$); T *has index* 2 *in* Σ^+ *or* T *equals* Σ^+ *according to whether* q *is odd or even.*[4]

Moreover, T_1 *and* T_2, *acting on an arbitrary* $\mathscr{L}_1$*-line or* $\mathscr{L}_2$*-line, respectively, are isomorphic to* $PSL(2, q)$ *in its usual representation over* $PG(1, q)$.

[4] The elements of T (resp. T_i) are called *translations* (resp. $\mathscr{L}_i$*-translations*) for their remarkable properties (see 3.2, 3.3).

The orbits of the groups T and Σ on the set of circles $\mathscr{C}$ are the equivalence classes with respect to the equivalence relation generated by the tangency relation.[5] *There are one or two orbits according to whether q is even or odd.*

2. Automorphisms

2.1. Let $(M, \mathscr{L}_1, \mathscr{L}_2, \mathscr{C})$ be an arbitrary Minkowski plane. The *residual plane* M_p with respect to a point $p \in M$ is the set $M - (L_1 \cup L_2)$, where L_1, L_2 are the lines through p, provided with all non-void subsets $K \cap M_p$ for $K \in \mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{C}$. Such a subset $K \cap M_p$ is called a *hyperbol* of M_p if K is a circle of M not containing p; it is called a *line* if K is a circle through p or a line of M (distinct from L_1, L_2). More precisely, if $K \in \mathscr{L}_1 \cup \mathscr{L}_2$ $(K \neq L_1, L_2)$, $K \cap M_p$ is an *isotropic line.*

2.2. RESULT (Heise and Karzel [6]). *The residual plane M_p at a point p of a Minkowski plane M, provided with its lines, is an affine plane.*

Let us note that $K \cap M_p$ and $K' \cap M_p$ are parallel iff either K and K' are tangent circles at p or they both belong to $\mathscr{L}_i$, for $i = 1$ or 2.

2.3. As a consequence of 2.2, each residual M_p can be extended to a projective plane $\bar{M}_p$. Let us denote by M_p^∞ the ideal line of M_p and by ∞_i the ideal point of the isotropic lines corresponding to $\mathscr{L}_i$-lines of M ($i = 1$ or 2). It is easily checked that, for each hyperbol H, $H \cup \{\infty_1, \infty_2\}$ is an oval in the projective plane $\bar{M}_p$.

2.4. Clearly, every automorphism of the Minkowski plane M, fixing a point p, induces a collineation in M_p (resp. in $\bar{M}_p$) preserving isotropic lines and hyperbols (resp. stabilizing $\{\infty_1, \infty_2\}$ and preserving hyperbols). The converse is true:

2.5. RESULT (Percsy [15]). *A collineation α of M_p, mapping each hyperbol on a hyperbol and each isotropic line on an isotropic line, induces an automorphism $\hat{\alpha}$ of the Minkowski plane M fixing p.*

The proof is straightforward; we only mention how $\hat{\alpha}$ acts on the lines L_1, L_2 through p. By Axioms M1 and M2 there is a natural one-to-one mapping σ from $(L_1 \cup L_2) - \{p\}$ on the set of isotropic lines: for every point $a \in L_i - \{p\}$, a^σ is the line $L_a \in \mathscr{L}_{i+1}$ containing it ($i \equiv 1$ or 2 mod 2). Since α induces a (one-to-one) mapping on the isotropic lines, we obtain, by using σ, a permutation of $L_1 \cup L_2$, which is precisely the action of $\hat{\alpha}$ on this set (obviously, we define $\hat{\alpha}(p) = p$).

[5] Two circles C and C' are equivalent in that sense iff there are circles $C_1, \ldots, C_n$ such that $C_1 = C$, $C_n = C'$ and C_i is tangent to C_{i+1} for $i = 1, \ldots, n-1$.

2.6. Let us denote by $\mathrm{Aut}(M)$ the full automorphism group of an arbitrary Minkowski plane M. We use the following notation:

(i) $\mathrm{Aut}^+(M)$ (resp. $\mathrm{Aut}^-(M)$ is the set of all automorphisms of M preserving (resp. interchanging) $\mathscr{L}_1$ and $\mathscr{L}_2$.

(ii) $\mathrm{Aut}_i(M)$ (for $i = 1$ or 2) is the subgroup of $\mathrm{Aut}^+(M)$ of all automorphism stabilizing each $\mathscr{L}_i$-line.

Clearly, $\mathrm{Aut}^+(M)$ is either a normal subgroup of index 2 in $\mathrm{Aut}(M)$ or $\mathrm{Aut}(M)$ itself.

2.7. PROPOSITION. *In a Minkowski plane M, the following properties hold for $i = 1$ and 2:*

(i) $\mathrm{Aut}_i(M)$ *is a normal subgroup of* $\mathrm{Aut}^+(M)$;

(ii) *if* $\alpha \in \mathrm{Aut}_i(M)$ *fixes a point x, it fixes the $\mathscr{L}_{i+1}$-line through x pointwise* ($i \equiv 1, 2 \bmod 2$);

(iii) $\mathrm{Aut}_i(M)$ *acts semi-regularly on* $\mathscr{C}$;[6]

(iv) *a non-trivial element of* $\mathrm{Aut}_i(M)$ *fixes at most two points on each $\mathscr{L}_i$-line;*

(v) *each element of* $\mathrm{Aut}_1(M)$ *commutes with each element of* $\mathrm{Aut}_2(M)$.

Proof. The proof is straightforward. For (iv) and (v), we use the property that an $\alpha \in \mathrm{Aut}_i(M)$, fixing a point p, induces a central collineation $\bar{\alpha}$ in $\bar{M}_p$ of centre ∞_i.

2.8. *Proof of Theorem A.* According to the notations of 1.3, let C be a circle $y = \phi(x)$, where $\phi \in PGL(K)$. Clearly, the symmetry σ_C with respect to C is the mapping

$$(x_1, x_2) \mapsto (\phi^{-1}(x_2), \phi(x_1)), \tag{1}$$

where $\phi \in PGL(K)$.

By Benz [2, Satz 1.2, p.297], (1) is an automorphism iff for all $\psi \in PGL(K)$,

$$\phi\psi^{-1}\phi \in PGL(K). \tag{2}$$

The latter condition is obviously satisfied if $K = F_q$ or N_q for $PGL(K)$ is then a group. If $K = Q_q^\sigma$, it is easily checked that it is not satisfied (we use the definition of $PGL(Q_q^\sigma)$ in 1.3 and the property that $\bar{\sigma}G_i\bar{\sigma}^{-1} = G_i$ *for* $i = 1, 2$).

3. TRANSLATIONS

The following result shows the importance, for our problem, of a certain kind of automorphism, called *translation* (Definition 3.2). The study of these

[6] As usually, this means that, given C, C' in $\mathscr{C}$, there is at most one element of $\mathrm{Aut}_i(M)$ mapping C on C'. In other words, only the identity stabilizes a given circle.

automorphisms leads to Theorem 3.6, which is proved by using the Hering–Kantor–Seitz–Shult classification [8].

3.1. RESULT (Dienst [4]). *Let M be a (not necessarily finite) Minkowski plane in which every symmetry with respect to a circle is an automorphism; denote by Σ the group generated by all circle-symmetries. Then, for all $p \in M$, M_p is a translation plane and every translation of M_p induces (in the sense of 2.5) an automorphism of M contained in Σ.*

The second part of this result does not appear explicitly in [4], but the author uses a theorem of Pickert which implies the entire property 3.1.

3.2. DEFINITION. An automorphism α of an (arbitrary) Minkowski plane M is called a *translation* if there exists a point p, fixed by α, such that α induces a translation in M_p. A translation belonging to $\mathrm{Aut}_i(M)$ is an $\mathscr{L}_i$-*translation* ($i = 1, 2$).

3.3. PROPOSITION. *For an automorphism α, the following properties are equivalent ($i \equiv 1, 2 \bmod 2$):*

(i) *α is an $\mathscr{L}_i$-translation;*

(ii) *α fixes a point q and induces a translation in M_q fixing ∞_i;*

(iii) *the fixed points of α are all the points on a line $K \in \mathscr{L}_{i+1}$;*

(iv) *α is in $\mathrm{Aut}_i(M)$ and there is an $\mathscr{L}_i$-line admitting a unique fixed point p.*

Proof. (i) $\Rightarrow$ (ii). This is a consequence of Definition 3.2.

(ii) $\Rightarrow$ (iii). α stabilizes each $\mathscr{L}_i$-line not containing q; so the line $L_i \in \mathscr{L}_i$ through q is also fixed and the line $L_{i+1} \in \mathscr{L}_{i+1}$ through q must be fixed pointwise (Axiom M2). There are no fixed point in M_q, nor on $L_i - \{q\}$ (for no $\mathscr{L}_{i+1}$-line, except L_{i+1}, is fixed, see 2.7(ii)).

(iii) $\Rightarrow$ (iv). Since K is fixed, α belongs to $\mathrm{Aut}^+(M)$. Consequently every $\mathscr{L}_i$-line must be fixed, since it contains a unique fixed point.

(iv) $\Rightarrow$ (i). In $\bar{M}_p$, α induces a central collineation of centre ∞_i and fixing ∞_{i+1}. This collineation must be a translation of M_p otherwise there would be an $\mathscr{L}_{i+1}$-line pointwise fixed, meeting L in a second fixed point (2.7(ii)).

3.4. COROLLARY. *Let p be a point and $L_i \in \mathscr{L}_i$ a line through p such that, for all $a, b \in L_i - \{p\}$, there exists an $\mathscr{L}_i$-translation fixing p and mapping a on b. Then the set $\mathscr{T}_i$ of all $\mathscr{L}_i$-translations fixing p is a group acting regularly on $L_i - \{p\}$. Moreover, if the hypotheses hold for $i = 1$ and $i = 2$, the groups $\mathscr{T}_i$ are abelian.*

Proof. Note that by 3.3(iii), all elements of $\mathscr{T}_i$ fix a point q not on L_i. The first assertion follows thus from 3.3(ii) and from similar properties of translations in affine planes (Dembowski [3, p.131 and no.15, p.122]).

The second part is a consequence of 3.3(ii) (where $q = p$) and Dembowski [3, p.131 and no.11, p.121].

3.5. Let us note that, for all $p, a, b \in M$, such that a, b are not on a line through p, there is at most one translation fixing p and mapping a on b. In view of Definition 3.2, this follows from a similar property of translations in affine planes (Dembowski [3, p.131 and no.15, p.122]).

According to 3.1, we are interested in Minkowski planes having all possible translations:

3.6. THEOREM. *Let M be a Minkowski plane such that for all $a, b, p \in M$, with a and b not on a line through p, there exists a translation fixing p and mapping a on b. Then, the automorphism group generated by all translations of M is the normal subgroup $T_1 \times T_2$ of* Aut(M), *where T_i denotes the group generated by all $\mathscr{L}_i$-translations, for $i = 1, 2$.*[7]

Moreover, if M has finite order q, then q is a prime power and for $i = 1, 2$, T_i, as a permutation group acting on an arbitrary $\mathscr{L}_i$-line, is isomorphic to $PSL(2, q)$ in its usual representation over a projective line.

3.7. LEMMA. *The conjugate, in* Aut(M), *of an $\mathscr{L}_i$-translation is an $\mathscr{L}_j$-translation, for some j.*

Proof. This follows immediately from Proposition 3.3(iii) which provides an intrinsic geometric characterization of $\mathscr{L}_i$-translations.

3.8. *Proof of the First Part of* 3.6. By 3.2 and the well-known properties of translations in affine planes (see, for instance, Dembowski [3, p. 131]), every translation of M is a product of an $\mathscr{L}_1$- and an $\mathscr{L}_2$-translation. Consequently, the group generated by all translations must be $T_1 \cdot T_2$ (see also 2.7(v)). Moreover, T_1 and T_2 are normal subgroups in $\mathrm{Aut}^+(M)$ (by 3.7 and 2.7(i)) and, clearly, $T_1 \cap T_2$ is the identity. Thus $T_1 \cdot T_2$ is actually a direct product and, by 3.7, must be normal in Aut(M).

The proof of the second assertion of Theorem 3.6 is based on the following part of the Hering–Kantor–Seitz–Shult classification of groups with a split BN-pair of rank 1. It is a consequence of Lemmas 3.10 to 3.13.

3.9. RESULT (Hering–Kantor–Seitz [8]). *Let G be a finite group doubly transitive on a set Ω, such that for each* $\mathrm{a} \in \Omega$, *the stabilizer G_a of* a *has a normal subgroup regular on $\Omega - \{\mathrm{a}\}$.*

Then G has a normal subgroup which acts on Ω as one of the following groups in its usual 2-transitive representation on $q + 1$ points:

(I) *a sharply 2-transitive group;*

(II) *$PSL(2, q)$;*

[7] $T_1 \times T_2$ denotes a direct product.

(III) *$Sz(\sqrt{q})$;*
(IV) *$PSU(3, \sqrt[3]{q})$;*
(V) *a group of Ree type.*
Moreover, each of these groups satisfies the above hypotheses.

3.10. LEMMA. *The group T_i, defined in* 3.6, *acting on an arbitrary $\mathscr{L}_i$-line L, is isomorphic to one of the groups listed in* 3.9 ($i = 1$ *or* 2).

Proof. By Corollary 3.4, the subgroup of all the $\mathscr{L}_i$-translations fixing a given point p on L is regular on $L - \{p\}$. Moreover, it is normal in the subgroup of T_i stabilizing p, for, by 3.3(iv), it is the set of all the elements of T_i fixing only the point p. Moreover, T_i is clearly doubly transitive, and result 3.9 therefore applies: T_i contains a normal subgroup N which is one of the groups (I)–(V).

Since each of these groups satisfies the hypothesis of 3.9, for all p, N_p contains a normal regular subgroup $S(p)$. The semi-regularity of $S(p)$ implies that it contains only $\mathscr{L}_i$-translations (Proposition 3.3(iv)) and its transitivity implies that every $\mathscr{L}_i$-translation fixing p belongs to it (Corollary 3.4). Therefore all $\mathscr{L}_i$-translations are in N.

Since T_i is generated by these translations, T_i must be N.

3.11. LEMMA. *T_i cannot be* (V). *If it is* (IV), *it is also* (I).

Proof. T_i contains no element fixing more than two points (Proposition 2.7(iv)). Therefore, it cannot be a Ree-type group (Ward [17, (5), p. 63]). Nor can it be $PSU(3, \sqrt[3]{q})$, unless it be $PSU(3, 2)$, which is sharply 2-transitive (see, for instance, Huppert [9, no. 10.12, p. 242 and no. 10.14, p. 245]).

3.12. LEMMA. *T_i is not* (III).

Proof. Assume that T_i is isomorphic to $Sz(\sqrt{q})$; then q must be even. Consequently, all $\mathscr{L}_i$-translations have order 2: this follows from 3.1 and from the well-known property that, in a translation plane, all translations have the same prime order (see, for instance, Dembowski [3, no. 13, p. 190]).

Therefore, the group S of all $\mathscr{L}_i$-translations fixing a given point p on L, is a 2-group. Since S is regular on $L - \{p\}$ (Corollary 3.4), its order equals the order q of a 2-Sylow subgroup in $Sz(\sqrt{a})$ (Lüneburg [10, p. 26]). Then S is actually a 2-Sylow, and its centre must be of order $\sqrt{q}$ (Lüneburg [10, (4.1.b), p. 26]). This is a contradiction, for S is abelian by Corollary 3.4.

3.13. LEMMA. *Let M_0 be a finite Minkowski plane and assume that* $\mathrm{Aut}_i(M_0)$ (*for $i = 1$ or* 2) *contains a subgroup G that is sharply* 2*-transitive on an $\mathscr{L}_i$-line L. Then M_0 is either $Mn(F_2)$ or $Mn(F_3)$ and G, acting on L, is isomorphic to $PSL(2, 2)$ or $PSL(2, 3)$, resp., in their usual* 2*-transitive representation.*

Proof. We may assume $i = 1$. Since G is sharply 2-transitive, one can associate to it an affine plane A by the following well-known method (Dembowski [3, p. 140]):

(i) A is the set $L \times L$;

(ii) two parallel classes of lines of A are given by the sets $\{a\} \times L$ and $L \times \{b\}$, for $a, b \in L$;

(iii) all other lines are subsets $\{(x^\alpha, x) \mid x \in L\}$, for some $\alpha \in G$.

We make use of another well-known method (Benz [2, p. 296]) to identify A with the set of points of M_0. Let C be a circle; a point $x \in M_0$ is associated to the point $(a, a') \in A$ obtained as follows: a (resp. a') is the intersection of L with the line through x (resp. x') meeting it, where x' is the unique point on C such that x and x' are on a line not intersecting L. Thus the lines $\{b\} \times L$ and $L \times \{b'\}$, for $b, b' \in L$, coincide with the lines of M_0, and all other lines of A correspond to the circles of M_0 that are in the orbit of C under G. The latter assertion follows from the hypothesis that G is contained in $\mathrm{Aut}_1(M_0)$.

Let p be a point not on C. Since A is an affine plane of order $q + 1$, there are q circles through p, besides the $\mathscr{L}_i$-lines through it, that coincide with (affine) lines of A. Any two of them meet only in p, and only one of them is parallel to C (as a line of A). Thus we get $q - 1$ circles, pairwise tangent at p, any of which meet C in a unique point.

Now consider the projective plane $\overline{(M_0)_p}$ of order q (see 2.3). C corresponds to an oval in this plane, admitting exactly $q - 1$ tangent lines through a point on $(M_0)_p^\infty$. By a result of Qvist (see Dembowski [3, no. 23, p. 148]), $q - 1$ must be less than 3; therefore, either $q = 2$, or $q = 3$.

In these cases, M_0 is isomorphic to $Mn(F_2)$ or $Mn(F_3)$, resp. This is a consequence from the unicity of the Minkowski planes of order 2 and 3: the number of circles in a Minkowski plane is $(q + 1)q(q - 1)$ (Heise and Karzel [6]) and the number of 'possible circles' (i.e. subsets of $q + 1$ points meeting every line in a unique point) is $(q + 1)!$. The two numbers are equal if $q = 2$ or 3. Moreover, any sharply 2-transitive group acting on three or four points is isomorphic to the symmetric group on three points or the alternating group on four points. These groups are isomorphic, respectively, to $PSL(2, 2)$ and $PSL(2, 3)$.

It is easily checked that both groups may actually happen as automorphisms groups of Minkowski planes in the way described in our Lemma 3.13.

4. Proof of Theorems B and C

4.1. We need the following result, which is a corollary, in the finite case, of a characterization of Minkowski planes over a Tits nearfield (Percsy [14, Lemma 2 and Result 1]).

4.2. RESULT (Percsy [14]). *Let M be a finite Minkowski plane and let i be either 1 or 2. If* $\mathrm{Aut}_i(M)$ *is transitive on the set of circles, then M is isomorphic to $Mn(F_q)$ or $Mn(N_q)$.*

Proof. Actually, the proof of this particular case does not appear in [14]; but as a consequence of [14, Lemma 2] and of Benz [2, p. 299], such a plane is associated to a finite sharply 3-transitive group. Thus, in view of 1.3, it must be either $Mn(F_q)$ or $Mn(N_q)$.

4.3. From now on, *let $(M, \mathscr{L}_1, \mathscr{L}_2, \mathscr{C})$ be a finite Minkowski plane in which every circle-symmetry is an automorphism.* By 3.1 and 3.6, $\mathrm{Aut}_i(M)$ (for $i = 1$ and 2) contains a subgroup T_i isomorphic, as a permutation group acting on an arbitrary $\mathscr{L}_i$-line, to $PSL(2, q)$, where q equals the order of M.

It is well known that $PSL(2, q)$ has order $\varepsilon\alpha$, where $\alpha = (q+1)q(q-1)$ and ε is either 1 or $\frac{1}{2}$, according to whether q is even or odd. Since α is also the number of circles in M (Heise and Karzel [6] or Percsy [13]), we can deduce the following from 2.7(iii): if q is even, T_i acts regularly on $\mathscr{C}$; if q is odd, T_i has two orbits on $\mathscr{C}$ of length $\alpha/2$. In the even case, Theorem B is a consequence of 4.2.

4.4. Accordingly, we may suppose q odd. Then, it is easily shown that T_1 and T_2 have the same orbits on $\mathscr{C}$. For if there is an $\mathscr{L}_i$-translation from a circle C on a circle C', by 3.3(ii), 3.3(iii) and 2.2, C and C' are tangent; the converse follows from 3.1 and 3.3(ii). Consequently, C and C' are in the same orbit under T_i iff there are $C_1, \ldots, C_n \in \mathscr{C}$, such that $C_1 = C$, $C_n = C'$ and C_i is tangent to C_{i+1} for $i = 1, \ldots, n-1$.

Let $\mathscr{C}_1$, $\mathscr{C}_2$ be the same orbits of T_1 and T_2 and let U be a given circle in $\mathscr{C}_1$. For every circle C, we define two permutations α_1^C and α_2^C of M:

(i) if $x \in U$, $\alpha_i^C(x)$ is the intersection of C and the $\mathscr{L}_i$-line through x ($i = 1, 2 \bmod 2$);

(ii) if $x \notin U$, the $\mathscr{L}_{i+1}$-line through x meets U in y; $\alpha_i^C(x)$ is the intersection of the $\mathscr{L}_i$-line through x and the $\mathscr{L}_{i+1}$-line through $\alpha_i^C(y)$.

4.5. LEMMA. (i) *For any $C \in \mathscr{C}$, if α_i^C maps every circle on a circle, then it belongs to* $\mathrm{Aut}_i(M)$.

(ii) *If $C \in \mathscr{C}_1$, α_i^C belongs to T_i. Conversely, every $\alpha \in T_i$ is the mapping α_i^D, where $D = U^\alpha$.*

(iii) *For any C, C', the mappings α_1^C, and $\alpha_2^{C'}$ commute.*

(iv) *For any C, C', $\alpha_1^C(C') = \alpha_2^{C'}(C)$.*

(v) *For any $C \in \mathscr{C}$: $\alpha_1^C\alpha_2^C = \alpha_2^C\alpha_1^C = \sigma_C\sigma_U$.*[8]

Proof. (i) By definition, α_i^C stabilizes every $\mathscr{L}_i$-line and maps an $\mathscr{L}_{i+1}$-line on an $\mathscr{L}_{i+1}$-line.

[8] According to 1.4, σ_U and σ_C denote the symmetries with respect to U and C.

(ii) By 2.7(iii), α_i^C must be the member of T_i that maps U on C.

(iii) If x is a point, $x_1 = \alpha_1^C(x)$ is on the $\mathscr{L}_1$-line through x and $x_2 = \alpha_2^{C'}(x)$ is on the $\mathscr{L}_2$-line through x. Since α_i^C maps an $\mathscr{L}_{i+1}$-line on an $\mathscr{L}_{i+1}$-line, $\alpha_1^C(x_2) = \alpha_2^{C'}(x_1)$.

(iv) This follows from (iii), for

$$\alpha_1^C(C') = \alpha_1^C\alpha_2^{C'}(U) = \alpha_2^{C'}\alpha_1^C(U) = \alpha_2^{C'}(C).$$

(v) Let x be a point and let x', x'' denote $\sigma_U(x)$ and $\sigma_C\sigma_U(x)$ respectively. Furthermore, let u_i and c_i be the intersections of U and C, respectively, with the $\mathscr{L}_i$-line through x' ($i = 1, 2$). (All these points need not be distinct.)

Clearly, $\alpha_i^C(u_i) = c_i$. Since $x = \sigma_U(x')$, the point $y = \alpha_1^C(x)$ is the intersection of the $\mathscr{L}_1$-line L_1 through x and the $\mathscr{L}_2$-line L_2 through c_1, and L_1 contains u_2. Similarly, $x'' = \sigma_C(x')$ implies that x'' is on L_2 and on the $\mathscr{L}_1$-line through c_2. Also $x'' = \alpha_2^C(y)$, for $c_2 = \alpha_2^C(u_2)$; therefore, $\alpha_2^C\alpha_1^C = \sigma_C\sigma_U$.

The remaining part of (iv) follows from (iii).

4.6. LEMMA. *For any $C \in \mathscr{C}_2$, α_i^C maps every circle on a circle.*

Proof. (1) We claim that α_i^C maps $\mathscr{C}_1$ onto $\mathscr{C}_2$.[9] Let us prove this for $i = 1$; the second case is similar. If $D \in \mathscr{C}_1$, α_2^D is an automorphism (Lemma 4.5(ii)) and $\alpha_1^C(D) = \alpha_2^D(C)$ by 4.5(iv). So $\alpha_1^C(D)$ is a circle. Conversely, for each $E \in \mathscr{C}_2$ there is an $\alpha \in T_2$ mapping C on E (see 4.4). But $\alpha = \alpha_2^D$ for some $D \in \mathscr{C}_1$ (4.5(ii)) and $E = \alpha_2^D(C) = \alpha_1^C(D)$ by 4.5(iv).

(2) Now let D be an element of $\mathscr{C}_2$: we have $\alpha_1^C(D) = \alpha_2^C\alpha_1^C(\alpha_2^C)^{-1}(D)$ (by 4.5(iii)). But $(\alpha_2^C)^{-1}(D)$ is a circle of $\mathscr{C}_1$ (by (1)), and $\alpha_2^C\alpha_1^C = \sigma_C\sigma_U$ is an automorphism (4.5(iv)), so $\alpha_1^C(D)$ must be a circle. One proves similarly that α_2^C maps $\mathscr{C}_2$ on $\mathscr{C}_1$.

4.7. *End of Proof of Theorem B.* It follows from 4.6, 4.5(i) and 4.5(ii) that all mappings α_1^C defined in 4.4 are elements of $\mathrm{Aut}_1(M)$. Therefore, result 4.2 applies.

4.8. *Proof of Theorem C.* (i) First we note that Σ^+ is the set of all products of an even number of circle-symmetries. It is therefore a normal subgroup of index 2 in Σ.

(ii) By 3.1 and 3.6 there is a normal subgroup $T = T_1 \times T_2$ in Σ such that T_i is isomorphic to $PSL(2, q)$ and is contained in $\mathrm{Aut}_i(M)$.

(iii) Given two circles C and U, we have

$$\alpha_1^C\alpha_2^C = \sigma_C\sigma_U,$$

where α_1^C, α_2^C are the mappings defined in 4.4. For odd order planes this

[9] The property that α_i^C is 'onto' is obvious here since the plane is finite, but in view of the generalization in § 5, we prove this lemma without the finiteness assumption.

follows from Lemma 4.5(v); the proof 4.5(v) also applies to even order planes without any change.

(iv) If the order of the plane is even, T_i is transitive on $\mathscr{C}$ (by 4.3) and it must be equal to $\mathrm{Aut}_i(M)$ (see 2.7(iii)). Thus, α_i^C is in T_i for every $C \in \mathscr{C}$ and, by (iii) and (i), T coincides with Σ^+. Theorem C is now obvious.

(v) From now on, suppose that the plane has odd order. By 4.7, $\mathrm{Aut}_i(M)$ is transitive on $\mathscr{C}$. So T_i has index 2 in it (see 4.3 and 2.7(iii)). It follows from 4.5(ii), 4.6 and 4.5(i) that, for any $C \in \mathscr{C}$, $\alpha_1^C\alpha_2^C$ belongs to $\bar{T} = T_1T_2 \cup T_1^*T_2^*$, where $T_i^* = \mathrm{Aut}_i(M) - T_i$. Clearly, $\bar{T}$ is a group, which contains T as a subgroup of index 2; by (i) and (ii), $\bar{T} = \Sigma^+$. Moreover, this implies that $\Sigma^+ \cap \mathrm{Aut}_i(M) = T_i$, i.e. T_i is the set of all the elements of Σ stabilizing every $\mathscr{L}_i$-line.

(vi) By 4.4 we know that T has two orbits $\mathscr{C}_1$, $\mathscr{C}_2$ on $\mathscr{C}$. Since $\mathscr{C}_1$, $\mathscr{C}_2$ are equivalence classes with respect to the equivalence relation generated by the tangency (4.4), they are sets of imprimitivity of $\mathrm{Aut}(M)$. But a symmetry σ_C, for $C \in \mathscr{C}_1$ for instance, fixes C, and thence, preserves $\mathscr{C}_1$ and $\mathscr{C}_2$. Therefore Σ is not transitive on $\mathscr{C}$, and this ends the proof.

5. What about infinite Minkowski planes?

The proof of Theorem B is based on the existence of sufficiently many translations (result 3.1) and on the characterization of Minkowski planes associated to sharply 3-transitive groups as planes in which $\mathrm{Aut}_i(M)$ is sufficiently large (Theorem 4.2). Both results hold for infinite Minkowski planes.

The crucial step, in the finite case (see 3.6), is to prove that the translations actually generate a 'half' of the group needed in 4.2. Unfortunately, only the first part of Theorem 3.6 is valid in general; clearly, the construction of $PSL(2, q)$ is not. But Lemmas 4.5 and 4.6 apply also to infinite planes; therefore our proof holds if we add the following axiom, which enables us to avoid the second part of 3.6:

THEOREM D. *Let M be a (possibly infinite) Minkowski plane in which every circle-symmetry is an automorphism. Assume that there are at most two equivalence classes with respect to the equivalence relation generated by the tangency relation. Then M is isomorphic to a Minkowski plane associated to a sharply 3-transitive group.*

In other words, M is a well-known plane: it can be described by means of coordinates over a Tits nearfield in a way very similar to 1.3 (see Percsy [14] or [15]).

Let us note finally that the hypotheses of Theorem D actually occur in infinite Minkowski planes: the number of the above-mentioned equivalence classes of $Mn(K)$ is one if K is, for instance, a perfect field of characteristic 2 (this follows from condition (OT) in Percsy [13, Theorem 4]) and two if K is the field of real numbers.

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