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OVOIDS AND SPREADS OF FINITE CLASSICAL POLAR SPACES

ABSTRACT. Let P be a finite classical polar space of rank $r, r \geqslant 2$. An ovoid O of P is a pointset of P, which has exactly one point in common with every totally isotropic subspace of rank r. It is proved that the polar space $W_n(q)$ arising from a symplectic polarity of $PG(n, q)$, n odd and $n > 3$, that the polar space $Q(2n, q)$ arising from a non-singular quadric in $PG(2n, q)$, $n > 2$ and q even, that the polar space $Q^-(2n+1, q)$ arising from a non-singular elliptic quadric in $PG(2n+1, q)$, $n > 1$, and that the polar space $H(n, q^2)$ arising from a non-singular Hermitian variety in $PG(n, q^2)$, n even and $n > 2$, have no ovoids.

Let S be a generalized hexagon of order n ($\geqslant 1$). If V is a pointset of order $n^3 + 1$ of S, such that every two points are at distance 6, then V is called an ovoid of S. If $H(q)$ is the classical generalized hexagon arising from $G_2(q)$, then it is proved that $H(q)$ has an ovoid iff $Q(6, q)$ has an ovoid. There follows that $Q(6, q)$, $q = 3^{2h+1}$, has an ovoid, and that $H(q)$, q even, has no ovoid.

A regular system of order m on $H(3, q^2)$ is a subset K of the lineset of $H(3, q^2)$, such that through every point of $H(3, q^2)$ there are m (>0) lines of K. B. Segre shows that, if K exists, then $m = q + 1$ or $(q + 1)/2$. If $m = (q + 1)/2$, K is called a hemisystem. The last part of the paper gives a very short proof of Segre's result. Finally it is shown how to construct the 4-(11, 5, 1) design out of the hemisystem with 56 lines ($q = 3$).

1. OVOIDS AND SPREADS

Let P be a finite classical polar space of rank (or index) r, $r \geqslant 2$ [3]. An ovoid O of P is a pointset of P, which has exactly one point in common with every totally isotropic subspace of rank r. A spread S of P is a set of maximal totally isotropic subspaces, which constitutes a partition of the pointset.

We shall use the following notation:

$W_n(q)$	the polar space arising from a symplectic polarity of $PG(n, q)$, n odd;
$Q(2n, q)$	the polar space arising from a non-singular quadric Q in $PG(2n, q)$;
$Q^+(2n + 1, q)$	the polar space arising from a non-singular hyperbolic quadric Q^+ [7] in $PG(2n + 1, q)$;
$Q^-(2n + 1, q)$	the polar space arising from a non-singular elliptic quadric Q^- [7] in $PG(2n + 1, q)$;
$H(n, q^2)$	the polar space arising from a non-singular Hermitian variety H [7] in $PG(n, q^2)$.

The following results are known:

(a) $W_n(q)$, n odd, has always a (regular) spread (the proof given in [16] for $n = 5$, extends to any odd n). $W_3(q)$ has an ovoid iff q is even [19]; every ovoid of $W_3(q)$, q even, is an ordinary ovoid of $PG(3, q)$ and every ordinary ovoid of $PG(3, q)$, q even, is an ovoid of some $W_3(q)$ [18].

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(b) $Q^+(3, q)$ has spreads and ovoids (trivial). Since $Q(4, q)$ is the dual of $W_3(q)$ [18], the polar space $Q(4, q)$ has always an ovoid, and has a spread iff q is even. $Q^-(5, q)$ has no ovoids [2] (see also Section 7). Since $Q^-(5, q)$ is the dual of $H(3, q^2)$ ([1], [20]) the polar space $Q^-(5, q)$ has always spreads (see (c)). $Q^+(5, q)$ has always ovoids (if we consider Q^+ as the Klein quadric, these ovoids correspond to the ordinary spreads of $PG(3, q)$). $Q^+(7, q)$ has a spread [25], from which it easily follows (by triality) that $Q^+(7, q)$ has an ovoid. $Q^+(4n + 1, q)$ has no spreads [8]. $Q(2n, q)$, $Q^-(4n + 1, q)$, $Q^+(m, q)$ and $Q^-(m, q)$, q even, m odd and $m \neq 4n + 1$, have always spreads [8].

(c) $H(3, q^2)$ has ovoids (any Hermitian curve on H is an ovoid of $H(3, q^2)$). Since $H(3, q^2)$ is the dual of $Q^-(5, q)$ ([1], [20]), the polar space $H(3, q^2)$ has no spread (see also Section 7).

We shall frequently use the following theorem [17]:

The only pointsets of $PG(m, q)$, $m > 2$, having exactly 1 or n (>1) distinct points in common with every hyperplane and having one point in common with at least one hyperplane, are the lines of $PG(m, q)$ and the (ordinary) ovoids of $PG(3, q)$.

2. The polar spaces $Q(2n, q)$, q even, and $W_n(q)$

THEOREM. *$W_n(q)$, n odd and $n > 3$, has no ovoid.*

Proof. Suppose O is an ovoid of $W_n(q)$, n odd and $n > 3$. We consider the intersection of O and a hyperplane $PG(n - 1, q)$ of $PG(n, q)$. Let x be the image of $PG(n - 1, q)$ with respect to the symplectic polarity π defining $W_n(q)$ (x is a point of $PG(n - 1, q)$). Suppose $x \in O$ and assume there is a point $y \in O$, $y \in PG(n - 1, q)$, $x \neq y$. There is a maximal totally isotropic subspace containing x and y, a contradiction since O is an ovoid. So $|PG(n - 1, q) \cap O| = 1$. Next we suppose that $x \notin O$, and let $y \in O \cap PG(n - 1, q)$ (y exists since any maximal totally isotropic subspace contained in $PG(n - 1, q)$, i.e. containing x, has one point in common with O). Then there is a maximal totally isotropic subspace containing x and y. Now we count in two ways the number of ordered pairs (maximal totally isotropic subspace R contained in $PG(n - 1, q)$, $y \in R \cap O$). Then we obtain: number of maximal totally isotropic subspaces containing x (i.e. contained in $PG(n - 1, q)) = |PG(n - 1, q) \cap O| \times$ (number of maximal totally isotropic subspaces containing x and y, $y \in O \cap PG(n - 1, q)$). Hence $|PG(n - 1, q) \cap O| =$ (number of maximal totally isotropic subspaces of a $W_{n-2}(q)$)/(number of maximal totally isotropic subspaces of a $W_{n-4}(q)) = (1 + q)(1 + q^2) \ldots (1 + q^{(n-1)/2})/(1 + q)(1 + q^2) \ldots (1 + q^{(n-3)/2}) = 1 + q^{(n-1)/2}$. So we have always $|PG(n - 1, q) \cap O| = 1$ or $1 + q^{(n-1)/2}$, and there is at least one hyperplane for which $|PG(n-1, q) \cap O| = 1$. From Section 1 it follows that $n = 3$, a contradiction.

COROLLARY. *The polar space $Q(2n, q)$, $n > 2$ and q even, has no ovoids.*

Proof. Suppose O is an ovoid of $Q(2n, q)$, $n > 2$ and q even. Let p be the nucleus [14] of Q and let $PG(2n-1, q)$ be a hyperplane which does not contain p. If we project $Q(2n, q)$ from p onto $PG(n-1, q)$, then there arises a polar space $W_{2n-1}(q)$, and the projection of O is an ovoid of $W_{2n-1}(q)$, a contradiction.

3. The polar space $Q^-(2n+1, q)$

THEOREM. *$Q^-(2n+1, q)$, $n > 1$, has no ovoid.*

Proof. Suppose O is an ovoid of $Q^-(2n+1, q)$, $n > 1$. We consider the intersection of O and a hyperplane $PG(2n, q)$ of $PG(2n+1, q)$. Let x be the pole of $PG(2n, q)$ with respect to the quadric Q^-. Suppose $x \in O$ (then $x \in PG(2n, q)$) and assume there is a point $y \in O$, $y \in PG(2n, q)$, $x \neq y$. Then there is a maximal totally isotropic subspace containing x and y, a contradiction since O is an ovoid. So $|PG(2n, q) \cap O| = 1$. Next we suppose that $x \notin O$, $x \in Q^-$ (then $x \in PG(2n, q)$). Now we count in two ways the number of ordered pairs (maximal totally isotropic subspace R contained in $PG(2n, q)$, $y \in R \cap O$). Then we obtain: number of maximal totally isotropic subspaces containing $x = |PG(2n, q) \cap O| \times$ (number of maximal totally isotropic subspaces containing x and y, $y \in O \cap PG(2n, q)$). Hence $|PG(2n, q) \cap O| =$ (number of maximal totally isotropic subspaces of a $Q^-(2n-1, q)$)/(number of maximal totally isotropic subspaces of a $Q^-(2n-3, q)$) $= \prod_{i=2}^{n} (q^i + 1)/\prod_{i=2}^{n-1} (q^i + 1) = q^n + 1$. Finally, let $x \notin Q^-$. Then $Q^- \cap PG(2n, q)$ is a non-singular quadric Q in $PG(2n, q)$. So $|PG(2n, q) \cap O| =$ (number of maximal totally isotropic subspaces of $Q(2n, q)$)/(number of maximal totally isotropic subspaces of $Q(2n, q)$ containing a given point) $= \prod_{i=1}^{n} (q^i + 1)/\prod_{i=1}^{n-1} (q^i + 1) = q^n + 1$. So we have always $|PG(2n, q) \cap O| = 1$ or $q^n + 1$, and there is at least one hyperplane for which $|PG(2n, q) \cap O| = 1$. From Section 1 it follows that $n = 1$, a contradiction.

4. The polar space $H(n, q^2)$, n even

THEOREM. *$H(n, q^2)$, n even and $n > 2$, has no ovoid.*

Proof. Suppose O is an ovoid of $H(n, q^2)$, n even and $n > 2$. We consider the intersection of O and a hyperplane $PG(n-1, q^2)$ of $PG(n, q^2)$. Let x be the image of $PG(n-1, q^2)$ with respect to the unitary polarity π defining H. Suppose $x \in O$ (then $x \in PG(n-1, q^2)$) and assume there is a point $y \in O$, $y \in PG(n-1, q^2)$, $x \neq y$. Then there is a maximal totally isotropic subspace containing x and y, a contradiction since O is an ovoid. So $|PG(n-1, q^2) \cap O| = 1$. Next we suppose that $x \in H$, $x \notin O$ (then $x \in PG(n-1, q^2)$). Now we count in two ways the number of ordered pairs (maximal totally isotropic subspace R contained in $PG(n-1, q^2)$, $y \in R \cap O$). Then we obtain: number of maximal totally isotropic subspaces containing $x = |PG(n-1, q^2) \cap O| \times$ (number of maximal totally isotropic subspaces containing x and y, $y \in O \cap PG(n-1, q^2)$). Hence $|PG(n-1, q^2) \cap O|$ = (number of maximal totally isotropic subspaces of a $H(n-2, q^2)$)/(number of maximal totally isotropic subspaces of a $H(n-4, q^2)$) $= (q^3+1) \times (q^5+1) \ldots (q^{n-1}+1)/(q^3+1)(q^5+1) \ldots (q^{n-3}+1) = q^{n-1}+1$. Finally let $x \notin H$ (then $x \notin PG(n-1, q^2)$). Then $H \cap PG(n-1, q^2)$ is a non-singular Hermitian variety of $PG(n-1, q^2)$. So $|PG(n-1, q^2) \cap H|$ = (number of maximal totally isotropic subspaces of $H(n-1, q^2)$)/(number of maximal totally isotropic subspaces of $H(n-1, q^2)$ containing a given point) $= (q+1)(q^3+1) \ldots (q^{n-1}+1)/(q+1)(q^3+1) \ldots (q^{n-3}+1) = q^{n-1}+1$. So we have always $|PG(n-1, q^2) \cap O| = 1$ or $q^{n-1}+1$, and there is at least one hyperplane for which $|PG(n-1, q^2) \cap O| = 1$. From Section 1 and $n > 3$ follows a contradiction.

5. The polar spaces $Q(2n, q)$, q odd, $H(n, q^2)$, n odd, and $Q^+(2n+1, q)$

Concerning ovoids we do not have an existence or non-existence theorem in the cases of the polar spaces $Q(2n, q)$ $(n > 2)$, q odd, $H(n, q^2)(n > 3)$, n odd, and $Q^+(2n+1, q)(n > 3)$. We only know that $Q(4, q)$, $H(3, q^2)$, $Q^+(5, q)$ and $Q^+(7, q)$ have ovoids. So the first cases to consider are $Q(6, q)$, q odd, $H(5, q^2)$ and $Q^+(9, q)$. The following remark may be useful: if the polar space $Q(2n, q)$ (resp. $H(n, q^2)$, resp. $Q^+(2n+1, q)$) has an ovoid O, then also the polar space $Q(2n-2, q)$ (resp. $H(n-2, q^2)$, resp. $Q^+(2n-1, q)$) has an ovoid (easy by considering the maximal totally isotropic subspaces of $Q(2n, q)$ (resp. $H(n, q^2)$, resp. $Q^+(2n+1, q)$) containing a fixed point $x \notin O$ of Q (resp. H, Q^+)).

6. The polar space $Q(6, q)$ and the classical generalized hexagon (of order q) arising from $G_2(q)$

Introduction. A generalized hexagon [5] of order n $(\geqslant 1)$ is an incidence structure $S = (P, B, \mathrm{I})$, with an incidence relation satisfying the following axioms.

(i) each point (resp. line) is incident with $n+1$ lines (resp. points);
(ii) $|P| = |B| = 1 + n + n^2 + n^3 + n^4 + n^5 = v$;

(iii) 6 is the smallest positive integer k such that S has a circuit consisting of k points and k lines.

As usual the distance of two elements $\alpha, \beta \in P \cup B$ is denoted by $\lambda(\alpha, \beta)$ or $\lambda(\beta, \alpha)$ [5].

If V is a set of points (resp. lines) such that $\lambda(x, y) = 6$ (resp. $\lambda(L, M) = 6$) for all distinct $x, y \in V$ (resp. $L, M \in V$), then $|V| \leqslant v/(n^2 + n + 1)$ or $|V| \leqslant n^3 + 1$. If $|V| = n^3 + 1$, then we say that V is an ovoid (resp. spread) of the hexagon S [5].

In [5] we remarked that the classical generalized hexagon $H(q)$ (of order q) arising from $G_2(q)$ has always a spread. We also proved that a generalized hexagon S of order q has an ovoid (resp. spread) if S admits a polarity. J. Tits informed us that it is possible to prove that the classical hexagon $H(q)$ of order q, $q = 3^{2h+1}$, admits a polarity [22].

In the presentation of J. Tits of the classical generalized hexagon of order q, the set P is the pointset of $Q(6, q)$ and the set B is a subset of the lineset of $Q(6, q)$ [21]. Moreover, for $x, y \in P$, $x \neq y$, we have $\lambda(x, y) \leqslant 4$ iff x and y are on a line of the polar space $Q(6, q)$ (see also [24]).

THEOREM. *$H(q)$ has an ovoid iff $Q(6, q)$ has an ovoid.*

Proof. Suppose that O is an ovoid of $H(q)$. If $x, y \in O$, $x \neq y$, are in a plane of $Q(6, q)$, then xy is a line of Q and so $\lambda(x, y) \leqslant 4$ in $H(q)$, a contradiction. So every plane of $Q(6, q)$ has at most one point in common with O. Since $|O| = q^3 + 1$, every plane of $Q(6, q)$ has exactly one point in common with O, and hence O is an ovoid of $Q(6, q)$.

Conversely, suppose that O is an ovoid of $Q(6, q)$. If x, $y \in O$, $x \neq y$, are at distance at most 4 in $H(q)$, then xy is a line of $Q(6, q)$ and so x and y are in a plane of $Q(6, q)$, a contradiction. Hence for all $x, y \in O$, $x \neq y$, we have $\lambda(x, y) = 6$ in $H(q)$. Since $|O| = q^3 + 1$, O is an ovoid of $H(q)$.

COROLLARY 1. *$Q(6, q)$, $q = 3^{2h+1}$, has an ovoid.*

Proof. $H(q)$, $q = 3^{2h+1}$, admits a polarity and consequently has an ovoid. Hence $Q(6, q)$, $q = 3^{2h+1}$, has an ovoid.

COROLLARY 2. *$H(q)$, q even, has no ovoid.*

Proof. Suppose O is an ovoid of $H(q)$, q even. Then O is an ovoid of $Q(6, q)$, q even. This is in contradiction with Section 2.

7. Hemisystems of the Polar Space $H(3, q^2)$

Introduction. A regular system of order m on $H(3, q^2)$ is a subset K of the lineset of $H(3, q^2)$, such that through every point of H there are m (>0)

lines of K. B. Segre shows that, if K exists, then either K is the set of all lines of $H(3, q^2)$ or $m = (q+1)/2$ [15]. In the latter case, K consists of $(q+1)(q^3+1)/2$ lines and is called a hemisystem [15]. The proof is restricted to q odd, but in [1] A. A. Bruen and J. W. P. Hirschfeld remark that with their definition of a quadric permutable with H, it also holds for q even. So, for q even, there are no regular systems on $H(3, q^2)$ other than the set of all lines. Another corollary is that $H(3, q^2)$ has no spread. In what follows we give a very short proof of Segre's result.

Let K be a regular system of order m, $0 < m < q+1$, of $H(3, q^2)$. If θ is an anti-isomorphism of $H(3, q^2)$ onto $Q^-(5, q)$, then K^θ is a pointset of Q^- which has exactly m points in common with every line of $Q^-(5, q)$. Now we define the incidence structure $S = (P, B, \mathrm{I}): P = Q^- - K^\theta$, B is the lineset of $Q^-(5, q)$, and I is the incidence of $Q^-(5, q)$.

THEOREM. *$S = (P, B, \mathrm{I})$ is a partial quadrangle* [4] *with parameters* $s = q - m$, $t = q^2$, $\mu = q^2 + 1 - m(q+1)$.

Proof. Every line of B is incident with $q - m + 1$ points of P, and every point of P is incident with $q^2 + 1$ lines of B. If $M \in B$, $x \in P$, $x \not\mathrm{I} M$, then M contains at most one point which is collinear (in S) with x. Finally, let us consider two non-collinear points x and y of S, and call μ the number of points of S collinear with both. The $q^2 + 1$ points of Q^- collinear (in $Q^-(5, q)$) with x and y are the points of an elliptic quadric in a $PG(3, q)$. Now we consider the $q+1$ hyperplanes $PG(4, q)$ containing $PG(3, q)$, and their intersections with K^θ. We obtain

$$\begin{aligned} &2((q^2+1)m - (q^2+1-\mu)) \\ &\quad + (q-1)\left(\frac{(q^2+1)(q+1)m}{q+1} - (q^2+1-\mu)\right) \\ &\quad + (q^2+1-\mu) = |K^\theta| = \frac{m(q^2+1)(q^3+1)}{q^2+1} \end{aligned}$$

or $\mu = q^2 + 1 - m(q+1)$ (and so $m \leqslant q - 1$). Hence S is a partial quadrangle with parameters $s = q - m$, $t = q^2$, $\mu = q^2 + 1 - m(q+1)$.

THEOREM. *If K is a regular system of order m, $0 < m < q+1$, then $m = (q+1)/2$. So the corresponding partial quadrangle has parameters $s = (q-1)/2$, $t = q^2$, $\mu = (q-1)^2/2$.*

Proof. Let K be a regular system of order m, $0 < m < q+1$, and let $S = (P, B, \mathrm{I})$ be the corresponding partial quadrangle. Then

$$\begin{aligned} |P| = v &= 1 + (t+1)s(1 + ts/\mu) \\ &= 1 + (q^2+1)(q-m)(1 + q^2(q-m)/(q^2+1-m(q+1))) \end{aligned}$$

[4]. Since $v = (q^3+1)(q+1-m)$, there results $m = (q+1)/2$.

Some Properties of the Pointset K^θ, with $m = (q+1)/2$

(a) Let $PG(4,q)$ be a hyperplane of $PG(5,q)$. If $PG(4,q)$ is not a tangent hyperplane of Q^-, then $|PG(4,q) \cap K^\theta| = (q+1)(q^2+1)/2$; if $PG(4,q)$ is tangent to Q^- at $x \in K^\theta$, then $|PG(4,q) \cap K^\theta| = ((q^2+1)(q-1)/2)+1$; if $PG(4,q)$ is tangent to Q^- at $x \notin K^\theta$, then $|PG(4,q) \cap K^\theta| = (q^2+1)(q+1)/2$. So $|PG(4,q) \cap K^\theta|$ takes only two values.

(b) Let $PG(3,q)$ be a three-space of $PG(5,q)$. If $PG(3,q) \cap Q^- = E$ is hyperbolic, then $|PG(3,q) \cap K^\theta| = (q+1)^2/2$; if E is degenerate, with vertex on K^θ, then $|PG(3,q) \cap K^\theta| = (q^2+1)/2$; if E is degenerate, with vertex not on K^θ, then $|PG(3,q) \cap K^\theta| = (q+1)^2/2$; if E is elliptic and if all the points of E are collinear (in $Q^-(5,q)$) with two points of K^θ, then $|PG(3,q) \cap K^\theta| = (q-1)^2/2$ (μ of the preceding theorem); if E is elliptic and if there is no point on K^θ which is collinear with every point of E, then $|PG(3,q) \cap K^\theta| = (q+1)^2/2$; if E is elliptic and if all the points of E are collinear with exactly one point of K^θ, then $|PG(3,q) \cap K^\theta| = (q^2+1)/2$ (considering the $q+1$ hyperplanes containing $PG(3,q)$, and their intersections with K^θ, we obtain

$$\left(\frac{(q^2+1)(q-1)}{2}+1-v\right)+\left(\frac{(q^2+1)(q+1)}{2}-v\right)$$
$$+(q-1)\left(\frac{(q^2+1)(q+1)}{2}-v\right)+v=\frac{(q+1)(q^3+1)}{2},$$

with $v = |PG(3,q) \cap K^\theta|$). So $|PG(3,q) \cap K^\theta|$ takes only three values.

(c) Let C be an irreducible conic on Q^-, and let $|C \cap K^\theta| = \gamma$. If π is the plane of C and if π' is the polar plane of π, then the irreducible conic $\pi' \cap Q^-$ is denoted by C'. Further, let $|C' \cap K^\theta| = \gamma'$. By considering the q^2+q+1 three-spaces containing π, and their intersections with K^θ, we obtain

$$\gamma'\left(\frac{q^2+1}{2}-\gamma\right)+(q+1-\gamma')\left(\frac{(q+1)^2}{2}-\gamma\right)$$
$$+\frac{q(q-1)}{2}\left(\frac{(q+1)^2}{2}-\gamma\right)+\frac{\gamma'(\gamma'-1)}{2}\left(\frac{(q-1)^2}{2}-\gamma\right)$$
$$+\frac{(q+1-\gamma')(q-\gamma')}{2}\left(\frac{(q+1)^2}{2}-\gamma\right)+\gamma'(q+1-\gamma')$$
$$\times\left(\frac{q^2+1}{2}-\gamma\right)+\gamma=\frac{(q+1)(q^3+1)}{2},$$

or $\gamma+\gamma' = q+1$.

The case $q = 3$

$q = 3$ is the only value where the set K^θ is known to exist. Here K^θ is the 56-cap of R. Hill [11] (a k-cap is a set of k points, no three of which are collinear). Moreover, in $PG(5, 3)$ there is no 57-cap, and any 56-cap is necessarily of the type described above [12]. The partial quadrangle $S = (P, B, \mathrm{I})$ has parameters $s = 1$, $t = 9$, $\mu = 2$ and is essentially the graph of Gewirtz [9]. If $S^* = (P, B^*, \mathrm{I}^*)$, with $B^* = \{L_x \parallel x \in P\}$, where L_x is the set of all points of P collinear (in S) with x, and I^* the natural incidence relation, then S^* is the 2-(56, 11, 2) design first mentioned as a design by Hall, Lane and Wales [10]. We remark that the 56-cap of R. Hill and the corresponding hemisystem of $H(3, 9)$ were also studied by A. A. Bruen and J. W. P. Hirschfeld in [1].

Finally we obtain as follows the unique 4-(11, 5, 1) design of Witt [23]. We consider L_x, with $x \in P$ (L_x is contained in a $PG(4, 3)$). Call B^{**} the set of all three-spaces having at least four points in common with L_x, and I^{**} the natural incidence relation.

THEOREM. $S^{**} = (L_x, B^{**}, \mathrm{I}^{**})$ *is the presentation of H.S.M. Coxeter* [6] *of the unique* 4-(11, 5, 1) *design.*

Proof. Let $L_x = \{x, x_1, \ldots, x_{10}\}$. Evidently no four points x, x_i, x_j, x_k, i, j, k distinct, are coplanar. Now let us suppose that x_i, x_j, x_k, x_l, i, j, k, l distinct, are in a plane π. Then x_i, x_j, x_k, x_l are the four points of an irreducible conic C on Q^-. If π' is the polar plane of π, and if $C' = \pi' \cap Q^-$, then $x \in C'$. So we have $|C \cap K^\theta| + |C' \cap K^\theta| \leqslant 3$, in contradiction with property (c) of the set K^θ. Consequently no four points of L_x are coplanar.

Let $PG(3, 3)$ be a three-space containing at least four points of L_x ($PG(3, 3)$ is contained in the $PG(4, 3)$ defined by L_x). If $PG(3, 3)$ contains x, then $PG(3, 3) \cap Q^-$ is degenerate, and so $|PG(3, 3) \cap L_x| = 5$. If $PG(3, 3)$ does not contain x, then $PG(3, 3) \cap Q^-$ is elliptic (since $PG(3, 3) \subset PG(4, 3)$ and Q^- is elliptic). As $|PG(3, 3) \cap P| \geqslant 4$ and as x is collinear with every point of $PG(3, 3) \cap Q^-$, there is no other point of P collinear with all points of $PG(3, 3) \cap Q^-$. Now from property (c) of the set K^θ it follows that $|PG(3, 3) \cap P| = |PG(3, 3) \cap L_x| = 5$. Hence every three-space containing at least four points of L_x contains exactly five points of L_x. We conclude that S^{**} is the presentation of H. S. M. Coxeter of the unique 4-(11, 5, 1) design [6].

Remark. The presentations of H. S. M. Coxeter of the unique 5-(12, 6, 1) design and the unique 4-(11, 5, 1) design were rediscovered by G. Pellegrino [13].

Acknowledgement: Further information on ovoids and spreads can be found in [26].

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