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Generalized Solutions of a Class of Nuclear-Space-Valued Stochastic Evolution Equations*

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Abstract. Generalized solutions are defined for stochastic evolution equations of the form $dY_t = A^* Y_t\ dt + dZ_t$ on the nuclear triple $\mathscr{S}(R^d) \subset L^2(R^d) \subset \mathscr{S}'(R^d)$, where A does not map $\mathscr{S}(R^d)$ into itself. One case which is treated in detail involves $A = -(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$. This example arises as the Langevin equation for the fluctuation limit of a system of particles migrating according to a symmetric stable process and undergoing critical branching in a random medium.

1. Introduction

Stochastic evolution equations of the form

$$dY_t = A^* Y_t\, dt + dZ_t \tag{1.1}$$

defined on nuclear triples $F \subset H \subset F'$ arise in many applications (e.g., [2], [5], [6], [9]-[12], [14], [16], [20]-[24], [28], and [31]-[34]). The operator A is assumed to generate a semigroup $\{T_t\}$ on the Hilbert space H and it is usually assumed that the nuclear space F is contained in the domain of A and is invariant under A and T_t. These assumptions are basic for the usual techniques employed in the study of such equations. A different situation arises when F is not invariant under A, because then A^* does not map F' into itself and therefore the notation $A^* Y_t$ has no meaning.

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An example of such an equation is

$$dY_t = \Delta_\alpha^* Y_t\,dt + dZ_t \tag{1.2}$$

defined on $\mathscr{S}(R^d) \subset L^2(R^d) \subset \mathscr{S}'(R^d)$, where $\mathscr{S}(R^d)$ and $\mathscr{S}'(R^d)$ are the usual Schwartz spaces and $\Delta_\alpha \equiv -(-\Delta)^{\alpha/2}, 0 < \alpha < 2$ (i.e., the generator of the spherically symmetric stable process on R^d). In this case $\varphi \in \mathscr{S}(R^d)$ does not imply that $\Delta_\alpha \varphi$ and $T_t\varphi$ are in $\mathscr{S}(R^d)$ (fast decay at infinity fails). This raises the question of giving an appropriate meaning to equations of the type (1.2). Our objective in this paper is to do this, in particular for (1.2) which was stated in [6] as the Langevin equation for the fluctuation limit of a particle system. This is achieved by means of appropriate intermediate spaces in the triple $\mathscr{S}(R^d) \subset L^2(R^d) \subset \mathscr{S}'(R^d)$.

In several of the papers cited above the operator A is Δ (Brownian motion), which maps $\mathscr{S}(R^d)$ into itself. Most of the evolution equations in these papers still hold with Δ replaced by Δ_α, and the same problem of interpretation of (1.2) will arise. The approach we propose here may be useful for these cases also.

An alternative procedure could be taken: rather than insisting on $\mathscr{S}(R^d)$ as a space of test functions we might construct an *ad hoc* nuclear space F with the desired invariance properties (e.g., [28], the method used in [20] and [21] does not work in our case because $(\lambda I - \Delta_\alpha)^{-r}$ is not Hilbert–Schmidt for any r). This could be done in our case [6], but F turns out to be too small, and therefore F' too large; and in this sort of problem, while generally there is no smallest adequate F', it is desirable to have a reasonably small F'. One reason for this is that the processes Y which satisfy equations of the type above often arise as limits of other F'-valued processes, and therefore a smaller F' means a stronger form of convergence. Another reason is that a smaller F' implies a smaller support of the distribution of the process Y, which yields finer information.

Section 2 is devoted to the nuclear and intermediate spaces we need and background material, in particular the regularization theorem [1], [18], which is one of our main tools.

In Section 3 we formulate definitions of generalized forms for equations of the type (1.1) and for their evolution (or variation of constants) solutions on $\mathscr{S}(R^d) \subset L^2(R^d) \subset \mathscr{S}'(R^d)$, and show that under these definitions an equation of this type has a unique solution and it is given in evolution form.

Section 4 is concerned with the processes Y and Z in [6] and in Section 5 we prove that for (1.2) in [6] the formulation given in Section 3 is applicable, thus justifying this equation which was written formally in that paper.

2. Preliminaries and Intermediate Spaces

We give some preliminary results we need and introduce intermediate spaces for the operator Δ_α. The space $\mathscr{S}(R^d)$ of C^∞-functions rapidly decreasing at infinity on R^d is topologized by either of the two following increasing sequences of norms:

$$\|\varphi\|_n = \left\{ \sum_{|k|=0}^{n} \int_{R^d} (1+|x|^2)^n |D^k\varphi(x)|^2\,dx \right\}^{1/2} \qquad \text{(Hilbert norm)},$$

$$|||\varphi|||_n = \max_{0\le|k|\le n} \sup_{x\in R^d} (1+|x|^2)^n |D^k\varphi(x)|, \qquad \varphi \in \mathscr{S}(R^d),$$

$n = 0, 1, 2, \ldots$, where $k = (k_1, \ldots, k_d)$, $|k| = k_1 + \cdots + k_d$, $D^k = \partial^{|k|}/\partial x_d^{k_1} \cdots \partial x_d^{k_d}$, and $|\cdot|$ is the usual norm on R^d. If $\mathscr{S}_n(R^d)$ denotes the $\|\cdot\|_n$-completion of $\mathscr{S}(R^d)$, then $\mathscr{S}(R^d) = \bigcap_{n=0}^{\infty} \mathscr{S}_n(R^d)$, and $\mathscr{S}(R^d)$ is a countably Hilbert nuclear space.

The space of tempered distributions $\mathscr{S}'(R^d)$ is the strong topological dual of $\mathscr{S}(R^d)$, and $\mathscr{S}'(R^d) = \bigcup_{n=0}^{\infty} \mathscr{S}'_n(R^d)$ where $\mathscr{S}'_n(R^d)$ is the dual of $\mathscr{S}_n(R^d)$ with dual norm denoted by $\|\cdot\|_{-n}$.

Let $\mathscr{D}([a, b])$ denote the space of C^∞-functions with supports contained in (a, b), $a < b$ in R, with its usual topology. The completed tensor product $\mathscr{S}(R^d) \hat{\otimes} \mathscr{D}([a, b])$ is nuclear [30]; this space is topologized by the norms

$$|||\Phi|||_n = \max_{0 \le |k| \le n} \sup_{x \in R^d, t \in [a,b]} (1 + |x|^2)^n |D^k \Phi(x, t)|,$$

$$\Phi \in \mathscr{S}(R^d) \otimes \mathscr{D}([a, b]), \quad n = 0, 1, \ldots,$$

where D^k acts also on the variable t. Separately continuous bilinear mappings from $\mathscr{S}(R^d) \times \mathscr{D}([a, b])$ into an arbitrary topological vector space are continuous [30].

We denote by $\langle \cdot, \cdot \rangle$ the duality on $(\mathscr{S}'(R^d), \mathscr{S}(R^d))$, as well as other dualities (the underlying spaces will be clear from the context).

Let $D([0, T], \mathscr{S}'(R^d))$ denote the space of functions from $[0, T]$ into $\mathscr{S}'(R^d)$ which are right-continuous and possess left limits, with a Skorohod-type topology [27].

Lemma 2.1 [3]. *For $x \in D([0, T], \mathscr{S}'(R^d))$ let $\tilde{x}$ be defined by*

$$\langle \tilde{x}, \Phi \rangle = \int_0^T \langle x(t), \Phi(\cdot, t) \rangle \, dt, \qquad \Phi \in \mathscr{S}(R^{d+1}).$$

Then $x \mapsto \tilde{x}$ is a continuous mapping from $D([0, T], \mathscr{S}'(R^d))$ into $\mathscr{S}'(R^{d+1})$. If Φ is restricted to $\mathscr{S}(R^d) \hat{\otimes} \mathscr{D}([a, b])$, $[0, T] \subset [a, b]$, then the mapping is continuous into $(\mathscr{S}(R^d) \hat{\otimes} \mathscr{D}([a, b]))'$.

An obvious consequence of this lemma is:

Corollary 2.2. *If X is a process in $D([0, T], \mathscr{S}'(R^d))$, then $\tilde{X}$ is an $\mathscr{S}'(R^{d+1})$-valued (or $(\mathscr{S}(R^d) \hat{\otimes} \mathscr{D}([a, b]))'$-valued) random variable.*

Lemma 2.3 [3]. *Let X be a process in $D([0, T], \mathscr{S}'(R^d))$, continuous at T a.s. Then the distributions of X and $\tilde{X}$ determine each other.*

Let F be a nuclear space and denote by $\mathscr{L}^0(\Omega, \mathscr{F}, P)$ the space of equivalence classes of real random variables on a complete probability space with the topology of convergence in probability (this topology is metrized by $\rho(X, Y) = E(|X - Y| \wedge 1)$). A *linear random functional* on F is a family $\{X_f, f \in F\}$ in $\mathscr{L}^0(\Omega, \mathscr{F}, P)$ which is linear and continuous on F.

Lemma 2.4 (Regularization Theorem [1], [18]). *A linear random functional on F has a unique regular version, i.e., there exists a unique F'-valued random variable X such that $\langle X, f \rangle = X_f$ a.s. for each $f \in F$.*

The regularization theorem will be applied when F is $\mathscr{S}(R^d)$ or $\mathscr{S}(R^d)\hat{\otimes}\mathscr{D}([a, b])$.

We now introduce the intermediate spaces. Let $C(R^d)$ denote the space of continuous functions on R^d, $C_c(R^d)$ the subset of $C(R^d)$ of functions with compact supports, and $C_0(R^d)$ the set of elements of $C(R^d)$ vanishing at infinity.

For $p > 0$ let $\varphi_p(x) = (1 + |x|^2)^{-p}$, $x \in R^d$, and define

$$C_p(R^d) = \{\varphi \in C(R^d): \|\varphi\|_p < \infty\},$$

where

$$\|\varphi\|_p = \sup_{x \in R^d} |\varphi(x)/\varphi_p(x)|$$

and

$$C_{p,0}(R^d) = \{\varphi \in C(R^d): \varphi/\varphi_p \in C_0(R^d)\}.$$

Then $C_{p,0}(R^d)$ and $C_p(R^d)$ are Banach spaces for the norm $\|\cdot\|_p$. Note that $\mathscr{S}(R^d) \subset C_{p,0}(R^d) \subset C_p(R^d)$ for all $p > 0$, and $C_{p,0}(R^d) \subset L^2(R^d)$ for $p > d/2$. In addition the inclusion $\mathscr{S}(R^d) \subset C_{p,0}(R^d)$ is continuous and dense for any $p > 0$, as well as the inclusion $C_{p,0}(R^d) \subset L^2(R^d)$ for $p > d/2$.

We denote by $C'_p(R^d)$ and $C'_{p,0}(R^d)$ the duals of $C_p(R^d)$ and $C_{p,0}(R^d)$, respectively, with dual norm designated by $\|\cdot\|_{-p}$. Hence for $p > d/2$ we have the following sequence of inclusions:

$$\mathscr{S}(R^d) \subset C_{p,0}(R^d) \subset L^2(R^d) \subset C'_{p,0}(R^d) \subset \mathscr{S}'(R^d).$$

For $p > 0$ let $\mathscr{M}_p(R^d)$ denote the space of nonnegative Radon measures μ on R^d such that $\langle \mu, \varphi_p \rangle < \infty$ equipped with the p-vague topology, i.e., the smallest topology making the maps $\mu \mapsto \langle \mu, \varphi \rangle$ continuous for all $\varphi \in C_c(R^d)^+ \cup \{\varphi_p\}$. The Lebesgue measure on R^d belongs to $\mathscr{M}_p(R^d)$ for $p > d/2$. We have $\mathscr{M}_p(R^d) \subset \mathscr{S}'_n(R^d)$ for some n depending on p, and $\mathscr{M}_p(R^d) \subset C'_p(R^d)$ with $\|\mu\|_{-p} = \langle \mu, \varphi_p \rangle$ for $\mu \in \mathscr{M}_p(R^d)$. On $\mathscr{M}_p(R^d)$ the topology induced by $\mathscr{S}'_n(R^d)$ is weaker than the p-vague topology, and the one induced by $C'_p(R^d)$ is stronger than the p-vague topology.

Set $\{S_t, t \geq 0\}$ denote the semigroup of operators on $L^2(R^d)$ determined by the spherically symmetric stable process on R^d with exponent α, $0 < \alpha \leq 2$ (e.g., [19]). The case $\alpha = 2$ corresponds to the Wiener process with variance parameter 2. The generator of $\{S_t\}$ is the fractional power of the Laplacian: $\Delta_\alpha \equiv -(-\Delta)^{\alpha/2}$ (e.g., [29]), and $\mathscr{S}(R^d) \subset \mathrm{Dom}(\Delta_\alpha)$. The spaces $C_p(R^d)$ and $\mathscr{M}_p(R^d)$ are in duality, and S_t is defined on $\mathscr{M}_p(R^d)$ by duality.

For $\alpha = 2$ the operators S_t and $\Delta_2 \equiv \Delta$ map $\mathscr{S}(R^d)$ into itself, but for $\alpha < 2$ they do not, as can be verifed by means of Fourier transforms. This is the reason why we cannot work with the triple $\mathscr{S}(R^d) \subset L^2(R^d) \subset \mathscr{S}'(R^d)$ alone and we introduce the intermediate spaces related to φ_p.

The next results, which follow from [15], show why these spaces are appropriate.

Lemma 2.5 [6]. *For each $t \geq 0$ and $p > d/2$, and additionally $p < (d + \alpha)/2$ in case $\alpha < 2$, S_t is a bounded linear operator from $(C_p(R^d), \|\cdot\|_p)$ into itself, and from $(\mathscr{M}_p(R^d), \|\cdot\|_{-p})$ into itself.*

Lemma 2.6 [6]. *For each $p > d/2$, and additionally $p < (d + \alpha)/2$ in case $\alpha < 2$, and any $\varphi \in C_p(R^d)$ such that $\lim_{|x|\to\infty} \varphi(x)/\varphi_p(x)$ exists, $t \mapsto S_t\varphi$ is a continuous curve in $(C_p(R^d), \|\cdot\|_p)$. Also, for any $\mu \in \mathscr{M}_p(R^d)$, $t \mapsto S_t\mu$ is p-vaguely continuous.*

Corollary 2.7. *For each $p > d/2$, with $p < (d + \alpha)/2$ in case $\alpha < 2$, S_t is a continuous linear mapping from $\mathscr{S}(R^d)$ into $C_{p,0}(R^d)$ for any $t \geq 0$, and $t \mapsto S_t\varphi$ is a continuous curve in $(C_{p,0}(R^d), \|\cdot\|_p)$ for any $\varphi \in \mathscr{S}(R^d)$.*

The following inequality is basic for proving the previous results and is also used below: for each $p > d/2$, with $p < (d + \alpha)/2$ in case $\alpha < 2$, given $T > 0$ there is a constant $C_T > 0$ such that

$$\|S_t\varphi\|_p \leq C_T\|\varphi\|_p \tag{2.1}$$

for $\varphi \in C_p(R^d)$ and $t \in [0, T]$.

Lemma 2.8. *For each $p > d/2$, with $p < (d + \alpha)/2$ in case $\alpha < 2$, Δ_α is a continuous linear mapping from $\mathscr{S}(R^d)$ into $C_{p,0}(R^d)$.*

Proof. We consider the case $\alpha < 2$ (the case $\alpha = 2$ is obvious). We have the representation (e.g., p. 166 of [17], in the case $d = 1$)

$$\Delta_\alpha\varphi(x) = K\int_{R^d} \psi(x, y)|y - x|^{-(d+\alpha)}\,dy, \qquad x \in R^d, \quad \varphi \in \mathscr{S}(R^d),$$

where

$$\psi(x, y) = \varphi(y) - \varphi(x) - \nabla\varphi(x)\cdot(y - x)(1 + |x - y|^2)^{-1}$$

($\cdot$ is the scalar product in R^d) and K is a constant. Therefore

$$\Delta_\alpha\varphi(x) = K[F(x) + G(x)],$$

where

$$F(x) = \int_{|y-x|<1} \psi(x, y)|y - x|^{-(d+\alpha)}\,dy$$

and

$$G(x) = \int_{|y-x|\geq 1} \psi(x, y)|y - x|^{-(d+\alpha)}\,dy.$$

Using Taylor's theorem we can obtain, since $\alpha < 2$,

$$\|F\|_p \leq \text{const}\{\||\nabla\varphi|\|_p + \|\xi_\varphi\|_p\},$$

where

$$\xi_\varphi(x) = \sup_{|z-x|<1}\ \max_{1\leq i,j\leq d} |\partial^2\varphi(z)/\partial z_i\,\partial z_j|.$$

On the other hand,

$$G(x) = \int_{|y-x|\geq 1} (\varphi(y) - \varphi(x))|y - x|^{-(d+\alpha)}\, dy,$$

hence

$$\|G\|_p \leq \text{const}\|\varphi\|_p + \sup_{x\in R^d} (1 + |x|^2)^p \int_{|y-x|\geq 1} |\varphi(y)|\,|y - x|^{-(d+\alpha)}\, dy.$$

Now, for any $q > 0$,

$$\int_{|y-x|\geq 1} |\varphi(y)|\,|y - x|^{-(d+\alpha)}\, dy \leq \|\varphi\|_q \int_{|y-x|\geq 1} (1 + |y|^2)^{-q}|y - x|^{-(d+\alpha)}\, dy$$

$$\leq \|\varphi\|_q \int_{R^d} (1 + |y|^2)^{-q} f(|y - x|)\, dy,$$

where

$$f(r) = \begin{cases} 1, & 0 \leq r \leq 1 \\ r^{-(d+\alpha)}, & r \geq 1. \end{cases}$$

By Lemma 4.1 of [8] we can show that

$$\int_{R^d} (1 + |y|^2)^{-q} f(|y - x|)\, dy \sim |x|^{-(d+\alpha)}$$

as $|x| \to \infty$ for $q \geq (d + \alpha)/2$. Then for such q, since $2p - d - \alpha < 0$,

$$\sup_{x\in R^d} (1 + |x|^2)^p \int_{R^d} (1 + |y|^2)^{-q} f(|y - x|)\, dy < \infty.$$

Hence

$$\|G\|_p \leq \text{const}(\|\varphi\|_p + \|\varphi\|_q), \qquad q \geq (d + \alpha)/2.$$

In conclusion, there is a constant $C > 0$ such that

$$\|\Delta_\alpha \varphi\|_p \leq C(\|\varphi\|_p + \||\nabla\varphi|\|_p + \|\xi_\varphi\|_p + \|\varphi\|_q), \qquad \varphi \in \mathscr{S}(R^d),$$

with $q \geq (d + \alpha)/2$. Therefore $\varphi \to 0$ in $\mathscr{S}(R^d)$ implies $\|\Delta_\alpha \varphi\|_p \to 0$ (use the norms $|\!|\!|\cdot|\!|\!|_n$ in $\mathscr{S}(R^d)$).

It remains to show that Δ_α maps $\mathscr{S}(R^d)$ into $C_{p,0}(R^d)$. But this follows from Corollary 2.7 and the fact that $C_{p,0}(R^d)$ is closed for $\|\cdot\|_p$. □

Remark. The above proof can be modified to yield the fact that $\|\Delta_\alpha \varphi_p\|_p < \infty$.

Let us summarize the previous results since they constitute the setting for our formulation of stochastic evolution equations of the type (1.2).

Proposition 2.9. *For $\alpha < 2$ and $d/2 < p < (d + \alpha)/2$ we have the inclusions*

$$\mathscr{S}(R^d) \subset C_{p,0}(R^d) \subset L^2(R^d) \subset C'_{p,0}(R^d) \subset \mathscr{S}'(R^d),$$

$\mathscr{S}(R^d)$ is continuously and densely embedded in $C_{p,0}(R^d)$, the operators Δ_α and S_t for each $t \geq 0$ are continuous linear mappings from $\mathscr{S}(R^d)$ into $C_{p,0}(R^d)$, and $t \mapsto S_t \varphi$ is a continuous curve in $C_{p,0}(R^d)$ for each $\varphi \in \mathscr{S}(R^d)$.

The results in this section are the tools for the proofs in the following sections. In general we omit computation details.

3. A Generalized Form of $\mathscr{S}'$-Valued Stochastic Evolution Equations

We will define a generalized form of the symbolic stochastic evolution equation

$$dY_t = A^* Y_t \, dt + dZ_t, \qquad t \in [0, T], \tag{3.1}$$

where Y and Z are $\mathscr{S}'(R^d)$-valued processes, Z is a semimartingale, $Z_0 = 0$, Y_0 and Z are independent, and A generates a semigroup $\{T_t\}$ on $L^2(R^d)$. We assume that $\mathscr{S}(R^d) \subset \mathrm{Dom}(A)$ but A and T_t do not necessarily map $\mathscr{S}(R^d)$ into itself.

We denote $\mathscr{H} = \mathscr{S}(R^d) \hat{\otimes} \mathscr{D}([-\delta, T])$ where $\delta > 0$ is fixed, and $\Phi_t \equiv \Phi(\cdot, t)$.

We start with some formal calculations in order to motivate our definitions. Applying (3.1) to $\varphi \in \mathscr{S}(R^d)$, multiplying by $f \in \mathscr{D}([-\delta, T])$, and integrating by parts we obtain

$$\int_0^T \langle Y_t, \varphi f'(t) + A\varphi f(t) \rangle \, dt = \langle Y_0, \varphi f(0) \rangle + \int_0^T \langle Z_t, \varphi f'(t) \rangle \, dt,$$

which extended to $\mathscr{H}$ *yields*

$$\int_0^T \left\langle Y_t, \frac{\partial}{\partial t} \Phi_t + A\Phi_t \right\rangle dt = -\langle Y_0, \Phi_0 \rangle + \int_0^T \left\langle Z_t, \frac{\partial}{\partial t} \Phi_t \right\rangle dt, \qquad \Phi \in \mathscr{H}. \tag{3.2}$$

Since $A\Phi_t$ may not lie in $\mathscr{S}(R^d)$, the term $\int_0^T \langle Y_t, A\Phi_t \rangle \, dt$ is not well defined (all the other terms are well defined; see Corollary 2.2). Nevertheless, (3.2) together with the special case of Proposition 2.9 gives us the basis of a definition.

Definition 3.1. Let Y and Z be processes in $D([0, T], \mathscr{S}'(R^d))$, defined on the same probability space $(\Omega, \mathscr{F}, P)$. Then Y is said to be a *generalized solution* of (3.1) if there exists a Banach space of real functions on R^d, denoted by $V(R^d)$, such that $\mathscr{S}(R^d) \subset V(R^d) \subset L^2(R^d)$, $\mathscr{S}(R^d)$ is continuously and densely embedded in $V(R^d)$, A is a continuous linear map from $\mathscr{S}(R^d)$ into $V(R^d)$, the expression $\int_0^T \langle Y_t, A\Phi_t \rangle \, dt$ is a random variable on $(\Omega, \mathscr{F}, P)$ for each $\Phi \in (\mathrm{Dom}\,(A) \cap V(R^d)) \hat{\otimes} \mathscr{D}([-\delta, T])$, and (3.2) holds for each such Φ (equality in $\mathscr{L}^0(\Omega, \mathscr{F}, P)$).

We will define similarly a generalized form of the evolution solution of (3.1), namely

$$Y_t = T_t^* Y_0 + \int_0^t T_{t-s}^* \, dZ_s, \qquad t \in [0, T]. \tag{3.3}$$

From (3.3) we have, for $\psi \in \mathscr{S}(R^d)$ and $g \in \mathscr{D}([-\delta, T])$,

$$\int_0^T \langle Y_t, \psi\rangle g(t)\, dt = \left\langle Y_0, \int_0^T T_t\psi g(t)\, dt\right\rangle + \int_0^T \int_0^t \langle dZ_s, T_{t-s}\psi\rangle g(t)\, dt.$$

Using formally integration by parts and Fubini's theorem the last term is transformed as follows:

$$\begin{aligned}\int_0^T \int_0^t \langle dZ_s, T_{t-s}\psi\rangle g(t)\, dt &= \int_0^T \left\langle dZ_s, \int_s^T T_{t-s}\psi g(t)\, dt\right\rangle \\ &= -\int_0^T \left\langle Z_s, -\psi g(s) - \int_s^T T_{t-s}A\psi g(t)\, dt\right\rangle ds \\ &= -\int_0^T \left\langle Z_s, \int_s^T T_{t-s}\psi g'(t)\, dt\right\rangle ds.\end{aligned}$$

Hence, extending to $\mathscr{H}$ we have formally

$$\begin{aligned}\int_0^T \int_0^t \langle dZ_s, T_{t-s}\Psi_t\rangle\, dt &= \int_0^T \left\langle dZ_s, \int_s^T T_{t-s}\Psi_t\, dt\right\rangle \\ &= -\int_0^T \left\langle Z_s, \int_s^T T_{t-s}\frac{\partial}{\partial t}\Psi_t\, dt\right\rangle ds, \qquad \Psi \in \mathscr{H}, \qquad (3.4)\end{aligned}$$

and therefore

$$\begin{aligned}&\int_0^T \langle Y_t, \Psi_t\rangle\, dt \\ &\quad = \left\langle Y_0, \int_0^T T_t\Psi_t\right\rangle dt - \int_0^T \left\langle Z_s, \int_s^T T_{t-s}\frac{\partial}{\partial t}\Psi_t\, dt\right\rangle ds, \qquad \Psi \in \mathscr{H}. \qquad (3.5)\end{aligned}$$

The left-hand side of (3.5) is well defined (Corollary 2.2) but the right-hand side is not because $T_t\Psi_r$ and $T_t(\partial\Psi_r/\partial r)$ may not lie in $\mathscr{S}(R^d)$. But again with the motivation of Proposition 2.9, (3.5) gives the basis of a definition.

Definition 3.2. Let Y and Z be processes in $D([0, T], \mathscr{S}'(R^d))$, defined on the same probability space $(\Omega, \mathscr{F}, P)$. Then Y is said to be the *generalized evolution solution* of (3.1) if there exists a Banach space of real functions on R^d, denoted by $V(R^d)$, such that $\mathscr{S}(R^d) \subset V(R^d) \subset L^2(R^d)$, $\mathscr{S}(R^d)$ is continuously and densely embedded in $V(R^d)$, T_t is a continuous linear map from $\mathscr{S}(R^d)$ into $V(R^d)$ for each $t \in [0, T]$, $t \mapsto T_t\psi$ is a continuous curve in $V(R^d)$ for each $\psi \in \mathscr{S}(R^d)$, the right-hand side of (3.5) is a random variable on $(\Omega, \mathscr{F}, P)$ for each $\Psi \in V(R^d) \hat{\otimes} \mathscr{D}([-\delta, T])$, and (3.5) holds for each such Ψ (equality in $\mathscr{L}^0(\Omega, \mathscr{F}, P)$).

Remarks 3.3. (a) Since $\mathscr{D}([-\delta, T])$ is nuclear, the π and ε completions of $(\mathrm{Dom}(A) \cap V(R^d)) \otimes \mathscr{D}([-\delta, T])$ are isomorphic [30]. Similarly for $V(R^d) \otimes \mathscr{D}([-\delta, T])$.

(b) In (3.2) all the terms except $\int_0^T \langle Y_t, A\Phi_t\rangle\, dt$ constitute $\mathscr{H}'$-valued random variables (Corollary 2.2). If this exceptional term can be shown to define also an

$\mathscr{H}'$-random variable, then by the Hahn–Banach theorem all the terms in (3.2) can be extended to $(\mathrm{Dom}(A) \cap V(R^d)) \hat{\otimes} \mathscr{D}([-\delta, T])$ (the seminorms on $V(R^d) \otimes_\pi \mathscr{D}([-\delta, T])$ are described in Proposition 43.1 of [30]). Moreover, since the fact that $\mathscr{S}(R^d)$ is dense in $\mathrm{Dom}(A) \cap V(R^d)$ implies that $\mathscr{H}$ is dense in $(\mathrm{Dom}(A) \cap V(R^d)) \otimes \mathscr{D}([-\delta, T])$ [30, Proposition 43.9], these extensions are unique. Thus it suffices to prove that

$$\left\{\int_0^T \langle Y_t, A\Phi_t \rangle \, dt, \Phi \in \mathscr{H}\right\} \tag{3.6}$$

defines an $\mathscr{H}'$-random variable in order for Definition 3.1 to make sense. Similarly, in order for Definition 3.2 to make sense, since Y_0 and Z are independent if suffices to prove that

$$\left\{\left\langle Y_0, \int_0^T T_t \Psi_t \, dt \right\rangle, \Psi \in \mathscr{H}\right\} \tag{3.7}$$

and

$$\left\{\int_0^T \left\langle Z_s, \int_s^T T_{t-s} \frac{\partial}{\partial t} \Psi_t \, dt \right\rangle, \Psi \in \mathscr{H}\right\} \tag{3.8}$$

define $\mathscr{H}'$-random variables. Now contrary to the other terms in (3.2) and (3.5), the families (3.6), (3.7), and (3.8) do not in general define $\mathscr{H}'$-random variables. It must be shown in each specific case that they do, and for this it is necessary to use the distributions of the processes Y and Z, the fact that $\mathscr{S}(R^d)$ is dense in $V(R^d)$, and the regularization theorem, as is illustrated in the following sections.

(c) The process Y is uniquely determined by $\{\int_0^T \langle Y_t, \Psi_t \rangle \, dt, \Psi \in \mathscr{H}\}$, assuming it is continuous at T a.s. (Lemma 2.3). Hence only one process Y can satisfy (3.5).

(d) The difference between our formulation and the analogous one in the deterministic theory (e.g., p. 131 of [29]), is that in ours some of the terms in (3.2) and (3.5) are not required to be well defined *a priori*, and they must be shown to be well defined in specific cases.

We now show that Definitions 3.1 and 3.2 are equivalent, assuming, on the basis of Remark 3.3(b), that all terms are $\mathscr{H}'$-random variables.

Proposition 3.4. *Provided that the conditions in Definitions* 3.1 *and* 3.2 *are satisfied and all the terms in* (3.2) *and* (3.5) *constitute* $\mathscr{H}'$*-valued random variables, the stochastic evolution equation* (3.1) *has a unique generalized solution and it is given by the generalized evolution solution.*

Proof. Let (3.2) be satisfied for $\Phi \in (\mathrm{Dom}(A) \cap V(R^d)) \hat{\otimes} \mathscr{D}([-\delta, T])$. To show that (3.5) holds for each $\Psi \in V(R^d) \hat{\otimes} \mathscr{D}([-\delta, T])$, since $\mathscr{S}(R^d)$ is dense in $V(R^d)$ and by the Hahn–Banach theorem (as in Remark 3.3(b)) it suffices to take Ψ of the form $\Psi = \psi \otimes g$, $\psi \in \mathscr{S}(R^d)$, $g \in \mathscr{D}([-\delta, T])$. Define

$$\Phi(x, s) = \int_s^T T_{t-s} \psi(x) g(t) \, dt, \qquad x \in R^d, \quad s \in [-\delta, T].$$

The conditions of the definitions imply $\Phi \in (\mathrm{Dom}(A) \cap V(R^d)) \hat{\otimes} \mathscr{D}([-\delta, T])$ (note that $\Phi(\cdot, T) = 0$). Hence (3.2) holds for this Φ. We have

$$\Phi_0 = \int_0^T T_t \psi g(t)\, dt$$

$$= \int_0^T T_t \Psi_t\, dt,$$

$$\frac{\partial}{\partial s} \Phi_s = -\psi g(s) - \int_s^T T_{t-s} A \psi g(t)\, dt$$

$$= \int_s^T T_{t-s} \psi g'(t)\, dt$$

$$= \int_s^T T_{t-s} \frac{\partial}{\partial t} \Psi_t\, dt,$$

and

$$\frac{\partial}{\partial s} \Phi_s + A\Phi_s = -\psi g(s) - \int_s^T T_{t-s} A \psi g(t)\, dt + A \int_s^T T_{t-s} \psi g(t)\, dt$$

$$= -\psi g(s) = -\Psi_s.$$

Substituting into (3.2) yields

$$-\int_0^T \langle Y_t, \Psi_t \rangle\, dt = -\left\langle Y_0, \int_0^T T_t \Psi_t\, dt \right\rangle + \int_0^T \left\langle Z_s, \int_s^T T_{t-s} \frac{\partial}{\partial t} \Psi_t\, dt \right\rangle ds,$$

which is (3.5).

Assume now that (3.5) is satisfied for $\Psi \in V(R^d) \hat{\otimes} \mathscr{D}([-\delta, T])$. To show that (3.2) holds for each $\Phi \in (\mathrm{Dom}(A) \cap V(R^d)) \hat{\otimes} \mathscr{D}([-\delta, T])$, since $\mathscr{S}(R^d)$ is dense in $\mathrm{Dom}(A) \cap V(R^d)$ and by the Hahn–Banach theorem it suffices to take Φ of the form $\Phi = \varphi \otimes f$, $\varphi \in \mathscr{S}(R^d)$, $f \in \mathscr{D}([-\delta, T])$. Define

$$\Psi(x, s) = \frac{\partial}{\partial s} \Phi(x, s) + A\Phi(x, s), \qquad x \in R^d, \quad s \in [-\delta, T].$$

The conditions of the definitions imply $\Psi \in V(R^d) \hat{\otimes} \mathscr{D}([-\delta, T])$. Hence (3.5) holds for this Φ. We have

$$\int_0^T T_t \Psi_t\, dt = \int_0^T T_t \varphi f'(t)\, dt + \int_0^T T_t A \varphi f(t)\, dt$$

$$= -\varphi f(0)$$

$$= -\Phi_0$$

and

$$\int_s^T T_{t-s} \frac{\partial}{\partial t} \Psi_t \, dt = \int_s^T T_{t-s} \varphi f''(t) \, dt + \int_s^T T_{t-s} A \varphi f'(t) \, dt$$
$$= -\varphi f'(s)$$
$$= -\frac{\partial}{\partial s} \Phi_s.$$

Substituting into (3.5) yields

$$\int_0^T \left\langle Y_t, \frac{\partial}{\partial t} \Phi_t + A\Phi_t \right\rangle dt = -\langle Y_0, \Phi_0 \rangle + \int_0^T \left\langle Z_t, \frac{\partial}{\partial t} \Phi_t \right\rangle dt,$$

which is (3.2). □

Proposition 3.4 is a result of the usual type for evolution equation (e.g., [4], [7], [20], and [22]), except for the fact that all the terms in (3.2) and (3.5) need not be well defined *a priori*, but may be well defined for particular processes.

4. The Processes Y and Z

In the following section we apply the previous theory to the system discussed in [6]. In that paper we studied the asymptotic fluctuations of a branching particle system in a random medium, where the particles migrate according to a spherically symmetric stable process in R^d with exponent α, $0 < \alpha \leq 2$. The branching law is critical and belongs to the domain of normal attraction of a stable law with exponent $1 + \beta$, $0 < \beta \leq 1$. The fluctuation of the system is described by an $\mathscr{S}'(R^d)$-valued process $Y^\varepsilon \equiv \{Y_t^\varepsilon, t \geq 0\}$ depending on a parameter $\varepsilon > 0$. It was shown that for $d > \alpha/\beta$, under certain assumptions on the initial distribution of particles the process Y^ε converges weakly in $D([0, \infty), \mathscr{S}'(R^d))$ to a process Y as $\varepsilon \to 0$. It was stated in [6] that Y satisfies the Langevin equation

$$dY_t = \Delta_\alpha^* Y_t \, dt + dZ_t,$$

where Z is an $\mathscr{S}'(R^d)$-valued stable process with independent increments. Our aim is to make this statement precise. We assume that $\alpha < 2$ since the case $\alpha = 2$ is well understood.

In the rest of this section we present some background on the processes Y and Z.

Part of the convergence proof in [6] involves showing that for $d/2 < p < (d + \alpha)/2$, and for $0 = t_0 < t_1 < \cdots < t_n$, $\varphi_0, \varphi_1, \ldots, \varphi_n \in C_p(R^d)$, and $n \geq 1$, the random vector $(\langle Y_{t_0}^\varepsilon, \varphi_0 \rangle, \ldots, \langle Y_{t_n}^\varepsilon, \varphi_n \rangle)$ converges weakly, the limit being denoted by $(Y_{t_0}(\varphi_0), \ldots, Y_{t_n}(\varphi_n))$. In the limit random vector the φ_k play the role of parameters; however, it is a consequence of the model that for each $t \geq 0$ the random variable $Y_t(\varphi)$ is a linear function of φ a.s.

The characteristic function of the limit random vector above satisfies the relation

$$E\exp\left\{-i\sum_{j=0}^{n}Y_{t_j}(\varphi_j)\right\}=E\exp\left\{-i\left[\sum_{j=0}^{n-1}Y_{t_j}(\varphi_j)+Y_{t_{n-1}}(S_{t_n-t_{n-1}}\varphi_n)\right]\right.$$
$$\left.+\gamma\left\langle\Lambda_{t_{n-1}},\int_{t_{n-1}}^{t_n}S_{t_n-r}(iS_{r-t_{n-1}}\varphi_n)^{1+\beta}\,dr\right\rangle\right\},\tag{4.1}$$

$\varphi_0,\ldots,\varphi_n\in C_p(R^d)$. Here γ is a positive constant, and $\Lambda_t\equiv S_t^*\Lambda$, where Λ is a deterministic element of $\mathscr{M}_p(R^d)$.

Iterating (4.1) we find

$$E\exp\left\{-i\sum_{j=0}^{n}Y_{t_j}(\varphi_j)\right\}$$
$$=E\exp\left\{-iY_0\left(\sum_{j=0}^{n}S_{t_j}\varphi_j\right)\right.$$
$$\left.+\gamma\sum_{j=0}^{n-1}\left\langle\Lambda_{t_j},\int_{t_j}^{t_{j+1}}S_{t_{j+1}-r}\left(iS_{r-t_j}\left(\sum_{k=j}^{n-1}S_{t_{k+1}-t_{j+1}}\varphi_{k+1}\right)\right)^{1+\beta}dr\right\rangle\right\},\tag{4.2}$$

$\varphi_0,\ldots,\varphi_n\in C_p(R^d)$. Moreover, (4.1) implies that $Y_{t_n}(\varphi_n)-Y_{t_{n-1}}(S_{t_n-t_{n-1}}\varphi_n)$ and $\{Y_{t_0}(\varphi_0),\ldots,Y_{t_{n-1}}(\varphi_{n-1})\}$ are independent for all $\varphi_0,\ldots,\varphi_n\in C_p(R^d)$, and therefore

$$E[\exp\{-iY_{t_n}(\varphi_n)\}\,|\,Y_{t_0}(\varphi_0),\ldots,Y_{t_{n-1}}(\varphi_{n-1});\ \varphi_0,\ldots,\varphi_{n-1}\in C_p(R^d)]$$
$$=\exp\left\{-iY_{t_{n-1}}(S_{t_n-t_{n-1}}\varphi_n)+\gamma\left\langle\Lambda_{t_{n-1}},\int_{t_{n-1}}^{t_n}S_{t_n-r}(iS_{r-t_{n-1}}\varphi_n)^{1+\beta}\,dr\right\rangle\right\},\tag{4.3}$$

$\varphi_n\in C_p(R^d)$.

Now, since for the $\mathscr{S}'(R^d)$-valued process Y the random vector $(\langle Y_{t_0},\varphi_0\rangle,\ldots,\langle Y_{t_n},\varphi_n\rangle)$ is distributed as $(Y_{t_0}(\varphi_0),\ldots,Y_{t_n}(\varphi_n))$ for $\varphi_0,\ldots,\varphi_n\in\mathscr{S}(R^d)$, we would like to write expressions corresponding to (4.2) and (4.3) for the process Y; namely,

$$E\exp\left\{-i\sum_{j=0}^{n}\langle Y_{t_j},\varphi_j\rangle\right\}$$
$$=E\exp\left\{-i\left\langle Y_0,\sum_{j=0}^{n}S_{t_j}\varphi_j\right\rangle\right.$$
$$\left.+\gamma\sum_{j=0}^{n-1}\left\langle\Lambda_{t_j},\int_{t_j}^{t_{j+1}}S_{t_{j+1}-r}\left(iS_{r-t_j}\left(\sum_{k=j}^{n-1}S_{t_{k+1}-t_{j+1}}\varphi_{k+1}\right)\right)^{1+\beta}dr\right\rangle\right\},\tag{4.4}$$

$\varphi_0,\ldots,\varphi_n\in\mathscr{S}(R^d)$, and (as a consequence of (4.3))

$$E[\exp\{-i\langle Y_t,\varphi\rangle\}\,|\,\langle Y_r,\psi\rangle,\ r\le s,\ \psi\in\mathscr{S}(R^d)]$$
$$=\exp\left\{-i\langle Y_s,S_{t-s}\varphi\rangle+\gamma\left\langle\Lambda_s,\int_s^t S_{t-r}(iS_{r-s}\varphi)^{1+\beta}\,dr\right\rangle\right\},\tag{4.5}$$

$0 \le s < t$, $\varphi \in \mathcal{S}(R^d)$. However, since $\sum_{j=0}^{n} S_{t_j}\varphi_j$ and $S_{t-s}\varphi$ lie in $C_{p,0}(R^d)$ but not necessarily in $\mathcal{S}(R^d)$, the expressions $\langle Y_0, \sum_{j=0}^{n} S_{t_j}\varphi_j\rangle$ and $\langle Y_s, S_{t-s}\varphi\rangle$ make no sense and they must be justifed.

The main assumption in [6] is that the fluctuations of the system at time 0 converge weakly and the limit Y_0 is such that $Y_0(\varphi)$ is defined for all $\varphi \in C_p(R^d)$. This suffices for the proof of convergence of finite-dimensional distributions. In order to prove tightness of $\{Y^\varepsilon\}_\varepsilon$ in $D([0, \infty), \mathcal{S}'(R^d))$ a stronger condition is needed. A simple condition which simplifies computations is that Y_0 is a $C'_p(R^d)$-valued random variable such that $E\|Y_0\|_{-p}^{1+\beta} < \infty$. (In most cases $Y_0 = 0$, so this is not an important restriction, see [6].) Therefore (4.4) is legitimate, as well as (4.5) with $s = 0$. In order to legitimize (4.5) for $s > 0$ we show that for fixed $s > 0$ the $\mathcal{S}'(R^d)$-valued random variable Y_s has an extension to $C_{p,0}(R^d)$, and therefore the expression $\langle Y_s, S_{t-s}\varphi\rangle$ is well defined for $\varphi \in \mathcal{S}(R^d)$; this is done by applying the regularization theorem on $\mathcal{S}(R^d)$ (Proposition 5.2). (We cannot assert that Y is a $C'_{p,0}(R^d)$-valued process because the regularization theorem does not hold on $C_{p,0}(R^d)$.) Note that (4.5) implies the Markov property of the process Y.

The $\mathcal{S}'(R^d)$-valued process Z has independent increments such that

$$E \exp\{-i(\langle Z_t, \varphi\rangle - \langle Z_s, \varphi\rangle)\} = \exp\left\{\lambda \int_s^t \langle \Lambda_r, (i\varphi)^{1+\beta}\rangle \, dr\right\},$$

$$0 \le s < t, \quad \varphi \in \mathcal{S}(R^d), \tag{4.6}$$

and $Z_0 = 0$.

For each $\varphi \in \mathcal{S}(R^d)$ the real process $\langle Z, \varphi\rangle$ is stable; hence it has a version in $D([0, \infty), R)$, and therefore by [26] Z has a version in $D([0, \infty), \mathcal{S}'(R^d))$. The process Y is in $D([0, \infty), \mathcal{S}'(R^d))$ since it arises as a weak limit in this space. (It should be possible to show directly from (4.4) that Y has a version in $D([0, \infty), \mathcal{S}'(R^d))$.)

There exists $n \ge 0$ such that $Y_t - S_t^* Y_0$ takes values in $\mathcal{S}'_n(R^d)$ for all $t \ge 0$. This follows from Theorem 3.1 in Chapter 3 of [13] because $\langle \Lambda, \int_0^t S_{t-r}(iS_r\varphi)^{1+\beta}\, dr\rangle$ is continuous in φ for some norm $\|\cdot\|_n$ for all $t \ge 0$ (see (4.5)), which can be shown using (2.1) and the usual relationships between the norms $\|\cdot\|_p$, $\|\cdot\|_n$, and $|||\cdot|||_n$. The same is true of the process Z.

In the case $\beta = 1$ the processes $Y - S^* Y_0$ and Z are Gaussian, and it can be shown by [25] that they have continuous versions, but for $\beta < 1$ they are discontinuous. Thus the fact that Y is continuous or not depends on whether the branching law has finite second moment or not, which corresponds to $\beta = 1$ and $\beta < 1$, respectively (see [6]), and the continuity or discontinuity of the particle motion ($\alpha = 2$ or $\alpha < 2$) had no bearing on this. In the case $\beta < 1$, Y has no fixed points of discontinuity.

Processes like Y, in the case $\beta < 1$, are called *infinite-dimensional stable Ornstein–Uhlenbeck processes*, meaning that they satisfy an equation of Langevin type and $\langle Y_t, \varphi\rangle$ has a stable distribution for each t and φ (more precisely $\langle Y_t, \varphi\rangle - \langle Y_0, S_t\varphi\rangle$ in our case). Note that Y does not have independent increments.

5. The Langevin Equation for the Process Y

We apply the formulation given in Section 3 to the processes Y and Z in order to justify the Langevin equation for Y. In addition we show that for fixed $s > 0$ the $\mathcal{S}'(R^d)$-random variable Y has an extension to $C_{p,0}(R^d)$, so that the expression $\langle Y_s, S_{t-s}\varphi\rangle$ in (4.5) is well defined for $\varphi \in \mathcal{S}(R^d)$.

The conditions on the operators Δ_α and S_t in Definitions 3.1 and 3.2 are satisfied with $V(R^d) = C_{p,0}(R^d)$, due to Proposition 2.9. By Remark 3.3(b) we must show that $\int_0^T \langle Y_t, \Delta_\alpha \Phi_t\rangle\, dt$, $\langle Y_0, \int_0^T S_t\Phi_t\, dt\rangle$ and $\int_0^T \langle Z_s, \int_s^T S_{t-s}\partial\Phi_t/\partial t\, dt\rangle\, ds$, $\Phi \in \mathcal{H} \equiv \mathcal{S}(R^d) \hat{\otimes} \mathcal{D}([-\delta, T])$ define $\mathcal{H}'$-random variables on the same space where Y and Z are defined.

Since we assumed that Y_0 is a $C_p'(R^d)$-random variable, and $\Phi \mapsto \int_0^T S_t\Phi_t\, dt$ is continuous from $\mathcal{H}$ into $C_{p,0}(R^d)$, then $\{\langle Y_0, \int_0^T S_t\Phi_t dt\rangle, \Phi \in \mathcal{H}\}$ defines an $\mathcal{H}'$-random variable due to Lemma 2.4.

Proposition 5.1.

$$\left\{\int_0^T \langle Y_t, \Delta_\alpha \Phi_t\rangle\, dt, \Phi \in \mathcal{H}\right\}$$

defines an $\mathcal{H}'$-random variable on the same space where Y is defined.

Proof. By the regularization theorem (Lemma 2.4) it suffices to define the expression $\int_0^T \langle Y_t, \Delta_\alpha\Phi_t\rangle\, dt$ in such a way that it is a linear random functional on $\mathcal{H}$ on the same probability space where Y is defined.

We show first that the integral $\int_0^T \langle Y_t, \varphi\rangle f(t)\, dt$ is well defined for $\varphi \in C_{p,0}(R^d)$, $d/2 < p < (d+\alpha)/2$, and $f \in \mathcal{D}([-\delta, T])$. The following fact is used (see [6]). If X is a real random variable such that $E|X|^{1+\theta} < \infty$ for some $\theta > 0$, then

$$E|X|^{1+\theta} \leq K^{1+\theta} + C(1+\theta)\int_K^\infty r^{1+\theta}\int_0^{1/r} [1 - \operatorname{Re} F(\lambda)]\, d\lambda\, dr, \tag{5.1}$$

where $C > 0$ is a constant, $F(\lambda)$ is the characteristic function of X, and $K > 0$ is arbitrary.

From (4.4) and (4.5) we have, for $\lambda \geq 0$,

$$E[\exp\{i\langle Y_t, \varphi\rangle\lambda\} \mid \langle Y_0, \psi\rangle, \psi \in \mathcal{S}(R^d)]$$
$$= \exp\{i\langle Y_0, S_t\varphi\rangle\lambda\} \exp\left\{\gamma\left\langle \Lambda, \int_0^t S_{t-r}(-iS_r\varphi)^{1+\beta}\, dr\right\rangle \lambda^{1+\beta}\right\}, \quad \varphi \in \mathcal{S}(R^d), \tag{5.2}$$

and

$$F_\varphi(\lambda) \equiv E\exp\{i\langle Y_t, \varphi\rangle\lambda\}$$
$$= E\exp\{i\langle Y_0, S_t\varphi\rangle\lambda\}$$
$$\times \exp\left\{\gamma\left\langle \Lambda, \int_0^t S_{t-r}(-iS_r\varphi)^{1+\beta}\, dr\right\rangle\lambda^{1+\beta}\right\}, \quad \varphi \in \mathcal{S}(R^d). \tag{5.3}$$

Hence

$$1 - \operatorname{Re} F_\varphi(\lambda) = E(1 - \operatorname{Re}[\exp\{iA\lambda\} \exp\{B\lambda^{1+\beta}\}]), \tag{5.4}$$

where

$$A = \langle Y_0, S_t\varphi \rangle \quad \text{and} \quad B = \gamma\left\langle \Lambda, \int_0^t S_{t-r}(-iS_r\varphi)^{1+\beta}\, dr \right\rangle.$$

We can write (5.4) as

$$1 - \operatorname{Re} F_\varphi(\lambda) = E(1 - \operatorname{Re}(1 + U)(1 + V)),$$

where

$$U = \exp\{iA\lambda\} - 1 \quad \text{and} \quad V = \exp\{B\lambda^{1+\beta}\} - 1.$$

Therefore

$$\begin{aligned} 1 - \operatorname{Re} F_\varphi(\lambda) &= E \operatorname{Re}(-V(1 + U) - U) \\ &\le E|V|\,|1 + U| - E \operatorname{Re} U \\ &\le |V| - E \operatorname{Re} U. \end{aligned}$$

It follows from (5.2) that $|\exp\{B\lambda^{1+\beta}\}| \le 1$, and therefore $|V| \le 2|B|\lambda^{1+\beta}$. Now, using (2.1),

$$\begin{aligned} |B| &\le \gamma \int_0^t \langle \Lambda, |S_{t-r}(-iS_r\varphi)^{1+\beta}| \rangle\, dr \\ &\le \text{const} \|\varphi\|_\infty^\beta \langle \Lambda, S_t|\varphi| \rangle \\ &\le \text{const} \|\varphi\|_\infty^\beta \|\Lambda\|_{-p} \|\varphi\|_p \\ &\le \text{const} \|\Lambda\|_{-p} \|\varphi\|_p^{1+\beta}, \end{aligned}$$

($\|\cdot\|_\infty$ is the supremum norm), where the constant is independent of t (for $t \in [0, T]$); hence

$$|V| \le \text{const} \|\varphi\|_p^{1+\beta} \lambda^{1+\beta}.$$

For the other term we have

$$\begin{aligned} -\operatorname{Re} U &= -\operatorname{Re} \exp\{iA\lambda\} + 1 \\ &= -\cos A\lambda + 1 \\ &= \lambda^{1+\beta}(1 - \cos A\lambda)/\lambda^{1+\beta} \\ &\le H(\varphi)\lambda^{1+\beta}, \end{aligned}$$

where

$$H(\varphi) = \sup_{0\le t\le T}\sup_{\lambda\ge 0}\lambda^{-(1+\beta)}(1-\cos\langle Y_0, S_t\varphi\rangle\lambda)$$
$$= \text{const}\sup_{0\le t\le T}|\langle Y_0, S_t\varphi\rangle|^{1+\beta}$$
$$\le \text{const}\|Y_0\|_{-p}^{1+\beta}\sup_{0\le t\le T}\|S_t\varphi\|_p^{1+\beta}$$
$$\le \text{const}\|Y_0\|_{-p}^{1+\beta}\|\varphi\|_p^{1+\beta}$$

(the constant depends also on β). Therefore, since $E\|Y_0\|_{-p}^{1+\beta} < \infty$,

$$1 - \operatorname{Re} F_\varphi(\lambda) \le \text{const}\|\varphi\|_p^{1+\beta}\lambda^{1+\beta}, \qquad \varphi \in \mathscr{S}(R^d).$$

Hence, taking $\theta < \beta$ in (5.1) we have

$$E|\langle Y_t, \varphi\rangle|^{1+\theta} \le K^{1+\theta} + \text{const}\int_K^\infty r^{1+\theta}\int_0^{1/r}\lambda^{1+\beta}\,d\lambda\,dr\|\varphi\|_p^{1+\beta},$$

so

$$E|\langle Y_t, \varphi\rangle|^{1+\theta} \le K^{1+\beta} + \text{const}\,K^{\theta-\beta}\|\varphi\|_p^{1+\beta}, \qquad \varphi\in\mathscr{S}(R^d), \tag{5.5}$$

for all $t \in [0, T]$, with $K > 0$ arbitrary.

Given $\varphi \in C_{p,0}(R^d)$ let $(\varphi_n)_n$ be a sequence in $\mathscr{S}(R^d)$ such that $\|\varphi_n - \varphi\|_p \to 0$ (by Proposition 2.9). By (5.5) we have for $f \in \mathscr{D}([-\delta, T])$, using Hölder's inequality,

$$E\left|\int_0^T\langle Y_t, \varphi_n\rangle f(t)\,dt - \int_0^T\langle Y_t, \varphi_m\rangle f(t)\,dt\right|^{1+\theta}$$
$$\le \text{const}\int_0^T E|\langle Y_t, \varphi_n - \varphi_m\rangle|^{1+\theta}|f(t)|^{1+\theta}\,dt$$
$$\le \text{const}[K^{1+\theta} + K^{\theta-\beta}(\|\varphi_n - \varphi_m\|_p^{1+\beta})]\sup_{0\le t\le T}|f(t)|^{1+\theta}.$$

Therefore, since $(\varphi_n)_n$ is a Cauchy sequence in $C_{p,0}(R^d)$,

$$\limsup_{m,n\to\infty} E\left|\int_0^T\langle Y_t, \varphi_n\rangle f(t)\,dt - \int_0^T\langle Y_t, \varphi_m\rangle f(t)\,dt\right|^{1+\theta}$$
$$\le \text{const}\,K^{1+\theta}\sup_{0\le t\le T}|f(t)|^{1+\theta}.$$

But K is arbitrary, so we may let $K \to 0$ and therefore $\left(\int_0^T\langle Y_t, \varphi_n\rangle f(t)\,dt\right)_n$ is a Cauchy sequence in probability. Hence it converges in probability and it can be shown that the limit is independent of the chosen sequence $(\varphi_n)_n$. We define $\int_0^T\langle Y_t, \varphi\rangle f(t)\,dt$ to be this limit.

Let us write $Y\{\varphi, f\} = \int_0^T\langle Y_t, \varphi\rangle f(t)\,dt$, $\varphi \in C_{p,0}(R^d)$, $f \in \mathscr{D}([-\delta, T])$. It follows from the procedure above that $Y\{\varphi, f\}$ is a bilinear form which is separately continuous in φ and f. Hence, by Proposition 2.9 $(\varphi, f) \mapsto Y\{\Delta_\alpha\varphi, f\}$ is a separately continuous bilinear form from $\mathscr{S}(R^d) \times \mathscr{D}([-\delta, T])$ into $\mathscr{L}^0(\Omega, \mathscr{F}, P)$, and therefore it is continuous. This form can be extended to $\mathscr{S}(R^d) \otimes \mathscr{D}([-\delta, T])$ and it is continuous on this space. We designate it by $\int_0^T \langle Y_t, \Delta_\alpha \Phi_t \rangle \, dt$ for $\Phi \in \mathscr{S}(R^d) \otimes \mathscr{D}([-\delta, T))$. Finally, by a procedure similar to the one above it is shown that $\int_0^T \langle Y_t, \Delta_\alpha \Phi_t \rangle \, dt$ can be extended to $\mathscr{H}$ (the integral is also defined as a limit in probability, obtained by means of a sequence $(\Phi_n)_n$ is $\mathscr{S}(R^d) \otimes \mathscr{D}([-\delta, T])$ converging to $\Delta_\alpha \Phi \in C_{p,0}(R^d) \hat{\otimes} \mathscr{D}([-\delta, T])$. This yields a continuous linear map from $\mathscr{H}$ into $\mathscr{L}^0(\Omega, \mathscr{F}, P)$ which is the linear random functional we need.

Finally we apply the regularization theorem, and we use the same symbol $\int_0^T \langle Y_t, \Delta_\alpha \Phi_t \rangle \, dt$ to designate the regular version. □

Proposition 5.2. *For fixed $s > 0$, $\langle Y_s, \cdot \rangle$ has a continuous linear extension to $C_{p,0}(R^d)$, and*

$$\{\langle Y_s, S_{t-s}\varphi \rangle, \ \varphi \in \mathscr{S}(R^d)\}$$

defines an $\mathscr{S}'(R^d)$-random variable on the same space where Y_s is defined.

Proof. The proof is the same as that of Proposition 5.1 up to (5.5). It is then shown, similarly as above, that if $(\varphi_n)_n$ is a sequence in $\mathscr{S}(R^d)$ converging to $\varphi \in C_{p,0}(R^d)$, then $(\langle Y_s, \varphi_n \rangle)_n$ is a Cauchy sequence in probability, therefore it has a limit in probability, which is denoted by $\langle Y_s, \varphi \rangle$ and is linear continuous in φ. Then by Proposition 2.9 $\varphi \mapsto \langle Y_s, S_{t-s}\varphi \rangle$ is a linear random functional on $\mathscr{S}(R^d)$; next we apply the regularization theorem (Lemma 2.4), and denote the regular version also by $\langle Y_s, S_{t-s}\varphi \rangle$.

Now we pass to the process Z. From (3.4) we have the formal expression

$$\int_0^T \left\langle dZ_s, \int_s^T S_{t-s}\Psi_t \, dt \right\rangle = -\int_0^T \left\langle Z_s, \int_s^T S_{t-s} \frac{\partial}{\partial t} \Psi_t \, dt \right\rangle ds, \qquad \Psi \in \mathscr{H}. \tag{5.6}$$

We will give definitions of the two sides of (5.6) as random variables on the same space where Z is defined, and show that they are equal in $\mathscr{L}^0(\Omega, \mathscr{F}, P)$. Only the right-hand side of (5.6) is needed in connection with Proposition 3.4, but the left-hand side is relevant because it has the form of a stochastic integral.

Since $\int_0^T S_{t-s}\Psi_t \, dt$ and $\int_s^T S_{t-s} \, \partial\Psi_t/\partial t \, dt$ belong to $C_{p,0}(R^d)$ for $\Psi \in \mathscr{H}$, we must first define $\langle Z_s, \varphi \rangle$ for fixed $s > 0$ and $\varphi \in C_{p,0}(R^d)$.

Proposition 5.3. *For fixed $s > 0$, $\langle Z_s, \cdot \rangle$ has a continuous linear extension to $C_{p,0}(R^d)$.*

Proof. The argument in the proof of Proposition 5.2 works also in this case due to the similarity of the structures of the characteristic functionals of Y_s and Z_s. (The present case is in fact simpler.) □

Note that expression (4.6) then holds also for $\varphi \in C_{p,0}(R^d)$.

Since Z is in $D([0, \infty), \mathscr{S}'(R^d))$, and $s \mapsto \int_s^T S_{t-s}\Psi_t \, dt$ and $s \mapsto \int_s^T S_{t-s}\partial\Psi_t/\partial t \, dt$ are continuous curves in $C_{p,0}(R^d)$ for $\Psi \in \mathscr{H}$, it is natural to define each of the terms in (5.6) as a limit of Stieltjes sums. The next result shows that this can be done.

Proposition 5.4. *Let $\Psi \in \mathscr{H}$ be fixed. Let $0 = t_1^n < t_2^n < \cdots < t_{N_n}^n = T$, $n = 1, 2, \ldots$, be an increasing sequence of partitions such that $\sup_k(t_{k+1}^n - t_k^n) \to 0$ as $n \to \infty$. Let*

$$A_n = \sum_{k=0}^{N_n - 1} \left\langle Z_{t_{k+1}^n} - Z_{t_k^n}, \int_{t_k^n}^T S_{t-t_k^n}\Psi_t\, dt \right\rangle$$

and

$$B_n = \sum_{k=0}^{N_n - 1} (t_{k+1}^n - t_k^n)\left\langle Z_{t_k^n}, \int_{t_k^n}^T S_{t-t_k^n}\partial\Psi_t/\partial t\, dt \right\rangle,$$

$n = 1, 2, \ldots$. Then A_n and B_n are random variables on the same space where Z is defined, they converge in probability as $n \to \infty$, and their limits coincide in $\mathscr{L}^0(\Omega, \mathscr{F}, P)$. The limits of A_n and B_n are independent of the particular sequence of partitions $\{t_k^n\}$, they are denoted by the left-hand side and the right-hand side of (5.6), respectively, and

$$\begin{aligned} E \exp&\left\{ i \int_0^T \left\langle dZ_s, \int_s^T S_{t-s}\Psi_t\, dt \right\rangle \right\} \\ &= E \exp\left\{ -i \int_0^T \left\langle Z_s, \int_s^T S_{t-s}\partial\Psi_t/\partial t\, dt \right\rangle ds \right\} \\ &= \exp\left\{ \gamma \left\langle \Lambda, \int_0^T S_s\left[i \int_s^T S_{t-s}\Psi_t\, dt \right]^{1+\beta} ds \right\rangle \right\}, \qquad \Psi \in \mathscr{H}. \end{aligned} \tag{5.7}$$

Proof. The extension given in Proposition 5.3 holds for each $s > 0$ a.s., due to the application of the regularization theorem. This causes no problem in the present proof because it involves only countably many time points.

We show that $(A_n)_n$ is a Cauchy sequence in probability and $A_n - B_n$ converges to 0 in probability. Hence both A_n and B_n converge to the same limit in probability. It suffices to prove that

$$E \exp\{iu(A_m - A_n)\} \to 1 \qquad \text{as} \quad n, m \to \infty, \qquad u \in R, \tag{5.8}$$

and

$$E \exp\{iu(A_n - B_n)\} \to 1 \qquad \text{as} \quad n \to \infty, \qquad u \in R. \tag{5.9}$$

Denote $\xi_s = \int_s^T S_{t-s}\Psi_t\, dt$. Assume $m > n$. For each $0 \le k \le N_n$ there is a k', $0 \le k' \le N_m$, such that $t_k^n = t_{k'}^m$. Then

$$A_m - A_n = \sum_{k=0}^{N_n - 1} \sum_{j=k'}^{(k+1)'-1} \langle Z_{t_{j+1}^m} - Z_{t_j^m}, \xi_{t_j^m} - \xi_{t_{k'}^m} \rangle.$$

Using the independence of the increments of Z and (4.6) we have

$$E \exp\{iu(A_m - A_n)\} = \exp\{u^{1+\beta}\gamma L_{m,n}\},$$

where

$$L_{n,m} = \left\langle \Lambda, \sum_{k=0}^{N_n-1} \sum_{j=k'}^{(k+1)'-1} \int_{t_j^m}^{t_{j+1}^m} S_r[i(\xi_{t_j^m} - \xi_{t_{k'}^m})]^{1+\beta}\, dr \right\rangle.$$

Hence to prove (5.8) it suffices to show that $L_{n,m} \to 0$ as n, $m \to \infty$. Using results in Section 2, in particular (2.1), we have

$$|L_{n,m}| \le \|\Lambda\|_{-p} \sum_{k=0}^{N_n-1} \sum_{j=k'}^{(k+1)'-1} \int_{t_j^m}^{t_{j+1}^m} \|S_r[i(\xi_{t_j^m} - \xi_{t_{k'}^m})]^{1+\beta}\|_p\, dr$$

and

$$\|S_r[i(\xi_{t_j^m} - \xi_{t_{k'}^m})]^{1+\beta}\|_p \le 2 \sup_{0\le t\le T} \|\xi_t\|_\infty^\beta C_T \|\xi_{t_j^m} - \xi_{t_{k'}^m}\|_p;$$

hence

$$|L_{n,m}| \le \text{const} \sum_{k=0}^{N_n-1} \sum_{j=k'}^{(k+1)'-1} (t_{j+1}^m - t_j^m)\|\xi_{t_j^m} - \xi_{t_{k'}^m}\|_p.$$

Now, for each $x \in R^d$, $s \mapsto \xi_s(x)$ is differentiable, so by the mean value theorem

$$\xi_{t_j^m}(x) - \xi_{t_{k'}^m}(x) = (t_j^m - t_{k'}^m)\xi'_r\,(x)$$

for some $r \in [t_{k'}^m, t_j^m]$, and $\sup_{0\le t\le T}\|\xi'_t\|_p < \infty$. Hence

$$|L_{n,m}| \le \text{const} \sum_{k=0}^{N_n-1} \sum_{j=k'}^{(k+1)'-1} (t_{j+1}^m - t_j^m)(t_j^m - t_{k'}^m)$$

$$\le \text{const}\; T \sup_k (t_{k+1}^n - t_k^n) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore (5.8) is proved.

The proof of (5.9) is similar, so we indicate only some of the main steps. Use summation by parts in B_n, do a Taylor expansion of ξ_s at $s = T$, do an integration by parts, and estimate the errors similarly as $L_{n,m}$.

We now prove (5.7). Using the independence of the increments of Z and (4.6) as before we obtain

$$E\exp\{iA_n\} = \exp\left\{\gamma\left\langle \Lambda, \sum_{k=0}^{N_n-1} \int_{t_k^n}^{t_{k+1}^n} S_r[i\xi_{t_k^n}]^{1+\beta}\, dr\right\rangle\right\}$$

$$= \exp\left\{\gamma\left\langle \Lambda, \sum_{k=0}^{N_n-1} (t_{k+1}^n - t_k^n) S_{t_k^n}[i\xi_{t_k^n}]^{1+\beta} + \varepsilon_n\right\rangle\right\},$$

where ε_n is the sum of the errors made by replacing S_r by $S_{t_k^n}$ for $r \in (t_k^n, t_{k+1}^n]$. It can be shown by estimates as above that $\varepsilon_n \to 0$ as $n \to \infty$, and since $s \mapsto S_s[i\xi_s]^{1+\beta}$ is continuous, then

$$\sum_{k=0}^{N_n-1} (t_{k+1}^n - t_k^n) S_{t_k^n}[i\xi_{t_k^n}]^{1+\beta} \to \int_0^T S_s\left[i\int_s^T S_{t-s}\Psi_t\, dt\right]^{1+\beta} ds \qquad \text{as} \quad n \to \infty.$$

Finally, the previous results are independent of the particular sequence of partitions used. □

From (5.7) and the Bochner–Minlos theorem we have

Corollary 5.5.

$$\left\{\int_0^T \left\langle Z_s, \int_s^T S_{t-s}\partial\Psi_t/\partial t\, dt\right\rangle ds, \Psi \in \mathscr{H}\right\}$$

defines an $\mathscr{H}'$-random variable on the same space where Z is defined.

We have shown that the conditions of Proposition 3.4 are satisfied, and we can now state our main result.

Theorem 5.6. *The processes Y and Z with paths in $D([0, T], \mathscr{S}'(R^d))$ whose finite-dimensional distributions are given by* (4.4) *and* (4.6), *respectively, satisfy the Langevin equation*

$$dY_t = \Delta_\alpha^* Y_t\, dt + dZ_t, \qquad t \in [0, T];$$

moreover, Y is the unique solution of this equation and it is given by the evolution solution

$$Y_t = S_t^* Y_0 + \int_0^t S_{t-s}^*\, dZ_s, \qquad t \in [0, T],$$

where these expressions are interpreted in the sense of Definitions 3.1 *and* 3.2.

Proof. The only fact which remains to be proved is that the equalities (3.2) and (3.5) are satisfied. Due to Proposition 3.4 we can prove either one of them, and the other one will then also hold.

Let $\mathscr{Y}$ denote the $\mathscr{H}'$-random variable defined by

$$\langle \mathscr{Y}, \Psi\rangle = \left\langle Y_0, \int_0^T S_t\Psi_t\, dt\right\rangle - \int_0^T \left\langle Z_s, \int_s^T S_{t-s}\frac{\partial}{\partial t}\Psi_t\, dt\right\rangle ds, \qquad \Psi \in \mathscr{H}$$

(see (3.5)). Since Y_0 and Z are independent we have

$$\begin{aligned} &E\exp\{-i\langle \mathscr{Y}, \Psi\rangle\} \\ &\quad = E\exp\left\{-i\left\langle Y_0, \int_0^T S_t\Psi_t\, dt\right\rangle\right\}\exp\left\{\gamma\int_0^T \left\langle \Lambda_s, \left(i\int_s^T S_{t-s}\Psi_t\, dt\right)^{1+\beta}\right\rangle ds\right\}. \end{aligned} \tag{5.10}$$

Using the same procedure as in the proof of Proposition 4.1 in [3] we may take Ψ of the form $\Psi = \sum_{j=0}^n \varphi_j\delta_{t_j}$, where $\varphi_0, \ldots, \varphi_n \in \mathscr{S}(R^d)$, $0 = t_0 < t_1 < \cdots < t_n \le T$, and δ_{t_j} is the Dirac distribution centered at t_j, $j = 0, \ldots, n$ (more precisely, we can take sequences in $\mathscr{D}([-\delta, T])$ which converge in $\mathscr{S}'(R)$ to these Dirac distributions). Then the right-hand side of (5.10) becomes

$$E\exp\left\{-i\left\langle Y_0, \sum_{j=0}^{n} S_{t_j}\varphi_j\right\rangle\right\}\exp\left\{\gamma\int_0^T\left\langle\Lambda_s, \left(i\sum_{s<t_j} S_{t_j-s}\varphi_j\right)^{1+\beta}\right\rangle ds\right\},$$

and

$$\int_0^T\left\langle\Lambda_s, \left(i\sum_{s<t_j} S_{t_j-s}\varphi_j\right)^{1+\beta}\right\rangle ds$$

$$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left\langle S^*_{s-t_k}\Lambda_{t_k}, \left(i\sum_{j=k}^{n-1} S_{t_{j+1}-s}\varphi_{j+1}\right)^{1+\beta}\right\rangle ds$$

$$= \sum_{k=0}^{n-1}\left\langle\Lambda_{t_k}, \int_{t_k}^{t_{k+1}} S_{s-t_k}\left(iS_{t_{k+1}-s}\sum_{j=k}^{n-1} S_{t_{j+1}-t_{k+1}}\varphi_{j+1}\right)^{1+\beta} ds\right\rangle$$

$$= \sum_{k=0}^{n-1}\left\langle\Lambda_{t_k}, \int_{t_k}^{t_{k+1}} S_{t_{k+1}-r}\left(iS_{r-t_k}\sum_{j=k}^{n-1} S_{t_{j+1}-t_{k+1}}\varphi_{j+1}\right)^{1+\beta} dr\right\rangle.$$

Hence the random variable $\langle\mathscr{Y}, \Psi\rangle$ is defined for all Ψ of the form $\Psi = \sum_{j=0}^{n}\varphi_j\delta_{t_j}$, and for each such Ψ its distribution coincides with the finite-dimensional distribution of the process Y with the same t_j and φ_j given by (4.4). Therefore $\langle\mathscr{Y}, \Psi\rangle = \int_0^T\langle Y_t, \Psi_t\rangle\, dt$ by Lemma 2.3 for all $\Psi\in\mathscr{H}$, since Y has no fixed points of discontinuity (assuming Y is in $D([0, T], \mathscr{S}'(R^d))$). □

Remark 5.7. In this proof we took Y_0 and Z as given on a probability space, we defined the $\mathscr{H}'$-random variable $\mathscr{Y}$ in terms of Y_0 and Z, and we showed that $\langle\mathscr{Y}, \Psi\rangle$ is of the form $\int_0^T\langle Y_t, \Psi_t\rangle\, dt$ for a process Y with finite-dimensional distributions given by (4.4). In order to complete the argument we should prove directly from (4.4) that Y has a version in $D([0, T], \mathscr{S}'(R^d))$, but we omit this since we know it is true from [6], and a proof from (4.4) is cumbersome. Another way of proving Theorem 5.6 is as follows. We take Y as given and denote by $\mathscr{Z}$ the $\mathscr{H}'$-random variable defined by

$$\langle\mathscr{Z}, \Phi\rangle = \langle Y_0, \Phi_0\rangle + \int_0^T\left\langle Y_t, \frac{\partial}{\partial t}\Phi_t + \Delta_\alpha\Phi_t\right\rangle dt, \qquad \Phi\in\mathscr{H}$$

(see (3.2)). The distribution of the right-hand side is obtained using (4.5), which requires interpreting the integral as a limit of Stieltjes sums, and then it is shown that $\langle\mathscr{Z}, \Phi\rangle$ is of the form $\int_0^T\langle Z_t, \partial\Phi_t/\partial t\rangle\, dt$ for a process Z with independent increments distributed according to (4.6). This proof turns out to be less simple. Note that the solution to the equation is "strong" in the sense that given one of the processes Y and Z on a probability space the other process is constructed (pathwise) from it so as to satisfy the equation which is also unique by Proposition 3.4.

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