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Norm inequalities for partitioned operators and an application

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1 Introduction

Let $\mathscr{H}$ be a complex separable Hilbert space and let $\mathscr{B}(\mathscr{H})$ be the space of bounded linear operators on $\mathscr{H}$. Let $P_1, \ldots, P_n$ be a mutually orthogonal family of projections in $\mathscr{H}$ such that $\oplus P_i = I$. Given T in $\mathscr{B}(\mathscr{H})$, let $T_{ij} = P_i T P_j$, $i,j = 1, 2, \ldots, n$. Making the usual identifications we can write T in a block-matrix form

$$T = [T_{ij}] \quad 1 \leqq i,j \leqq n. \tag{1}$$

Let $\mathscr{I}_p$ denote the Schatten p-class of operators and $\|A\|_p$ the Schatten p-norm of an operator A in $\mathscr{I}_p$, $1 \leqq p \leqq \infty$. Here, $\|\cdot\|_\infty$ denotes the usual operator norm and $\mathscr{I}_\infty$ the ideal of compact operators. See [9], [11] or [12] for comprehensive accounts of these norm ideals.

One of the objects of this paper is to study relations between the norm of T and those of its block matrix entries T_{ij}. We shall prove the following

Theorem 1. *Let T be an operator in the class $\mathscr{I}_p$, for some $1 \leqq p \leqq \infty$, having a block-matrix decomposition (1). Then*

$$n^{4/p-2} \sum_{i,j} \|T_{ij}\|_p^2 \leqq \|T\|_p^2 \leqq \sum_{i,j} \|T_{ij}\|_p^2, \tag{2}$$

for $2 \leqq p \leqq \infty$; and

$$\sum_{i,j} \|T_{ij}\|_p^2 \leqq \|T\|_p^2 \leqq n^{4/p-2} \sum_{i,j} \|T_{ij}\|_p^2, \tag{3}$$

for $1 \leqq p \leqq 2$.

The inequalities (2) are also valid for noncompact operators, if $\|\cdot\|_\infty$ is understood to be the operator norm on $\mathscr{B}(\mathscr{H})$.

Theorem 2. *Let T be an operator in the class $\mathscr{I}_p$, for some $1 \leqq p < \infty$, having a block-matrix decomposition (1). Then*

$$n^{2-p} \|T\|_p^p \leqq \sum_{i,j} \|T_{ij}\|_p^p \leqq \|T\|_p^p, \tag{4}$$

for $2 \leqq p < \infty$; *and*

$$\|T\|_p^p \leqq \sum_{i,j} \|T_{ij}\|_p^p \leqq n^{2-p} \|T\|_p^p \tag{5}$$

for $1 \leqq p \leqq 2$.

In [7] Bhatia and Holbrook gave an elementary proof of one set of the famous Clarkson–McCarthy inequalities by reinterpreting them and then extending their validity to a larger class of norms. We follow a similar strategy here. These more general norms are defined in Sect. 2. In Sect. 3 we prove some results from which the above theorems follow as corollaries.

Though these inequalities seem elementary they are likely to have several applications. In Sect. 4 we use one of them to show that a large class of norm ideals has the Radon–Riesz Property.

2. Q-norms and Q^*-norms

We shall assume the basic facts about norm ideals of Hilbert space operators, for which the reader could turn to [9, 11] or [12]. Each such ideal corresponds to a *unitarily invariant* or *symmetric* norm. Each such norm is a *symmetric gauge function* of the *singular values* of an operator. Recall that if A is a compact operator the singular value of A, by definition, are the eigenvalues of the positive operator $|A| = (A^*A)^{1/2}$ enumerated as $s_1(A) \geqq s_2(A) \geqq \cdots$. There is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on operators.

Thus, the l_p-norms on sequences give rise to the Schatten p-norms on operators

$$\|A\|_p = \left(\sum_j s_j^p(A)\right)^{1/p}, \quad 1 \leqq p \leqq \infty, \tag{6}$$

where $\|A\|_\infty$ denotes the usual operator norm. The concept of a *Q-norm* was introduced in [4]. A unitarily invariant norm $\|\cdot\|_Q$ is a Q-norm if there exists another unitarily invariant norm $\|\cdot\|_{\hat{Q}}$ such that

$$\|A\|_Q^2 = \|A^*A\|_{\hat{Q}}. \tag{7}$$

Note that

$$\|A\|_{2p}^2 = \|A^*A\|_p, \quad 1 \leqq p \leqq \infty. \tag{8}$$

Hence for $p \geqq 2$ the Schatten p-norms are Q-norms. We shall say that a unitarily invariant norm is a Q^*-norm if it is the dual of a Q-norm. The Schatten p-norms for $1 \leqq p \leqq 2$ are examples of Q^*-norms.

Next consider the family

$$\|A\|_{(k),p} = \left(\sum_{j=1}^k s_j^p(A)\right)^{1/p}, \quad 1 \leqq p < \infty;\ k = 1, 2, \ldots. \tag{9}$$

Each of these defines a unitarily invariant norm. For $p = 1$ these reduce to the familiar class of *Ky Fan norms* defined as

$$\|A\|_{(k)}=\sum_{j=1}^{k} s_j(A), \quad k=1,2,\dots. \tag{10}$$

For $p \geqq 2$ each of the norms defined by (9) is a Q-norm, because a relation analogous to (8) holds. The duals of these norms can be quite complicated. The dual norm of (10) is

$$\|A\|'_{(k)}=\max\left\{\|A\|_\infty, \frac{1}{k}\|A\|_1\right\}, \quad k=1,2,\dots. \tag{11}$$

For $1<p<\infty$ and $k=2,3,\dots$ the dual norm of (9) can be described as follows. Let q be the conjugate index of p: $\frac{1}{p}+\frac{1}{q}=1$; and for each positive integer r let $\{s_j(A)\}_1^r$ denote the finite sequence consisting of the r largest singular values of A. Then

$$\|A\|'_{(k),p}=\max_{1\leqq r\leqq k-1}\ \max_{0\leqq\lambda<k^{-1/p}}\left\{(1-(k-r)\lambda^p)^{1/p}\|\{s_j(A)\}_1^r\|_q+\lambda\sum_{j=r+1}^{\infty} s_j(A)\right\}. \tag{12}$$

This was shown by Ando in response to a question by one of the authors. By what has been said above each of the norms (12) is a Q^*-norm when $p \geqq 2$, being the dual of a Q-norm.

Often, it turns out that inequalities which are true for all p-norms are also true for all unitarily invariant norms; while those which are true only when $p \geqq 2$ are also true for all Q-norms because the crucial property of the norms involved is their quadratic character (7). See [1,4,6,7] for illustrations of this phenomenon. Results of this paper also fall into this pattern.

This extension, however, first demands a reinterpretation of the original inequalities as in [5,7]. A useful viewpoint is to go to direct sums as follows. Given two sequences of numbers $x=\{x_1,x_2,\dots\}$ and $y=\{y_1,y_2,\dots\}$ define a new sequence $x \vee y$ as $\{x_1,y_1,x_2,y_2,\dots\}$. Now if $|||\cdot|||$ is a unitarily invariant norm corresponding to the symmetric gauge function Φ, define

$$|||A\oplus B|||=\Phi(\{s_j(A)\}\vee\{s_j(B)\}). \tag{13}$$

This idea can be extended to n-tuples of operators in an obvious way. Let us denote $A\oplus\cdots\oplus A$, where the operator A is repeated n times in this direct sum, by $\bigoplus_{n\text{ copies}} A$. It is easy to see that

$$\left\|\bigoplus_{n\text{ copies}} A\right\|_p=n^{1/p}\|A\|_p, \quad 1\leqq p<\infty, \tag{14}$$

and

$$\left\|\bigoplus_{n\text{ copies}} A\right\|_\infty=\|A\|_\infty. \tag{15}$$

3. Proof of the main results and their generalizations

For the sake of brevity we will not repeatedly mention that whenever we use the notations $\|T\|_Q$, $\|T\|_{Q^*}$ or $|||T|||$ we are assuming that the operator T belongs to an ideal $\mathscr{I}_Q$, $\mathscr{I}_{Q^*}$ or $\mathscr{I}_{|||\cdot|||}$ associated with a Q-norm, a Q^*-norm or a unitarily invariant norm respectively.

Proposition 3. *Let $T=[T_{ij}]$ be an operator in an $n\times n$ block-matrix form. Then*

$$\|T\|_Q^2 \leqq \sum_{i,j} \|T_{ij}\|_Q^2, \tag{16}$$

for all Q-norms, and

$$\|T\|_{Q^*}^2 \geqq \sum_{i,j} \|T_{ij}\|_{Q^*}^2, \tag{17}$$

for all Q^*-norms.

Proof. Let R_k denote the matrix obtained by retaining the kth row of the operator matrix $[T_{ij}]$ and replacing all its other rows by zeros. Note that $R_k^* R_m = 0$ if $k \neq m$. So

$$T^*T = \left(\sum_{k=1}^n R_k\right)^* \left(\sum_{k=1}^n R_k\right) = \sum_{k=1}^n R_k^* R_k.$$

Hence, for every unitarily invariant norm

$$|||T^*T||| \leqq \sum_{k=1}^n |||R_k^* R_k|||. \tag{18}$$

Hence, for every Q-norm,

$$\|T\|_Q^2 \leqq \sum_{k=1}^n \|R_k\|_Q^2. \tag{19}$$

Since $|||T||| = |||T^*|||$ for every unitarily invariant norm, we also have

$$\|T\|_Q^2 \leqq \sum_{k=1}^n \|C_k\|_Q^2, \tag{20}$$

where C_k denotes the matrix obtained by retaining the kth column of $[T_{ij}]$ and replacing the other columns by zeros.

Now, first apply (19) and then (20). We get the inequality (16). The inequality (17) follows from (16) by a familiar duality argument as used in [7]. □

The second inequality in (2) and the first inequality in (3) are special cases of the above proposition.

In [10] one of the present authors proved, in the special case when $n=2$, the second of the inequalities (4) and the first of the inequalities (5), using the Clarkson–McCarthy inequalities. It turns out that, suitably reinterpreted, they can be generalised to Q-norms and Q^*-norms respectively, and in proving this more general version the force of the Clarkson–McCarthy inequalities is not needed.

We will use the following Lemma which is an extension of Theorem 1 of [7].

Lemma 4. *Let $A_1,\dots,A_n$ be positive operators. Then for every unitarily invariant norm*

$$\frac{1}{n}|||(\sum A_i)\oplus\cdots\oplus(\sum A_i)||| \leqq |||A_1\oplus\cdots\oplus A_n|||$$
$$\leqq |||(\sum A_i)\oplus 0\cdots\oplus 0|||, \tag{21}$$

where, each of the direct sums involves n summands.

Proof. This can be proved by a slight modification of the proof in [7]. Let X be the $n\times n$ matrix

$$X=\begin{bmatrix} A_1^{1/2} & A_2^{1/2} & \cdots & A_n^{1/2} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then XX^* has as its top left entry $\sum A_i$ and all its other entries are zero. On the other hand X^*X has $A_1, A_2,\dots,A_n$ as its diagonal entries. Since $|||XX^*|||=|||X^*X|||$ and since the norm of the diagonal part of a matrix is always smaller than the norm of the whole matrix [7,9], the second inequality in (21) follows. The first one is proved by noting that the block diagonal matrix each of whose diagonal entries is $\sum_{i=1}^{n} A_i$ can be written as sum of n block diagonal matrices $\operatorname{diag}(A_{\sigma(1)}, A_{\sigma(2)},\dots,A_{\sigma(n)})$ obtained by cyclically permuting the indices $(1,2,\dots,n)$. By unitary invariance each of these n matrices has the same norm. So, the triangle inequality gives the desired result. □

For p-norms, the above Lemma says: if $A_1,\dots,A_n$ are positive operators then for $1\leqq p<\infty$, we have

$$n^{1-p}\left\|\sum_{i=1}^{n} A_i\right\|_p^p \leqq \sum_{i=1}^{n}\|A_i\|_p^p \leqq \left\|\sum_{i=1}^{n} A_i\right\|_p^p. \tag{22}$$

Suppose we are now given n^2 positive operator A_{ij}, $i,j=1,2,\dots,n$. Then by considering the direct sum of the n matrices

$$X_i=\begin{bmatrix} A_{i1}^{1/2} & A_{i2}^{1/2} & \cdots & A_{in}^{1/2} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$i=1,2,\dots,n$ and applying the same argument as above we get

$$|||A_{11}\oplus A_{12}\oplus\cdots\oplus A_{nn}||| \leqq \left|\left|\left|\left(\sum_{j=1}^{n} A_{1j}\right)\oplus\left(\sum_{j=1}^{n} A_{2j}\right)\oplus\cdots\oplus\left(\sum_{j=1}^{n} A_{nj}\right)\oplus 0\cdots\oplus 0\right|\right|\right|, \tag{23}$$

for every unitarily invariant norm. Here it is to be understood that the left hand side involves n^2 direct summands A_{ij} and the right hand side involves the direct sum of n^2 terms the first n of which are the row sums of the matrix $[A_{ij}]$ and the remaining $n^2 - n$ are zero. Using this we can prove:

Proposition 5. *Let T be an operator with block decomposition $T = [T_{ij}]$, $1 \leqq i,j \leqq n$. Then*

$$\| T \|_Q \geqq \left\| \bigoplus_{i,j} T_{ij} \right\|_Q, \tag{24}$$

for all Q-norms; and

$$\| T \|_{Q^*} \leqq \left\| \bigoplus_{i,j} T_{ij} \right\|_{Q^*}, \tag{25}$$

for all Q^*-norms.

Proof. The matrix T^*T has for its diagonal entries the terms $\sum_{j=1}^{n} T_{ji}^* T_{ji}$, $i = 1, 2, \ldots, n$. As already mentioned in the proof of Lemma 4, for every unitarily invariant norm, the diagonal part of a matrix has norm no larger than that of the original matrix. Use this first and then the inequality (23) to get

$$|||T^*T||| \geqq \left|\left|\left| \bigoplus_{i,j} T_{ij}^* T_{ij} \right|\right|\right|. \tag{26}$$

By the definition of Q-norms, (24) follows from (26). By a duality argument we obtain (25) from this. □

Notice that the second inequality in (4) is a special case of (24) while the first inequality in (5) is a special case of (25).

The function $f(t) = t^r$ on the positive half-line is convex when $r \geqq 1$ and concave when $r \leqq 1$. Choose $r = 2/p$. When $p \geqq 2$ the first inequality in (2) follows from the second inequality in (4) by concavity of the map $f(t)$. The second inequality in (3) now follows from this by duality. Equivalently, we could have first derived this latter inequality, using the convexity of the map $f(t) = t^{2/p}$, from the first inequality in (5). Since the dual of $\mathscr{I}_1$ is $\mathscr{B}(H)$ this explains our remark that the inequalities (2) are true for arbitrary operators. Now choose $r = p/2$. The first inequality in (4) then follows from the second one in (2), while the second inequality in (5) follows from the first one in (3).

Both our Theorems 1 and 2 have been established. We now give examples to show that they are sharp.

For each n consider the $n \times n$ matrix T with all entries 1. Then $\| T \|_p = n$, for all p. Let T_{ij} be the ij entry of T, i.e. each $T_{ij} = 1$. Then the first of the inequalities (4) and the second of the inequalities (5) become equalities.

For each n, let $P_1, \ldots, P_n$ be mutually orthogonal rank one projections in the Hilbert space $\mathbb{C}^n$. Let T be a block-matrix operator whose rows are $(P_{\sigma(1)}, P_{\sigma(2)}, \ldots, P_{\sigma(n)})$ where σ varies over the cyclic permutations of $(1, 2, \ldots, n)$. Then T is a unitary matrix of order n^2. Hence $\| T \|_p = n^{2/p}$ for all p. Each of the entries T_{ij} is one of the projections P_k chosen above. So $\| T_{ij} \|_p = 1$, for all i,j and for all

p. This shows that the first inequality in (2) and the second inequality in (3) are sharp.

The remaining inequalities in Theorems 1 and 2 are obviously sharp.

Both these theorems can be extended in various directions; we will examine that in a subsequent paper.

4. The Radon–Riesz property of Q^*-ideals

Let X be a Banach space. We say X has the *Radon–Riesz property* (RRP) if whenever x_n is a sequence in X such that x_n converges weakly to an element x of X and $\|x_n\| \to \|x\|$ then $\|x_n - x\| \to 0$.

Arazy [2, 3] and Simon [12, 13] have obtained several interesting results concerning the RRP for norm ideals. In particular, the Schatten p-ideals have the RRP when $1 \leqq p < \infty$. For $1 < p < \infty$ this is a consequence of the uniform convexity of these ideals. However, the ideal $\mathscr{I}_1$ is not uniformly convex but has the RRP. This follows from the general theory developed by Arazy. However, Arazy [3] also gave an interesting special proof for the ideal $\mathscr{I}_1$ using the first of our inequalities (3) which he proved for $p = 1$. The same idea works for all Q^*-norms as shown below.

We will use the simple fact that in the case of norm ideals the weak Banach space convergence in the definition of RRP may be replaced by weak operator convergence. In other words, a norm ideal $\mathscr{I}$ associated with the unitarily invariant norm $|||\cdot|||$ has the RRP whenever weak operator convergence of a sequence A_n in $\mathscr{I}$ to an element A of $\mathscr{I}$ together with the convergence of $|||A_n|||$ to $|||A|||$ implies $|||A_n - A||| \to 0$, see [3, 13].

Theorem 6. *The ideals $\mathscr{I}_{Q^*}$ corresponding to Q^*-norms all have the Radon–Riesz property.*

Proof. The proof is an imitation of Arazy's proof for the ideal $\mathscr{I}_1$. We give it here for the reader's convenience.

We will use a coordinate-free version of our inequality (17): if P and $P^\perp$ are complementary orthogonal projections in $\mathscr{H}$ then

$$\|T\|_{Q^*}^2 \geqq \|PTP\|_{Q^*}^2 + \|PTP^\perp\|_{Q^*}^2 + \|P^\perp TP\|_{Q^*}^2 + \|P^\perp TP^\perp\|_{Q^*}^2. \tag{27}$$

Let A_n be a sequence in $\mathscr{I}_{Q^*}$ which converges in the weak operator sense to an element A of $\mathscr{I}_{Q^*}$. By Lemma 2(c) of Simon [13] there exists an increasing sequence of finite rank projections P_n with strong limit I such that $P_nA_nP_n$ converges to A in the Q^*-norm topology. (This is true, in general, for all norm ideals). Using the inequality (27) and then the Cauchy–Schwarz inequality we get

$$\|A_n\|_{Q^*}^2 \geqq \|P_nA_nP_n\|_{Q^*}^2 + \tfrac{1}{3}[\|P_nA_nP_n^\perp\|_{Q^*} + \|P_n^\perp A_nP_n\|_{Q^*} + \|P_n^\perp A_nP_n^\perp\|_{Q^*}]^2,$$

for each n. (There is a little error in [3] in this step with the last square missing). Hence, we can write

$$\begin{aligned}\|A_n - A\|_{Q^*} &\leqq \|P_nA_nP_n - A\|_{Q^*} + \|P_nA_nP_n^\perp\|_{Q^*} + \|P_n^\perp A_nP_n\|_{Q^*} + \|P_n^\perp A_nP_n^\perp\|_{Q^*} \\ &\leqq \|P_nA_nP_n - A\|_{Q^*} + [3(\|A_n\|_{Q^*}^2 - \|P_nA_nP_n\|_{Q^*}^2)]^{1/2}.\end{aligned}$$

The right hand side of the above inequality goes to zero as $n \to \infty$. This proves the Theorem. □

We conclude with some remarks:

1. Arazy [3] has proved that the norm ideal $\mathscr{I}_\Phi$ associated with a symmetric gauge function Φ has the RRP iff the corresponding sequence space l_Φ has it. So the sequence spaces corresponding to Q^*-norms also have the RRP. As the examples in Sect. 2 shows, some of these norms can be quite complicated; so a direct proof of the RRP for them may be quite involved.

2. As remarked earlier, the RRP of the ideals $\mathscr{I}_p$ for $1 < p < \infty$ is a consequence of their uniform convexity. This, in turn, follows from the Clarkson–McCarthy inequalities. One part of these is easier to prove – in fact a proof as easy as the ones given here exists [7]. However, this easier part is the one used to derive the uniform convexity of $\mathscr{I}_p$ for $2 \leqq p < \infty$. The uniform convexity of $\mathscr{I}_p$ for $1 < p < 2$ is a consequence of the "harder" Clarkson–McCarthy inequalities usually proved by complex interpolation methods. (See the discussion in [8].) Our proof of Theorem 6 which includes the case of $\mathscr{I}_p$ for $1 \leqq p \leqq 2$ is thus much simpler.

3. By our Theorem 6 and Lemma 1 of Simon [13] each ideal $\mathscr{I}_{Q^*}$ has the following property. If A, B in $\mathscr{I}_{Q^*}$ are such that $\|A\|_{Q^*} = \|B\|_{Q^*}$ and $s_j(A) \leqq s_j(B)$ for all j, then $s_j(A) = s_j(B)$ for all j.

For applications of the contents of this section see [12, p. 40–43].

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References

1. Ando, T., Bhatia, R.: Eigenvalue inequalities associated with the Cartesian decomposition. Linear Multilinear Algebra **22**, 133–147 (1987)
2. Arazy, J.: Basic sequences, embeddings and the uniqueness of the symmetric structure in unitary matrix spaces. J. Funct. Anal. **40**, 302–340 (1981)
3. Arazy, J.: More on convergence in unitary matrix spaces, Proc. Am. Math. Soc. **83**, 44–48 (1981)
4. Bhatia, R.: Some inequalities for norm ideals. Commun. Math. Phys. **111**, 33–39 (1987)
5. Bhatia, R.: Some inequalities for norm ideals II. J. Indian Math. Soc. **50**, 131–138 (1986)
6. Bhatia, R.: Perturbation inequalities for the absolute value map in norm ideals of operators. J. Oper. Theory **19**, 129–136 (1988)
7. Bhatia, R., Holbrook, J.A.R.: On the Clarkson–McCarthy inequalities. Math. Ann. **28**, 7–12 (1988)
8. Fack, T., Kosaki, H.: Generalized s-numbers of τ-measurable operators. Pac. J. Math. **123**, 269–300 (1986)
9. Gohberg, I.C., Krein, M.G.: Introduction to the theory of linear nonselfadjoint operators. Providence RI: American Math. Soc. 1969
10. Kittaneh, F.: On the continuity of the absolute value map in the Schatten classes. Linear Algebra Appl. **118**. 61–68 (1989)
11. Schatten, R.: Norm ideals of completely continuous operators. Berlin Heidelberg New York: Springer 1960
12. Simon, B.: Trace ideals and their applications. Cambridge: Cambridge University Press 1979
13. Simon, B.: Convergence in trace ideals. Proc. Am. Math. Soc. **83**, 39–43 (1981)

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