Math. Ann. 287, 73–105 (1990)

Mathematische Annalen

Completely positive measures and Feller semigroups

Philippe Clément[1] and Jan Prüss[2]

[1] Faculteit voor Wiskunde en Informatica, Technische Universiteit Delft, 2628 BL Delft, The Netherlands
[2] Fachbereich 17 Mathematik und Informatik, Universität-GHS Paderborn, Warburger Strasse 100, D-4790 Paderborn, Federal Republic of Germany

0. Introduction

In recent years it has become apparent that the class of completely positive kernels introduced by Clément and Nohel [8, 9] plays a prominent role in the theory of Volterra equations in Banach spaces. Recall that a kernel $a_1 \in L^1_{\text{loc}}(\mathbb{R}_+)$ is called *completely positive* if there are $k_0 \geqq 0$ and $k_1 \in L^1_{\text{loc}}(\mathbb{R}_+)$ nonnegative and nonincreasing, such that

$$k_0 a_1(t) + \int_0^t a_1(t-\tau) k_1(\tau) d\tau = 1 \ , \quad t > 0 \ . \tag{0.1}$$

An equivalent of this definition is in terms of the solutions $s(t, \mu)$ of the parameter dependent equation

$$s(t) + \mu \int_0^t a_1(t-\tau) s(\tau) d\tau = 1 \ , \quad t, \mu > 0 \ ; \tag{0.2}$$

$a_1(t)$ is completely positive iff $s(t, \mu)$ is nonnegative and nonincreasing in $t > 0$, for each $\mu > 0$. It follows from the results of Friedman [14] that every kernel $a_1(t) > 0$, which is nonincreasing and such that $\log a_1(t)$ is convex, is completely positive; see also Miller [27]. In particular, every completely monotonic function $a_1 \in L^1_{\text{loc}}(\mathbb{R}_+)$ is completely positive, since it is wellknown that such functions are log-convex; this follows also from Reuter's theorem [33].

The concept of complete positivity has been successfully applied to the study of abstract Volterra equations of the form

$$u(t) + \int_0^t a_1(t-\tau) B u(\tau) d\tau = f(t) \ , \quad t > 0 \ , \tag{0.3}$$

where B denotes an m-accretive (linear or nonlinear) operator in a Banach space X; cf. Clément [5], Clément and Nohel [8], Clément and Mitidieri [7], Gripenberg [17], Prüss [29–31]. Equations (0.3) arise in several applications, like the theory of viscoelastic materials (cf. e.g. Prüss [29]), or the theory of heat conduction in materials with memory; cp. Section 6 for the latter. If $f(t)$ is of the form

$$f(t)=x+\int_0^t a_1(t-\tau)g(\tau)d\tau \ , \quad t>0 \ , \tag{0.4}$$

then by virtue of (0.1), (0.3) is equivalent to the problem

$$\frac{d}{dt}\,(k_0u(t)+\int_0^t k_1(t-\tau)u(\tau)d\tau)+Bu(t)=k_1(t)x+g(t), u(0)=x \ . \tag{0.5}$$

This motivates the study of operators A defined by

$$(Au)(t)=\frac{d}{dt}\left(k_0u(t)+\int_0^t k_1(t-\tau)u(\tau)d\tau\right) , \quad t>0 \ , \tag{0.6}$$

in certain function spaces, where $k_0 \geqq 0$, and $k_1 \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is nonnegative and nonincreasing. It has been shown that operators A of the form (0.6) are m-accretive in $L^p(0, T; X)$ and in $L^p(\mathbb{R}_+; X)$, $1 \leqq p < \infty$, where X denotes any Banach space; cf. Clément [5] and Gripenberg [17]. The semigroup $T_p(\tau)$ generated by A admits a representation of the form

$$(T_p(\tau)f)(t)=\int_0^t f(t-s)d_sw(s,\tau) \ , \quad t,\tau>0 \ , \tag{0.7}$$

where the measures $d_tw(\cdot,\tau)$ are nonnegative and finite.

The structure of these measures $d_tw(\cdot,\tau)$ – in particular their supports and their regularity properties – have been studied in the recent paper Prüss [29]. There it was observed that they lead to another characterization of completely positive kernels. Namely, $a_1(t)$ is completely positive iff the transport equation

$$\begin{aligned} &w(t,\tau)+\int_0^t a_1(t-s)\,\frac{\partial}{\partial\tau}\,w(s,\tau)ds=0 \\ &w(t,0)=1, w(0,\tau)=0 \ , \quad t,\tau>0 \end{aligned} \tag{0.8}$$

admits a solution $w(t,\tau)\geqq 0$ which is nondecreasing in t and nonincreasing in τ. $w(t,\tau)$ and the solution $s(t,\mu)$ of (0.2) are related by

$$s(t,\mu)=-\int_0^\infty e^{-\mu\tau}d_\tau w(t,\tau) \ , \quad t,\mu>0 \ , \tag{0.9}$$

i.e. $s(t,\cdot)$ is the Laplace transform of the nonnegative finite measure $-d_\tau w(t,\cdot)$. Prüss [29] contains also another more practical characterization of complete positivity by means of Laplace transforms; namely $a_1(t)$ is completely positive iff the function $\varphi(\lambda)=1/\hat{a}_1(\lambda)$ satisfies

$$\varphi(\lambda)>0 \quad \text{and} \quad \varphi'(\lambda) \quad \text{is completely monotonic for } \lambda>0 \ . \tag{0.10}$$

Observe also that the Laplace transforms of the measures $d_tw(\cdot,\tau)$ are given by

$$h(\lambda,\tau)=\exp(-\tau\varphi(\lambda)) \ , \quad \lambda,\tau>0 \ , \tag{0.11}$$

which yields just another equivalence. $a_1(t)$ is completely positive iff $h(\lambda,\tau)\leqq 1$, and $h(\cdot,\tau)$ is completely monotonic for all $\tau>0$.

Functions $\varphi(\lambda)$ which are subject to (0.10) have already been considered by Bochner [4] and Feller [13], and are called *Bernstein functions* by Berg and Forst [2]. Their relations to infinitely divisible probability measures via (0.11) and to translation invariant positive C_0-semigroups of contractions via (0.7) are well-known; cf. e.g. Hille-Phillips [18], Feller [13] and the more recent monograph of Berg and Forst [2]. Every Bernstein function admits the representation

$$\varphi(\lambda)=1/\widehat{da}(\lambda), \lambda>0 , \tag{0.12}$$

where da is a unique nonnegative measure on $\mathbb{R}_+$. Berg and Forst [2] call this measure a *potential measure*, and in case it is absolutely continuous w.r. to Lebesgue-measure, its Radon-Nikodym derivative $a_1(t)=a'(t)$ is a completely positive kernel as defined above; in the notation of Kingman [21], $a_1(t)$ is the multiple of a *p-standard function*. In the sequel we shall call a measure da on $\mathbb{R}_+$ *completely positive*, if $\varphi(\lambda)$ defined by (0.12) is a Bernstein function.

It is one purpose of this paper to present a unified approach to completely positive measures via Bernstein's theorem, simplifying and extending this way existing proofs. This approach is based on a new representation of Bernstein functions which directly employs k_0 and k_1 appearing in (0.1); this is the main result of Sect. 1. In Sect. 2 we then derive the characterizations of a completely positive measure da in terms of φ, h, s, and also in terms of the complete symbol of (0.8)

$$\sigma(\lambda, \mu)=\frac{1}{\lambda}\cdot\frac{\widehat{da}(\lambda)}{1+\mu\widehat{da}(\lambda)}=\frac{1}{\lambda}\cdot\frac{1}{\mu+\varphi(\lambda)} , \quad \lambda>0 , \tag{0.13}$$

while Sect. 3 is devoted to the study of the solution $w(t, \tau)$ of (0.8), the central subject of the theory. As a first consequence of this approach, we obtain in Sect. 4 a complete description of the domains of the generators of Feller-semigroups $T_p(\tau)$ in $L^p(\mathbb{R})$, $1\leqq p<\infty$, induced by the measures $d_t w(\cdot, \tau)$. This description is in terms of the functions $f\in L^p(\mathbb{R})$ itself, rather than the classical one which is in terms of their Fourier transforms. We also compute the dual semigroups, and study the dependence of $T_p(\tau)$ on the measure $da(t)$. Since these semigroups have a representation in terms of the measures $d_t w(\cdot, \tau)$, it is not difficult to extend them in the "tensor product sense" to the spaces $L^p(\mathbb{R}_+; X)$, where X denotes a general Banach space; this is done in the first part of Sect. 5.

Since for functions f of the form (0.4), (0.3) is equivalent to (0.5), it is natural to look at (0.3) as an equation of the form

$$\mathscr{A}u+\mathscr{B}u=g \tag{0.14}$$

in $L^p(0, T; X)$ or in $L^p(\mathbb{R}_+; X)$. Recently, it has been shown by Dore and Venni [12] that the boundedness of the imaginary powers of $\mathscr{A}$ and $\mathscr{B}$ significantly influence the behaviour of the solutions of (0.14), in particular their regularity. For this reason the second part of Sect. 5 is devoted to the study of the boundedness of $\mathscr{A}^{i\gamma}$, $\gamma\in\mathbb{R}$. Via the method of transference developed by Coifman and Weiss [11], certain results on "tensor product extensions" of positive linear operators, and by means of an abstract multiplier theorem due to McConnell [26], we prove that the negative generator $\mathscr{A}_p$ of the Feller semigroup $\mathscr{T}_p(\tau)$ induced by a completely positive measure admits bounded imaginary powers for $p \in (1, \infty)$ and the estimate

$$|\mathscr{A}_p^{i\gamma}| \leqq C_p(1+|\gamma|^2)e^{|\gamma|\frac{\pi}{2}} , \quad \gamma \in \mathbb{R} \tag{0.15}$$

holds, provided the Banach space X is ζ-convex; see McConnell [26] or Dore and Venni [12] or Prüss and Sohr [32] for brief explanations of this concept. In particular, any space $L^p(\Omega)$, $p \in (1, \infty)$, where $(\Omega, \mathscr{M}, \mu)$ is a σ-finite measure space, is ζ-convex. If X is a Hilbert space and $p=2$, or if $k_1(t)$ is completely monotonic, (0.15) can be sharpened; in fact, we prove that (0.15) holds with $\pi/2$ replaced by

$$\Theta_A = \sup\{|\arg \varphi(\lambda)| : \operatorname{Re} \lambda > 0\} \leqq \frac{\pi}{2} . \tag{0.16}$$

This is important for applications as shown in Sect. 6. It would be interesting to know whether (0.15) with $\pi/2$ replaced by Θ_A always holds, or whether there are counterexamples.

Finally, Sect. 6 is devoted to the equations of linear heat conduction in materials with memory (cf. Nunziato [28])

$$\begin{aligned} & b_0 u_t(t,x) + \frac{\partial}{\partial t} \int_{-\infty}^{t} b_1(t-s)u(s,x)ds \\ & \qquad = c_\infty \Delta u(t,x) + \frac{\partial}{\partial t} \int_{-\infty}^{t} c_1(t-s)\Delta u(s,x)ds + f(t,x) , \\ & \qquad t \in \mathbb{R} , \quad x \in \Omega , \\ & u(t,x) = 0 , \quad t \in \mathbb{R} , \quad x \in \partial\Omega ; \end{aligned} \tag{0.17}$$

here $\Omega \subset \mathbb{R}^n$ denotes a smooth bounded domain, and $u(t,x)$ represents the temperature of the material point $x \in \Omega$ at time t. The numbers $b_0 \geqq 0$, $c_\infty \geqq 0$ and the functions $b_1(t)$, $c_1(t)$ reflect the heat conduction properties of the material under consideration. We rewrite (0.17) as an abstract equation of the form

$$\mathscr{B}u + \mathscr{C}\mathscr{A}u = f$$

in the Banach space $L^p(\mathbb{R}, L^q(\Omega))$, $1 < p, q < \infty$, and apply the results of Dore and Venni [12] and Prüss and Sohr [32] to obtain wellposedness of (0.17). Here the estimates (0.15) and its improvement involving (0.16) are crucial. Up to now there is no other approach to the maximal regularity of the solutions of (0.17) in the $L^p - L^q$ framework; here maximal regularity means that for $f \in L^p(\mathbb{R}, L^q(\Omega))$ we obtain $u \in L^p(\mathbb{R}, W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))$ and some extra time regularity, e.g. $u \in W^{1,p}(\mathbb{R}, L^q(\Omega))$ in case $b_0 > 0$ holds. Note that since $L^p([0,T], L^q(\Omega)) \hookrightarrow L^p(\mathbb{R}_+, L^q(\Omega) \hookrightarrow L^p(\mathbb{R}, L^q(\Omega))$, our results also apply to the initial value problem for (0.17) with initial value $u(0,x)=0$ on finite intervals $[0,T]$ or on the halfline $\mathbb{R}_+$.

1. Bernstein functions

The theorem of Bernstein on completely monotonic functions will play a fundamental role in what follows. Before we state this result, recall the definition of a completely monotonic function on $\mathbb{R}_+$

Definition 1.1. A C^∞-function $f:(0,\infty)\to\mathbb{R}$ is called *completely monotonic* if

$$(-1)^n f^{(n)}(\lambda)\geqq 0 \quad \text{for all } \lambda>0 \quad \text{and} \quad n\in\mathbb{N}_0 \ . \tag{1.1}$$

We shall denote the class of completely monotonic functions by $\mathscr{CM}$.

Bernstein's theorem characterizes completely monotonic functions as Laplace transforms of positive measure supported on $\mathbb{R}_+$.

Theorem 1.2 (Bernstein). *A function $f:(0,\infty)\to\mathbb{R}$ is completely monotonic if and only if there exists a unique function $g:[0,\infty)\to\mathbb{R}$, nondecreasing, left-continuous, with $g(0)=0$ such that*

$$f(\lambda)=\int_0^\infty e^{-\lambda t}dg(t) \ , \quad \lambda>0 \ . \tag{1.2}$$

Moreover

$$(-1)^n f^{(n)}(\lambda)=\int_0^\infty t^n e^{-\lambda t}dg(t) \ , \quad \lambda>0 \ , \quad n\in\mathbb{N}_0 \tag{1.3}$$

and

$$(-1)^n f^{(n)}(0^+)=\int_0^\infty t^n dg(t) \ , \quad n\in\mathbb{N}_0 \ . \tag{1.4}$$

In the sequel we shall denote by $BV(\mathbb{R}_+)$ the space of functions of bounded variation on $\mathbb{R}_+$ normalized by $g(0)=0$ and left-continuity, by $BV_{\mathrm{loc}}(\mathbb{R}_+)$ the space of functions which are of bounded variation on each compact interval $J\subset\mathbb{R}_+$ and normalized in the same way. We shall use the notation

$$\widehat{dg}(\lambda)=\int_0^\infty e^{-\lambda t}dg(t) \ , \quad \lambda>0 \tag{1.5}$$

for $g\in BV_{\mathrm{loc}}(\mathbb{R}_+)$, whenever the integral exists in the Lebesgue-Stieltjes sense, and similarly

$$\hat{k}(\lambda)=\int_0^\infty e^{-\lambda t}k(t)dt \ , \quad \lambda>0 \ , \tag{1.6}$$

whenever $k\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

It is wellknown that the class $\mathscr{CM}$ is closed under pointwise addition, multiplication and convergence; see [35]. However, it is not true that the composition of two functions $f, g\in\mathscr{CM}$ belongs to $\mathscr{CM}$ again, in general. For the composition, the class of Bernstein functions appears naturally.

Definition 1.3 (Berg-Forst). A C^∞-function $\varphi:(0,\infty)\to\mathbb{R}$ is called a *Bernstein function* if φ is positive and φ' is completely monotonic.

Observe that Berg and Forst [2, p. 61] allow φ to be nonnegative, but the only function we exclude in our definition is the function $\varphi\equiv 0$.

Proposition 1.4 (Feller). *Let f be a completely monotonic function and φ, ψ be Bernstein functions. Then*

(i) *$f \circ \varphi$ is completely monotonic.*

(ii) *$\psi \circ \varphi$ is a Bernstein function.*

This proposition is taken from [13, p. 441], for a converse, see [4]. It is easy to verify that the class of Bernstein functions is closed under pointwise addition, multiplication with positive numbers and pointwise convergence. From Bernstein's theorem one can deduce the following representation of Bernstein functions.

Theorem 1.5. *A function φ is a Bernstein function if and only if there exist unique nonnegative constants α, β and a unique nonnegative measure γ on $(0, \infty)$, not all zero, satisfying $\int_0^\infty \frac{s}{1+s}\, d\gamma(s) < \infty$ such that*

$$\varphi(\lambda) = \alpha + \beta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda s})\, d\gamma(s) \ , \quad \lambda > 0 \ . \tag{1.7}$$

For a proof of this theorem see for example [13]. This representation is not quite appropriate for our purposes; however, a slight modification of it appears to be of central importance in the following sections.

Theorem 1.6. *A function φ is a Bernstein function iff there exist unique nonnegative constants k_0, k_∞ and a unique nonnegative, nonincreasing, left-continuous function $k_1 : (0, \infty) \to \mathbb{R}$ satisfying $\lim_{t\to\infty} k_1(t) = 0$ and $\int_0^1 k_1(t)dt < \infty$, not all identically zero, such that*

$$\varphi(\lambda) = \lambda\left(k_0 + \frac{k_\infty}{\lambda} + \hat{k}_1(\lambda)\right) \ , \quad \textit{for all } \lambda > 0 \ . \tag{1.8}$$

We give a proof of Theorem 1.6 which is based on Bernstein's theorem only, although it is possible to derive it from Theorem 1.5.

Proof. Suppose $\varphi(\lambda)$ has the form (1.8) where k_0, k_∞ and k_1 are as in the Theorem. Then $\varphi(\lambda)$ is well defined for $\lambda > 0$, is positive and belongs to C^∞. To prove that φ is a Bernstein function, it is therefore sufficient to show that $f(\lambda) := (\lambda\hat{k}_1)'(\lambda)$ is completely monotonic. Assume first $k_1(0^+) < \infty$. Then by setting $k_1(0) = 0$ we have $k_1 \in BV(\mathbb{R}_+)$, hence

$$l(t) := -\int_0^t s\,dk_1(s) = -\int_{0^+}^t s\,dk_1(s)$$

is nondecreasing since k_1 is nonincreasing on $(0, \infty)$. We obtain

$$f(\lambda) = (\widehat{dk_1})'(\lambda) = -\widehat{t\,dk_1}(\lambda) = \widehat{dl}(\lambda)$$

which is completely monotonic by Theorem 1.2. If $k_1(0^+) = \infty$, we approximate k_1 by $k_{1,\varepsilon}(t) := k_1(t+\varepsilon)$, $\varepsilon > 0$, let $\varphi_\varepsilon(\lambda) = \lambda\left(k_0 + \frac{k_\infty}{\lambda} + \hat{k}_{1,\varepsilon}(\lambda)\right)$. Then φ_ε is a Bernstein function and $\lim_{\varepsilon\downarrow 0} \varphi_\varepsilon(\lambda) = \varphi(\lambda)$, for every $\lambda > 0$, since

$$\hat{k}_{1,\varepsilon}(\lambda) = e^{\lambda\varepsilon}\hat{k}_1(\lambda) - \int_0^\varepsilon e^{\lambda(\varepsilon - t)} k_1(t)dt \ .$$

Conversely, assume that φ is a Bernstein function. Then

$$k_\infty := \varphi(0^+) \tag{1.9}$$

exists, is nonnegative and finite. Subtracting k_∞ we may assume w.l.o.g. that $k_\infty = 0$. Since φ is positive and concave

$$k_0 := \lim_{\lambda\to\infty} \frac{\varphi(\lambda)}{\lambda} = \inf_{\lambda>0} \frac{\varphi(\lambda)}{\lambda} \tag{1.10}$$

exists, is nonnegative and finite, and so we may also assume $k_0 = 0$. Next observe that $\frac{\varphi(\lambda)}{\lambda}$ is also completely monotonic since $\varphi(\lambda) = \lambda \int_0^1 \varphi'(t\lambda)dt$, $\lambda > 0$. Thus by Bernstein's theorem, there exists $k \in BV_{\text{loc}}(\mathbb{R}_+)$, nondecreasing with $k(0^+) = 0$, such that $\varphi(\lambda) = \lambda \widehat{dk}(\lambda)$. On the other hand, since φ' is completely monotonic, we have $\varphi'(\lambda) = \widehat{dl}(\lambda)$ for some $l \in BV_{\text{loc}}(\mathbb{R}_+)$ nondecreasing. Then

$$\frac{\hat{l}}{\lambda} = \frac{\varphi'}{\lambda^2} = \frac{1}{\lambda^2}(\lambda^2\hat{k})' = 2\frac{\hat{k}}{\lambda} + \hat{k}' = 2\frac{\hat{k}}{\lambda} - \widehat{tk}$$

implies

$$tk(t) = 2\int_0^t k(s)ds - \int_0^t l(s)ds\ , \qquad t > 0\ . \tag{1.11}$$

From this identity and $k(0^+) = 0$, it follows that $k \in W^{1,1}[0,T]$, for every $T > 0$; we define

$$k_1 := k'\ . \tag{1.12}$$

This defines k_1 a.e. only, however differentiating (1.11) we obtain

$$tk_1(t) = k(t) - l(t)\ , \qquad t > 0 \tag{1.13}$$

which defines k_1 everywhere as a left-continuous function, and in addition we have $tk_1 \in BV_{\text{loc}}(\mathbb{R}_+)$; in particular $k_1 \in BV[\varepsilon, T]$, for every $0 < \varepsilon < T < \infty$.

We now show that k_1 is nonincreasing. For this purpose, let $p \in \mathscr{C}(0,\infty)$ be nonnegative with compact support; we then have

$$0 \leq \int_0^\infty p(t)dl(t) = \int_0^\infty p(t)dk(t) - \int_0^\infty p(t)d(tk_1(t)) = -\int_0^\infty p(t)tdk_1(t)\ .$$

Since p has been arbitrary, and the space of continuous functions with compact support in $(0,\infty)$ is dense in $\mathscr{C}(0,\infty)$ w.r. to the compact-open topology this implies that $k_1(t)$ is nonincreasing. Finally, we have

$$\lim_{t\to\infty} k_1(t) = \lim_{\lambda\downarrow 0} \lambda\hat{k}_1(\lambda) = \lim_{\lambda\downarrow 0} \varphi(\lambda) = 0. \quad \square$$

Later on, we shall need a decomposition of $k_1(t)$ into one part $k_2(t)$ which is integrable and another part $k_3(t)$ which is of bounded variation on $\mathbb{R}_+$. We choose the following

$$k_1(t) = k_2(t) + k_3(t)\ , \qquad t > 0\ , \tag{1.14}$$

where

$$k_2(t) = \max(k_1(t) - k_1(1), 0)\ , \qquad t > 0\ , \tag{1.15}$$

and

$$k_3(t) = \min(k_1(t), k_1(1)) , \qquad t>0 . \tag{1.16}$$

Then representation (1.8) becomes

$$\varphi(\lambda) = \lambda k_0 + \lambda \hat{k}_2(\lambda) + k_\infty + \widehat{dk}_3(\lambda) , \qquad \lambda > 0 . \tag{1.17}$$

As a direct consequence of (1.17) we obtain

Corollary 1.7. *Let φ be a Bernstein function. Then φ admits a continuous extension to $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqq 0\}$ which is holomorphic on the interior of $\mathbb{C}_+$.*

2. Completely positive measures

Let φ be a Bernstein function. Then by Proposition 1.4, $1/\varphi$ is completely monotonic, therefore by Bernstein's theorem, there exists a function $a \in BV_{loc}(\mathbb{R}_+)$ such that $\widehat{da} = 1/\varphi$. On the other hand, by Theorem 1.6, $\varphi(\lambda) = \lambda \widehat{dk}(\lambda)$, where

$$k(t) = k_0 + k_\infty t + \int_0^t k_1(s)ds , \qquad t > 0 \tag{2.1}$$

is positive, nondecreasing and concave. Thus $\widehat{da}(\lambda)\widehat{dk}(\lambda) = \frac{1}{\lambda}$, $\lambda > 0$, which implies that the measure da is a solution to the Volterra integral equation

$$\int_0^t k(t-s)da(s) = t , \qquad t > 0 . \tag{2.2}$$

Conversely, if $k : (0, \infty) \to \mathbb{R}$ is positive, nondecreasing and concave, then k can be written in the form (2.1). Indeed $k_0 := k(0^+)$ which exists and is nonnegative; $k_\infty := \lim_{t\to\infty} \frac{k(t)}{t} = \inf_{t>0} \frac{k(t)}{t}$ which exists and is nonnegative since k is positive and concave. Then $k(t) - k_0 - k_\infty t$ is the primitive of a nondecreasing function k_1, which can be choosen left-continuous, and is such that $\lim_{t\to\infty} k_1(t) = 0$. By Theorem 1.6, $\varphi(\lambda)$ defined by (1.8) is a Bernstein function. Therefore, the measure da with Laplace-transform $\widehat{da}(\lambda) = 1/\varphi(\lambda)$ is a solution to (2.2). Observe that da is the only measure solution of (2.2). Indeed, integrating twice the equation,

$$\int_0^t k(t-s)db(s) = 0 , \qquad t > 0 , \tag{2.3}$$

where $b \in BV_{loc}(\mathbb{R}_+)$, and using Titchmarsh's theorem [36], we see that the only solution of (2.3) is $db = 0$. We summarize this in the following theorem which is due to Gripenberg [16].

Theorem 2.1. *Let $k : (0, \infty) \to \mathbb{R}$ be a positive, nondecreasing concave function. Then there exists a unique function $a \in BV_{loc}(\mathbb{R}_+)$ satisfying (2.2). Moreover, the function a is nondecreasing and satisfies $\widehat{da}(\lambda) < \infty$ for $\lambda > 0$.*

Remarks 2.2. (i) In case $k_0 > 0$ or equivalently $k(0^+) > 0$, (2.2) reduces to the simple Volterra equation of second kind

$$k_0 a_1 + (k_\infty + k_1) * a_1 = 1 \ , \qquad t > 0 \ , \tag{2.4}$$

which has a unique solution in $W^{1,1}_{loc}(\mathbb{R}_+)$ and therefore $a(t) = \int_0^t a_1(s)ds$ is the unique solution of (2.2). The positivity of a_1 was observed first by Friedman [14]; see also Levin [24]. Reuter [33] proved that a_1 is even completely monotonic if k_1 has this property.

(ii) The case where $k_0 = 0$ has been obtained in Gripenberg [16] where Fourier transform arguments are used. When $k_1(0^+) < \infty$, the measure da has a jump part and when $k_1(0^+) = \infty$, da has no jump part, but the singular part may still be nontrivial as is shown in [16].

In case the solution of (2.2) is absolutely continuous, $a_1 = a'$ is called a completely positive function in Clément and Nohel [9]. More generally we introduce

Definition 2.3. Let $a \in BV_{loc}(\mathbb{R}_+)$. The measure da is called *completely positive* if there is a function $k : (0, \infty) \to \mathbb{R}$, positive, nondecreasing and concave such that (2.2) is satified.

Observe that a completely positive measure is Laplace-transformable for $\lambda > 0$, $\widehat{da}(\lambda)$ is positive and completely monotonic. In the sequel we shall need the functions $s(t, \mu)$, $\mu \in \mathbb{C}$, which are defined as the solutions in $BV_{loc}(\mathbb{R}_+)$ of the equation

$$s(t) + \mu \int_0^t s(t-\tau) da(\tau) = 1 \ , \qquad t > 0 \ , \qquad \mu > 0 \ . \tag{2.5}$$

The main result of this section is the following theorem which contains several characterizations of completely positive measures.

Theorem 2.4. *Suppose $a \in BV_{loc}(\mathbb{R}_+)$ is Laplace transformable and such that $\widehat{da}(\lambda) > 0$ for $\lambda > 0$. Then the following assertions are equivalent:*

(i) *da is completely positive;*

(ii) *$\varphi(\lambda) = 1/\widehat{da}(\lambda)$ is a Bernstein function, i.e.* $\dfrac{-\widehat{da}'(\lambda)}{\widehat{da}^2(\lambda)}$ *is completely monotonic;*

(iii) *$\psi_\tau(\lambda) = e^{-\tau/\widehat{da}(\lambda)}$ is completely monotonic for every $\tau > 0$;*

(iv) *$\varphi_\mu(\lambda) = \dfrac{\widehat{da}(\lambda)}{1 + \mu\widehat{da}(\lambda)}$ is completely monotonic for every $\mu > 0$;*

(v) *the solution $s(t, \mu)$ of (2.5) is positive and nonincreasing for every $\mu > 0$.*

Remarks 2.5. (i) The equivalence between (ii), (iii), (iv) is well-known; see [13, 2]. A measure da satisfying (iii) is called a *potential kernel* in Berg and Forst [2], and if it is absolutely continuous, it's derivative $a_1 = a'$ is equivalent with a multiple of a *p-standard function* (see Kingman [21]).

(ii) For the case of absolutely continuous a, the equivalence (i) and (v) was observed in Clément and Nohel [9]; see Clément and Mitidieri [7].

(iii) Property (ii) is used in an essential way in Prüss [29–31] to establish various properties of solutions of abstract Volterra equations in Banach spaces.

Proof of Theorem 2.4. (i)$\Leftrightarrow$(ii) has already been established.

(ii)$\Rightarrow$(iii) If φ is a Bernstein function, then by Proposition 1.4. $e^{-\tau\varphi}\in\mathscr{CM}$.

(iii)$\Rightarrow$(iv) If $e^{-\tau\varphi}\in\mathscr{CM}$ for each $\tau>0$, then since φ is positive, we have

$$\frac{\widehat{da}(\lambda)}{1+\mu\widehat{da}(\lambda)}=\frac{1}{\mu+\varphi(\lambda)}=\int_0^\infty e^{-\tau\varphi(\lambda)}e^{-\mu\tau}d\tau\ ,\qquad \mu>0\ ,$$

hence this function belongs to $\mathscr{CM}$ for each $\mu>0$.

(iv)$\Rightarrow$(ii) Since $\varphi\in C^\infty$ and is positive, it is sufficient to prove $\varphi'\in\mathscr{CM}$. We have $\frac{1}{\mu+\varphi}\in\mathscr{CM}$ for every $\mu>0$, hence $\frac{\mu^2\varphi'}{(\mu+\varphi)^2}\in\mathscr{CM}$ for every $\mu>0$ and by taking the limit as $\mu\to\infty$, we obtain $\varphi'\in\mathscr{CM}$.

(ii)$\Rightarrow$(v) For every $\mu>0$, $\frac{1}{\varphi+\mu}\in\mathscr{CM}$ by Proposition 1.4. By Bernstein's theorem there exists $r_\mu\in BV_{\text{loc}}(\mathbb{R}_+)$, nondecreasing such that $\widehat{dr}_\mu(\lambda)=\frac{1}{\varphi(\lambda)+\mu}$. Note that $r_\mu\in BV(\mathbb{R}_+)$ since $r_\mu(\infty)=\frac{1}{\varphi(0^+)+\mu}\leqq\frac{1}{\mu}$. Set $s_\mu(t)=1-\mu r_\mu(t)$. Then clearly s_μ is nonnegative and nonincreasing. We claim that $s_\mu(t)=s(t,\mu)$. Indeed, by using $\varphi=1/\widehat{da}$, $\hat{s}_\mu(\lambda)=\frac{1}{\lambda}-\mu\hat{r}_\mu(\lambda)=\frac{1}{\lambda}(1-\mu\widehat{dr}_\mu(\lambda))=\frac{1}{\lambda}\cdot\frac{1}{1+\mu\widehat{da}(\lambda)}$. Thus s_μ satisfies (2.5) and by uniqueness $s_\mu(t)=s(t,\mu)$.

(v)$\Rightarrow$(ii) Since s_μ is positive, nonincreasing, by Theorem 1.6, $\varphi_\mu(\lambda)=\lambda\mu\hat{s}_\mu(\lambda)$ is a Bernstein function for every $\mu>0$. On the other hand,

$$\varphi_\mu(\lambda)=\frac{\mu}{1+\mu\widehat{da}(\lambda)}=\frac{\mu\varphi(\lambda)}{\mu+\varphi(\lambda)}\to\varphi(\lambda)\ ,$$

as $\mu\to\infty$. Hence $\varphi(\lambda)=\frac{1}{\widehat{da}(\lambda)}$ is a Bernstein function. □

Corollary 2.6. *Let da be completely positive and let $s(t,\mu)$ denote the solution of (2.5), $\mu>0$. Then $s(\cdot,\mu)\in BV(\mathbb{R}_+)$ and satisfies also the equation*

$$\mu s(t,\mu)dt+(dk*ds(\cdot,\mu))(t)=dk(t)\ ,\qquad t>0\ ,\qquad \mu>0\ . \tag{2.6}$$

The function $\varphi_\mu(\lambda)=\lambda\hat{s}(\lambda,\mu)$ is a Bernstein function. Define $r(t,\mu)$ by

$$r(t,\mu)=\mu^{-1}(1-s(t,\mu))\ ,\qquad \mu>0\ ,\qquad t\geqq 0\ . \tag{2.7}$$

Then $r(\cdot,\mu)\in BV(\mathbb{R}_+)$, is nondecreasing, $r(0+,\mu)=0$ and $r(\cdot,\mu)$ satisfies

$$\mu r(t,\mu)+\frac{d}{dt}(dk*r(\cdot,\mu))(t)=1\ ,\qquad t>0\ ,\qquad \mu>0 \tag{2.8}$$

and

$$\widehat{dr}(\lambda,\mu)=\frac{\widehat{da}(\lambda)}{1+\mu\widehat{da}(\lambda)}\ ,\qquad \lambda,\mu>0\ . \tag{2.9}$$

The measures $dr(\cdot,\mu)$ are completely positive.

Remarks 2.7. (i) Observe that if da is completely positive and $a_0>0$ then $a_0\delta_0+da$ is also completely positive, where δ_0 denotes the Dirac measure at 0. This is immediate from Theorem 2.4 (v); see also Berg and Forst [2].

(ii) If a is positive, increasing and concave, then it follows from Remark 2.2 (i) that $s(t,\mu)\geqq 0$ for every $\mu, t>0$. Replace k by a and a_1 by s in (2.4) to see this.

(iii) Concerning the decreasingness of s, when $a(t)$ is absolutely continuous ($da=a_1\,dt$), it follows from a result of Friedman [14], see also Miller [27], that if a_1 is positive and $\log a_1$ is convex, then $s(\cdot,\mu)$ is decreasing for every $\mu>0$. Therefore the class of positive, decreasing and log-convex locally integrable kernels is contained in the class of completely positive kernels. It was proven by Hirsch [19] that this class is contained in the class of potential kernels which are by Theorem 2.5 also completely positive. Note that completely monotonic kernels are positive, decreasing, and log convex, hence completely positive whenever they are locally integrable. This is also a consequence of Reuter's theorem [33].

(iv) In contrast to the class of positive, decreasing log-convex functions, the class of completely positive measures is not closed under addition.

We conclude this section with a corollary which reformulates Proposition 1.4 in terms of completely positive measures.

Corollary 2.8. *Suppose da is a completely positive measure and $b\in BV_{\mathrm{loc}}(\mathbb{R}_+)$ is Laplace transformable and nondecreasing. Then there is a unique nonnegative Laplace transformable measure dc such that*

$$\widehat{dc}(\lambda)=\widehat{db}(1/\widehat{da}(\lambda)) , \quad \lambda>0 . \tag{2.10}$$

If moreover db is completely positive, then dc is completely positive as well.

Remark 2.9. Corollary 2.8 is called *subordination principle* in Feller [13], and Berg and Forst [2] and *chain rule* in Prüss [31], where this principle is used to derive existence and regularity for resolvents of Volterra equations in Banach spaces from semigroups or cosine families.

3. Translation invariant Feller semigroups

Let da be a completely positive measure. By Corollary 2.8, there exist functions $w(\cdot,\tau)\in BV(\mathbb{R}_+)$, nondecreasing and less than one, for $\tau\geqq 0$, such that

$$\widehat{dw}(\lambda,\tau)=e^{-\tau/\widehat{da}(\lambda)} , \quad \lambda>0 \qquad \tau\geqq 0 . \tag{3.1}$$

From the relation

$$e^{-(\tau_1+\tau_2)/\widehat{da}}=e^{-\tau_1/\widehat{da}}\cdot e^{-\tau_2/\widehat{da}}$$

it follows that dw satisfies the semigroup property

$$\int_0^t w(t-s,\tau_1)\,dw(s,\tau_2)=w(t,\tau_1+\tau_2) , \quad t\geqq 0 , \quad \tau_1,\tau_2\geqq 0 \tag{3.2}$$

and

$$w(t,0)=1 , \quad t>0 . \tag{3.3}$$

These measures $\{dw(\cdot,\tau)\}_{\tau\geq 0}$ give rise to a semigroup of operators $\{T_0(\tau)\}_{\tau\geq 0}$ on $C_0(\mathbb{R})$ *defined by*

$$(T_0(\tau)f)(t)=\int_0^\infty f(t-s)dw(s,\tau)\ , \qquad t\in\mathbb{R}\ , \qquad \tau\geqq 0\ , \tag{3.4}$$

where $f\in C_0(\mathbb{R})$.

It is well-known (see Hille-Phillips [18]), that the semigroup $\{T_0(\tau)\}_{\tau\geq 0}$ is strongly continuous for $\tau\geqq 0$ on $C_0(\mathbb{R})$ equipped with the sup-norm and enjoys the following properties:

(F1) $T_0(\tau)$ are contractions for all $\tau\geqq 0$.

(F2) $T_0(\tau)$ are positive for all $\tau\geqq 0$ with respect to the standard cone $C_0^+(\mathbb{R})$ in $C_0(\mathbb{R})$.

(F3) $T_0(\tau)$ commutes with the group of translations on $\mathbb{R}$.

Such semigroups are called *translation invariant Feller semigroups* in Berg and Forst [2]. Since the support of the measures $dw(\cdot,\tau)$ is contained in $\mathbb{R}_+$, we have the additional property

(F4) $(T_0(\tau)f)(t)=0$ for all $\tau\geqq 0$ and $t\leqq 0$ whenever $f(t)=0$ for all $t\leqq 0$.

It is also known that the converse is true, namely that every translation invariant Feller semigroup on $C_0(\mathbb{R})$ satisfying (F4) is of the form (3.4) where the measures $dw(\cdot,\tau)$ satisfy (3.1) for some Bernstein function $\varphi=1/\widehat{da}$, i.e. for some completely positive measure da, thanks to Theorem 2.4 (see Berg and Forst [2]).

Next we consider the map $w:\mathbb{R}_+\to BV(\mathbb{R}_+)$ defined by $w(\tau)=w(\cdot,\tau)$, $\tau\geqq 0$. The space $BV(\mathbb{R}_+)$ becomes a commutative Banach algebra with unit e, where the multiplication $\bullet$ is defined by

$$(a\bullet b)(t)=\int_0^t a(t-s)db(s)\ , \qquad t\geqq 0 \tag{3.5}$$

and the norm is given by

$$\|a\|=\mathrm{Var}\,[a;\mathbb{R}_+] \tag{3.6}$$

see Gel'fand et al. [15].

The identities (3.2) and (3.3) show that w forms a semigroup in $BV(\mathbb{R}_+)$, and $\|w(\tau)\|\leqq 1$, for every $\tau\geqq 0$. We denote by $BV^+(\mathbb{R}_+)$ the closed convex cone of nondecreasing functions in $BV(\mathbb{R}_+)$; then $w(\tau)\in BV^+(\mathbb{R}_+)$, for every $\tau\geqq 0$. Consider $BV(\mathbb{R}_+)$ as a closed subspace of $BV(\mathbb{R}):=\{v:\mathbb{R}\to\mathbb{R}|v$ of bounded variation, left-continuous, $v(-\infty)=\lim_{t\to-\infty}v(t)=0\}$, equipped with the variation norm, by extending $w\in BV(\mathbb{R}_+)$ by 0 to $\mathbb{R}_-$. Then $BV(\mathbb{R}_+)$ is a subspace of the dual $C_0(\mathbb{R})^*$ and therefore inherits the weak*-topology of $C_0(\mathbb{R})^*$. Then the semigroup w is weak*-continuous on $\mathbb{R}_+$. On the other hand, we also have the identity

$$\frac{\partial}{\partial\tau}(\widehat{da}\,e^{-\tau/\widehat{da}})+e^{\tau/\widehat{da}}=0\ , \qquad \tau>0\ ,$$

from which by (3.1) and dividing by λ, we obtain

$$\begin{aligned}&\frac{\partial}{\partial\tau}(\widehat{da}\cdot\hat{w}(\tau))+\hat{w}(\tau)=0\ , \qquad \tau>0\ , \qquad \lambda>0\\ &\hat{w}(0)=1/\lambda\ , \qquad \lambda>0\ .\end{aligned} \tag{3.7}$$

Defining $e_\lambda(t) := e^{-\lambda t}$, $\lambda > 0$, $t \geqq 0$, and assuming first $a \in BV(\mathbb{R}_+)$, identity (3.7) becomes after an integration

$$\langle a \bullet w(\tau), e_\lambda \rangle + \int_0^\tau \langle w(\sigma), e_\lambda \rangle d\sigma = \langle a, e_\lambda \rangle \ , \quad \tau \geqq 0 \ \text{ and } \ \lambda > 0 \tag{3.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $BV(\mathbb{R})$ and $C_0(\mathbb{R})$. As we will show below, this implies that $a \bullet w$ is weakly*-continuously differentiable and

$$\begin{aligned} \text{weak*} \ \frac{d}{d\tau} (a \bullet w)(\tau) + w(\tau) = 0 \ , \quad \tau \geqq 0 \\ w(0) = e \ . \end{aligned} \tag{3.9}$$

In case $a \in BV_{\text{loc}}(\mathbb{R}_+)$ only, one can also define the multiplication (3.5) for $a, b \in BV_{\text{loc}}(\mathbb{R}_+)$, and (3.9) still makes sense in the topology generated by the continuous functions with compact support $C_c(\mathbb{R})$. It turns out that the existence of a solution in $BV^+(\mathbb{R}_+)$ of (3.9) such that $\|w(\tau)\| \leqq 1$, $\tau \geqq 0$, gives another characterization of completely positive measures. This is the content of the next theorem. In order to state it we need the following definition.

Let $a \in BV_{\text{loc}}(\mathbb{R}_+)$, and consider (3.9).

Definition 3.1. A function $w: \mathbb{R}^+ \to BV(\mathbb{R}^+)$ is called a *solution* of (3.9) if for each $p \in C_c(\mathbb{R})$, the function $\varphi(\tau) = \langle w(\tau), p \rangle$ is continuous, $\psi(\tau) = \langle a \bullet w(\tau), p \rangle$ is continuously differentiable and

$$\begin{aligned} \frac{d}{d\tau} \langle a \bullet w(\tau), p \rangle + \langle w(\tau), p \rangle = 0 \ , \quad \tau \geqq 0 \\ w(0) = e \ . \end{aligned} \tag{3.10}$$

We have

Theorem 3.2. *Let $a \in BV_{\text{loc}}(\mathbb{R}_+)$ be Laplace transformable. Then da is completely positive if and only if (3.9) posesses a solution $w \in BV^+(\mathbb{R}^+)$ satisfying $\|w(\tau)\| \leqq 1$, for all $\tau \geqq 0$. If this is the case, the solution is unique.*

Proof. Necessity: We only have to prove that $w(\tau)$ satisfies (3.10). For this purpose we rewrite (3.8) as

$$\langle e_\varepsilon(a \bullet w(\tau)), p \rangle + \int_0^\tau \langle e_\varepsilon w(\sigma), p \rangle d\sigma = \langle e_\varepsilon a, p \rangle \ , \quad \tau > 0 \ , \tag{3.11}$$

where $p = e_\lambda$ for every $\lambda > 0$, and $\varepsilon > 0$ is fixed. Here we used the notation

$$(e_\varepsilon a)(t) := \int_0^t e_\varepsilon(s) da(s) \ , \quad t \geqq 0 \ .$$

Since the exponential polynomials are dense in $C_c(\mathbb{R})$, we have (3.11) with $p \in C_c(\mathbb{R})$, hence also (3.10).

Sufficiency: Let w be a solution of (3.9) and let $a \in BV_{\text{loc}}$ be Laplace transformable. We can rewrite (3.10) as (3.11) with $p \in C_c(\mathbb{R})$, for every $\varepsilon > 0$. Since $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, (3.11) also holds for $p = e_\lambda$, $\lambda > 0$, hence (3.8) holds for every $\lambda > 0$. From the convolution theorem for the Laplace transform we obtain

$$\widehat{da}\cdot\widehat{dw}(\sigma)+\int_0^\tau \widehat{dw}(\sigma)d\sigma=\widehat{da}$$

for every $\lambda>0$. Set $\alpha=\widehat{da}(\lambda)$ and $\varphi(\tau)=\widehat{dw}(\tau)$. Then we have $\alpha\varphi'(\tau)+\varphi(\tau)=0$, $\varphi(0)=1$, and therefore, $\alpha\neq 0$ by continuity of φ and $\varphi(\tau)=e^{-\tau/\alpha}$, $\tau\geqq 0$. Assuming $\|w(\tau)\|\leqq 1$, for all $\tau\geqq 0$, we obtain $\alpha>0$. Assuming also $w\in BV^+(\mathbb{R}_+)$, we have $e^{-\tau/\widehat{da}}\in\mathscr{CM}$ for all $\tau>0$, by Bernstein's theorem, hence by Theorem 2.4 da is completely positive. Since $\widehat{da}(\lambda)>0$, w is only solution. □

We explicit some properties of the solution w in the next corollary.

Corollary 3.3. *Let da be completely positive, and let w denote the solution of* (3.9). *Then*

(i) *$w(t,\tau):=w(\tau)(t)$ is Borel measurable on $\mathbb{R}_+\times\mathbb{R}_+$.*

(ii) *$w(t,\tau)$ is nondecreasing in t and*

$$\lim_{t\to\infty} w(t,\tau)=e^{-\tau/a(\infty)}\ ,\qquad \lim_{t\to 0+} w(t,\tau)=e^{-\tau/a(0^+)}\ ,\qquad \tau>0$$

(iii) *$w(t,\tau)$ is nonincreasing and right-continuous in τ and*

$$\lim_{\tau\to\infty} w(t,\tau)=0\ ,\qquad \lim_{\tau\to 0} w(t,\tau)=1\ ,\qquad t>0\ .$$

(iv) *If $s(t,\mu)$ denotes the solution of* (2.5), *then*

$$s(t;\mu)=-\int_0^\infty e^{-\mu\tau}d_\tau w(t,\tau)\ ,\qquad t,\mu>0\ , \tag{3.12}$$

in particular $s(t;\mu)$ is completely monotonic w.r. to $\mu>0$.

Proof. (i) follows from the Post-Widder inversion formula for the Laplace transform by virtue of

$$w(t,\tau)=\lim_{h\to 0} w(t-h,\tau)=\lim_{h\to 0}\lim_{n\to\infty}\frac{1}{n!}(-d/d\lambda)^n\widehat{dw}\left(\frac{n}{t-h},\tau\right)\ ,$$

which holds for every $t,\tau>0$; cf. Widder [35].

(ii) follows from Bernstein's theorem.

(iii) The semigroup property (3.2) implies

$$w(t,\tau_1+\tau_2)=\int_0^t w(t-s,\tau_1)dw(s,\tau_2)\leqq\int_0^t dw(s,\tau_2)=w(t,\tau_2)\ , \tag{3.13}$$

for all $t,\tau_1,\tau_2\geqq 0$, since $w(\cdot,\tau)$ is nonnegative, nondecreasing and less than one. Therefore $w(t,\cdot)$ is nonincreasing for $t\geqq 0$. On the other hand, since $dw(\cdot,h)\to\delta_0(\cdot)$ as $h\to 0$ and $w(\cdot,\tau)$ is left-continuous, we obtain

$$w(t,\tau+h)=\int_0^t w(t-s,\tau)dw(s,h)\to w(t,\tau)$$

as $h\to 0$, hence $w(t,\cdot)$ is also right-continuous. $w(t,\tau)\to 0$ for $\tau\to\infty$ follows from $\hat{w}(\lambda,\tau)=e^{-\tau/\widehat{da}(\lambda)}/\lambda\to 0$ for $\tau\to\infty$.

(iv) To prove (3.12) observe that

$$-\int_0^\infty e^{-\mu\tau}d_\tau w(t,\tau) = -e^{-\mu\tau}w(t,\tau)|_0^\infty - \mu\int_0^\infty e^{-\mu\tau}w(t,\tau)d\tau$$

$$= 1-\mu\int_0^\infty e^{-\mu\tau}w(t,\tau)d\tau \ ,$$

hence

$$\left(-\int_0^\infty e^{-\mu\tau}d_\tau w(t,\tau)\right)\hat{}\,(\lambda) = \frac{1}{\lambda} - \mu\int_0^\infty\int_0^\infty e^{-\mu\tau}e^{-\lambda t}w(t,\tau)d\tau dt$$

$$= \frac{1}{\lambda} - \mu\int_0^\infty e^{-\mu\tau}\hat{w}(\lambda,\tau)d\tau$$

$$= \frac{1}{\lambda} - \frac{\mu}{\lambda}\int_0^\infty e^{-\mu\tau}e^{-\tau/\widehat{da}(\lambda)}d\tau$$

$$= \frac{1}{\lambda} - \frac{\mu}{\lambda}\,\frac{1}{\mu+1/\widehat{da}(\lambda)} = \frac{1}{\lambda}\cdot\frac{1}{1+\mu\widehat{da}(\lambda)} = \hat{s}(\lambda,\mu) \ ;$$

from this identity (3.12) follows by uniqueness of the Laplace transform. □

4. Generators of Feller semigroups in L^p

Let da be a completely positive measure and let w denote the solution of (3.9). We already noted in Sect. 3 that w induces a Feller semigroup on $C_0(\mathbb{R})$ given by

$$(T_0(\tau)f)(t) = \int_0^\infty f(t-s)d_s w(s,\tau) \ , \quad t\in\mathbb{R} \ , \quad \tau \geqq 0 \ . \tag{4.1}$$

Since the positive Borel measures $d_s w(\cdot,\tau)$ are bounded by one, this semigroup extends to each homogeneous Banach space $Y\subset L^1_{\text{loc}}(\mathbb{R})$ (see e.g. Katznelson [20] for the definition of homogeneous spaces), in particular in $L^p(\mathbb{R})$, $1\leqq p<\infty$ and $BUC(\mathbb{R})$ as well as $AP(\mathbb{R})$, etc. In each of these spaces the semigroup is strongly continuous, contractive, positive and translation invariant, and the analogue of (F4) in Sect. 3 holds for each of these spaces. The operators $T(\tau)$ are also well-defined in $L^\infty(\mathbb{R})$ as well as in $BC(\mathbb{R})$, all the above properties remain true except for the strong continuity. In $L^\infty(\mathbb{R})$, weak*-continuity holds and in $BC(\mathbb{R})$ even continuity with respect to the compact open topology. Let $p\in[1,\infty)$. We shall denote by $\{T_p(\tau)\}_{\tau\geqq 0}$ the semigroup by (4.1) in $L^p(\mathbb{R})$ and by A_p their negative infinitesimal generators. In the next theorem we give a characterization of A_p in terms of the function k which appears in the definition of a completely positive measure. For this purpose, we recall the decomposition (1.14) ~ (1.16); with this we can rewrite (2.8) as

$$r(t) + \frac{d}{dt}(dl_1 * r)(t) + (dl_2 * r)(t) = 1 \ , \quad t>0 \tag{4.2}$$

where $r(t)=r(t;1)$, $l_1(t)=k_0+\int_0^t k_2(\tau)d\tau$ and $l_2(t)=k_\infty+k_3(t)$. Note that $dl_1 * r\in W^{1,\infty}(\mathbb{R}_+)$. We can now state

Theorem 4.1. *Let da be a completely positive measure,* $1 \leqq p < \infty$ *and* $\{T_p(\tau)\}_{\tau \geqq 0}$ *be the semigroup as defined above. Then for its negative generator* A_p*, we have the following representation.*

$$D(A_p) = \{f \in L^p(\mathbb{R}) | k_0 f + k_2 * f \in W^{1,p}(\mathbb{R})\} \tag{4.3}$$

$$A_p f = \frac{d}{dt}(k_0 f + k_2 * f) + k_\infty f + dk_3 * f \tag{4.4}$$

with $k(t)$ *from* (2.1), (2.2) *and* k_2, k_3 *defined by* (1.15), (1.16); *here*

$$(k_2 * f)(t) = \int_{-\infty}^{t} k_2(t-s) f(s) ds = \int_0^\infty f(t-s) k_2(s) ds \ ,$$

and

$$(dk_3 * f)(t) = \int_0^\infty f(t-s) dk_3(s) \ , \qquad t \in \mathbb{R} \ .$$

Proof. Fix $p \in [1, \infty)$ and let $r(t) = r(t, 1)$, $t \geqq 0$, with $r(t, \mu)$ as defined in Corollary 2.6. Since $\mathrm{Var}\,[r; \mathbb{R}_+] \leqq 1$, the operator $(Rf)(t) = \int_0^\infty f(t-\sigma) dr(\sigma)$ is well-defined for every $f \in L^p(\mathbb{R})$, and R is a contraction in $L^p(\mathbb{R})$. We first show that $R = (I + A_p)^{-1}$. Let $f \in C_c(0, \infty)$. Then we can use the Laplace transform and obtain

$$\widehat{Rf}(\lambda) = \widehat{dr}(\lambda) \hat{f}(\lambda) = \frac{\widehat{da}(\lambda)}{1 + \widehat{da}(\lambda)} \cdot \hat{f}(\lambda)$$

by Corollary 2.6. On the other hand, since $(I + A_p)^{-1} f = \int_0^\infty e^{-\tau} T_p(\tau) f d\tau$, we obtain

$$((I + A_p)^{-1} f)\hat{}\,(\lambda) = \int_0^\infty e^{-\sigma} (T_p(\sigma) f)\hat{}\,(\lambda) d\sigma$$

$$= \left(\int_0^\infty e^{-\sigma} e^{-\sigma / \widehat{da}(\lambda)} d\sigma \right) \hat{f}(\lambda) = \frac{\widehat{da}(\lambda)}{1 + \widehat{da}(\lambda)} \hat{f}(\lambda) \ .$$

Hence $Rf = (I + A_p)^{-1} f$ for every $f \in C_c(0, \infty)$. Since both R and $(I + A_p)^{-1}$ are translation invariant, we have $Rf = (I + A_p)^{-1} f$ for every $f \in C_c(\mathbb{R})$. Since both R and $(I + A_p)^{-1}$ are bounded and $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, we have $R = (I + A_p)^{-1}$. Next define A by $D(A) = \{f \in L^p(\mathbb{R}) | k_0 f + k_2 * f \in W^{1,p}(\mathbb{R})\}$ and $Af = (k_0 f + k_2 * f)' + k_\infty f + dk_3 * f$. We want to show that $R = (I + A)^{-1}$, from which $A = A_p$ follows. We first claim $R(I + A) = I_{D(A)}$. Let $f \in D(A)$, then $dl_1 * f \in W^{1,p}(\mathbb{R})$, hence

$$dr * (dl_1 * f)' = (dr * dl_1 * f)' = d(dr * dl_1) * f = (de - dr - dl_2 * dr) * f$$

by using (4.2) and noting that $dr * dl_1 \in BV(\mathbb{R}_+)$. Here $e(t) = 1$ for $t > 0$ and 0 or $t \leqq 0$ as before. This implies the claim. Next we prove $(I + A)R = I$. Let $g \in L^p(\mathbb{R})$ and let $f = dr * g$. Then $f \in L^p(\mathbb{R})$ as well as $dl_1 * f$. Moreover, convolving (4.2) with g we obtain

$$d(dl_1 * dr) * g = g - f - dl_2 * f \ ,$$

which implies $dl_1 * f \in W^{1,p}(\mathbb{R})$ and $(I + A)Rg = g$. This completes the proof of the theorem. □

Remarks 4.2. (i) The characterization of $D(A_p)$ given in Hille-Phillips [18] is in terms of Fourier transforms, i.e. $f \in D(A_p)$ if and only if $\frac{1}{\widehat{da}(i\varrho)} \cdot \tilde{f}(\varrho)$ is the Fourier transform of a L^p function. Note that our characterization is in terms of the function itself instead of its transform.

(ii) The m-accretiveness of the operator A_p has been observed already in Clément [5] in the case $k_1 \in L^1(\mathbb{R}_+)$ and k_1 positive, decreasing, and log convex whenever $k_0 = 0$.

Next we consider the adjoint of the semigroups $\{T_p(\tau)\}_{\tau \geqq 0}$; for this purpose we make the usual identification $(L^p(\mathbb{R}))^* = L^q(\mathbb{R})$ where $\frac{1}{p}+\frac{1}{q}=1$ with the pairing $\langle f, g\rangle = \int_{\mathbb{R}} f(t)g(t)dt$ for $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. We shall make use of the reflection operator P defined by

$$(Pf)(t) = f(-t) \ , \quad t \in \mathbb{R} \ , \quad f \in L^p(\mathbb{R}) \ . \tag{4.5}$$

Let $l \in BV(\mathbb{R})$, $p \in [1, \infty)$ and $L_p \in B(L^p(\mathbb{R}))$ be defined by

$$(L_p f)(t) = \int_{\mathbb{R}} f(t-\sigma)dl(\sigma) \ , \quad t \in \mathbb{R} \ , \quad f \in L^p(\mathbb{R}) \ . \tag{4.6}$$

Then one verifies that L_p^*, the adjoint of L_p in $L^q(\mathbb{R})$, is given by $L_p^* = PL_qP$. Therefore we have:

Proposition 4.3. *Let da be completely positive and let $\{T_p(\tau)\}_{\tau \geqq 0}$ denote the semigroup in $L^p(\mathbb{R})$ given by* (4.1), $1 < p < \infty$. *Then*

(i) $T_p^*(\tau) = PT_q(\tau)P$, $\tau \geqq 0$,

(ii) $(\mu + A_p)^{-1*} = P(\mu + A_q)^{-1}P$, $\mu > 0$

(iii) $A_p^* = PA_qP$.

Remark 4.4. Assertion (i) obviously holds also for $p = 1$, with $q = \infty$. Let use define A_∞ by (4.4) where $p = \infty$. Then one verifies that assertions (ii) and (iii) also hold for $p = 1, q = \infty$. This implies that A_∞ is m-accretive, however $D(A_\infty)$ is in general not dense in $L^\infty(\mathbb{R})$.

We consider next the question of continuous dependence of the semigroups $\{T_p(\tau)\}_{\tau \geqq 0}$ on $L_p(\mathbb{R})$, associated with a completely positive measure da, with respect to da.

Theorem 4.5. *Let $p \in [1, \infty)$, let $\{da_n\}_{n \in \mathbb{N}}$ and da be completely positive and let $\varphi_n = 1/\widehat{da}_n$, $\varphi = 1/\widehat{da}$ denote their Bernstein functions, $\{T_{p,n}(\tau)\}_{\tau \geqq 0}$, $\{T_p(\tau)\}_{\tau \geqq 0}$ the associated semigroups and $A_{p,n}$, A_p their negative generators. Suppose moreover that*

$$\lim_{n \to \infty} a_n(\infty) = a(\infty) \tag{4.7}$$

where $a_n(\infty)$ and $a(\infty)$ may be infinite. Then the following assertions are equivalent:

(i) $\lim_{n \to \infty} a_n(t) = a(t)$, *for every $t > 0$ such that $a(s)$ is continuous at $s = t$.*

(ii) $\lim_{n \to \infty} \int_{\mathbb{R}} p(t)da_n(t) = \int_{\mathbb{R}} p(t)da(t)$, *for every $p \in C_c(\mathbb{R})$.*

(iii) $\lim_{n \to \infty} \varphi_n(\lambda) = \varphi(\lambda)$, *for every $\lambda > 0$.*

(iv) $\lim_{n\to\infty} (\mu+A_{p,n})^{-1}f=(\mu+A_p)^{-1}f$, *for every* $f\in L^p(\mathbb{R})$, $\mu>0$

(v) $\lim_{n\to\infty} T_{p,n}(\tau)f=T_p(\tau)f$, *for every* $f\in L^p(\mathbb{R})$, *uniformly on bounded* τ*-intervals.*

Proof. The equivalence (i)⇔(ii)⇔(iii) is well-known, see e.g. Feller [13].

The equivalence (iv)⇔(v) is called Trotter-Kato theorem in semigroup theory (see Yosida [36]).

(ii)⇒(iv) follows from the density of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$, and from (ii)⇔(iii) for $r(\cdot,\mu)$.

(iv)⇒(iii). Set $f(t)=e^{-t}$, $t\geqq 0$, and $f(t)=0, t<0$, and let $g_n=(1+A_{p,n})^{-1}f$ as well as $g=(1+A_p)^{-1}f$. Then $g_n\to g$ in $L^p(\mathbb{R})$ implies $\hat{g}_n(\lambda)\to\hat{g}(\lambda)$, for all $\lambda>0$. Finally observe that $\hat{g}_n(\lambda)=(1+\lambda)^{-1}\widehat{da}_n(\lambda)/(1+\widehat{da}_n(\lambda))$ and similarly for $\hat{g}$. □

Remarks 4.6. (i) In assertions (iv) and (v), $L^p(\mathbb{R})$ can be replaced by $C_0(\mathbb{R})$ or $BUC(\mathbb{R})$.

(ii) If (iv) or (v) holds for $L^1(\mathbb{R})$ or $BUC(\mathbb{R})$ then (4.7) holds.

In Sect. 3 we observed that translation invariant Feller semigroups satisfying (F4) in $C_0(\mathbb{R})$ are characterized by a completely positive measure. In the next proposition, we show that this is also true for $L^p(\mathbb{R})$, $1\leqq p<\infty$.

Proposition 4.7. *Let* $p\in[1,\infty)$, *and suppose* $\{T(\tau)\}_{\tau\geqq 0}$ *is a translation invariant* C_0*-semigroup of contractions in* $L^p(\mathbb{R})$ *such that* $T(\tau)$ *is positive for each* $\tau>0$, $T(\cdot)\not\equiv I$ *and* (F4) *holds. Then there is a completely positive measure da such that* $T(\tau)$ *is represented by* (4.1).

Proof. Let $T(\tau)$ be a translation invariant C_0-semigroup of positive contractions in $L^p(\mathbb{R})$ such that (F4) holds. Then by Theorem 3.6.1 in Larsen [23], there exists a family of nonnegative finite measures μ_τ such that

$$(T(\tau)f)(t)=\int_{-\infty}^{\infty} f(t-s)d\mu_\tau(s)\ ,\qquad f\in L^p(\mathbb{R})\ ,\qquad t\in\mathbb{R}\ .$$

(F4) implies $\operatorname{supp}\mu_\tau\subset\mathbb{R}_+$, therefore we may employ Laplace transforms. Let $h(\lambda,\tau)=\int_0^\infty e^{-\lambda t}d\mu_\tau(t)$; then the semigroup property of $T(\tau)$ implies the relation

$$h(\lambda,\tau_1+\tau_2)=h(\lambda,\tau_1)\cdot h(\lambda,\tau_2)\ ,\qquad \tau_1,\tau_2\geqq 0\ ,\qquad \lambda>0$$

and

$$h(\lambda,0)=1\ ,\qquad \lambda>0\ .$$

Strong continuity of $T(\tau)$ implies continuity of $h(\lambda,\cdot)$ for each fixed $\lambda>0$, and therefore there is a function $\varphi(\lambda)$ such that

$$h(\lambda,\tau)=e^{-\tau\varphi(\lambda)}\ ,\qquad \tau\geqq 0\ ,\qquad \lambda>0\ .$$

From the proof of Theorem 3.6.1 in Larsen [23] it also follows that $\operatorname{Var}[\mu_\tau;\mathbb{R}]\leqq 2$ for each $\tau>0$, hence $h(\lambda,\tau)\leqq 2$ and this yields $\varphi(\lambda)\geqq 0$ for all $\lambda>0$. Since the measures μ_τ are also nonnegative, $h(\lambda,\tau)=e^{-\tau\varphi(\lambda)}$ is completely monotonic for each $\tau>0$, and so $\varphi(\lambda)$ is a Bernstein function, by Theorem 2.4; note that $\varphi(\lambda)\not\equiv 0$ since $T(\tau)\not\equiv I$. Thus by Theorem 2.4 again, $\varphi(\lambda)=1/\widehat{da}(\lambda)$ for some completely positive measure, and by uniqueness of the Laplace transform, $T(\tau)$ is represented by (4.1). □

We conclude this section with

Corollary 4.8. *Let da be a completely positive measure and let A_p be defined as in Theorem 4.2; for some $p\in[1,\infty)$. Then the following assertions hold.*

(i) $N(A_p)=\{0\}$.

(ii) *A_p has a bounded inverse iff $a\in BV(\mathbb{R}_+)$ and $A_p^{-1}f=da*f, f\in L^p(\mathbb{R})$.*

(iii) *$k_0>0$ iff $a\in W^{1,\infty}(\mathbb{R}_+)$, and then $D(A_p)=W^{1,p}(\mathbb{R})$*

(iv) *A_p is bounded iff $k_0=0$ and $k_1(0^+)<\infty$.*

Proof. (i) Let $1\leqq p\leqq 2$. If $f\in N(A_p)\backslash\{0\}$, then $g:=(1+A_p)^{-1}f=f$; taking the Fourier transform and using the representation $g(t)=\int_{\mathbb{R}} f(t-s)\widehat{dr}(s)$, $t\in\mathbb{R}$, we obtain $\widehat{dr}(i\varrho)\tilde{f}(\varrho)=\tilde{f}(\varrho)$ a.e. on $\mathbb{R}$ where $\tilde{f}$ denotes the Fourier transform of f. Since $f\neq 0$, this implies $\widehat{dr}(i\varrho)=1$ on a set of positive measure, hence $\widehat{dr}(\lambda)=1$ for $\lambda>0$, contradicting $\widehat{dr}(\lambda)=\dfrac{\widehat{da}(\lambda)}{1+\widehat{da}(\lambda)}<1$ for $\lambda>0$. For $p\in(1,2]$, this implies $\overline{R(A_p)}=L^p(\mathbb{R})$, hence $N(A_p^*)=\{0\}$. From Proposition 4.3 we obtain $N(A_q)=\{0\}$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$, i.e. $N(A_p)=\{0\}$ for all $p\in[1,\infty)$.

(ii) It follows from (i) that A_p^{-1} exists. By using the Laplace transform, one obtains $A_p^{-1}f=da*f$ for every $f\in C_c(\mathbb{R})$. If $da\in BV(\mathbb{R}_+)$ then clearly; A_p^{-1} is bounded conversely, if A_p^{-1} is bounded, by Theorem 3.6.1 in Larsen [23] A_p^{-1} is represented as convolution with a bounded measure on $\mathbb{R}$, hence $\mathrm{Var}\,[a;\mathbb{R}_+]<\infty$ by uniqueness of the Fourier transform.

(iii) The equivalence can be seen from (2.2). If $k_0>0$, the solution of $k_0r+k_2*r=k_2$ belongs to $L^1(\mathbb{R})$ by the Paley-Wiener lemma (see e.g. Gel'fand et al. [15]), since $k_2\in L^1(\mathbb{R}_+)$, is positive, nonincreasing. It follows that $D(A_p)=W^{1,p}(\mathbb{R})$.

(iv) If $k_0=0$ and $k_1(0^+)<\infty$, then clearly A_p is bounded. Conversely, if A_p is bounded then by interpolation and by Proposition 4.3, A_2 has to be bounded and therefore $\left|\dfrac{1}{\widehat{da}(\lambda)}\right|$ is uniformly bounded for $\mathrm{Re}\,\lambda>0$. On the other hand

$$\frac{1}{\widehat{da}(\lambda)}=\lambda k_0+k_\infty+\lambda\hat{k}_1(\lambda)\geqq\lambda k_0+\int_0^{1/\lambda}\lambda e^{-\lambda t}k_1(t)\,dt$$

$$\geqq\lambda k_0+k_1(1/\lambda)(1-e^{-1})\ ,\qquad \lambda>0\ .$$

This implies $k_0=0$ and $k_1(0^+)<\infty$. □

5. "Tensor product" extension and imaginary powers

As is well-known, every positive linear bounded operator $L:L^p(\Omega)\to L^p(\Omega)$, $p\in[1,\infty)$, where Ω denotes a σ-finite measure space, possesses a unique "tensor product extension" $\mathscr{L}:L^p(\Omega;X)\to L^p(\Omega;X)$, where X is a general Banach space. More precisely, we have

Lemma 5.1. *Let $(\Omega,\mathscr{M},\mu)$ be a σ-finite measure space, let X be a Banach space with norm $\|\cdot\|$, $p\in[1,\infty)$, and let $L:L^p(\Omega)\to L^p(\Omega)$ be a bounded linear operator which is* *positive with the respect to the standard cone $L^p_+(\Omega)$. Then there exists a unique bounded linear operator $\mathscr{L}: L^p(\Omega; X) \to L^p(\Omega; X)$ satisfying*

$$\mathscr{L}(f \cdot x) = (Lf) \cdot x \quad \text{for all } f \in L^\infty(\Omega) \quad \text{and all } x \in X . \tag{5.1}$$

Moreover we have

$$\|\mathscr{L}\|_p = |L|_p , \tag{5.2}$$

and $\mathscr{L}$ is leaving invariant the closed convex cone $L^p(\Omega; K)$ induced by any given closed, convex cone K in X.

For the proof see (e.g.) Clément and Egberts [6]. We can apply this lemma to the semigroup $\{T_p(\tau)\}_{\tau \geqq 0}$ defined in Sect. 4. However, representation (4.1) gives a direct way to "extend" this semigroup to $L^p(\Omega; X)$, by using the Bochner integral instead of the Stieltjes integral. We have

Proposition 5.2. *Let da be a completely positive measure, $p \in [1, \infty)$ and let $dw(\cdot, \tau)$ and $dr(\cdot, \mu)$ denote the measures defined in Theorem 3.2 and Corollary 2.6. Then $\{\mathscr{T}_p(\tau)\}_{\tau \geqq 0}$ in $L^p(\mathbb{R}; X)$ defined by*

$$(\mathscr{T}_p(\tau) f)(t) = \int_0^\infty f(t-s) d_s w(s, \tau) , \qquad t \in \mathbb{R} , \qquad \tau \geqq 0 \tag{5.3}$$

forms a strongly continuous translation invariant semigroup of contractions in $L^p(\mathbb{R}; X)$. If $\mathscr{A}_p$ denotes the negative generator of $\{\mathscr{T}_p(\tau)\}_{\tau \geqq 0}$ then we have

$$((\mu + \mathscr{A}_p)^{-1} f)(t) = \int_0^\infty f(t-s) d_s r(s, \mu) . \tag{5.4}$$

Moreover,

$$\begin{aligned} \mathscr{D}(\mathscr{A}_p) &= \{ f \in L^p(\mathbb{R}; X) | k_0 f + k_2 * f \in W^{1,p}(\mathbb{R}; X) \} \\ \mathscr{A}_p f &= \frac{d}{dt} (k_0 f + k_2 * f) + k_\infty f + dk_3 * f , \end{aligned} \tag{5.5}$$

for $f \in \mathscr{D}(\mathscr{A}_p)$, where k_0, k_1, k_2, k_3, k_∞, are as in Theorem 4.1.

Proof. Since Var $[w(\cdot, \tau); \mathbb{R}_+] \leqq 1$, $\mathscr{T}_p(\tau)$ are contractions. The semigroup property follows from (3.2) and (3.3). Since the set

$$\mathscr{E} := \left\{ \sum_{i=1}^n f_i \cdot x_i | f_i \in L^p(\mathbb{R}),\ x_i \in X,\ i = 1, \ldots, n \right\} \tag{5.6}$$

is dense in $L^p(\mathbb{R}; X)$ and $\{\mathscr{T}_p(\tau)\}_{\tau \geqq 0}$ is strongly continuous on $\mathscr{E}$, it follows from the Banach-Steinhaus theorem that $\{\mathscr{T}_p(\tau)\}_{\tau \geqq 0}$ is strongly continuous. Define $\mathscr{A}_p$ by (5.5). As in the proof of Theorem 4.1, it follows that $\mathscr{A}_p$ is the negative generator of $\{\mathscr{T}_p(\tau)\}_{\tau \geqq 0}$. □

Remark 5.3. (i) It is clear from (5.4) that $\{\mathscr{T}_p(\tau)\}_{\tau \geqq 0}$ is analytic on some sector if $\{T_p(\tau)\}_{\tau \geqq 0}$ is analytic there.

(ii) It also follows that if K is a closed convex subset of X, then $L^p(\mathbb{R}, K)$ is invariant under $\{\mathscr{T}_p(\tau)\}_{p \geqq 0}$, if $a(\infty) = \infty$, and in case $a(\infty) < \infty$, if one assumes also $0 \in K$.

(iii) Since $|T_p(\tau)|_p = \|\mathscr{T}_p(\tau)\|_p$ for all $\tau > 0$, the type $\omega_0(\mathscr{A}_p)$ of $\mathscr{A}_p$ satisfies

$$\omega_0(\mathscr{A}_p) = \omega_0(A_p) = -\frac{1}{a(\infty)} = -k_\infty \ .$$

This follows from $\mathrm{Var}\,[w(\cdot,\tau);\mathbb{R}^+] \leqq e^{-\tau/a(\infty)}$ and $s(A_p) \geqq \frac{-1}{a(\infty)}$, where $s(A_p)$ denotes the spectral bound of A_p.

As an application of the subordination principle, Corollary 2.8, we consider the fractional powers of $\mathscr{A}_p$. Let da be completely positive and $\alpha \in (0,1)$. Since we have $(\widehat{da})^\alpha = \frac{1}{z^\alpha} \circ (1/\widehat{da})$ and $\frac{1}{\lambda^\alpha} = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{\wedge}(\lambda)$ is the transform of a completely monotonic, locally integrable, hence completely positive function, by Corollary 2.8, there are completely positive measures da_α such that

$$\widehat{da}_\alpha(\lambda) = (\widehat{da}(\lambda))^\alpha \ , \qquad \lambda > 0 \ .$$

Note that $a_\alpha \in BV(\mathbb{R}_+)$ iff $k_\infty > 0$ iff $a(\infty) < \infty$. These completely positive measures define the fractional powers $\mathscr{A}_p^\alpha$ which in general are unbounded, and which are negative generators of analytic semigroups. As a special case, if $a(t) = t$, then $\mathscr{A}_p = $ "$\frac{d}{dt}$", i.e. $k_0 = 1$, $k_1 = k_\infty = 0$, hence $\widehat{da}_\alpha(\lambda) = \lambda^{-\alpha}$ and $a_\alpha(t) = \int\limits_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)}\,ds = \frac{t^\alpha}{\Gamma(\alpha+1)}$, $t > 0$.

The study of purely imaginary powers is more difficult since the representing kernels are no longer completely positive. However, since the operators $\mathscr{A}_p$ are negative generators of positive contraction semigroups on $L^p(\mathbb{R}; X)$, one can prove the boundedness of $\mathscr{A}^{i\gamma}$, $\gamma \in \mathbb{R}$, and obtain explicit bounds in terms of

$$\theta_0 = \sup_{\mathrm{Re}\,\lambda > 0} |\arg \varphi(\lambda)| \ . \tag{5.7}$$

We show that $\theta_0 \leqq \pi/2$.

Proposition 5.4. *Let da be a completely positive measure. Then* $\mathrm{Re}\,\widehat{da}(\lambda) > 0$ *for all* $\mathrm{Re}\,\lambda > 0$.

Proof. We show that $\mathrm{Re}\,1/\widehat{da}(\lambda) > 0$. By (2.2), we have

$$\mathrm{Re}\,1/\widehat{da}(\lambda) = \mathrm{Re}\,(\lambda k_0 + k_\infty + \lambda \hat{k}_1(\lambda)) = k_0\,\mathrm{Re}\,\lambda + k_\infty + \mathrm{Re}\,(\lambda \hat{k}_1(\lambda)) \ .$$

If $k_1(0^+) < \infty$, then $k_1 \in BV(\mathbb{R}_+)$ and

$$\mathrm{Re}\,[\lambda \hat{k}_1(\lambda)] = \mathrm{Re}\,\widehat{dk_1}(\lambda) = k_1(0^+) + \mathrm{Re}\int\limits_{0^+}^\infty e^{-\lambda t}\,dk_1(t) \geqq k_1(0^+) + \int\limits_{0^+}^\infty dk_1(t) = 0 \ .$$

If $k_1(0^+) = \infty$, then we approximate k_1 by $k_{1,\varepsilon}(t) = k_1(t+\varepsilon)$, $\varepsilon > 0$, and obtain again $\mathrm{Re}\,[\lambda \hat{k}_1)] \geqq 0$. This shows that $\mathrm{Re}\,\widehat{da}(\lambda) \geqq 0$ for $\mathrm{Re}\,\lambda > 0$, however we cannot have $\mathrm{Re}\,\widehat{da}(\lambda_0) = 0$, since this function is harmonic. This completes the proof. □

One method to obtain boundedness of $\mathscr{A}^{i\gamma}$ is to use multiplier theory.

We begin with the scalar valued case, $X = \mathbb{R}$. For $p = 2$, this is an easy consequence of Plancherel's theorem, indeed the multiplier associated with $A_2^{i\gamma}$ is given by $\varphi(\lambda)^{i\gamma}$, $\gamma \in \mathbb{R}$ and $\operatorname{Re}\lambda \geqq 0$. Hence

$$\|A_2^{i\gamma}\|_2 \leqq \sup_{\operatorname{Re}\lambda>0} |\varphi(\lambda)^{i\gamma}| = e^{|\gamma|\theta_0} \leqq e^{|\gamma|\frac{\pi}{2}} , \tag{5.8}$$

which is best possible. The same holds true when X is a Hilbert space.

Next we consider the case $p \in (1, \infty)$, $p \neq 2$ and $X = \mathbb{R}$. The Mikhlin multiplier theorem (see e.g. Bergh and Löfström [3]) yields the estimate:

$$\|A_p^{i\gamma}\|_p \leqq M_p \cdot \sup_{\operatorname{Re}\lambda>0} (|\varphi(\lambda)^{i\gamma}| + |\lambda(\varphi(\lambda)^{i\gamma})'|) .$$

Since $(\varphi(\lambda)^{i\gamma})' = i\gamma \dfrac{\varphi'(\lambda)}{\varphi(\lambda)} \varphi(\lambda)^{i\gamma}$, this gives

$$\|A_p^{i\gamma}\|_p \leqq M_p' \cdot (1+|\gamma|) e^{\theta_0|\gamma|} , \qquad \gamma \in \mathbb{R} , \tag{5.9}$$

provided that

$$C_1 = \sup_{\operatorname{Re}\lambda>0} \left|\lambda \frac{\varphi'(\lambda)}{\varphi(\lambda)}\right| < \infty . \tag{5.10}$$

In the Banach valued case, when the space X is ζ-convex (or equivalently UMD), then by McConnell's multiplier theorem ([26]), we obtain the following estimate

$$\|A_p^{i\gamma}\| \leqq M_p'' \cdot \sup_{\operatorname{Re}\lambda>0} (|\varphi(\lambda)^{i\gamma}| + |\lambda(\varphi(\lambda)^{i\gamma})'| + |\lambda^2(\varphi(\lambda)^{i\gamma})''|) .$$

This yields

$$\|A_p^{i\gamma}\| \leqq M_p''' \cdot (1+|\gamma|^2) e^{\theta_0|\gamma|} , \qquad \gamma \in \mathbb{R} , \tag{5.11}$$

provided (5.10) holds and also

$$C_2 = \sup_{\operatorname{Re}\lambda>0} \left|\lambda^2 \frac{\varphi''(\lambda)}{\varphi(\lambda)}\right| < \infty . \tag{5.12}$$

Unfortunately, Condition (5.10) is not satisfied for all completely positive measures, as the Bernstein function $\varphi(\lambda) = 1 - e^{-\lambda}$ shows. However for completely positive measures of the form

$$da(t) = a_0\delta_0 + a_1(t)dt , \tag{5.13}$$

where $a_0 \geqq 0$, and $a_1 \in \mathscr{CM}$, it can be shown that conditions (5.10) and (5.12) hold.

Proposition 5.5. *Let da be of the form* (5.13) *with $a_0 \geqq 0$ and $a_1 \in \mathscr{CM} \cap L^1(0,1)$. Then we have*

$$|\lambda^n \widehat{da}^{(n)}(\lambda)| \leqq n! |\widehat{da}(\lambda)| , \qquad \textit{for } \operatorname{Re}\lambda > 0 , \qquad n \in \mathbb{N} . \tag{5.14}$$

Proof. Since $a_0 \geqq 0$ and $a_1 \in \mathscr{CM}$, thanks to Bernstein theorem there is $b \in BV_{\mathrm{loc}}(\mathbb{R}_+)$ nondecreasing such that $\widehat{da}(\lambda) = \int_0^\infty \frac{1}{s+\lambda} db(s)$. Differentiating this relation we obtain

$$(-\lambda)^n \widehat{da}^{(n)}(\lambda) = n! \int_0^\infty \frac{\lambda^n}{(s+\lambda)^{n+1}} db(s) , \qquad \text{for } \lambda \in \mathbb{C} \backslash \mathbb{R}_- .$$

Since $\left|\frac{\lambda}{s+\lambda}\right| \leqq 1$ for $\operatorname{Re}\lambda > 0$ and $s \geqq 0$, we obtain with $\lambda = \sigma + i\varrho$

$$|\lambda^n \widehat{da}^{(n)}(\lambda)|^2 \leqq n!^2 \left(\int_0^\infty \frac{\sqrt{\sigma^2+\varrho^2}}{(s+\sigma)^2+\varrho^2} \, db(s) \right)^2$$

$$\leqq n!^2 \int_0^\infty \frac{db(s)}{(s+\sigma)^2+\varrho^2} \cdot \int_0^\infty \frac{\sigma^2+\varrho^2}{(s+\sigma)^2+\varrho^2} \, db(s)$$

$$\leqq n!^2 \left(\left(\int_0^\infty \frac{\sigma+s}{(s+\sigma)^2+\varrho^2} \, db(s) \right)^2 + \varrho^2 \left(\int_0^\infty \frac{db(s)}{(s+\sigma)^2+\varrho^2} \right)^2 \right)$$

$$\leqq n!^2 ((\operatorname{Re} \widehat{da}(\lambda))^2 + (\operatorname{Im} \widehat{da}(\lambda))^2) = n!^2 |\widehat{da}(\lambda)|^2 \ . \quad \square$$

Note that Inequality (5.14) is sharp, as the example $\widehat{da}(\lambda) = 1/\lambda$, i.e. $a_1(t) = 1$, $a_0 = 0$, shows. Estimate (5.14) implies (5.10) and (5.12), since

$$\frac{\varphi'}{\varphi} = -\frac{\widehat{da}'}{\widehat{da}} \quad \text{and} \quad \frac{\varphi''}{\varphi} = -\frac{\widehat{da}''}{\widehat{da}} + 2\left(\frac{\widehat{da}'}{\widehat{da}}\right)^2 .$$

We consider now another method, the transference method, which leads to the desired result in the general case of a completely positive measure. This method is based on the following result.

Theorem 5.6. *Let X be a Banach space, $1 < p < \infty$, $\{S(\tau)\}_{\tau \geqq 0}$ be a strongly continuous semigroup of positive contractions on $L^p(\Omega)$, where $(\Omega, \mathcal{M}, \mu)$ is a σ-finite measure space. Let $\{\mathcal{S}(\tau)\}_{\tau \geqq 0}$ be its "tensor product" extension defined in Lemma 5.1 to $L^p(\Omega; X)$, and let $\mathcal{A}$ denote its negative generator. Let $b \in L^1(\mathbb{R}_+)$ and denote by $\mathcal{B}$ the convolution operator on $L^p(\mathbb{R}; X)$ defined by $\mathcal{B} f = b * f$. Then the operator defined by*

$$\hat{b}(\mathcal{A}) f = \int_0^\infty b(\tau) \mathcal{S}(\tau) f \, d\tau \ , \qquad f \in L^p(\Omega; X) \tag{5.15}$$

is bounded in $L^p(\Omega; X)$ and the following estimate holds.

$$\|\hat{b}(\mathcal{A})\|_p \leqq \|\mathcal{B}\|_p \tag{5.16}$$

Here $\|\mathcal{B}\|_p$ denotes the norm of $\mathcal{B}$ in $L^p(\mathbb{R}; X)$.

Remark 5.7. This theorem is an extension of a beautiful result of Coifman and Weiss [11] for the case $X = \mathbb{R}$. The extension is made possible by Lemma 5.1 where the positivity of the semigroups $(S(\tau))_{\tau \geqq 0}$ is used. For the sake of completeness, we give a proof in Appendix A.

If the space X is ζ-convex, then we can combine Theorem 5.6 with McConnell's multiplier theorem to obtain the following result.

Theorem 5.8. *Let X be a ζ-convex Banach space, $1 < p < \infty$ and $\{\mathcal{S}(\tau)\}_{\tau \geqq 0}$, and $\mathcal{A}$ be as in Theorem 5.6. Assume also that $N(\mathcal{A}) = \{0\}$ (hence also $\overline{R(\mathcal{A})} = L^p(\Omega; X)$, since X is reflexive). Then the imaginary powers $\mathcal{A}^{i\gamma}$ are bounded in $L^p(\Omega; X)$ for $\gamma \in \mathbb{R}$, and* *we have the estimate:*

$$\|\mathscr{A}^{i\gamma}\|_p \leqq M_p \cdot (1+\gamma^2)e^{|\gamma|\frac{\pi}{2}}, \qquad \gamma \in \mathbb{R} \tag{5.17}$$

where M_p depends only on p and X.

Remark 5.9. (i) Comparing Estimate (5.17) with (5.11) and (5.8) we see that for specific semigroups, (5.17) is not sharp, and so (5.8) and (5.11) are still useful.

(ii) If X is a Hilbert space, (5.8) and (5.17) can be used to obtain better estimates in L^p via interpolation. In particular, $\pi/2$ can be replaced by $\pi/2-\varepsilon$ when $\theta_0<\pi/2$. This is important for applications, see Sect. 6, and Prüss and Sohr [32].

Proof of Theorem 5.8. It is known that a closed linear densely defined operator B in a Banach space Y satisfying for some constant $M\geqq 1$

$$\begin{gathered}(-\infty,0)\subset\varrho(B)\,, \quad N(B)=0\,, \quad \overline{R(B)}=Y\,,\\ \|(\mu+B)^{-1}\|\leqq M/\mu \quad \text{for all} \quad \mu>0\,,\end{gathered} \tag{5.18}$$

admits fractional powers B^z of any order $z\in\mathbb{C}$, not necessarily bounded. For $|\mathrm{Re}\,z|<1$, $z\neq 0$, $x\in D(B)\cap R(B)$, $B^z x$ is given by

$$\begin{aligned}B^z x=\frac{\sin\pi z}{\pi}\Big\{&z^{-1}x-(1+z)^{-1}B^{-1}x\\ &+\int_0^1 t^{z+1}(t+B)^{-1}B^{-1}x\,dt+\int_1^\infty t^{z-1}(t+B)^{-1}Bx\,dt\Big\}\,,\end{aligned} \tag{5.19}$$

cf. Komatsu [22]. Note that the integrals in (5.19) are absolutely convergent by estimate (5.18). Recall also that if the operator B additionally is the negative generator of a C_0-semigroup $\{T(\tau)\}_{\tau\geqq 0}$ of negative type then

$$B^{-z}x=\int_0^\infty \frac{\tau^{z-1}}{\Gamma(z)}\,T(\tau)x\,d\tau\,, \qquad x\in Y\,, \qquad 0<\mathrm{Re}\,z<1 \tag{5.20}$$

see Komatsu [22]. Observe that for $\mu\geqq 0$, the operators $\mu+\mathscr{A}$ satisfy (5.18) and for $\mu>0$, they generate a semigroup of negative type $\{e^{-\mu\tau}\mathscr{S}(\tau)\}_{\tau\geqq 0}$ in $Y=L^p(\Omega;X)$. Let $\gamma\in\mathbb{R}\backslash\{0\}$ be fixed, and define

$$b_\mu(\tau)=e^{-\mu\tau}\frac{\tau^{i\gamma+\mu-1}}{\Gamma(i\gamma+\mu)}\,, \qquad \tau>0\,, \qquad \mu>0\,. \tag{5.21}$$

To prove the boundedness of $\mathscr{A}^{i\gamma}$ we use the approximations $(\mu+\mathscr{A})^{i\gamma-\mu}$, $\mu>0$. Thanks to (5.20) with $B=\mu+\mathscr{A}$ and $z=-i\gamma+\mu$, we have $(\mu+\mathscr{A})^{i\gamma-\mu}=\hat{b}_\mu(\mathscr{A})$; note that $b_\mu\in L^1(\mathbb{R}_+)$. From Theorem 5.6 together with McConnell's multiplier theorem, we obtain the following uniform bound on $\hat{b}_\mu(\mathscr{A})$:

$$\begin{aligned}\|\hat{b}_\mu(\mathscr{A})\|_p&\leqq M_p\sup_{\mathrm{Re}\,\lambda>0}(|\hat{b}_\mu(\lambda)|+|\lambda\hat{b}_\mu'(\lambda)|+\lambda^2\hat{b}_\mu''(\lambda)|)\\ &\leqq M_p\cdot c\cdot(1+\gamma^2)e^{|\gamma|\frac{\pi}{2}}\,,\end{aligned} \tag{5.22}$$

where c is independent of γ and μ.

Next we prove the convergence of $\hat{b}_\mu(\mathscr{A})f$ to $\mathscr{A}^{i\gamma}f$ as $\mu\to 0+$ for $f\in D(\mathscr{A})\cap R(\mathscr{A})$, a dense subset in $L^p(\Omega;X)$. From Banach-Steinhaus' theorem the result then follows. For this purpose, we rewrite (5.19) as

$$B^z x=\frac{\sin \pi z}{\pi}\{z^{-1}x-(1+z)^{-1}B^{-1}\, x$$

$$+\int_0^1 t^{-z}(1+tB)^{-1}Bx dt+\int_1^\infty t^{-z-2}(1+tB)^{-1}B^{-1}x dt\} . \tag{5.23}$$

Replace B by $\mu+\mathscr{A}$ and z by $i\gamma-\mu$, in (5.23), and observe that, by (5.18), $(\mu+\mathscr{A})^{-1}f\to\mathscr{A}^{-1}f$ as $\mu\to 0+$, since $f\in R(\mathscr{A})$. We also have $(1+t\mu+t\mathscr{A})^{-1}\to(1+t\mathscr{A})^{-1}$ for each $t>0$ and therefore by Lebesgue's dominated convergence theorem, we obtain

$$\mathscr{A}^{i\gamma}f=\lim_{\mu\downarrow 0}(\mu+\mathscr{A})^{i\gamma-\mu}f , \quad f\in D(\mathscr{A})\cap R(\mathscr{A}) .$$

This completes the proof of the theorem. □

6. Applications to heat flow with memory

In this section we shall discuss an application of the results presented above to the problem of heat flow in materials with memory.

We consider a model introduced by Nunziato [28], see also Clément and Nohel [9] and Lunardi [25]. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary and consider the equation

$$b_0 u_t(t,x)+\frac{\partial}{\partial t}\int_{-\infty}^{t} b_1(t-s)u(s,x)ds$$

$$=c_\infty \Delta u(t,x)+\frac{\partial}{\partial t}\int_{-\infty}^{t} c_1(t-s)\Delta u(s,x)ds+f(t,x) ,$$

$$t\in\mathbb{R} , \quad x\in\Omega . \tag{6.1}$$

$$u(t,x)=0 , \quad t\in\mathbb{R} , \quad x\in\partial\Omega ,$$

where $u(t,x)$ is the temperature of the point $x\in\Omega$ at time $t\in\mathbb{R}$, and $f(t,x)$ is the heat supply. If $b_1=c_1=0$ and b_0, c_∞ are positive, (6.1) reduces to the ordinary heat equation. Our basic assumptions on b_0, b_1, c_∞, c_1 are

$$b_1 \text{ is positive, of positive type and integrable, } b_0\geqq 0 , \quad b_0+\int_0^\infty b_1(\tau)d\tau>0 ; \tag{6.2}$$

and

$$c_\infty>0,\ c_1 \text{ is positive and of positive type} . \tag{6.3}$$

Here we are interested in solutions in $L^p(\mathbb{R};L^q(\Omega))$, with $1<p, q<\infty$, having the maximal regularity property, i.e. if $f\in L^p(\mathbb{R};L^q(\Omega))$, then $u\in L^p(\mathbb{R};W^{2,q}(\Omega)$

$\cap W_0^{1,q}(\Omega))$. For this purpose we rewrite (6.1) as an equation of the form

$$\mathscr{B}u+\mathscr{C}\mathscr{A}u=f \tag{6.4}$$

in the Banach space $E=L^p(\mathbb{R};L^q(\Omega))$, where the operators are defined as follows. We introduce an operator A by means of

$$D(A)=W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega),\, Av=-\Delta v \quad \text{for} \quad v\in D(A)\ . \tag{6.5}$$

Let $\mathscr{A}$ denote its pointwise extension to E, where $D(\mathscr{A})=L^p(\mathbb{R};D(A))$, and $D(A)$ is equipped with the graph norm. Note that $\mathscr{A}$ is closed, $-\mathscr{A}$ generates an analytic semigroup of negative type and the imaginary powers $\mathscr{A}^{i\gamma}$, $\gamma\in\mathbb{R}$. are bounded and satisfy the estimate

$$\|\mathscr{A}^{i\gamma}\|\leqq C_\varepsilon e^{|\gamma|\varepsilon}\ ,\qquad \gamma\in\mathbb{R}\ ,$$

for every $\varepsilon>0$. Let $\mathscr{B}$ be defined by

$$\begin{aligned} &D(\mathscr{B})=\{u\in E|b_0u+b_1*u\in W^{1,p}(\mathbb{R};L^q(\Omega))\}\ ,\\ &\mathscr{B}u=\frac{d}{dt}(b_0u+b_1*u)\ ,\qquad \text{for } u\in\mathscr{D}(\mathscr{B})\ . \end{aligned} \tag{.6.6}$$

Then $\mathscr{B}$ is a closed densely defined operator in E.

For simplicity we also assume $c_1\in L^1(\mathbb{R})$; like in Theorem 4.1 this assumption can easily be omitted, but we shall not do this here.

Define $\mathscr{C}$ by means of

$$\begin{aligned} &D(\mathscr{C})=\{u\in E|c_1*u\in W^{1,p}(\mathbb{R};L^q(\Omega))\}\ ,\\ &\mathscr{C}u=c_\infty u+\frac{d}{dt}(c_1*u)\ ,\qquad \text{for } u\in D(\mathscr{C})\ . \end{aligned} \tag{6.7}$$

Then $\mathscr{C}$ is a closed, linear densely defined operator in E. Assume like in Clément and Nohel [9], Lunardi [25], that b_1 and c_1 are nonincreasing. It follows that the functions defined by

$$\begin{cases} k_b(t)=b_0+\int_0^t b_1(s)ds \\ \qquad\qquad\qquad\qquad\qquad t>0 \\ k_c(t)=c_\infty t+\int_0^t c_1(s)ds \end{cases}$$

are positive, nondecreasing and concave, and therefore give rise to completely positive measures da_b and da_c, by Theorem 2.1. Using Theorem 4.1 and Proposition 5.2, the operators $\mathscr{B}$ and $\mathscr{C}$ generate contraction semigroups in E which are positive with respect to the usual cone E^+ of positive functions. Corollary 4.8 implies $N(\mathscr{B})=N(\mathscr{C})=0$ and $R(\mathscr{B})$, $R(\mathscr{C})$ dense in E; since $c_\infty>0$, $\mathscr{C}$ even has a bounded inverse, cf. Remark 5.3 (iii).

Applying Theorem 5.8, with $X=L^q(\Omega)$ which is ζ-convex (see Prüss and Sohr [32]), we have

$$\|\mathscr{B}^{i\gamma}\|\leqq M(1+\gamma^2)e^{|\gamma|\pi/2}\ ,\qquad \gamma\in\mathbb{R} \tag{6.8}$$

and the same holds for $\mathscr{C}$. Since the operator $\mathscr{C}^{-1}$ is bounded, the operator $\mathscr{D}$ defined by $\mathscr{D}=\mathscr{C}\mathscr{A}$ is closed. $\mathscr{D}$ has also bounded imaginary powers by Corollary 3 of Prüss and Sohr [32], and Estimate (6.8) holds for $\mathscr{D}$ with $\pi/2$ replaced by $\pi/2+\varepsilon$. Since (6.4) is of the form

$$\mathscr{B}u+\mathscr{D}u=f \tag{6.9}$$

and the resolvents of $\mathscr{B}$ and $\mathscr{D}$ commute, we can apply Theorem 4 of Prüss and Sohr [32], provided that we can show that $\theta_{\mathscr{B}}+\theta_{\mathscr{D}}<\pi$, where $\theta_{\mathscr{B}}$ is the type of the group $\{\mathscr{B}^{i\gamma}\}_{\gamma\in\mathbb{R}}$, similarly for $\theta_{\mathscr{D}}$. We only know $\theta_{\mathscr{B}}+\theta_{\mathscr{D}}<\pi+\varepsilon$, however, if $\mathscr{C}$ is bounded, i.e. $c_1(0^+)<\infty$, we have $\theta_{\mathscr{C}}<\pi/2$, since $\mathscr{C}-c_\infty I$ is m-accretive, by using the Dunford integral representation for the imaginary powers. Hence $\theta_{\mathscr{D}}<\pi/2$ as well, since ε can be choosen arbitrary small. When $\mathscr{C}$ is unbounded, and if $\hat{c}_1(\lambda)$ satisfies (5.10) and (5.12) and as well as

$$\sup_{\operatorname{Re}\lambda>0}|\arg(c_\infty+\lambda\hat{c}_1(\lambda))|<\pi/2\ , \qquad \operatorname{Re}\lambda>0 \tag{6.10}$$

or $b_0=0$, $\hat{b}_1(\lambda)$ satisfies (5.10) and (5.12), and

$$\sup_{\operatorname{Re}\lambda>0}|\arg\lambda\hat{b}_1(\lambda)|<\pi/2 \tag{6.11}$$

holds, then from (5.11) and (5.7) we have

$$\theta_{\mathscr{C}}<\frac{\pi}{2}\ \text{i.e.}\ \theta_{\mathscr{D}}<\frac{\pi}{2}\ , \qquad \text{resp.}\ \theta_{\mathscr{B}}<\frac{\pi}{2}\ .$$

Summarizing we obtain

Theorem 6.1. *Let $b_0\geqq 0$, $c_\infty>0$ and $b_1, c_1\in L^1(\mathbb{R}_+)$ be nonnegative and nonincreasing, and let $b_0+\int_0^\infty b_1(t)dt>0$. Assume in addition $c_1(0^+)<\infty$ or (6.10) or (6.11). Then for every $f\in L^p(\mathbb{R};L^q(\Omega))$, (6.1) possesses one and only one solution u satisfying $u\in L^p(\mathbb{R};W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega))$, b_0u+b_1*u, $c_1*\Delta u\in W^{1,p}(\mathbb{R};L^q(\Omega))$, and there is a constant $M>0$ such that*

$$\|u\|+\|\nabla u\|+\|\nabla^2 u\|+\left\|\frac{d}{dt}(b_0u+b_1*u)\right\|+\left\|\frac{d}{dt}(c_1*\Delta u]\right\|\leqq M\|f\|\ ,$$

for $f\in L^p(\mathbb{R};L^q(\Omega)$, where $\|\cdot\|$ denotes the norm in $L^p(\mathbb{R};L^q(\Omega))$.

Since the operator $\mathscr{C}$ has a bounded inverse, (6.4) is equivalent to

$$\mathscr{C}^{-1}\mathscr{B}u+\mathscr{A}u=\mathscr{C}^{-1}f\ . \tag{6.12}$$

In general $\mathscr{C}^{-1}\mathscr{B}$ is not closed but it is closable and its closure is $\mathscr{B}\mathscr{C}^{-1}$ (note that $\mathscr{B}$ and $\mathscr{C}^{-1}$ commute). Hence a solution $u\in\mathscr{D}(\mathscr{A})$ such that $\mathscr{C}^{-1}u\in\mathscr{D}(\mathscr{B})$ of

$$\mathscr{B}\mathscr{C}^{-1}u+\mathscr{A}u=\mathscr{C}^{-1}f \tag{6.13}$$

can be considered as a mild solution of (6.1). Note that if $\mathscr{C}$ is bounded, then $\mathscr{C}^{-1}\mathscr{B}$ is closed and the two notions of solutions coincide.

The operator $\mathscr{B}\mathscr{C}^{-1}$ has the function

$$\varphi(\lambda)=\frac{\lambda(b_0+\hat{b}_1(\lambda))}{c_\infty+\lambda\hat{c}_1(\lambda)}\ , \qquad \lambda>0 \tag{6.14}$$

as its symbol; we observe that $\operatorname{Re}\varphi(\lambda)>0$ for $\operatorname{Re}\lambda>0$ holds. Indeed a simple computation shows that

$$\operatorname{Re}\varphi(i\varrho)=\alpha(\varrho)\cdot(\varrho^2\operatorname{Re}\hat{c}_1(i\varrho)(b_0+\operatorname{Re}\hat{b}_1(i\varrho))+(-\varrho\operatorname{Im}\hat{b}_1(i\varrho))(c_\infty-\varrho\operatorname{Im}\hat{c}_1(i\varrho)))$$

where $\alpha(\varrho)=|c_\infty+\hat{c}_1(i\varrho)|^{-2}$. Since b_1 and c_1 are of positive type we have, $\operatorname{Re}\hat{b}_1(i\varrho)$, $\operatorname{Re}\hat{c}_1(i\varrho)\geqq 0$, $\varrho\in\mathbb{R}$, and since both functions are also nonincreasing we also have $-\varrho\operatorname{Im}\hat{b}_1(i\varrho)$, $-\varrho\operatorname{Im}\hat{c}_1(i\varrho)\geqq 0$, $\varrho\in\mathbb{R}$. This shows $\operatorname{Re}\varphi(i\varrho)\geqq 0$, therefore $\operatorname{Re}\varphi(\lambda)>0$, for $\operatorname{Re}\lambda>0$ by the maximum principle. Summarizing, we obtain:

Theorem 6.2. *Let $b_0\geqq 0$, $c_\infty>0$, and let $b_1, c_1\in L^1(\mathbb{R}_+)$ be nonnegative, nonincreasing, of positive type, and such that $b_0+\int_0^\infty b_1(t)dt>0$. Then for $f\in L^2(\mathbb{R};L^2(\Omega))$, there is a unique mild solution u of (6.1), such that $u\in\mathscr{D}(\mathscr{A})$ and $\mathscr{C}^{-1}u\in\mathscr{D}(\mathscr{B})$, i.e. $u\in L^2(\mathbb{R};W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega))$, $v=da_c*u\in L^2(\mathbb{R};L^2(\Omega))$, $b_0v+b_1*v\in W^{1,2}(\mathbb{R};L^2(\Omega))$, where da_c denotes the completely positive measure defined by c_∞ and c_1. Moreover, there exists a constant $M>0$, such that*

$$\|u\|+\|\nabla u\|+\|\nabla^2 u\|+\left\|\frac{d}{dt}(b_0v+b_1*v)\right\|\leqq M\|f\| , \quad \textit{for all } f\in L^2(\mathbb{R};L^2(\Omega)) .$$

If in addition, $\hat{b}_1$ and $\hat{c}_1$ satisfy (5.10) *and* (5.12) *then the theorem is true in $L^p(\mathbb{R};L^q(\Omega))$, for all $1<p, q<\infty$.*

Proof. We only need to consider the case $p, q\neq 2$. Since the operator $\mathscr{B}\mathscr{C}^{-1}$ has φ as its symbol we can apply McConnell's theorem which shows that $\mathscr{B}\mathscr{C}^{-1}$ has bounded imaginary powers and estimate (6.8) holds for $\mathscr{B}\mathscr{C}^{-1}$. From Theorem 4 in Prüss and Sohr [32] the result follows.

Remarks 6.3. (i) First recall that $\hat{b}_1$ and $\hat{c}_1$ satisfy (5.10) and (5.12) when b_1 and c_1 are completely monotonic, see Proposition 5.5.

(ii) Since c_1 is nonincreasing, we always have $\operatorname{Re}(c_\infty+\lambda\hat{c}_1(\lambda))\geqq c_\infty>0$, thus the second condition (6.10) is equivalent to

$$\overline{\lim_{\varrho\to\infty}}\,|\arg(c_\infty+i\varrho\hat{c}_1(i\varrho))|<\pi/2 .$$

In case $c_1(0_+)=\infty$, this condition holds if $c_1(t)\sim c_0t^{\alpha-1}/\Gamma(\alpha)$ for some $\alpha\in(0,1)$; in fact then $\hat{c}_1(i\varrho)\sim c_0(i\varrho)^{-\alpha}$, hence

$$\overline{\lim_{\varrho\to\infty}}\,|\arg(c_\infty+i\varrho\hat{c}_1(i\varrho))=\overline{\lim_{\varrho\to\infty}}\,|\arg(i\varrho)^{1-\alpha}|=(1-\alpha)\frac{\pi}{2}<\frac{\pi}{2} .$$

(iii) Observe that Theorem 6.1 gives much more time-regularity than Theorem 6.2; in particular, if $b_0>0$ we obtain $u\in W^{1,p}(\mathbb{R};L^q(\Omega))$ rather then $v=da_c*u\in W^{1,p}(\mathbb{R};L^q(\Omega))$. For the case $c_1(0+)<\infty$, Theorem 6.1 even includes Theorem 6.2.

(iv) When φ defined in (6.14) is a Bernstein function, then the operator $\mathscr{B}\mathscr{C}^{-1}$ is the negative generator of a positive contraction semigroup on $L^p(\mathbb{R};L^q(\Omega))$. Therefore Theorem 6.2 holds also in case $\hat{b}_1$, $\hat{c}_1$ do not satisfy (5.10) and (5.12) since in the proof McConnell's theorem can be replaced by Theorem 5.8. On the other hand, since $\mathscr{C}^{-1}$ is a positive operator, when φ is a Bernstein function, the mild solution u is positive whenever f has this property. This was observed by Lunardi [25], where sufficient conditions on the kernels are given for φ to be a Bernstein function; see also Clément and Nohel [9].

(v) For the sake of simplicity we considered $A = -\Delta$ with Dirichlet boundary conditions only. Other boundary conditions can be treated similarly. For example, if the non flux condition is imposed at the boundary, i.e. $c_\infty \frac{\partial u}{\partial n} + \frac{d}{dt} c_1 * \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ or equivalently $\frac{\partial u}{\partial n} = 0$ since the operator $\mathscr{C}$ is invertible, then the operator A defined by

$$D(A) = \{u \in W^{2,q}(\Omega) \,\Big|\, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$$

$Au = -\Delta u$, is still m-accretive, but has a nontrivial kernel. Then one can solve (6.1) by projecting to the range of A, since $L^q(\Omega) = N(A) \oplus R(A)$ holds.

(vi) In a forthcoming paper, we shall use the results of this section to study a semilinear version of (6.1) with critical growth.

7. Appendix A

Proof of Theorem 5.6. As in Coifman and Weiss [11], we will deduce Theorem 5.6 from its discrete counterpart. Let X be a Banach space. Given a sequence $\{b_k\}_{k \in \mathbb{Z}}$ in $l^1(\mathbb{Z})$ we denote by $\mathscr{B}$ the convolution operator in $l^p(\mathbb{Z}; X)$, $1 < p < \infty$, defined by

$$(\mathscr{B}u)_k = (b * u)_k = \sum_{l \in \mathbb{Z}} b_{k-l} u_l \ , \tag{7.1}$$

where $u \in l^p(\mathbb{Z}; X)$ and by $\|\mathscr{B}\|_p$ its norm in $l^p(\mathbb{Z}; X)$.

Lemma A.1. *Let $(\Omega, \mathscr{M}, \mu)$ be a σ-finite measure space, $1 < p < \infty$, X be a Banach space and $\mathscr{U} : L^p(\Omega; X) \to L^p(\Omega; X)$ be a bounded linear bijective operator satisfying:*

$$c = \sup_{k \in \mathbb{Z}} \|\mathscr{U}^k\|_p < \infty \ . \tag{7.2}$$

Let $b \in l^1(\mathbb{Z})$. Then we have the estimate:

$$\Big\| \sum_{k \in \mathbb{Z}} b_k \mathscr{U}^k f \Big\|_p \leq c^2 \|\mathscr{B}\|_p \|f\|_p \quad \text{for all } f \in L^p(\Omega; X) \ . \tag{7.3}$$

Proof. It is sufficient to prove (A.3) for $b \in c_{00}$, the space of finite sequences. Let N be such that $b_k = 0$ for $|k| > N$. Let $\varepsilon > 0$ and $f \in L^p(\Omega; X)$ be given. Choose M so large that $\frac{2M+2N+1}{2M+1} \leq 1 + \varepsilon$. Since for all $g \in L^p(\Omega; X)$ and $l \in \mathbb{Z}$,

$$\|g\|_p \leq c \|\mathscr{U}^l g\|_p \ , \quad \text{we obtain}$$

$$\Big\| \sum_{k=-N}^{N} b_k \mathscr{U}^{-k} f \Big\|_p^p \leq \frac{c^p}{2M+1} \sum_{l=-M}^{M} \Big\| \sum_{k=-N}^{N} b_k \mathscr{U}^{l-k} f \Big\|_p^p \ .$$

Let $\chi(n)=1$ if $|n|\leqq M+N$, 0 otherwise; then

$$\sum_{l=-M}^{M}\left\|\sum_{k=-N}^{N} b_k\mathcal{U}^{l-k}f\right\|_p^p=\sum_{l=-M}^{M}\left\|\sum_{k\in\mathbb{Z}} b_k\chi(l-k)\mathcal{U}^{l-k}f\right\|_p^p$$

$$\leqq\sum_{l\in\mathbb{Z}}\int_\Omega\left|\sum_{k\in\mathbb{Z}} b_k\chi(l-k)(\mathcal{U}^{l-k}f)(\omega)\right|^p d\mu(\omega)$$

$$=\int_\Omega\sum_{l\in\mathbb{Z}}\left|\sum_{k\in\mathbb{Z}} b_k\chi(l-k)(\mathcal{U}^{l-k}f)(\omega)\right|^p d\mu(\omega)$$

$$\leqq\|\mathcal{B}\|_p^p\int_\Omega\sum_{k\in\mathbb{Z}}\chi(k)|(\mathcal{U}^k f)(\omega)|^p d\mu(\omega)$$

$$=\|\mathcal{B}\|_p^p\sum_{k=-(M+N)}^{M+N}\|\mathcal{U}^k f\|_p^p\leqq c^p(1+\varepsilon)(2M+1)\|\mathcal{B}\|_p^p\|f\|_p^p$$

letting $\varepsilon\to 0+$, the result follows. □

Next we consider the analogue of Theorem 4.16 in Coifman and Weiss [11].

Theorem A.2. *Let $(\Omega,\mathcal{M},\mu)$ be a σ-finite measure space, $1<p<\infty$, X be a Banach space and $T: L^p(\Omega)\to L^p(\Omega)$ be a positive, linear contraction. Let $\mathcal{T}: L^p(\Omega;X)\to L^p(\Omega;X)$ be its extension as defined in Lemma 5.1, and let $b\in l^1(\mathbb{Z})$, such that $b_k=0$, for $k<0$. Then we have the estimate:*

$$\left\|\sum_{k=0}^{\infty} b_k\mathcal{T}^n f\right\|_p\leqq\|\mathcal{B}\|_p\|f\|_p \tag{7.4}$$

for all $f\in L^p(\Omega;X)$.

Proof. It follows from the dilation theorem of Akcoglu and Sucheston [1], that there exist a measure space Σ, a positive invertible isometry $U: L^p(\Sigma)\to L^p(\Sigma)$, a positive isometric imbedding $D: L^p(\Omega)\to L^p(\Sigma)$, a projection $P: L^p(\Sigma)\to L^p(\Sigma)$ with norm 1, such that

$$DT^n=PU^nD\ ,\qquad n\in\mathbb{N}_0\ . \tag{7.5}$$

By Lemma 5.1, we extend all these operators to the X-valued case, thanks to positivity! From Lemma A.1, we obtain

$$\left\|\sum_{k=0}^{\infty} b_k\mathcal{T}^k f\right\|_p=\left\|\mathcal{D}\sum_{k=0}^{\infty} b_k\mathcal{T}^k f\right\|_p$$

$$=\left\|\sum_{k=0}^{\infty} b_k\mathcal{D}\mathcal{T}^k f\right\|_p=\left\|\sum_{k=0}^{\infty} b_k\mathcal{P}\mathcal{U}^k\mathcal{D}f\right\|_p$$

$$\leqq\left\|\sum_{k=0}^{\infty} b_k\mathcal{U}^k\mathcal{D}f\right\|_p\leqq\|\mathcal{B}\|_p\|\mathcal{D}f\|_p=\|\mathcal{B}\|_p\|f\|_p\ .\quad\square$$

We can now deduce Theorem 5.6 from Theorem A.2, as in [11]. Observe that we may restrict our attention to the case where b has compact support. We shall construct sequences $\{b_N(j)\}$ such that

$$\lim_{N\to\infty} \sum_{j=0}^{\infty} b_N(j)\mathcal{T}^j(1/N)f = \int_0^\infty b(\tau)\mathcal{T}(\tau)f\,d\tau\ , \tag{7.6}$$

for every $f \in L^p(\Omega; X)$, and

$$\|\mathcal{B}_N\|_p \leqq \|\mathcal{B}\|_p\ , \quad \text{for all } N \geqq 0\ , \tag{7.7}$$

where $\mathcal{B}_N$ denotes the convolution operators in $l^p(\mathbb{Z}; X)$ associated with the sequences $\{b_N(j)\}$ and $\mathcal{B}$ the convolution operator with kernel b in $L^p(\mathbb{R}; X)$.

Then (7.6) and (7.7) imply

$$\|\hat{b}(\mathcal{A})\| \leq \varlimsup_{N\to\infty} \left\| \sum_{j=0}^{\infty} b_N(j)\mathcal{T}^j(1/N)f \right\|_p$$

$$\leqq \varlimsup_{N\to\infty} \|\mathcal{B}_N\|_p \|f\|_p \leqq \|\mathcal{B}\|_p \|f\|_p\ .$$

The sequences $\{b_N(j)\}$ are defined as in [11] by means of

$$b_N(j) = \int_{-1/N}^{1/N} b\left(\frac{j}{N}+s\right)(1-N|s|)\,ds$$

$$= \int_0^1 \int_0^1 \frac{1}{N}\, b\left(\frac{j+t-s}{N}\right) ds\,dt\ . \tag{7.8}$$

In order to prove (7.6), observe that if $g \in C(\mathbb{R}_+; L^p(\Omega; X))$ is piecewise linear with nodes at $\{j/N\}_{j=0}^{\infty}$, then

$$\int_0^\infty b(\tau)g(\tau)\,d\tau = \sum_{j=0}^{\infty} b_N(j)g(j/N)\ .$$

Thus, if g_N denotes the continuous piecewise linear function with nodes at $\{j/N\}_{j=0}^{\infty}$ such that $g_N(j/N) = \mathcal{T}(1/N)^j f$, then

$$\left\| \int_0^\infty b(\tau)\mathcal{T}(\tau)f\,d\tau - \sum_{j=0}^{\infty} b_N(j)g_N(j/N) \right\|_p$$

$$= \left\| \int_0^\infty b(\tau)(\mathcal{T}(\tau)f - g_N(\tau))\,d\tau \right\|_p \to 0 \quad \text{as } N \to \infty\ ,$$

i.e. (7.6) holds.

To prove (7.7) we introduce linear operators $\mathcal{P}_N : l^p(\mathbb{Z}; X) \to L^p(\mathbb{R}; X)$ and $\mathcal{Q}_N : L^p(\mathbb{R}; X) \to l^p(\mathbb{Z}; X)$ by means of

$$(\mathcal{P}_N g)(t) = N^{1/p} \sum_{j\in\mathbb{Z}} \chi_{[j/N,(j+1)/N)}(t)g(j)\ , \qquad t \in \mathbb{R}$$

and

$$(\mathcal{Q}_N f)(j) = N^{1/q} \int_{j/N}^{(j+1)/N} f(t)\,dt\ , \qquad j \in \mathbb{Z}\ .$$

It is then easy to show that $\|\mathscr{P}_N\|=1$ and $\|\mathscr{Q}_N\|\leqq 1$ for all N. For the convolution operators $\mathscr{B}_N$ we now have the representation

$$\mathscr{B}_N g = b_N * g = \mathscr{Q}_N(b * \mathscr{P}_N g) = \mathscr{Q}_N \mathscr{B} \mathscr{P}_N g \quad ;$$

this identity obviously implies (7.7). □

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Received May 17, 1989; in revised form August 31, 1989