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Néron models in the setting of formal and rigid geometry

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Let X_K be a smooth K-scheme of finite type, where K is the field of fractions of a discrete valuation ring R. A Néron model of X_K is a smooth R-model X which satisfies the so-called Néron mapping property: If Z is a smooth R-scheme, any K-morphism $f_K : Z_K \to X_K$ between generic fibres extends uniquely to an R-morphism $f : Z \to X$. In his epochal paper [12], Néron has shown the existence and quasi-compactness of such models for abelian varieties.

In the present paper, which contains parts of the doctoral thesis of the second author, the theory of Néron models is transferred to the context of formal and rigid geometry, interpreting rigid spaces X_K over a complete and discretely valued field K as generic fibres of appropriate formal R-schemes X, so-called formal R-models; cf. [4]. The definition of formal Néron models is quite analogous to the one we have in the classical case, although we do not require that a Néron model U of a rigid space X_K is a formal R-model of X_k itself; it is just a formal R-model of a suitable open rigid subspace $U_K \subset X_K$. As main result we show that Néron's existence theorem remains valid for rigid groups with a bounded set of points $X_K(K^{sh})$, where K^{sh} is the field of fractions of a strict henselization of R. However, we do not restrict ourselves to quasi-compact Néron models and investigate also the connection between a Néron model $\mathfrak{X}$ (or better, Néron lft-model in the terminology of [7]) of a finite type K-group scheme $\mathfrak{X}_K$ and the Néron model U of its associated rigid K-group X_K. As we show, one passes from $\mathfrak{X}$ to U by means of formal completion, at least if $\mathfrak{X}$ is quasi-compact or if $\mathfrak{X}_K$ (and hence $\mathfrak{X}$) are commutative. If one wants to extend this relationship to the general case, it seems that our definition of Néron models has to be relaxed slightly, so that it better corresponds to the definition of Néron lft-models in the scheme case. Namely, instead of requiring that the generic fibre U_K of U is an open rigid subspace of X_K, one has to ask for a monomorphism $U_K \to X_K$, which is an open immersion on quasi-compact open parts of U_K.

To mention a possible application, Néron models of rigid groups may be considered as a first step towards general structural theorems for rigid groups, just as we have them in conjunction with semi-abelian reduction for abelian varieties and their rigid uniformizations. On the other hand, Néron models of rigid groups are useful in computing component groups of ordinary Néron models. In this paper we just consider the easy case of an abelian variety $\mathfrak{X}_K$, admitting a split uniformization $X_K = E_K/M_K$ with uniformizing group E_K and a split lattice M_K. Then the Néron model U of the rigid K-group X_K is just the quotient E/M of the Néron models E of E_K and M of M_K so that the component group Φ_U of U is isomorphic to the quotient Φ_E/Φ_M, where the component group Φ_M coincides with the special fibre of M. Since U is the formal completion of the classical Néron model $\mathfrak{X}$ of $\mathfrak{X}_K$, we see that the component group $\Phi_{\mathfrak{X}}$ is isomorphic to Φ_E/Φ_M.

1 Definitions and statement of the existence theorem

In the following, let R be a complete discrete valuation ring, K its field of fractions, and k its residue field. Then, if $\mathfrak{X}$ is an R-scheme, it consists of two fibres, the generic fibre $\mathfrak{X}_K$ and the special fibre $\mathfrak{X}_k$. Furthermore, $\mathfrak{X}$ is called an **R-model** of its generic fibre $\mathfrak{X}_K$. The problem of constructing a Néron model for a smooth K-scheme $\mathfrak{X}_K$ of locally finite type consists in finding a "good" R-model $\mathfrak{X}$ of $\mathfrak{X}_K$.

In rigid geometry, the setting is quite similar. The analogue of an R-scheme of locally finite type is a formal R-scheme of locally tf (topologically finite) type; in this paper we will just say **formal R-scheme**, assuming tacitly that it is of locally tf type. Local parts of such a formal R-scheme X are of type Spf $R\langle\zeta_1,\ldots,\zeta_n\rangle/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal in the restricted power series ring $R\langle\zeta_1,\ldots,\zeta_n\rangle$. Certainly, a formal R-scheme X has a special fibre $X_k = X \otimes_R k$, but, in our situation, it also has a generic fibre X_K; cf. [4]. The latter is a classical rigid K-space in the sense of [13] or [2]. Locally, on any open affine part Spf $A \subset X$, the generic fibre of X is given by the rigid K-space Sp $A \otimes_R K$. Similarly as before, X is called a **formal R-model** of X_K.

In order to deal with Néron models, it must be pointed out that models of rigid spaces have to be viewed from a slightly different way. The reason is that for a formal R-scheme X of locally tf type, any point of the generic fibre X_K specializes into a point of the special fibre X_k; see [4, 3.4]. On the other hand, Néron models of (ordinary) K-schemes X_K live from the fact that one can modify R-models by removing closed parts from the special fibre, leaving the generic fibre intact.

Definition 1.1 *Let X_K be a smooth rigid K-space. A* **(formal) Néron model of** *X_K consists of a smooth formal R-scheme U, whose generic fibre U_K is an open rigid subspace of X_K, and which satisfies the following universal mapping property:*

Given a smooth formal R-scheme Z and a morphism of rigid K-spaces $f_K : Z_K \to X_K$, *it extends uniquely to a morphism of formal R-schemes* $f : Z \to U$.

Of course, in a more precise way, we would have to say that $f_K : Z_K \to X_K$ restricts to a morphism of rigid K-spaces $Z_K \to U_K$ and that the latter extends uniquely to a morphism of formal R-schemes $f : Z \to U$. Also note that the uniqueness of such extensions is automatic; see for example [4, assertion (b) in the proof of 4.1]. It is clear that the Néron model U of X_K, if it exists, is unique and that the formation of U is compatible with étale base change on R. Furthermore, U will be separated if X_K is separated (use [4, 4.7]). Dealing with group objects, U is a formal R-group scheme if X_K is a rigid K-group. Also note that formal R-group schemes and rigid K-groups are automatically separated.

To state the main result to be proved in this paper, let R^{sh} be a strict henselization of R, and let K^{sh} be the field of fractions of R^{sh}. Although R^{sh} might not be complete, we can consider R^{sh}-valued points of formal R-schemes, using the fact that R^{sh} is a direct limit of complete discrete valuation rings which are étale and, hence, finite over R. Similarly, there is the notion of K^{sh}-valued points of rigid K-spaces.

Theorem 1.2 *A smooth rigid K-group* X_K *admits a quasi-compact formal Néron model U if and only if the group* $X_K(K^{sh})$ *of its* K^{sh}*-valued points is bounded; i.e., contained in a quasi-compact rigid subspace of* X_K.

The only if part is trivial. To prove the if part, the first step is to construct a weak Néron model, just as in the classical case.

Definition 1.3 *Let* X_K *be a smooth rigid K-space. A* **weak (formal) Néron model of** X_K *is a smooth formal R-scheme U, whose generic fibre* U_K *is an open rigid subspace of* X_K, *and which has the property that the cononical map* $U(R^{sh}) \to X_K(K^{sh})$ *is bijective.*

It is clear that any formal Néron model satisfies the mapping property required for a weak Néron model, whereas a converse of this assertion is true for groups:

Criterion 1.4 *Let* X_K *be a smooth rigid K-group, and let U be a smooth formal R-group scheme whose generic fibre* U_K *is a retrocompact open rigid subgroup of* X_K. *Then U is a Néron model of* X_K *if and only if it is a weak Néron model of* X_K.

Recall that an open rigid subspace $U_K \subset X_K$ is called **retrocompact** if the inclusion map $U_K \to X_K$ is quasi-compact; i.e., if $U_K \cap V_K$ is quasi-compact for any quasi-compact open rigid subspace $V_K \subset X_K$. This is a technical condition which is automatically satisfied if U_K is quasi-compact and X_K is quasi-separated. We will prove the criterion 1.4 in Sect. 2, using it later in Sect. 5 to derive the assertion of Theorem 1.2. More precisely, in the situation of 1.2, we will first construct a weak Néron model of X_K and then modify it in such a way that it becomes a formal R-group scheme.

2 Weak Néron models and their mapping property

Just as for ordinary R-schemes, there is the notion of R-rational or R-birational maps between formal R-schemes X and Y; we will only consider the case, where X and Y are *smooth*, which is enough for our purposes. By an R-rational map $X \dashrightarrow Y$ we understand an equivalence class of R-morphisms $X' \to Y$, where X' is R-dense open in X. Any R-rational map $f : X \dashrightarrow Y$ has a domain of definition $\operatorname{dom}(f)$, and there is a well-defined morphism $\operatorname{dom}(f) \to Y$ in case Y is separated.

Let us start by recalling some technical facts, which will be needed.

Lemma 2.1 *Consider a flat morphism $u : X' \to X$, as well as an R-rational map $f : X \dashrightarrow Y$ of smooth formal R-schemes X', X, Y. Assume that Y is separated. Then $f \circ u$ is an R-rational map satisfying*

$$\operatorname{dom}(f \circ u) = u^{-1}(\operatorname{dom} f) \, .$$

In particular, f is defined everywhere if u is faithfully flat and $f \circ u$ is defined everywhere.

Proof. Reduce modulo powers of a uniformizing element $\pi \in R$ and apply [7, 2.5/5]. □

Lemma 2.2 *Let X, Y be flat formal R-schemes and $f_K : X_K \to Y_K$ a K-morphism between associated generic fibres. Assume that the special fibre X_k is reduced.*
(i) *If X is non-empty, there is a non-empty open part $X' \subset X$, such that $f_K|_{X'_K}$ extends to a morphism of formal R-schemes $f : X' \to Y$.*
(ii) *If Y is affine, assertion* (i) *is true for $X = X'$; i.e., f_K extends to a morphism of formal R-schemes $f : X \to Y$.*

Proof. In order to verify assertion (i), we may assume that X is quasi-compact. Then it follows from [4, 2.5 and 4.1] that there is an admissible formal blowing-up $X' \to X$ of some coherent open ideal $\mathcal{J}$ on X such that $f_K : X_K \to Y_K$ extends to a morphism of formal R-schemes $f : X' \to Y$. If we divide $\mathcal{J}$ by an appropriate power of a uniformizing element π of R, we can assume that $\mathcal{J}$ is not contained in $\pi \mathcal{O}_X$. Since X_k is reduced, the ideal $\pi \mathcal{O}_X$ equals its radical. So the center of the blowing-up $X' \to X$ is strictly contained in the special fibre X_k, and there is a non-empty open part $V \subset X$, over which the blowing-up of $\mathcal{J}$ is an isomorphism. Restricting $f : X' \to Y$ to the inverse image of V with respect to the blowing-up $X' \to X$, we get the desired extension of f_K.

In the situation of assertion (ii), we may assume that both, X and Y are affine, say $X = \operatorname{Spf} A$ and $Y = \operatorname{Spf} B$. Then we have to show that any K-homomorphism $\varphi_K : B \otimes_R K \to A \otimes_R K$ maps the subring $B \subset B \otimes_R K$ into the subring $A \subset A \otimes_R K$. However, the latter is clear, since φ_K is contractive with respect to the supremum semi-norm, and since $A \otimes_R k$ is reduced, so that A consists of all elements of $A \otimes_R K$ having supremum semi-norm $\leqq 1$. □

We can draw some interesting conclusions from assertion (i) of 2.2.

Proposition 2.3 *Let X_K be a rigid K-group which extends to a smooth formal R-group scheme X. Then X is unique, up to canonical isomorphism.*

Proposition 2.4 *Let X, Y be formal R-group schemes, where X is smooth, and let $f_K : X_K \to Y_K$ be a morphism of rigid K-groups. Then f_K extends uniquely to a morphism of formal R-group schemes $f : X \to Y$.*

Proofs of 2.3 and 2.4 Since 2.3 is a consequence of 2.4, we need only verify 2.4. Using 2.2(i), we know that f_K extends to an R-morphism $X' \to Y$ on some non-empty open part $X' \subset X$. If X contains enough R-valued points, we can use translations in order to show that f_K extends to a morphism of R-group schemes $f : X \to Y$ which, automatically, is unique. In the general case, we must replace the ground field K by a finite separable extension in order to extend f_K to the identity component or some other component of X. Faithfully flat descent, applied to the situation obtained after reducing modulo powers of a uniformizing element $\pi \in R$, shows then that the extension is defined over R. □

The assertion of 2.4 says that we can view the category of smooth formal R-group schemes as a full subcategory of the category of rigid K-groups. This is why we will sometimes make no difference in our notation between a smooth formal R-group scheme X and its associated rigid K-group X_K.

Next we want to show that weak Néron models satisfy a mapping property which is similar to the one of Néron models.

Proposition 2.5 *Let X_K be a smooth rigid K-space, and let U be a smooth formal R-model of some retrocompact open rigid subspace $U_K \subset X_K$. Then the following are equivalent:*

(i) U is a weak Néron model of X_K; i.e., U (is smooth and) the canonical map $U(R^{sh}) \to X_K(K^{sh})$ is bijective.

(ii) Any rigid K-morphism $f_K : Z_K \to X_K$, where Z_K is the generic fibre of a smooth formal R-scheme Z, extends uniquely to an R-rational map $f : Z \dashrightarrow U$.

Proof. We only have to show that condition (i) implies condition (ii), the converse is trivial. So assume (i) and consider a rigid K-morphism $f_K : Z_K \to X_K$ with Z being a smooth formal R-model of Z_K; we may assume that Z is affine and irreducible. Applying 2.2(i) and replacing Z by some non-empty open part, there is an open affinoid subspace $V_K \subset X_K$ with $f_K(Z_K) \subset V_K$. Since $V_K \cap U_K$ is quasi-compact by our assumption on U_K, we can find a formal R-model V of V_K, containing an open formal subscheme V', whose generic fibre coincides with $V_K \cap U_K$; see [4, 4.4]. Using 2.2(i), we can restrict Z again and thereby assume that $f_K : Z_K \to V_K$ extends to an R-morphism $f : Z \to V$. Now Z, as a smooth formal scheme over R, contains an R^{sh}-valued point a.

Then $f(a)$ is an R^{sh}-valued point of V', since its generic fibre belongs to U_K. Thus, replacing Z by $f^{-1}(V')$, we can assume that f_K maps Z_K into U_K. Applying 2.2(i) once more, we see that f_K extends to an R-rational map $f : Z \dashrightarrow U$. The uniqueness of f is automatic; see [4, statement (b) in the proof of 4.1]. □

Next we want to adapt an extension theorem of Weil for rational maps into group schemes to our situation; for the corresponding result which is used in the case of ordinary Néron models, see [7, 4.4/1].

Theorem 2.6 *Let U be a smooth formal R-group scheme, whose generic fibre U_K is an open rigid subgroup of some rigid K-group X_K. Furthermore, consider a smooth formal R-scheme Z and a K-morphism $v_K : Z_K \to X_K$, and assume that v_K extends to an R-rational map $v : Z \dashrightarrow U$. Then $v_K(Z_K) \subset U_K$ and v is defined everywhere; i.e., is a morphism of formal R-schemes.*

Proof. We may assume that Z is quasi-compact and connected. Consider the morphism

$$w_K : Z_K \times_K Z_K \to X_K, \quad (z_1, z_2) \mapsto v_K(z_1)v_K(z_2)^{-1},$$

as well as its R-rational extension

$$w : Z \times_R Z \dashrightarrow U, \quad (z_1, z_2) \mapsto v(z_1)v(z_2)^{-1}.$$

Let V (resp. W) be the domain of definition of v (resp. w), where $V \times_R V \subset W$. We want to show that W contains the diagonal Δ of $Z \times_R Z$. Proceeding indirectly, let us assume that the latter is not the case. Then there exists a closed point $z \in \Delta - W$.

Let $U' \subset U$ be an affine open formal subscheme containing the unit section of U. Since $w|_{W \cap \Delta}$ factors through the unit section and, hence, through U', there is a formal open neighborhood $W' \subset W$ of $W \cap \Delta$ such that $w(W') \subset U'$. Then we have

$$W' \cap \Delta = W \cap \Delta \supset (V \times_R V) \cap \Delta$$

and, identifying Δ with Z, we see that $W' \cap \Delta$ is R-dense in Δ since V is R-dense in Z. Now consider the closed subset $(Z \times_R Z) - W' \subset Z \times_R Z$, and let F be the union of all its irreducible components which do not contain z. Then $Y = (Z \times_R Z) - F$ is an open formal subscheme of $Z \times_R Z$ which contains W' and the point z. If d is the relative dimension of Z over R, there are functions $f_1, \dots, f_{2d-1} \in \mathcal{O}_{Y,z}$ vanishing at z, such that, locally at z,

(i) *the closed formal subscheme $\Delta \cap Y$ of Y is defined by $f_1, \dots, f_d$,*
(ii) *the functions $f_1, \dots, f_{2d-1}$ define a closed formal subscheme $M \subset Y$ of relative dimension* 1,
(iii) *writing $N = Y - W'$, we have $M \cap N = \{z\}$ (use that $W' \cap \Delta$ is R-dense in Δ).*

Let $Y' \subset Y$ be an affine formal neighborhood of z such that the above is true on Y'. If we switch to the associated rigid situation, we see that w_K maps $Y'_K \cap \Delta_K$ onto the unit section of X_K. Then, by [10, 1.6], there is a tubular neighborhood $Y'_K(\varepsilon^{-1}f_1, \ldots, \varepsilon^{-1}f_d)$ of $Y'_K \cap \Delta_K$, which is mapped by w_K into U'_K. In particular, w_K maps the "Hartogs figure"

$$Y'_K(\varepsilon^{-1}f_1, \ldots, \varepsilon^{-1}f_{2d-1}) \cup (Y' - N)_K$$

into U'_K, and the continuation theorem [1, Sect. 3], or [11, Theorem 7], shows that, in fact, all of Y'_K is mapped into U'_K. But then, by 2.2(ii), the morphism $Y'_K \to U'_K$ extends to a morphism of formal R-schemes $Y' \to U'$, and we see that Y' is contained in the domain of definition of $w : Z \times_R Z \dashrightarrow U$. In particular, w is defined at z which, however, contradicts the fact that $z \in \Delta - W$. Thus, we have shown $\Delta \subset W$.

Now, in order to show that $v : Z \dashrightarrow U$ is defined everywhere, it is enough to construct a faithfully flat morphism $f : Z' \to Z$ of smooth formal R-schemes such that $v \circ f$ is defined everywhere; cf. 2.1. Setting $Z' = W \cap (Z \times_R V)$, we can consider the morphism $f : Z' \to Z$ which is induced from the projection of $Z \times_R Z$ onto the first factor. Then f is smooth and, in particular, flat. Furthermore, f is surjective. To verify this, we may look at special fibres and, thus, think in terms of k-schemes. Fixing a point $z \in Z$, we can apply the base change $k \hookrightarrow k(z)$ and thereby assume that z is a k-valued point of Z. Then $W \cap (z \times Z)$ is an open neighborhood of (z,z) in $z \times Z$, and it must meet $z \times V$, since V is dense in Z. Hence, $Z' \cap (z \times Z)$ is non-empty and is mapped by f onto z.

It remains to show that $v \circ f$ is defined everywhere. But this is clear, since it coincides on $V \times_R V$ with the morphism

$$Z' \to U, \quad (z, z') \mapsto w(z, z')v(z') .$$

So $v \circ f$ is defined everywhere, and the same is true for v. □

As a corollary, we see that 2.5 and 2.6 imply the assertion of the criterion 1.4. We want to use the criterion in order to determine the formal Néron model of the multiplicative group $\mathbb{G}_{m,K}$.

Example 2.7 Consider the multiplicative group $\mathbb{G}_{m,K}$ as a rigid K-group and the formal multiplicative group $\bar{\mathbb{G}}_{m,R}$ as an open rigid subgroup of $\mathbb{G}_{m,K}$. Then, if $\pi \in R$ is a uniformizing element, $\mathbb{G} = \bigcup_{n \in Z} \pi^n \cdot \bar{\mathbb{G}}_{m,R}$ makes sense as a smooth formal R-group scheme and as an open rigid subgroup of $\mathbb{G}_{m,K}$. Since G contains all K^{sh}-valued points of $\mathbb{G}_{m,K}$, we see by 1.4 that it is the Néron model of $\mathbb{G}_{m,K}$. We can describe the component group Φ_G of G in an intrinsic way. If Y is the group of characters of $\mathbb{G}_{m,K}$, there is a canonical pairing $\Phi_G \times Y \to \mathbb{Z}$ by evaluating characters $y \in Y$ on components of $G \subset \mathbb{G}_{m,K}$ and taking values in the value group $\mathbb{Z}$ of K. The pairing gives rise to an identification $\Phi_G = Y^*$, where $Y^* = \mathrm{Hom}(Y, \mathbb{Z})$ is the dual of Y.

3 The construction of weak Néron models

Just as in the case of ordinary Néron models, the first step of the proof of 1.2 is the construction of weak Néron models via Néron's smoothening process. Since this process, as presented in [7, Chap. 3], carries over almost literally from R-schemes to formal R-schemes, we restrict ourselves to just giving a sketch. Of course, we will use freely standard facts about smoothness; for example, see [5, Sect. 1], for smoothness in terms of formal schemes and [6, Sect. 2] for smoothness in terms of classical rigid spaces.

The smoothening process, in its classical sense, involves blowing-ups with centers in the special fibres of R-schemes. The corresponding notion in the formal scheme setting is the one of *formal blowing-up* with centers in special fibres of formal R-schemes; see [4, Sect. 2], where, more generally, (admissible) formal blowing-ups of coherent open ideals on quasi-compact formal R-schemes are defined. If $X' \to X$ is such a formal blowing-up with center $Y_k \subset X_k$ and corresponding ideal $\mathcal{J} \subset \mathcal{O}_X$, the open formal subscheme of X', where $\mathcal{J}\mathcal{O}_{X'}$ is generated by a uniformizing element $\pi \in R$, is called the **dilatation** of Y_k in X; it is denoted by X'_π. As in the classical case, see [7, 3.2/1], X'_π is uniquely characterized as a flat formal R-scheme over X, whose special fibre $(X'_\pi)_k$ lies over Y_k and which satisfies the following mapping property:

If $v : Z \to X$ is a morphism of formal R-schemes with Z R-flat and v_k factoring through Y_k, then v lifts uniquely to a morphism of formal R-schemes $Z \to X'_\pi$.

In particular, in place of v we can consider an R^{sh}-valued point of X. Since a formal blowing-up $X' \to X$ is an isomorphism over the complement of its center, it follows that the canonical map $X'(R^{sh}) \to X(R^{sh})$ is bijective. An alternative way to see this is by using [4, 3.3]. Now let us formulate the assertion of the smoothening process.

Theorem 3.1 *Let X be a quasi-compact formal R-scheme, whose generic fibre X_K is smooth. Then there is a morphism of formal R-schemes $f : X' \to X$, which is the composition of a sequence of formal blowing-ups with centers in the corresponding special fibres, such that any R^{sh}-valued point of X factors through the smooth locus of X'.*

Since, by 2.2(ii), any K^{sh}-valued point of X_K extends uniquely to an R^{sh}-valued point of X, we see:

Corollary 3.2 *In the situation of* 3.1, *the smooth locus of X' is a weak Néron model of $X_K = X'_K$.*

Furthermore, if we use the existence of formal R-models of quasi-compact and quasi-separated rigid K-spaces, see [4, 4.1], we obtain an existence assertion for weak Néron models:

Corollary 3.3 *Let X_K be a smooth rigid K-space, which is quasi-separated. Assume that the set of K^{sh}-valued points of X_K is bounded. Then X_K admits a weak Néron model, which is quasi-compact.*

To sketch the proof of 3.1, let $\Omega^1_{X/R}$ be the $\mathcal{O}_X$-module of differential 1-forms on X. Fixing an R^{sh}-valued point a of X, let $a^*\Omega^1_{X/R}$ be its pull-back to R^{sh}. We will view $a^*\Omega^1_{X/R}$ as a true R^{sh}-module and write $\delta(a)$ for the length of its torsion part. The latter measures the defect of smoothness of X at a. In fact, one shows as in [7, 3.3/1]:

The point a factors through the smooth locus of X if and only if $\delta(a) = 0$.

Using the Jacobi criterion, one can characterize $\delta(a)$ in terms of minors of Jacobi matrices and then show as in [7, 3.3/3] that $\delta(a)$ is bounded as a function on $X(R^{sh})$.

Next, write k_s for the residue field of R^{sh}, so that k_s is a separable algebraic closure of k. Consider the following condition for closed subschemes $Y_k \subset X_k$:

(N) *The family of all k_s-valued points of Y_k, which lift to R^{sh}-valued points of X, is schematically dense in Y_k.*

If $Y_k \subset X_k$ satisfies this condition, it follows that Y_k is geometrically reduced and, in particular, that its smooth locus is dense in Y_k; cf. [7, 3.3/4]. Now we can formulate the key lemma, which allows to lower the defect of smoothness of X.

Lemma 3.4 *In the situation of* 3.1, *let $Y_k \subset X_k$ be a closed subscheme satisfying condition* (N), *and let U_k be an open subscheme of Y_k such that U_k is smooth and the pull-back $\Omega^1_{X/R}|_{U_k}$ is locally free. Let $X'_\pi \to X$ be the dilatation of Y_k in X. Then, if $a \in X(R^{sh})$ specializes into a point $a_k \in U_k(k_s)$, its unique lifting $a' \in X'_\pi(R^{sh})$ satisfies*

$$\delta(a') \leqq \max\{0, \delta(a) - 1\} .$$

In particular, we have $\delta(a') < \delta(a)$ if a is not contained in the smooth locus of X and specializes into a point of U_k.

For the proof of this assertion, one follows literally the same argumentation, as given in [7, 3.3/5], just replacing polynomials by restricted power series. This being done, one uses the stratification technique of [7, Sect. 3.4] in order to derive the assertion of 3.1 from the lemma. Also here, the procedure is by literal translation. This concludes our sketch of proof of 3.1.

4 From weak Néron models to birational group laws

Having constructed weak Néron models, the next step in the proof of 1.2 consists in selecting so-called minimal components from those models, where the minimality is defined using orders of invariant differential forms. To make the machinery of [7, Sect. 4.3] work, we need some technical results.

Lemma 4.1 *Let Z be a smooth formal R-scheme. Then, for any generic point ζ of the special fibre Z_k, the local ring $\mathcal{O}_{Z,\zeta}$ is a discrete valuation ring. Any uniformizing element $\pi \in R$ is a uniformizing element for $\mathcal{O}_{Z,\zeta}$.*

Proof. Let π be a uniformizing element of R. Then $\mathcal{O}_{Z,\zeta}/(\pi)$ is isomorphic to $\mathcal{O}_{Z_k,\zeta}$, and the latter is a field, due to the fact that Z_k is smooth over k. Thus, π generates a maximal ideal in $\mathcal{O}_{Z,\zeta}$, and we see that $\mathcal{O}_{Z,\zeta}$, being an integral domain, is a discrete valuation ring. □

Lemma 4.2 *Let X be a formal R-scheme with geometrically reduced special fibre, and let $U \subset X$ be an R-dense open subscheme. Then any two sections*

$$h' \in \Gamma(U, \mathcal{O}_X), \quad h_K \in \Gamma(X_K, \mathcal{O}_{X_K}),$$

coinciding on U_K, extend uniquely to a section $h \in \Gamma(X, \mathcal{O}_X)$.

Proof. See [6, 5.4]. □

Lemma 4.3 *Let Z be a smooth and irreducible formal R-scheme and ζ the generic point of the special fibre Z_k. Furthermore, consider a line bundle $\mathcal{L}$ on Z, and a global section f_K of $\mathcal{L}_K$, the line bundle induced from $\mathcal{L}$ on the generic fibre Z_K of Z. Assume that f_K does not vanish identically on Z_K, and let π be a uniformizing element of R.*
(i) *There is a unique integer $n \in \mathbb{Z}$ such that $\pi^{-n} f_K$ extends to a generator of $\mathcal{L}_\zeta$ as $\mathcal{O}_{Z,\zeta}$-module.*
(ii) *If n is as in* (i), *$\pi^{-n} f_K$ extends to a global section f of $\mathcal{L}$. Furthermore, if f_K has no zeros on Z_K, the same is true for f on Z.*

In the above situation, the integer n is called the **order of f_K at $\zeta \in Z_k$**; we write $n = \mathrm{ord}_\zeta f_K$.

Proof of 4.3 We may work locally on Z and thereby assume $\mathcal{L} = \mathcal{O}_Z$ with Z being affine. Writing

$$\mathcal{O}_{Z,\zeta} = \varinjlim_{i \in I} A_i ,$$

where $(\mathrm{Spf} A_i)_{i \in I}$ is the system of all affine open formal neighborhoods of ζ in Z, all restriction maps $A_i \longrightarrow A_j$ are injective, since all maps $A_i \otimes_R k \longrightarrow A_j \otimes_R k$ are injective, due to the fact that Z is irreducible. Thus, f_K induces a non-zero element $f_{K,\zeta}$ in

$$\mathcal{O}_{Z,\zeta} \otimes_R K = \varinjlim_{i \in I} (A_i \otimes_R K).$$

Using 4.1, there is a well-defined integer $n \in \mathbb{Z}$, such that $\pi^{-n} f_{K,\zeta}$ is a unit in $\mathcal{O}_{Z,\zeta}$. This shows assertion (i).

We just have seen that $\pi^{-n} f_K$ extends to a section f of $\mathcal{O}_Z$ on some open formal neighborhood $Z' \subset Z$ of ζ. But then, using 4.2, we see that $\pi^{-n} f_K$ is defined on all of Z. If f_K has no zeros on Z_K, the same reasoning applies to $\pi^n f_K^{-1}$, and it follows that $\pi^{-n} f_K$ is invertible on Z. □

Let us return now to the situation of 1.2. So we consider a smooth rigid K-group X_K, whose set of K^{sh}-valued points is bounded; let $d = \dim X_K$. As in the case of ordinary K-group schemes, there is a left-invariant differential form ω of degree d on X_K, which generates the differential module $\Omega^d_{X_K/K}$; see [7, 4.2/3]. Furthermore, ω is unique, up to a constant in K^*. For any open rigid subspace $U_K \subset X_K$ admitting a smooth formal R-model U, we can restrict ω to U_K and talk about the order of ω at generic points of the special fibre U_k or, in other words, at irreducible components of U. We write $\mathrm{ord}_U\omega$ for this order if U is irreducible.

For the rest of this section, let us fix X_K and ω as before, and let $\pi \in R$ be a uniformizing element. Frequently, we will have to consider a smooth and irreducible formal R-model U of some open rigid subspace $U_K \subset X_K$. To have a simple language, let us call U a **weak Néron component** of X_K. For example, if U is a weak Néron model of X_K, the connected components of U are weak Néron components of X_K.

Lemma 4.4 *Let U', U'' be two weak Néron components of X_K, and let $f : U' \dashrightarrow U''$ be an R-rational map which, on generic fibres, extends to an isomorphism $f_K : X_K \xrightarrow{\sim} X_K$. In particular, there must be a unit $a \in \Gamma(X_K, \mathcal{O}_{X_K})$ satisfying $f_K^*(\omega) = a\omega$. Assume for some point $x_K \in X_K(K^{sh})$, which extends to a point $x \in U'(R^{sh})$, that $a(x_K)$ is a unit in $R^{sh} \subset K^{sh}$. Then:*
(i) $n' := \mathrm{ord}_{U'}\omega \geqq \mathrm{ord}_{U''}\omega =: n''$.
(ii) *If V' is the domain of definition of f, the morphism $V' \to U''$ given by f is an open immersion if and only if $n' = n''$.*

Proof. We know from 4.3 that $\pi^{-n'}\omega$ generates $\Omega^d_{U'/R}$ and $\pi^{-n''}\omega$ generates $\Omega^d_{U''/R}$. Thus, there is a section $b \in \Gamma(V', \mathcal{O}_{U'})$ satisfying $f^*(\pi^{-n''}\omega) = b\pi^{-n'}\omega$ on V'. Then we have $b = \pi^{n'-n''}a$ on V'_K, and it follows from 4.2, that b extends to a section on U'. Writing $n = \mathrm{ord}_{U'}a$, we see from 4.3 that $\pi^{-n}a$ extends to a unit on U'. However, $a(x_K)$ being a unit in R^{sh}, we must have $n = 0$. Thus

$$n' - n'' = \mathrm{ord}_{U'}b \geqq 0\,,$$

which proves (i).

It remains to verify assertion (ii). The morphism $v : V' \to U''$ is étale if and only if the associated map $v^*\Omega^d_{U''/R} \to \Omega^d_{V'/R}$ is bijective; use [5, 1.2] and [7, 2.2/10]. The latter is the case if and only if b is invertible over V' and, hence, over U'; i.e., if and only if $n' - n'' = 0$. On the other hand, we claim that $v : V' \to U''$ is étale if and only if it is an open immersion. To verify this, assume that v is étale. Then v is flat and, thus, open. So we may assume that $v : V' \to U''$ is surjective and, hence, faithfully flat. As a consequence, its generic fibre $v_K : V'_K \to U''_K$ is surjective and, hence, an isomorphism. In order to show that v is an isomorphism also, it is enough to show that $v_K^{-1} : U''_K \to V'_K$ extends to an R-morphism $U'' \to V'$. To do this, we may assume that U'' and V' are affine. But then 2.2(ii) implies that v_K^{-1} extends to an R-morphism $U'' \to V'$, and the latter is an inverse of v. □

Considering a special case, we can apply the assertion of 4.4 to the identity map $\mathrm{id}_K : X_K \to X_K$ in place of f_K. Let us call two weak Néron components U', U'' of X_K **equivalent** if the identity map id_K extends to an R-birational map $U' \dashrightarrow U''$. Clearly, if U' and U'' are equivalent, we must have $\mathrm{ord}_{U'}\omega = \mathrm{ord}_{U''}\omega$.

Proposition 4.5 *There is a largest integer n_0 such that $\mathrm{ord}_U\omega \geqq n_0$ for all weak Néron components U of X_K. We call U* **minimal** *or, more precisely,* **ω-minimal** *if $\mathrm{ord}_U\omega = n_0$. Up to equivalence, there are only finitely many weak Néron components of X_K, which are minimal.*

Proof. Choose a weak Néron model U of X_K, and let $U_i, i = 1 \dots r$, be its components. Then, if U' is an arbitrary weak Néron component of X_K, we see from 2.5 that the identity on X_K extends to an R-rational map $U' \dashrightarrow U_i$ for some i. Thus, by 4.4(i), the first assertion of the proposition holds for

$$n_0 = \min_{i=1,\dots,r} \{\mathrm{ord}_{U_i}, \omega\} .$$

If U' is minimal, then, by 4.4(i), U_i is minimal also, and 4.4 (ii) shows that $U' \dashrightarrow U_i$ is R-birational in this case. Thus, up to equivalence, the minimal weak Néron components of X_K are given by the finitely many minimal components of any weak Néron model U of X_K. □

It remains to say that the notion of ω-minimality is independent of the choice of ω, since ω, as a left-invariant differential form on a smooth rigid K-group, is unique, up to a constant in K^*.

Lemma 4.6 *Let R' be a complete discrete valuation ring with field of fractions K', such that the extension R'/R is étale. Let ω' be the left-invariant differential form on $X_{K'} = X_K \otimes_K K'$ induced from ω.*

(i) *If U is a weak Néron component of X_K, then $U \otimes_R R'$ decomposes into finitely many weak Néron components U_i' of $X_{K'}$, where $\mathrm{ord}_{U_i'}\omega' = \mathrm{ord}_U\omega$ for all i.*

(ii) *If U is a weak Néron model of X_K, then $U \otimes_R R'$ is a weak Néron model of $X_{K'}$.*

(iii) *If U is a smooth formal R-model of some open rigid subspace $U_K \subset X_K$, whose components represent all ω-minimal weak Néron components of X_K, the same is true for $U \otimes_R R'$ in terms of ω'-minimal weak Néron components of $X_{K'}$.*

Proof. Assertion (i) is true, since π is a uniformizing element for R and R', whereas (ii) follows directly from the definition of weak Néron models. Finally, (iii) is a combination of (i) and (ii). □

Next we want to show that we can apply the assertion of 4.4 to the case where f_K is a translation on X_K.

Proposition 4.7 *Consider a point $g : T_K \to X_K$ of X_K with values in some rigid K-space T_K. Denote by τ_g (resp. τ_g') the left (resp. right) translation with g on*

X_K. Then $\tau_g^*\omega = \omega$, and there is an invertible global section $\chi(g)$ in $\mathcal{O}_{T_K}$ such that $\tau_g'^*\omega = \chi(g)\omega$. Varying g, we see that χ defines a character $X_K \to \mathbb{G}_{m,K}$.

In particular, if $g \in X_K(K^{sh})$ and the set of these points is bounded (which we are assuming in the situation of 1.2), then $\chi(g)$ is a unit in R^{sh}.

Proof. Clearly, we have $\tau_g^*\omega = \omega$, since ω is left-invariant. On the other hand, $\tau_g'^*\omega$ is left-invariant so that there is an invertible global section $\chi(g)$ in $\mathcal{O}_{T_K}$ satisfying $\tau_g'^*\omega = \chi(g)\omega$. Varying g, we see that χ is a functorial homomorphism from the points of X_K to the points of $\mathbb{G}_{m,K}$, the latter being viewed as a rigid K-group; thus, χ is a character $X_K \to \mathbb{G}_{m,K}$.

Now look at the group of K^{sh}-valued points of X_K which, by our assumption, is contained in some quasi-compact open rigid subspace of X_K. Then $\chi(X_K(K^{sh}))$ is contained in some quasi-compact open rigid subspace of $\mathbb{G}_{m,K}$ and, thus, must be contained in the subgroup of "units" of $\mathbb{G}_{m,K}$; i.e., in the subgroup which is induced by the formal multiplicative group $\hat{\mathbb{G}}_{m,R}$. □

Later, in Sect. 5, we will finish the proof of 1.2 by constructing a weak Néron model of X_K, which is a formal R-group scheme. The latter is done by removing non-minimal components from a weak Néron model of X_K and taking its "closure" in the sense of generating groups. A first step in this direction is the construction of an R-birational group law.

Proposition 4.8 *Let X_K be a smooth rigid K-group such that the set of its K^{sh}-valued points is bounded. Choose a weak Néron model V of X_K and denote by U the smooth formal R-scheme consisting of all minimal components of V. Then the group structure on X_K extends to an R-birational group law on U.*

More precisely, the multiplication $m_K : X_K \times_K X_K \to X_K$ extends to an R-rational map $m : U \times_R U \dashrightarrow U$ such that the universal translations

$$\Phi : U \times_R U \dashrightarrow U \times_R U, \quad (x, y) \mapsto (x, m(x, y))$$

$$\Psi : U \times_R U \dashrightarrow U \times_R U, \quad (x, y) \mapsto (m(x, y), y)$$

are R-birational. Furthermore, m is associative.

Proof. Let $d = \dim X_K$, and fix a non-trivial left-invariant differential form $\omega \in \Omega^d_{X_K/K}$. Then, writing p_1, p_2 for the projections of $X_K \times_K X_K$ onto its factors, $\omega^{\wedge 2} = p_1^*\omega \wedge p_2^*\omega$ is a non-trivial left-invariant differential form on $X_K \times_K X_K$. Furthermore, it is clear that $V \times_R V$ is a weak Néron model of $X_K \times_K X_K$ and that $U \times_R U$ is the open part consisting of all $\omega^{\wedge 2}$-minimal components. Now consider the universal left and right translations

$$\Phi_K : X_K \times_K X_K \to X_K \times_K X_K, \quad (x, y) \mapsto (x, m_K(x, y)),$$

$$\Psi_K : X_K \times_K X_K \to X_K \times_K X_K, \quad (x, y) \mapsto (m_K(x, y), y),$$

as well as their inverses, which are given by

$$(x, y) \mapsto (x, m_K(x^{-1}, y)), \quad \text{resp.} \quad (x, y) \mapsto (m_K(x, y^{-1}), y).$$

By 2.5, the morphism Φ_K extends to an R-rational map $U \times_R U \dashrightarrow V \times_R V$. Since $\Phi_K^*(\omega^{\wedge 2}) = \omega^{\wedge 2}$, we see from 4.4 that it is targeted to $U \times_R U$ so that Φ_K extends to an R-rational map $\Phi : U \times_R U \dashrightarrow U \times_R U$. Applying 4.4 again, it follows that Φ is an open immersion if we restrict the map to any component of $U \times_R U$. Applying the same reasoning to Φ_K^{-1}, it follows that Φ is R-birational.

In a similar way we can show that Ψ_K extends to an R-birational map $\Psi : U \times_R U \dashrightarrow U \times_R U$. Viewing $\omega_2 = p_2^*\omega$ as a left-invariant differential form on $X_K \times_K X_K$, relatively over the second factor X_K, we see that $\Psi_K^*(\omega_2)$ is again a relative left-invariant differential form on $X_K \times_K X_K$. Thus, there is an invertible global section a in $\mathcal{O}_{X_K}$ satisfying $\Psi_K^*\omega_2 = a\omega_2$. Viewing everything over the base K, we can write $\Psi_K^*(\omega^{\wedge 2}) = a\omega^{\wedge 2}$. Now, in each connected component of U, we can choose an R^{sh}-valued point x. Using the right translation τ'_{x_K} on X_K, we see from 4.7 that $a(x_K)$ is a unit in R^{sh}. So 4.4 is applicable as before, and Ψ_K extends to an R-birational map $\Psi : U \times_R U \dashrightarrow U \times_R U$.

Finally, composing Φ with the projection of $U \times_R U$ onto the second factor and Ψ with the projection of $U \times_R U$ onto the first factor, we obtain two R-rational maps $U \times_R U \dashrightarrow U$ extending m_K. Thus, they must coincide, and, writing m for this map, we see that Φ and Ψ are as stated in the assertion. From the associativity of m_K one concludes that m is associative. □

5 End of construction of Néron models

In this section, we want to finish the proof of 1.2. It basically remains to show that an R-birational group law, as obtained in 4.8, can be enlarged in such a way that it becomes a group law in the sense of formal R-group schemes. To do this, we can, in principle, follow the general procedure, as explained in [7, Chap. 5]; we may even use the general result of M. Artin in [8, exp. XVIII, 3.13]. However, things can be substantially simplified by using the fact that, on the generic fibre, the birational group law is part of a rigid K-group.

An R-birational group law $m : U \times_R U \dashrightarrow U$ on a smooth and quasi-compact formal R-scheme U is called **strict** if m is defined on a U-dense open formal subscheme $W \subset U \times_R U$ such that the universal translations Φ and Ψ define isomorphisms from W onto U-dense open formal subschemes $\Phi(W)$ and $\Psi(W)$ of $U \times_R U$. In this context, an open formal subscheme $W \subset U \times_R U$ is called U-dense if it is U-dense with respect to the two projections p_1, p_2 of $U \times_R U$ onto its factors; i.e., if for any $u \in U$ the fibre $p_i^{-1}(u)$ contains $W \cap p_i^{-1}(u)$ as a dense open subset.

Lemma 5.1 *Let $m : U \times_R U \dashrightarrow U$ be an R-birational group law on a smooth and quasi-compact formal R-scheme U. Then there is an R-dense open formal subscheme $U' \subset U$ such that m restricts to a strict R-birational group law on U'.*

Proof. Looking at special fibres, m induces a birational group law on u_k which, using [7, 5.2/2], can be restricted to a strict birational group law. □

Theorem 5.2 *Let X_K be a rigid K-group, and let U be a smooth and quasi-compact formal R-scheme, whose generic fibre U_K is an open rigid subspace of X_K. Assume that the group multiplication m_K of X_K extends to an R-birational group law m on U. Then there is a smooth and quasi-compact formal R-scheme $\bar{U}$ such that*
(i) $\bar{U}$ contains U as an R-dense open formal subscheme,
(ii) m extends to a morphism $\bar{m} : \bar{U} \times_R \bar{U} \to \bar{U}$ of formal R-schemes, which defines a structure of a formal R-group scheme on $\bar{U}$.

Any $\bar{U}$ of this type is unique up to canonical isomorphism, and its generic fibre $\bar{U}_K$ is in a natural way an open rigid subgroup of X_K.

Proof. Any formal R-scheme $\bar{U}$ can be identified with the projective limit of ordinary schemes $\varprojlim_i \bar{U} \otimes_R (R/(\pi^{i+1}))$, where π is a uniformizing element of R. Thus, for the uniqueness of associated formal group laws, we can use the corresponding uniqueness assertion in the scheme case; cf. [7, 5.1/3]. Furthermore, to show the existence of $(\bar{U}, \bar{m})$, we claim it is enough to consider the case, where m defines a strict R-birational group law on U. In fact, using 5.1 we can find an R-dense open formal subscheme $U' \subset U$ such that m restricts to a strict R-birational group law m' on U'. Now, assume that the assertion of the theorem holds for (m', U'); let $(\bar{U}', \bar{m}')$ be the associated formal R-group scheme. Then 2.6 shows that the R-rational map $U \dashrightarrow U'$ given by the inclusion $U \supset U'$ extends to a morphism $\iota : U \to \bar{U}'$. Let ω be a differential form generating $\Omega^d_{\bar{U}'/R}$. It follows that $\iota^*\omega$ generates the differential module $\Omega^d_{U/R}$ over U' and over the generic fibre U_K. But then, using 4.2 as in the proof of 4.3(ii), $\iota^*\omega$ must generate $\Omega^d_{U/R}$ over all of U. So $\iota : U \to \bar{U}'$ is étale, and we see as in the proof of 4.4(ii) that ι is an open immersion.

Thus, in the following, we can assume that m defines a strict R-birational group law on U. Let $W \subset U \times_R U$ be a U-dense open formal subscheme such that, on W, the universal translations $\Phi, \Psi : U \times_R U \dashrightarrow U \times_R U$ are open immersions onto U-dense open parts of $U \times_R U$. For any $a \in U(R)$, set

$$W_a = p_2(W \cap (a \times_R U)), \quad W'_a = p_1(W \cap (U \times_R a)),$$

where p_i is the projection of $U \times_R U$ onto its i-th factor. Then W_a and W'_a are R-dense open formal subschemes of U. Furthermore, the left translation with a_K on X_K extends to an open immersion $\tau_a : W_a \to U$ and, likewise, the right translation with a_K on X_K extends to an open immersion $\tau'_a : W'_a \to U$; let $\tilde{W}_a = \tau_a(W_a)$ and $\tilde{W}'_a = \tau'_a(W'_a)$. In particular, we can interpret τ_a and τ'_a as R-birational maps $U \dashrightarrow U$.

On a smooth formal R-scheme like U, one does not know much about the existence of R-valued points. However, the set of R^{sh}-valued points is dense. To verify this, just use the fact that the points with values in the separable closure of k lie dense in X_k and apply the lifting property for smooth morphisms. In the following, we will need the existence of enough R-valued points. So, without changing notations, we will replace R by the completion $\bar{\bar{R}}$ of R^{sh} and thereby assume that R itself is *strictly henselian*. At the end a descent argument is used in order to handle the general case.

Identifying the set of R-valued points $U(R)$ with $U_K(K)$, we can consider the group H which is generated by $U(R)$; it corresponds to the subgroup of $X_K(K)$ which is generated by $U_K(K)$. Writing the group law as multiplication, any $a \in H$ is of type $a = a_1^{\pm 1} \cdot \ldots \cdot a_r^{\pm 1}$ with $a_1, \ldots, a_r \in U(R)$. Since the left translation with each a_i is defined as an R-birational map $U \dashrightarrow U$, we see that the left translation with a, which is defined on X_K, extends to an R-birational map $U \dashrightarrow U$; the latter will be denoted by τ_a. In a similar way, there is the right translation τ'_a on U. Now, the crucial point of our proof consists in showing the following assertion for points $a \in H$:

The left translation $\tau_a : U \dashrightarrow U$ is defined at a point $u \in U(R)$ if and only if $a_K \cdot u_K \in U_K$. It is an open immersion on its domain of definition.

The only if part is clear. So assume $a_K \cdot u_K \in U_K$. Then this point gives rise to an R-valued point of U, which, in a suggestive way, will be denoted by au. Now let τ'_a be defined on some R-dense open formal subscheme $V \subset U$ and choose an R-valued point

$$z \in \tau_a'^{-1}(W'_u) \cap V \cap W'_{au} .$$

Writing za instead of $\tau'_a(z)$, we have

$$za \in W'_u, \quad z \in W'_{au}$$

or, equivalently,

$$u \in W_{za}, \quad au \in W_z .$$

Then consider

$$U(u) = (\tau_{za})^{-1}(\tilde{W}_{za} \cap \tilde{W}_z) \cap W_{za} ,$$

$$U(au) = \tau_z^{-1}(\tilde{W}_{za} \cap \tilde{W}_z) \cap W_z$$

as R-dense open neighborhoods of u, resp. au in U. In fact, these sets satisfy

$$\tau_{za}(U(u)) = \tau_z(U(au)) ,$$

and the points $\tau_{za}(u) \in \tau_{za}(U(u)), \tau_z(au) \in U(au)$ coincide, since they coincide on the generic fibre.

Looking at generic fibres, we have

$$z_K \cdot a_K \cdot U(u)_K = z_K \cdot U(au)_K$$

and, hence,

$$a_K \cdot U(u)_K = U(au)_K .$$

If we combine 4.2 with the fact that the left translation τ_a is defined on the generic fibre as well as on an R-dense open part of U, we see, in particular, that τ_a is defined on $U(u)$ and, hence, at u. Since the same reasoning applies to τ_a^{-1}, our claim above is justified.

For any point $a \in H$, we can interpret U as a formal R-model of $a_K U_K$ via the left translation $U_K \tilde{\to} a_K U_K$ with a_K. Suggestively, this formal model of $a_K U_K$ is denoted by aU. Now, due to the assertion we just proved, there are canonical gluing data for the translates $aU, a \in H$. For example, to glue U and a translate aU, consider the left translation $\tau_a : U \dashrightarrow U$, which is an open immersion on its domain of definition $U' \subset U$, as we have seen. Writing $aU' \subset aU$ for the open formal subscheme induced from $U' \subset U$, the canonical isomorphism $\tau_a : U' \tilde{\to} \tau_a(U')$ gives rise to an isomorphism $aU' \tilde{\to} \tau_a(U')$, which is the one we use for gluing aU to U. Then, again due to the assertion of above, we see that the gluing data on all translates aU extend the gluing data we have on generic fibres. In particular, the cocycle condition for triple overlaps is satisfied. Thus,

$$\bar{U} = \bigcup_{a \in H} aU$$

is a well-defined formal R-scheme. By our construction, it has the property that all left translations $\tau_a, a \in H$, extend to isomorphisms $\bar{U} \tilde{\to} \bar{U}$.

We want to show that $\bar{U}$ is the formal R-scheme we are looking for. So we claim that the R-rational multiplication m on U extends to an R-morphism $\bar{m} : \bar{U} \times_R \bar{U} \to \bar{U}$. To justify this claim, start with a closed point in $U \times_R U$. Using [4, 3.5], we can extend it to a point c of $U \times_R U$ with values in a discrete valuation ring R' which is finite flat over R; let $c = a \times b$ with $a, b \in U(R')$. Now, using R'/R as base change, consider the R'-dense open subscheme

$$\tau_a'^{-1}(W_b') \cap W_a' \subset U_{R'} .$$

Its image under the projection $U_{R'} \to U$ is an R-dense open formal subscheme of U. Thus, there is a point $z \in U(R)$ such that $z_{R'} \in \tau_a'^{-1}(W_b') \cap W_a'$. Writing za instead of $\tau_a'(z_{R'})$, we have $za \in W_b'$ so that $\bar{m}_{R'}$ is defined at $za \times b$. But then, due to the construction of $\bar{U}$, we see that $\bar{m}_{R'}$ is defined at $c = a \times b$, and 2.1 shows that $\bar{m}$ is defined at c.

Next we show that all right translations $\tau_a', a \in H$, are isomorphisms on $\bar{U}$. To see that τ_a' is defined at some closed point $u \in \bar{U}$, we may apply a left translation and thereby assume $u \in U$. Proceeding similarly as before, we can choose a point $z \in U(R)$ such that $\tau_z(u) \in W_a'$. But then, clearly, τ_a' is defined at u. The same is true for $\tau_{a^{-1}}'$.

Finally, to show that $\bar{m} : \bar{U} \times_R \bar{U} \dashrightarrow \bar{U}$ is defined everywhere, it is enough to show that $\bar{m}$ is defined on all products of type $aU \times_R bU$ with $a, b \in H$ or, using the associativity of $\bar{m}$, on all products of type $(aUb) \times_R U$. However, the latter is clear, since $\bar{m}$ is defined on $U \times_R U$ and since $\bar{U}$ is invariant under left and right translations with points in H.

In particular, our argumentation shows, that the universal translations $\Phi, \Psi : U \times_R U \dashrightarrow U \times_R U$ extend to morphisms

$$\bar{\Phi}, \bar{\Psi} : \bar{U} \times_R \bar{U} \to \bar{U} \times_R \bar{U} .$$

Since a similar reasoning applies to their inverses, we thereby see that $\bar{m}$ defines on $\bar{U}$ the structure of a formal R-group scheme. By [8, exp. $\mathrm{VI_B}$, corollaire 3.6] the identity component of $\bar{U}$ is quasi-compact. Hence, since $\bar{U}$ contains U as a quasi-compact open part, $\bar{U}$ itself is quasi-compact, and it follows from our construction that the associated rigid K-group $\bar{U}_K$ is canonically an open rigid subgroup of X_K. Thus, we have proved the assertion of 5.2 for a strictly henselian base R.

If R is not strictly henselian, we have to replace R by the completion $\bar{R}$ of its strict henselization R^{sh}. Thus, to end the proof of 5.2, it remains to show that any solution $(\bar{U}_{\bar{R}}, \bar{m}_{\bar{R}})$, we have found over $\bar{R}$, descends to a solution over R. Choosing a uniformizing element $\pi \in R$, let us introduce the residue rings $R_i = R/(\pi^{i+1})$ and $\bar{R}_i = \bar{R}/(\pi^{i+1})$ for $i \in \mathbb{N}$, and set $\bar{R}\hat{\otimes}_R\bar{R} = \varprojlim_i \bar{R}_i \otimes_R \bar{R}_i$. Then, pulling back $(\bar{U}_{\bar{R}}, \bar{m}_{\bar{R}})$ with respect to the two projections from Spf $\bar{R}\hat{\otimes}_R\bar{R}$ onto Spf$\bar{R}$, we get two solutions of a problem obtained from (U, m) via base change Spf $\bar{R}\hat{\otimes}_R\bar{R} \rightarrow$ Spf$R$. However, both solutions are canonically isomorphic after reducing modulo powers of π, due to the uniqueness assertion of [7, 5.1/3]. Thus, using a similar argument on triple products, it follows that we have descent data on all solutions $(\bar{U}, \bar{m}) \otimes_R R_i$, and that these are related to each other by reduction modulo powers of π. To see that all descent data are effective, is it enough to know that they are effective after reducing modulo π. However, $R/(\pi) \rightarrow \bar{R}/(\pi)$ is an extension of fields, and in this case the effectiveness of descent data on group schemes of finite type is well-known. So any solution $(\bar{U}_{\bar{R}}, \bar{m}_{\bar{R}}) \otimes_R R_i$ descends to a solution of $(U, m) \otimes_R R_i$ over R_i and it follows that $(\bar{U}_{\bar{R}}, \bar{m}_{\bar{R}})$ descends to a solution of (U, m) over R. □

Finally, to obtain the assertion of 1.2, let us go back to the R-birational group law (U, m) constructed in 4.8, and let $(\bar{U}, \bar{m})$ be the associated formal R-group scheme, whose existence has been proved above. Applying the criterion 1.4, it is enough to show that $(\bar{U}, \bar{m})$ is a weak Néron model of X_K. So let R' be a complete discrete valuation ring, which is finite étale over R, and consider a point $a_K \in X_K(K')$, where K' is the field of fractions of R'. Then, on $X_K \otimes_K K'$, we can consider the left translation τ_K with a_K. Using 4.4 and 4.6, it extends to an R'-rational map $U \otimes_R R' \dashrightarrow U \otimes_R R'$ and, hence, to an R'-rational map $\tau : \bar{U} \otimes_R R' \dashrightarrow \bar{U} \otimes_R R'$. The latter is a morphism due to 2.6.

But then we can compose the unit section Spf$R' \rightarrow \bar{U} \otimes_R R'$ with τ and, furthermore, with the projection $\bar{U} \otimes_R R' \rightarrow \bar{U}$, thereby obtaining an R'-valued point a of $\bar{U}$ extending a_K. Thus, $\bar{U}$ is a weak Néron model of X_K, and the proof of 1.2 is complete.

6 Formal Néron models versus ordinary ones

As before, let R be a complete discrete valuation ring with field of fractions K and residue field k. Starting with a smooth K-group scheme $\mathfrak{X}_K$ of finite type, we can ask if there is a Néron model $\mathfrak{X}$ of $\mathfrak{X}_K$, by which we mean a (quasi-compact) Néron model in the sense of [7, 1.2/1] or a (not necessarily quasi-compact) Néron lft-model in the sense of [7, 10.1/1]. On the other hand, we can pass from $\mathfrak{X}_K$ to the rigid K-group X_K, which is associated to $\mathfrak{X}_K$ via Serre's GAGA functor, and try to construct a formal Néron model U of X_K. As we will see, the relationship between X and U is quite simple, at least if we restrict ourselves to quasi-compact Néron models. We know from [7, 1.3/1] that $\mathfrak{X}_K$ admits a quasi-compact Néron model if and only if the set of its K^{sh}-valued points $\mathfrak{X}_K(K^{sh})$ is bounded in the sense of [7, 1.1/2], and we have shown the analogous fact for formal Néron models in 1.2. Thus, since we can identify $\mathfrak{X}_K(K^{sh})$ with $X_K(K^{sh})$ and since boundedness in both situations amounts to the same, we obtain the following fact:

Proposition 6.1 *Let $\mathfrak{X}_K$ be a smooth K-group scheme of finite type and X_K the associated rigid K-group. Then $\mathfrak{X}_K$ admits a quasi-compact Néron model if and only if X_K admits a quasi-compact formal Néron model.*

Next we want to show that a formal Néron model of X_K can be obtained from a Néron model of $\mathfrak{X}_K$ by means of formal completion.

Theorem 6.2 *Let $\mathfrak{X}_K$ be a smooth K-group scheme of finite type with Néron model $\mathfrak{X}$. Let X_K be the rigid K-group associated to $\mathfrak{X}_K$ and $\bar{X}$ the formal completion of $\mathfrak{X}$ along the special fibre $\mathfrak{X}_k$. Then, if $\mathfrak{X}$ is quasi-compact or if $\mathfrak{X}_K$ is commutative, the canonical morphism $\bar{X}_K \to X_K$ of rigid K-groups defines $\bar{X}_K$ as a retrocompact open rigid subgroup of X_K, and $\bar{X}$ is a formal Néron model of X_K.*

Proof. There are canonical bijections

$$X_K(K^{sh}) \xrightarrow{\sim} \mathfrak{X}_K(K^{sh}) \xrightarrow{\sim} \mathfrak{X}\,(R^{sh}) \xrightarrow{\sim} \bar{X}(R^{sh})\,,$$

and we see from the criterion 1.4 that $\bar{X}$ is a formal Néron model of X_K, provided the canonical morphism $\bar{X}_K \to X_K$ defines $\bar{X}_K$ as a retrocompact open rigid subgroup of X_K. Due to the construction of associated rigid K-spaces, the latter is clear if $\mathfrak{X}$ is quasi-compact or, to mention an example which is needed below, if $\mathfrak{X}$ is the Néron model of a split torus $\mathfrak{X}_K$.

So it remains to show that $\bar{X}_K$ is a retrocompact open rigid subgroup of X_K. To do this, we can assume that $\mathfrak{X}_K$ is commutative and, since the formation of Néron models is compatible with extensions R'/R of ramification index 1, see [7, 10.1/3], that R is strictly henselian. We will use the criteria of [7, 10.2/1 and 10.2/2], which say in our situation that $\mathfrak{X}_K$ admits a Néron model $\mathfrak{X}$ if and only if $\mathfrak{X}_K$ does not contain subgroups of type $\mathfrak{G}_a$ and that such an $\mathfrak{X}$ is quasi-compact if and only if $\mathfrak{X}_K$ does not contain subgroups of type $\mathfrak{G}_m$. Proceeding similarly as in the proofs of [7, 10.1/7 and 10.2/2], choose a maximal torus $\mathfrak{T}_K$ in $\mathfrak{X}_K$ and consider the associated exact sequence

$$0 \to \mathfrak{T}_K \to \mathfrak{X}_K \to \mathfrak{H}_K \to 0\,.$$

Since $\mathfrak{X}_K$ cannot contain subgroups of type $\mathfrak{G}_a$, we see that $\mathfrak{H}_K$ cannot contain subgroups of type $\mathfrak{G}_a$ and $\mathfrak{G}_m$ and, hence, that $\mathfrak{H}_K$ admits a quasi-compact Néron model $\mathfrak{H}$.

Let us first look at the case, where the torus $\mathfrak{T}_K$ is *split*. Viewing $\mathfrak{X}_K$ as a $\mathfrak{T}_K$-torsor over $\mathfrak{H}_K$, we can use the fact that such a torsor is locally trivial and, thus, is given by a set of primitive line bundles on $\mathfrak{H}_K$. Since the local rings of $\mathfrak{H}$ are factorial, these line bundles extend to primitive line bundles on the identity component $\mathfrak{H}^0$. Proceeding as in the proof of [7, 10.1/7] and using the fact that each connected component of $\mathfrak{H}$ admits an R-valued point since $R = R^{sh}$, the above exact sequence gives rise to an exact sequence between associated Néron models

$$0 \to \mathfrak{T} \to \mathfrak{X} \to \mathfrak{H} \to 0$$

and, hence, to an exact sequence between the corresponding formal completions

$$0 \to \bar{T} \to \bar{X} \to \bar{H} \to 0 .$$

Switching to the associated rigid situation and identifying $\bar{T}, \bar{X}, \bar{H}$ with their associated rigid K-groups, we have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \bar{T} & \to & \bar{X} & \to & \bar{H} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & T_K & \to & X_K & \to & H_K & \to & 0 , \end{array}$$

where the vertical maps are open immersions, except possibly for $\bar{X}_K \to X_K$, which is an open immersion on any quasi-compact open rigid subspace of $\bar{X}_K$ and, thus, in particular, a monomorphism. Since the components of $\mathfrak{X}$ are quasi-compact, the same is true for the connected components of $\bar{X}$. In particular, we can view the connected components of $\bar{X}$ as open rigid subspaces of X_K, and $\bar{X}$ itself at least as a subset of X_K. Thus, to see that $\bar{X}$ is a retrocompact open rigid subgroup of X_K, it remains to show that any quasi-compact open rigid subspace of X_K meets only finitely many connected components of $\bar{X}$.

Since $\bar{H}$ is quasi-compact, the above sequences say that there is a finite union $\bar{X}'$ of connected components of $\bar{X}$ such that, writing $\mathbb{Z}^r$ for a set of representatives of R-valued points of the components of $\bar{T}$, we have $\bar{X} = \mathbb{Z}^r \cdot \bar{X}'$. Now, using the fact that X_K is locally trivial as a T_K-torsor over H_K, consider a quasi-compact open rigid subspace $V_K \subset H_K$ such that, over V_K, the torsor X_K is trivial and, thus, of type $T_K \times_K V_K$. Then $\bar{X}'' = \bar{X}' \cap (T_K \times_K V_K)$ is quasi-compact, and we have

$$\bar{X} \cap (T_K \times_K V_K) = \mathbb{Z}^r \cdot \bar{X}'' .$$

From this it follows that any quasi-compact part of $T_K \times_K V_K$ can meet only finitely many translates of $\bar{X}''$ by points in $\mathbb{Z}^r$ and, varying V_K over a suitable admissible open covering of H_K, we see that $\bar{X}$ is a retrocompact open rigid subgroup of X_K, as claimed.

If the torus $\mathfrak{T}_K$ is not split, we can find a finite separable extension K'/K such that $\mathfrak{T}_K$ splits over K. Then, using [14, Chap. IV, Proposition 4.1.4] and proceeding as in the proof of [7, 10.2/2], the group $\mathfrak{X}_K \otimes_K K'$ does not contain subgroups of type $\mathbb{G}_a$, since this is true for $\mathfrak{X}_K$. Consequently, the group $\mathfrak{H}_K \otimes_K K'$ still admits a quasi-compact Néron model and, replacing K' by some finite unramified extension, we can assume that, in addition, the components of $\mathfrak{H}_K \otimes_K K'$ are split. Thus, we are reduced to the case, where there exists a finite separable extension K'/K such that the assertion of 6.2 holds for the group $\mathfrak{X}_K \otimes_K K'$ and its Néron model $\mathfrak{X}'$, which exists. In general, $\mathfrak{X}'$ will be different from $\mathfrak{X} \otimes_R R'$, where R' is the integral closure of R in K'. To obtain the assertion of 6.2 for $\mathfrak{X}$, we consider the canonical map $\mathfrak{X} \otimes_R R' \to \mathfrak{X}'$ and write $\mathfrak{Y}$ for the union of all components $\mathfrak{X}_i$ of $\mathfrak{X}$ (which are split by our assumption on the residue field of K) such that $\mathfrak{X}_i \otimes_R R'$ is mapped into $\mathfrak{X}'^0$, the identity component of $\mathfrak{X}'$. The latter is quasi-compact. Writing $\mathfrak{R}_{R'/R}$ for the Weil restriction with respect to the base change R'/R, the morphism $\mathfrak{Y} \otimes_R R' \to \mathfrak{X}'^0$ corresponds to a morphism $\mathfrak{Y} \to \mathfrak{R}_{R'/R}(\mathfrak{X}'^0)$. Then this map factors through the schematic closure of the closed subgroup $\mathfrak{X}_K \subset \mathfrak{R}_{K'/K}(\mathfrak{X}_K \otimes_K K')$, where we view $\mathfrak{R}_{K'/K}(\mathfrak{X}_K \otimes_K K')$ as the generic fibre of $\mathfrak{R}_{R'/R}(\mathfrak{X}'^0)$. Let $\mathfrak{Z}$ be a group smoothening of this group; cf. [7, 7.1/5]. Then $\mathfrak{Z}$ is quasi-compact, and it follows that $\mathfrak{Y} \to \mathfrak{R}_{R'/R}(\mathfrak{X}'^0)$ factors through $\mathfrak{Z}$. On the other hand, using the Néron mapping property for $\mathfrak{X}$, there is a canonical map $\mathfrak{Z} \to \mathfrak{X}$ which must factor through $\mathfrak{Y}$. Thereby we see that $\mathfrak{Y} \to \mathfrak{Z}$ is an isomorphism and, thus, that $\mathfrak{Y}$ is quasi-compact since the same is true for $\mathfrak{Z}$.

As a result, we have shown that the canonical map from the component group of $\mathfrak{X} \otimes_R R'$ to the component group of $\mathfrak{X}'$ has finite kernel. From this one easily concludes that $\mathfrak{X} \otimes_R R'$ induces a retrocompact open rigid subgroup of X'_K, the rigid K'-group associated to $\mathfrak{X}_K \otimes_K K'$, and that, likewise, $\mathfrak{X}$ gives rise to a retrocompact open rigid subgroup $\bar{X}$ in X_K. □

As an application of 6.2, we want to describe the formal Néron model of an abelian variety, having a split uniformization in the sense of rigid geometry.

Example 6.3 Consider an exact sequence of smooth K-group schemes of finite type

$$0 \to \mathfrak{T}_K \to \mathfrak{E}_K \to \mathfrak{B}_K \to 0\,,$$

where $\mathfrak{T}_K \simeq \mathbb{G}^d_{\mathrm{m},K}$ is a split torus, and where $\mathfrak{B}_K$ is an abelian variety with good reduction. As in the proof of 6.2, the associated sequence between Néron models

$$0 \to \mathfrak{T} \to \mathfrak{E} \to \mathfrak{B} \to 0$$

is exact, and the same is true for the sequence

$$0 \to \mathfrak{T}^0 \to \mathfrak{E}^0 \to \mathfrak{B} \to 0$$

between identity components. So $\mathfrak{E}^0$ is an extension of the abelian scheme $\mathfrak{B}$ by the split torus $\mathfrak{T}^0 = \mathbb{G}^d_{\mathrm{m},R}$. Furthermore, the component group of $\mathfrak{E}$ is the same as the one of $\mathfrak{T}$, namely the dual Y^* of the group of characters Y of $\mathfrak{T}$; cf. 2.7.

Next, let us switch to the associated rigid situation, let T_K, E_K, B_K be the rigid K-groups associated to the above K-group schemes, and, using 6.2, let T, E, B be their formal Néron models. Furthermore, let us consider a split lattice $M_K \subset E_K$ of rank d; i.e., a closed rigid subgroup, which is isomorphic to $\mathbb{Z}^d$ as a constant group. Then the quotient $A_K = E_K/M_K$ makes sense as a proper rigid K-group. Writing $M \subset E$ for the closed formal subgroup scheme induced from M_K or, in other words, for the Néron model of M_K, we can construct the quotient E/M as a formal R-group scheme. Using the criterion 1.4, it follows that E/M coincides with the formal Néron model A of A_K. Hence, the projection $E \longrightarrow A$ restricts to an isomorphism $E^0 \xrightarrow{\sim} A^0$ between identity components. Furthermore, the group of connected components of $A \simeq E/M$ equals the quotient Φ_E/Φ_M, where $\Phi_E = Y^*$ is the group of connected components of E and Φ_M is the group of connected components of M; the latter coincides with the special fibre of M. In particular, if A_K is algebraizable and, thus, an abelian variety, we have determined the component group of the classical Néron model of A_K. A similar description has been given in [9, Chap. III, 8.2].

Finally, let us return to the general assertion of 6.2. Presumably it is not possible to avoid the commutativity assumption for the Néron model $\mathfrak{X}$ of $\mathfrak{X}_K$ in the non-quasi-compact case. However, in order to extend 6.2 to any type of Néron model of $\mathfrak{X}_K$, we modify the notion of formal Néron models slightly.

Definition 6.4 *A* **(formal) Néron quasi-model** *of a smooth rigid K-group X_K consists of a smooth formal R-group scheme U with generic fibre U_K and of a morphism of rigid K-groups $\iota_K : U_K \longrightarrow X_K$ such that the following conditions are satisfied:*

(i) *ι_K, restricted to any quasi-compact open part of U_K, is an open immersion of rigid K-spaces; we say that ι_K is a* **quasi-open immersion**.

(ii) *The pair (U, ι_K) satisfies the Néron mapping property; i.e., given a smooth formal R-scheme Z and a morphism of rigid K-spaces $f_K : Z_K \longrightarrow X_K$, there is a unique morphism of formal R-schemes $g : Z \longrightarrow U$ satisfying $f_K = \iota_K \circ g_K$.*

Any formal Néron model is a formal Néron quasi-model, and both notions coincide in the quasi-compact case. To deal with Néron quasi-models, the following analogue of the criterion 1.4 is useful:

Criterion 6.5 *Let X_K be a smooth rigid K-group, and consider a smooth formal R-group scheme U together with a quasi-open immersion $\iota_K : U_K \longrightarrow X_K$ of rigid K-groups. Then the following are equivalent:*

(i) *U is a Néron quasi-model of X_K.*

(ii) *Given a smooth formal R-scheme Z and a morphism of rigid K-spaces $v_K : Z_K \longrightarrow X_K$, there is a unique rational map $w : Z \dashrightarrow U$ satisfying $v_K = \iota_K \circ w_K$.*

Proof. To show that condition (ii) implies condition (i), just realize that the assertion of 2.6 remains valid, if we consider a quasi-open immersion $U_K \to X_K$ instead of an open immersion $U_K \hookrightarrow X_K$; the proof is unchanged. □

Now let us state the analogue of 6.2 for Néron quasi-models.

Theorem 6.6 *Let $\mathfrak{X}_K$ be a smooth K-group scheme of finite type, and let X_K be the associated rigid K-group.*
(i) $\mathfrak{X}_K$ admits a Néron model if and only if X_K admits a formal Néron quasi-model.
(ii) If $\mathfrak{X}$ is a Néron model of $\mathfrak{X}_K$, its formal completion $\bar{X}$ is a Néron quasi-model of X_K via the canonical morphism $\bar{X}_K \to X_K$.

Proof. We start with assertion (ii). Let Z be a smooth formal R-scheme, and consider a morphism of rigid K-spaces $v_K : Z_K \to X_K$. Then, by 6.5, we have only to show that v_K extends to an R-rational map $Z \dashrightarrow \bar{X}$. In particular, we can replace Z by an R-dense open part. Furthermore, we can assume that Z is irreducible. If ζ is the generic point of the special fibre Z_k, it follows from 4.1 that the local ring $R' = \mathcal{O}_{Z,\zeta}$ is a discrete valuation ring; in fact, the extension R'/R is of ramification index 1 in the sense of [7, 3.6/1].

Now consider a finite affine open covering of $\mathfrak{X}_K$. It induces an admissible open covering of X_K and, thus, by pull-back, an admissible open covering of Z_K. Using an argument as in the proof of 2.2(i), we can replace Z by an R-dense open part and thereby assume that $v_K : Z_K \to X_K$ maps Z_K into an open part $V_K \subset X_K$ which is the analytification of an affine open part $\mathfrak{V}_K = \operatorname{Spec} \mathfrak{A}_K$ of $\mathfrak{X}_K$. Assuming Z affine, $Z = \operatorname{Spf} C$, the morphism v_K gives rise to the composition of homomorphisms

$$\mathfrak{A}_K \to C_K \to R' \otimes_R K ,$$

and the latter corresponds to a morphism $\operatorname{Spec}(R' \otimes_R K) \to \mathfrak{X}_K$ of K-schemes. Since the formation of Néron models commutes with a base change R'/R of ramification index 1, cf. [7, 10.1/3], we see that this morphism extends uniquely to an R-morphism $\operatorname{Spec} R' \to \mathfrak{X}$ or, if we shrink Z if necessary, to an R-morphism $\operatorname{Spec} C \to \mathfrak{X}$. Since C is complete, the latter gives rise to a morphism of formal R-schemes $Z = \operatorname{Spf} C \to \bar{X}$, which is the extension of v_K we are looking for. This settles assertion (ii).

Finally, it remains to verify the if part of assertion (i), namely, that $\mathfrak{X}_K$ admits a Néron model $\mathfrak{X}$ if X_K admits a Néron quasi-model X. This is done by constructing $\mathfrak{X}$ via algebraization from X, proceeding in the spirit of the paper [3], in particular, using [3, 1.6] (the notion of *open immersion*, mentioned at that place, corresponds to our notion of *quasi-open immersion*). We give only a sketch. As in [3, 3.5], one first constructs an R-dense open part $\mathfrak{X}'$ of the future Néron model $\mathfrak{X}$ and shows by means of [3, 1.6] that the group law of X induces an R-birational group law on $\mathfrak{X}'$. Then one can pass to the associated R-group scheme $\mathfrak{X}$ or, at least, to its identity component, cf. [7, 5.1/5], and use 6.5 in conjunction with [3, 1.6] to show that $\mathfrak{X}$ is a Néron model of $\mathfrak{X}_K$. □

References

1. Bartenwerfer, W.: Der Kontinuitätssatz für reindimensionale k-affinoide Räume. Math. Ann. **193**, 139–170 (1971)
2. Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean Analysis. Grundlehren Bd. 261. Berlin Heidelberg New York: Springer (1984)
3. Bosch, S., Lütkebohmert, W.: Néron models from the rigid analytic viewpoint. J. Reine Angew. Math. **365**, 69–84 (1986)
4. Bosch, S., Lütkebohmert, W.: Formal and rigid geometry I. Rigid spaces. Math. Ann. **295**, 291–317 (1993)
5. Bosch, S., Lütkebohmert, W.: Formal and rigid geometry II. Flattening techniques. Math. Ann. **296**, 403–429 (1993)
6. Bosch, S., Lütkebohmert, W., Raynaud, M.: Formal and rigid geometry III. The relative maximum principle. To appear in Math. Ann.
7. Bosch, S., Lütkebohmert, W., Raynaud, M.: Néron Models. Ergebnisse der Math., 3. Folge, Bd. 21. Berlin Heidelberg New York: Springer (1990)
8. Demazure, M., Grothendieck, A.: Séminaire de Géométrie Algébrique. Schémas en groupes (SGA 3). Lect. Notes Math. 151–153. Berlin Heidelberg New York: Springer (1970)
9. Faltings, G., Chai, Ch.-L.: Degeneration of abelian varieties. Ergeb. Math., 3. Folge, Bd. 22. Berlin Heidelberg New York: Springer (1990)
10. Kiehl, R.: Die De Rham Kohomologie algebraischer Mannigfaltigkeiten über einem bewerteten Körper. Publ. Math. Inst. Hautes Étud. Sci. **33** (1967)
11. Lütkebohmert, W.: Fortsetzbarkeit k-meromorpher Funktionen. Math. Ann. **220**, 273–284 (1976)
12. Néron, A.: Modéles minimaux des variétés abéliennes. Publ. Math., Inst. Hautes Étud. Sci. **21** (1964)
13. Tate, J.: Rigid analytic spaces. Invent. Math. **12**, 257–289 (1971)
14. Tits, J.: Lectures on algebraic groups. Yale University, New Haven (1967)