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Bilinear forms and extremal Kähler vector fields associated with Kähler classes

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Introduction

In this paper, we fix once for all an n-dimensional compact complex connected manifold X with a Kähler class $\gamma \in H^{1,1}(X,\mathbb{R}) := H^{1,1}(X) \cap H^2(X,\mathbb{R})$. For any complex variety Y, we denote by $\mathrm{Aut}^0(Y)$ the identity component of the group of holomorphic automorphisms of Y. Then the Albanese map of X to the Albanese variety $\mathrm{Alb}(X)$ induces a Lie group homomorphism

$$a_X : \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(\mathrm{Alb}(X))(\cong \mathrm{Alb}(X)),$$

and the identity component $G := \mathrm{Ker}^0 a_X$ of the kernel of a_X is a linear algebraic group (see [4]). Let R_u be the unipotent radical of G, and by setting $K_{\mathbb{C}} := G/R_u$, we have a reductive algebraic group $K_{\mathbb{C}}$ which is a complexification of a maximal compact subgroup K of G/R_u. Then the Chevalley decomposition allows us to obtain an algebraic group isomorphism $\imath : K_{\mathbb{C}} \cong \imath(K_{\mathbb{C}}) \subset G$, unique up to conjugacy in G, such that it gives a splitting to the exact sequence $1 \to R_u \to G \to K_{\mathbb{C}} \to 1$, i.e., G is written as a semidirect product

$$G = \tilde{K}_{\mathbb{C}} \ltimes R_u ,$$

where we put $\tilde{K}_{\mathbb{C}} := \imath(K_{\mathbb{C}})$ and $\tilde{K} := \imath(K)$ for the splitting $\imath$. Since K is just the image of $\tilde{K}$ under the projection of G onto G/R_u, it is easily seen that the pair $(K, \imath)$ is uniquely determined by $\tilde{K}$. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{k}_{\mathbb{C}}$ be the Lie algebras of G, K, $K_{\mathbb{C}}$, respectively. Put $\tilde{\mathfrak{k}} := \imath_*\mathfrak{k}$ and $\tilde{\mathfrak{k}}_{\mathbb{C}} := \imath_*\mathfrak{k}_{\mathbb{C}}$. We now take a $\tilde{K}$-invariant Kähler metric ω in the class γ, and write

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}}$$

in terms of a system $(z^1, z^2, \ldots, z^n)$ of holomorphic local coordinates on X. In this paper, a Kähler metric and the associated Kähler form are used interchangeably. If we move $\imath$ and K, then our ω runs through $\mathcal{M}_\gamma\,(\neq \phi)$, where

$\mathcal{M}_\gamma$ denotes the set of all Kähler metrics in the class γ such that the associated groups of the isometries, when intersected with G, are maximal compact in G. Let $\square_\omega := \sum_{\alpha,\beta} g^{\bar\beta\alpha}\partial^2/\partial z^\alpha \partial z^{\bar\beta}$ denote the complex Laplacian for functions on the Kähler manifold (X, ω). To each complex-valued smooth function φ on X, we associate a complex vector field $\mathrm{grad}^{(1,0)}_\omega \varphi$ on X of type $(1,0)$ by

$$\mathrm{grad}^{(1,0)}_\omega \varphi := \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta=1}^{n} g^{\bar\beta\alpha} \partial_{\bar\beta}\varphi \frac{\partial}{\partial z^\alpha} .$$

Let $\mathcal{H}_\omega$ be the space of all complex smooth functions φ on X such that $\mathrm{grad}^{(1,0)}_\omega \varphi$ is in $\tilde{\mathfrak{k}}_{\mathbb{C}}$ and that $\int_X \varphi\, \omega^n/n! = 0$. Then we have $\mathfrak{k}_{\mathbb{C}} \cong \mathcal{H}_\omega$ by associating to each $\mathcal{Y} \in \mathfrak{k}_{\mathbb{C}}$ a function $h^\omega_{\mathcal{Y}}$ in $\mathcal{H}_\omega$, called the **Hamiltonian function** for $\mathcal{Y}$, by

$$\tilde{\mathcal{Y}} = \mathrm{grad}^{(1,0)}_\omega h^\omega_{\mathcal{Y}} ,$$

where $\tilde{\mathcal{Y}} := \imath_* \mathcal{Y}$. Note that, if $\mathcal{Y} \in \mathfrak{k}$, then $h^\omega_{\mathcal{Y}}$ is a real-valued function on X. We now define a symmetric $\mathbb{C}$-bilinear form $B_{\tilde K,\omega} : \mathfrak{k}_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}} \to \mathbb{C}$ by

$$B_{\tilde K,\omega}(\mathcal{Y}, \mathcal{Z}) = \int_X h^\omega_{\mathcal{Y}} h^\omega_{\mathcal{Z}}\, \omega^n/n! ,$$

where the restriction of $B_{\tilde K,\omega}$ to $\mathfrak{k}$ is obviously positive definite. In particular, $B_{\tilde K,\omega}$ is a nondegenerate $\mathbb{C}$-bilinear form. We shall first show that

Theorem A. *For a given class γ, the bilinear form $B_{\tilde K,\omega}$ depends neither on the choice of a maximal compact subgroup $\tilde K$ in G, nor on the choice of a $\tilde K$-invariant Kähler metric ω in the class γ.*

Hence, we write $B_{\tilde K,\omega} : \mathfrak{k}_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}} \to \mathbb{C}$ simply as $B_\gamma : \mathfrak{k}_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}} \to \mathbb{C}$. For the reductive Lie algebra $\mathfrak{k}_{\mathbb{C}}$, its commutator subalgebra $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]$ is written as a direct sum $\bigoplus_{\ell=1}^p \mathfrak{s}_\ell$ of complex simple Lie algebras $\mathfrak{s}_\ell$. Therefore,

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{z} \oplus [\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}] = \mathfrak{z} \oplus \left(\bigoplus_{\ell=1}^{p} \mathfrak{s}_\ell \right) ,$$

where $\mathfrak{z}$ is the center of the Lie algebra $\mathfrak{k}_{\mathbb{C}}$. Let $B_\ell : \mathfrak{s}_\ell \times \mathfrak{s}_\ell \to \mathbb{C}$ be the Killing form for $\mathfrak{s}_\ell$, and let $B_{\mathfrak{z}} : \mathfrak{z} \times \mathfrak{z} \to \mathbb{C}$ be the restriction of the bilinear form B_γ to $\mathfrak{z}$. Then the structure of the bilinear form B_γ is given by the following:

Theorem B. *For some negative real constants a_ℓ, $B_\gamma = B_{\mathfrak{z}} \oplus \left(\bigoplus_{\ell=1}^p a_\ell B_\ell\right)$. In particular, $B_{\mathfrak{z}}$, a_1, $a_2, \ldots, a_p$ are invariants of the Kähler class γ on X.*

Consider the scalar curvature $\sigma_\omega := \sum_{\alpha,\beta} g^{\bar\beta\alpha} R_{\alpha\bar\beta}$ of the Kähler metric ω, where $R_{\alpha\bar\beta} := -\partial_\alpha \partial_{\bar\beta} \log \omega^n$. Note that $H\sigma_\omega := n c_1(X)\gamma^{n-1}[X]/\gamma^n[X]$ is the harmonic part of σ_ω. Let $C^\infty(X)$ be the space of all complex smooth functions on X endowed with the Hermitian inner product $(\varphi_1, \varphi_2)_{L^2(X,\omega)} := \int_X \varphi_1\, \bar\varphi_2\, \omega^n/n!$. Let $\mathrm{pr} : C^\infty(X) \to \mathcal{H}_\omega$ be the orthogonal projection. Now, we put

$$\tilde{\mathcal{V}}_\omega := \mathrm{grad}^{(1,0)}_\omega \mathrm{pr}(\sigma_\omega) \in \mathfrak{k}_{\mathbb{C}} .$$

If the vector field $\mathrm{grad}^{(1,0)}_{\omega}\sigma_{\omega}$ is holomorphic, i.e., $\tilde{\mathscr{V}}_{\omega} = \mathrm{grad}^{(1,0)}_{\omega}\sigma_{\omega}$, then ω is called an **extremal Kähler metric** (see [2, 3]), and as observed by Calabi in [3], any extremal Kähler metric in the class γ is always in $\mathscr{M}_{\gamma}$. As long as ω is in $\mathscr{M}_{\gamma}$, by abuse of terminology, we call $\tilde{\mathscr{V}}_{\omega}$ an **extremal Kähler vector field** even when ω is not an extremal Kähler metric. Since $\mathrm{pr}(\sigma_{\omega})$ is real-valued, $\tilde{\mathscr{V}}_{\omega}$ belongs to $\tilde{\mathfrak{k}}$, so that we can define an element $\mathscr{V}_{\omega}$ in $\mathfrak{k}$ by $\tilde{\mathscr{V}}_{\omega} = \iota_{*}\mathscr{V}_{\omega}$. Slightly modifying the Futaki character, let us now define a Lie algebra character $F_{\gamma} : \mathrm{Lie}(G/R_u)(= \mathfrak{k}_{\mathbb{C}}) \to \mathbb{C}$ by

$$F_{\gamma}(\mathscr{Y}) := (\sqrt{-1})^{-1}\int_X (\tilde{\mathscr{Y}} f_{\omega})\omega^n/n!, \quad \mathscr{Y} \in \mathfrak{k}_{\mathbb{C}},$$

where f_{ω} is a real-valued function in $C^{\infty}(X)$ satisfying $\sigma_{\omega} - H\sigma_{\omega} = \square_{\omega} f_{\omega}$. Note that F_{γ} is independent of the choice of ω in $\mathscr{M}_{\gamma}$. Now, for every ω in $\mathscr{M}_{\gamma}$, we shall show the following uniqueness of extremal Kähler vector fields:

Theorem C. *$F_{\gamma}(\mathscr{Y}) = B_{\gamma}(\mathscr{Y}, \mathscr{V}_{\omega})$ for all $\mathscr{Y} \in \mathfrak{k}_{\mathbb{C}}$. Hence, if we identify $(\mathfrak{k}_{\mathbb{C}})^{*}$ with $\mathfrak{k}_{\mathbb{C}}$ by the nondegenerate bilinear form $B_{\gamma} : \mathfrak{k}_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}} \to \mathbb{C}$, then F_{γ} coincides with $\mathscr{V}_{\omega}$.*

Corollary D. *The element $\mathscr{V}_{\omega}$ in $\mathfrak{k}_{\mathbb{C}}$ belongs to the center $\mathfrak{z}$, and is independent of the choice of ω in $\mathscr{M}_{\gamma}$. In particular, $F_{\gamma}(\mathscr{V}_{\omega})(= B_{\gamma}(\mathscr{V}_{\omega}, \mathscr{V}_{\omega}))$ is independent of the choice of ω in γ, and is an invariant of the Kähler class γ.*

Corollary E. *For any ω_1, ω_2 in $\mathscr{M}_{\gamma}$, there exists a $g \in R_u$ such that $g_{*}\tilde{\mathscr{V}}_{\omega_1} = \tilde{\mathscr{V}}_{\omega_2}$.*

For each $\mathscr{Y} \in \mathfrak{k}_{\mathbb{C}}$, let $\tilde{\mathscr{Y}}_{\mathbb{R}} := \iota_{*}\mathscr{Y} + \overline{\iota_{*}\mathscr{Y}}$ denote the real vector field on X associated with $\tilde{\mathscr{Y}} := \iota_{*}\mathscr{Y}$. Then for $\gamma = c_1(X)$, we obtain the following periodicity of extremal Kähler vector fields:

Theorem F. *If $\omega \in \mathscr{M}_{c_1(X)}$, then $\exp(2\pi m\tilde{\mathscr{V}}_{\omega\mathbb{R}}) = \mathrm{id}_X$ for some integer $m > 0$.*

Corollary G. *If $\omega \in \mathscr{M}_{c_1(X)}$, then both $\max_X \mathrm{pr}(\sigma_{\omega})$ and $\min_X \mathrm{pr}(\sigma_{\omega})$ are rational numbers independent of ω.*

This paper consists of three sections and two appendices. We organize these as follows. Section 1 is devoted to the study of the bilinear form B_{γ} and in particular, we prove Theorems A and B. Then in Sect. 2, we show the uniqueness of extremal Kähler vector fields, so that we prove Theorem C, Corollaries D and E. In Sect. 3, results in [6, 14] together with Theorem C will allow us to obtain the periodicity of extremal Kähler vector fields for $\gamma = c_1(X)$, and it in particular proves Theorem F and Corollary G. In Appendix 1, we complete the arguments in Sect. 1 by showing how a result in [13] can be applicable to Sect. 1. Finally in Appendix 2, another direct proof for the independence of the bilinear form $B_{\hat{K},\omega}$ on the choice of ω in γ will be given.

1 The symmetric $\mathbb{C}$-bilinear form B_γ on $\mathfrak{k}_{\mathbb{C}}$

(1.1) In view of the isomorphism $H^{1,1}(X,\mathbb{R}) \cong H^1(X, |\mathcal{O}^*|^2)$, we consider the real line bundle $\mathcal{L}$ on X associated with the Kähler class γ (cf. [13]). For the nowhere vanishing section τ to $\mathcal{L}^*$ as in Appendix 1, the holomorphic $\tilde{K}_{\mathbb{C}}$-action on X lifts to a quasi-holomorphic $\tilde{K}_{\mathbb{C}}$-action on $\mathcal{L}$ such that the associated infinitesimal action of $\tilde{\mathfrak{k}}_{\mathbb{C}}$ on $\mathcal{L}$ satisfies

$$(\sqrt{-1}\,\tau)^{-1}\tilde{\mathcal{Y}}\tau = h^{\omega}_{\mathcal{Y}}, \quad \mathcal{Y} \in \mathfrak{k}_{\mathbb{C}} .$$

We now choose a maximal algebraic torus $T_{\mathbb{C}}(\cong \mathbb{G}^r_{\mathrm{m}})$ in $K_{\mathbb{C}}$ such that its maximal compact subgroup $T(\cong (S^1)^r)$ is contained in K. Let $\mathfrak{t}$ be the real Lie subalgebra of $\mathfrak{k}$ associated with the subgroup T of K. Put $\tilde{T} := \iota(T)$ and $\tilde{\mathfrak{t}} := \iota_*\mathfrak{t}$. Then the moment map $\mu_\omega : X \to \mathfrak{t}^*$ (cf. [13, Sect. 4]) associated to the $\tilde{T}$-action on X sends each $x \in X$ to the associated element $\mu_\omega(x) \in \mathfrak{t}^*$ defined by

$$\langle \mu_\omega(x), \mathcal{Y}\rangle := h^{\omega}_{\mathcal{Y}}(x), \quad \mathcal{Y} \in \mathfrak{t} .$$

Note that the image $\mu(X)$ in $\mathfrak{t}^*$ of X under the map μ is a compact convex polyhedron independent of the choice of the $\tilde{K}$-invariant Kähler metric ω in the class γ (cf. [1, 7]; see also [14]). Let $\mu_{\omega*}(\omega^n/n!)$ be the measure on $\mu_\omega(X)$ obtained as the push-forward, by the map μ_ω, of the measure $\omega^n/n!$ on X. Then by [13, Corollary 5.2], the measure $\mu_{\omega*}(\omega^n/n!)$ is also independent of the choice of ω. Therefore, the restriction of the bilinear form $B_{\tilde{K},\omega}$ to $\mathfrak{t}$ (hence to $\mathfrak{t}_{\mathbb{C}}$),

$$B_{\tilde{K},\omega}(\mathcal{Y}_1,\mathcal{Y}_2) = \int_X h^{\omega}_{\mathcal{Y}_1} h^{\omega}_{\mathcal{Y}_2}\omega^n/n! = \int_{\mu_\omega(X)} y_1 y_2 \mu_*(\omega^n/n!), \quad \mathcal{Y}_1, \mathcal{Y}_2 \in \mathfrak{t} ,$$

is independent of the choice of ω, where the inclusion $\mu_\omega(X) \subset \mathfrak{t}^*$ allows us to regard $\mathcal{Y}_1$ and $\mathcal{Y}_2$ as functions on $\mu_\omega(X)$, denoted respectively by y_1 and y_2, such that $\mu^*_\omega y_1 = h^{\omega}_{\mathcal{Y}_1}$ and $\mu^*_\omega y_2 = h^{\omega}_{\mathcal{Y}_2}$.

Remark. For every positive integer m, the symmetric $\mathbb{C}$-multilinear form $B^{(m)}_{\tilde{K},\omega}$: $\mathfrak{t}^m \to \mathbb{C}$ defined by $B^{(m)}_{\tilde{K},\omega}(\mathcal{Y}_1,\mathcal{Y}_2,\ldots,\mathcal{Y}_m) = \int_X h^{\omega}_{\mathcal{Y}_1} h^{\omega}_{\mathcal{Y}_2}\ldots h^{\omega}_{\mathcal{Y}_m}\omega^n/n!$, $\mathcal{Y}_j \in \mathfrak{t}$, is more generally independent of the choice of ω. Moreover, by the argument in (1.2) below, $B^{(m)}_{\tilde{K},\omega}$ is independent also of the choice of $\tilde{K}$.

(1.2) In place of $\tilde{K}$, we choose another maximal compact subgroup $\tilde{K}'$ of G. Then there exists an element g of G such that $\tilde{K}' = \mathrm{Ad}(g^{-1})\tilde{K} = g^{-1}\tilde{K}g$. Since G is connected, the form $\omega' := g^*\omega$ is cohomologous to ω, and is therefore in the class γ. Note also that

$$h^{\omega'}_{\mathcal{Y}} = g^* h^{\omega}_{\mathcal{Y}}, \quad \mathcal{Y} \in \mathfrak{k}_{\mathbb{C}} ,$$

and it implies $B_{\tilde{K}',\omega'}(\mathcal{Y},\mathcal{Z}) = B_{\tilde{K},\omega}(\mathcal{Y},\mathcal{Z})$ for all $\mathcal{Y},\mathcal{Z} \in \mathfrak{k}_{\mathbb{C}}$ by

$$\int_X h^{\omega'}_{\mathcal{Y}} h^{\omega'}_{\mathcal{Z}}\omega'^n/n! = \int_X (g^*h^{\omega}_{\mathcal{Y}})(g^*h^{\omega}_{\mathcal{Z}})(g^*\omega^n)/n! = \int_X h^{\omega}_{\mathcal{Y}} h^{\omega}_{\mathcal{Z}}\omega^n/n! .$$

Thus, in proving Theorems A and B, we may fix a $\tilde{K}$ once for all, and it suffices to show (i) the independence of $B_{\tilde{K},\omega}$ on ω and (ii) the identity in Theorem B.

(1.3) In view of (1.2), we fix a $\tilde{K}$ once for all. Note that both $\imath$ and K are uniquely determined by $\tilde{K}$. We now take $\varphi_1, \varphi_2, \varphi_3 \in C^\infty(X)$ and $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3 \in \mathfrak{k}_{\mathbb{C}}$. Put $[\varphi_1, \varphi_2] = \sqrt{-1}\sum_{\alpha,\beta} g^{\bar{\beta}\alpha}(\partial_\alpha\varphi_1\partial_{\bar{\beta}}\varphi_2 - \partial_{\bar{\beta}}\varphi_1\partial_\alpha\varphi_2)$, called the Poisson bracket of φ_1 and φ_2 relative to ω. Recall the following standard fact (see for instance [13]):

$$h^{\omega}_{[\mathcal{Y}_1,\mathcal{Y}_2]} = [h^{\omega}_{\mathcal{Y}_1}, h^{\omega}_{\mathcal{Y}_2}]\,; \tag{1.3.1}$$

$$\int_X [\varphi_1, \varphi_2]\varphi_3\, \omega^n/n! = \int_X \varphi_1[\varphi_2, \varphi_3]\omega^n/n!\,. \tag{1.3.2}$$

Combining (1.3.1) and (1.3.2), we obtain

$$\begin{aligned} B_{\tilde{K},\omega}([\mathcal{Y}_1,\mathcal{Y}_2],\mathcal{Y}_3) &= \int_X h^{\omega}_{[\mathcal{Y}_1,\mathcal{Y}_2]} h^{\omega}_{\mathcal{Y}_3}\, \omega^n/n! = \int_X [h^{\omega}_{\mathcal{Y}_1}, h^{\omega}_{\mathcal{Y}_2}] h^{\omega}_{\mathcal{Y}_3}\, \omega^n/n! \\ &= \int_X h^{\omega}_{\mathcal{Y}_1}[h^{\omega}_{\mathcal{Y}_2}, h^{\omega}_{\mathcal{Y}_3}]\, \omega^n/n! = \int_X h^{\omega}_{\mathcal{Y}_1} h^{\omega}_{[\mathcal{Y}_2,\mathcal{Y}_3]}\omega^n/n! \\ &= B_{\tilde{K},\omega}(\mathcal{Y}_1, [\mathcal{Y}_2, \mathcal{Y}_3])\,. \end{aligned}$$

Hence, if $\mathcal{Y}_3 \in \mathfrak{z}$, then $B_{\tilde{K},\omega}([\mathcal{Y}_1,\mathcal{Y}_2],\mathcal{Y}_3) = 0$ by $[\mathcal{Y}_2,\mathcal{Y}_3] = 0$. Therefore, the bilinear form $B_{\tilde{K},\omega}$ is written in the form

$$B_{\tilde{K},\omega} = (B_{\tilde{K},\omega})_{|\mathfrak{z}} \oplus (B_{\tilde{K},\omega})_{|[\mathfrak{k}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}]}\,. \tag{1.3.3}$$

Put $\mathfrak{h} := [\mathfrak{k},\mathfrak{k}]$ and $\mathfrak{m} := \sqrt{-1}\mathfrak{h}$. We further set $\mathfrak{m}_{\mathbb{C}} := \mathfrak{h} + \mathfrak{m} = [\mathfrak{k}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}] = \bigoplus_{\ell=1}^{p}\mathfrak{s}_\ell$. In view of the identity $B_{\tilde{K},\omega}([\mathcal{Y}_1,\mathcal{Y}_2],\mathcal{Y}_3) = B_{\tilde{K},\omega}(\mathcal{Y}_1,[\mathcal{Y}_2,\mathcal{Y}_3])$ above, the restriction of the symmetric bilinear form $B_{\tilde{K},\omega}$ to $\mathfrak{m}$ is ad($\mathfrak{h}$)-invariant, hence (cf. [9, p. 257]),

$$(B_{\tilde{K},\omega})_{|\mathfrak{m}} = \sum_{\ell=1}^{p} a_\ell B_{\ell_{|\mathfrak{m}}}$$

for some constants $a_\ell \in \mathbb{R}$ possibly depending on the choice of ω. Complexifying this, we now obtain

$$(B_{\tilde{K},\omega})_{|[\mathfrak{k}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}]} = \sum_{\ell=1}^{p} a_\ell B_\ell\,. \tag{1.3.4}$$

Since the restriction of $B_{\tilde{K},\omega}$ to $\mathfrak{k}$ is positive definite, and since B_ℓ is the Killing form for $\mathfrak{s}_\ell$, it follows that a_ℓ are all negative. Finally, the proof of Theorems A and B is reduced to showing that $(B_{\tilde{K},\omega})_{|\mathfrak{z}}$ in (1.3.3) and all a_ℓ in (1.3.4) are independent of the choice of ω with $\tilde{K}$ fixed once for all. But such independence is straightforward from (1.1), and this completes the proof of Theorems A and B.

2 Uniqueness of extremal Kähler vector fields

(2.1) We shall first prove Theorem C and Corollary D. Let $(,)_\omega$ denote the pointwise Hermitian pairing, relative to the Kähler metric ω, on the space of all smooth differentiable 1-forms on X. Then, for $\mathscr{Y} \in \mathfrak{k}_{\mathbb{C}}$, we obtain (see also LeBrun and Simanca [11]):

$$\begin{aligned} F_\gamma(\mathscr{Y}) &= (\sqrt{-1})^{-1}\int_X (\tilde{\mathscr{Y}} f_\omega)\,\omega^n/n! = (\sqrt{-1})^{-1}\int_X (\mathrm{grad}^{(1,0)}_\omega h^\omega_{\mathscr{Y}})(f_\omega)\,\omega^n/n! \\ &= -\int_X \sum_{\alpha,\beta} g^{\bar\beta\alpha}\partial_{\bar\beta} h^\omega_{\mathscr{Y}} \partial_\alpha f_\omega\,\omega^n/n! = -\int_X (\bar\partial h^\omega_{\mathscr{Y}}, \bar\partial f_\omega)_\omega \omega^n/n! \\ &= \int_X h^\omega_{\mathscr{Y}}(\Box_\omega f_\omega)\,\omega^n/n! = \int_X h^\omega_{\mathscr{Y}}(\sigma_\omega - H\sigma_\omega)\omega^n/n! = \int_X h^\omega_{\mathscr{Y}}\sigma_\omega\omega^n/n! \\ &= \int_X h^\omega_{\mathscr{Y}}\ \mathrm{pr}(\sigma_\omega)\omega^n/n! = \int_X h^\omega_{\mathscr{Y}} h^\omega_{\mathscr{V}_\omega}\omega^n/n! = B_\gamma(\mathscr{Y}, \mathscr{V}_\omega)\,, \end{aligned}$$

which completes the proof of Theorem C. Since $B_\gamma([\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}], \mathscr{V}_\omega) = F_\gamma([\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]) = \{0\}$, Theorem B yields $\mathscr{V}_\omega \in \mathfrak{z}$. Then, Corollary D is straightforward from Theorem C.

(2.2) Next, we shall prove Corollary E. Let ω_1, ω_2 be extremal Kähler metrics in the same class γ. Then by Corollary D, there exists a unique element $\mathscr{V}$ in the center $\mathfrak{z}$ for the Lie algebra $\mathfrak{k}_{\mathbb{C}}$ of $K_{\mathbb{C}} = G/R_u$ such that

$$\mathscr{V}_{\omega_1} = \mathscr{V} = \mathscr{V}_{\omega_2}\,.$$

Therefore, we have isomorphisms $\iota_1 : K_{\mathbb{C}} \cong \iota_1(K_{\mathbb{C}}) \subset G$ and $\iota_2 : K_{\mathbb{C}} \cong \iota_2(K_{\mathbb{C}}) \subset G$ such that $\tilde{\mathscr{V}}_{\omega_1} = (\iota_1)_*\mathscr{V}$ and $\tilde{\mathscr{V}}_{\omega_2} = (\iota_2)_*\mathscr{V}$. Since these isomorphisms coincide up to conjugacy in G, there exists an element g of G such that $\iota_2 = \mathrm{Ad}(g) \circ \iota_1$. Note that $G = \iota_1(K_{\mathbb{C}}) \cdot R_u = R_u \cdot \iota_1(K_{\mathbb{C}})$. Then we can write g as $g' \cdot \iota_1(k)$ for some $g' \in R_u$ and $k \in K_{\mathbb{C}}$. Therefore,

$$\begin{aligned} \tilde{\mathscr{V}}_{\omega_2} &= (\iota_2)_*\mathscr{V} = \{\mathrm{Ad}(g) \circ \iota_1\}_*\mathscr{V} = \{\mathrm{Ad}(g') \circ \mathrm{Ad}(\iota_1(k)) \circ \iota_1\}_*\mathscr{V} \\ &= g'_*\{\mathrm{Ad}(\iota_1(k)) \circ \iota_1\}_*\mathscr{V} = g'_*\{(\iota_1)_*(\mathrm{Ad}(k)\mathscr{V})\} \\ &= g'_*(\iota_1)_*\mathscr{V} = g'_*\tilde{\mathscr{V}}_{\omega_1}\,. \end{aligned}$$

3 Periodicity of extremal Kähler vector fields

(3.1) Let $T, T_{\mathbb{C}}$ be as in (1.1). Write $T_{\mathbb{C}} = \mathbb{G}_m^r = \{(z_1, z_2, \ldots, z_r); z_i \in \mathbb{C}^*$ for all $i\}$. Then by setting $\mathscr{X}_i := \sqrt{-1}z_i\partial/\partial z_i$ and $\mathscr{X}_i^* := (\sqrt{-1}\,z_i)^{-1}dz_i$, we can regard $\mathfrak{t}, \mathfrak{t}^*$ as $\sum_{i=1}^r \mathbb{R}\mathscr{X}_i$, $\sum_{i=1}^r \mathbb{R}\mathscr{X}_i^*$ respectively. Then $\mathfrak{t}$ and $\mathfrak{t}^*$ admit natural $\mathbb{Z}$-structures

$$\sum_{i=1}^r \mathbb{Z}\mathscr{X}_i = \mathfrak{t}_{\mathbb{Z}} = \{\mathscr{Y} \in \mathfrak{t}; \exp(2\pi\mathscr{Y}) = 1\}\,,$$

$$\sum_{i=1}^{r}\mathbb{Z}\mathscr{X}_i^* = \mathfrak{t}_{\mathbb{Z}}^* = \{\theta \in \mathfrak{t}^*; <\theta,\ \mathfrak{t}_{\mathbb{Z}}> \subset \mathbb{Z}\}\,.$$

We then put $\mathfrak{t}_{\mathbb{Q}} := \mathfrak{t}_{\mathbb{Z}} \otimes \mathbb{Q} = \sum_{i=1}^{r}\mathbb{Q}\mathscr{X}_i$ and $\mathfrak{t}_{\mathbb{Q}}^* := \mathfrak{t}_{\mathbb{Z}}^* \otimes \mathbb{Q} = \sum_{i=1}^{r}\mathbb{Q}\mathscr{X}_i^*$. Recall the following fact:

Fact (see [14]) *Suppose $\gamma \in H^2(X,\mathbb{Q})$. Then the restriction of B_γ to $\mathfrak{t}$ is defined over $\mathbb{Q}$, i.e., $B_\gamma(\mathscr{X}_i,\mathscr{X}_j) \in \mathbb{Q}$ for all i and j.*

Hence, if $\gamma \in H^2(X,\mathbb{Q})$, then Theorem B is stated in a more refined way. Namely, if $\gamma \in H^2(X,\mathbb{Q})$, then a_1, a_2, $\ldots a_p$ are all rational numbers, and B_3 is defined over $\mathbb{Q}$.

(3.2) We now consider the case where $\gamma = c_1(X)_{\mathbb{Q}}$. Then the G-action on X naturally lifts to a bundle G-action on the anticanonical bundle K_X of X. Note that the associated infinitesimal action of $\mathfrak{g}$ on K_X (or more generally on the space of smooth differential forms on X) is just the Lie differentiation. Consider the complex Lie algebra $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$ for $T_{\mathbb{C}}$. For each $\mathscr{Y} \in \mathfrak{t}_{\mathbb{C}}$, let

$$L_{\tilde{\mathscr{Y}}} := d \circ i_{\tilde{\mathscr{Y}}} + i_{\tilde{\mathscr{Y}}} \circ d$$

denote the Lie differentiation with respect to $\tilde{\mathscr{Y}}$. Note also that, by $\gamma = c_1(X)_{\mathbb{Q}}$, the nowhere vanishing section τ to $\mathscr{L}^*$ in Appendix 1 is naturally regarded as a smooth volume form on X. In order to prove the periodicity, we need the following:

Lemma. *If $\gamma = c_1(X)_{\mathbb{Q}}$, then $F_\gamma(\mathscr{Y}) = \int_X(\sqrt{-1}\,\tau)^{-1}(L_{\tilde{\mathscr{Y}}}\tau)\omega^n/n!$ for all $\mathscr{Y} \in \mathfrak{t}_{\mathbb{C}}$.*

Proof. Consider the real-valued smooth function $f_\omega := \log(\tau/\omega^n)$ on X. It satisfies the identity $\sigma_\omega - H\sigma_\omega = \Box_\omega f_\omega$, since

$$\bar{\partial}\partial \log \omega^n - \bar{\partial}\partial \log \tau = \partial\bar{\partial} f_\omega, \text{i.e., } R_{\alpha\bar{\beta}} - g_{\alpha\bar{\beta}} = \partial_\alpha\partial_{\bar{\beta}} f_\omega\,.$$

Then by $\omega^n = \exp(-f_\omega)\tau$, we obtain the required equality as follows:

$$\begin{aligned}\sqrt{-1}n!F_\gamma(\mathscr{Y}) &= \int_X(\tilde{\mathscr{Y}} f_\omega)\omega^n = \int_X(\tilde{\mathscr{Y}} f_\omega)\omega^n + \int_X L_{\tilde{\mathscr{Y}}}\{\exp(-f_\omega)\tau\} \\ &= \int_X \exp(-f_\omega)L_{\tilde{\mathscr{Y}}}\tau = \int_X \tau^{-1}(L_{\tilde{\mathscr{Y}}}\tau)\omega^n\,.\end{aligned}$$

(3.3) *Proof of Theorem F.* Let $\omega \in \mathscr{M}_{c_1(X)}$. Put $\mathscr{X}_i' := (\sqrt{-1}\,)^{-1}\mathscr{X}_i$. Since ω is T-invariant, so is τ. In particular, we have $L_{\tilde{\mathscr{X}}_{i\mathbb{R}}}\tau = 0$, where as in the introduction, $\tilde{\mathscr{Y}}_{\mathbb{R}}$ denotes the real vector field $\imath_*\mathscr{Y} + \overline{\imath_*\mathscr{Y}}$ on X for every $\mathscr{Y} \in \mathfrak{t}_{\mathbb{C}}$. Therefore,

$$L_{\tilde{\mathscr{X}}'_{i\mathbb{R}}}\tau = (\sqrt{-1}\,)^{-1}(2L_{\tilde{\mathscr{X}}_i}\tau)\,.$$

Hence, by [6, (5.4.a)], the above lemma in (3.2) yields

$$F_\gamma(\mathscr{X}_i) = \frac{1}{2}\int_X \tau^{-1}(L_{\tilde{\mathscr{X}}'_{i\mathbb{R}}}\tau)\omega^n/n! \in \mathbb{Q}\,,$$

for all i. Then by the above fact in (3.1), we see from Theorem C that $\mathscr{V}_\omega \in \mathfrak{t}_{\mathbb{Q}}$, i.e., $\mathscr{V}_\omega \in \sum_{i=1}^r \mathbb{Q}\mathscr{Z}_i$. Therefore, for some positive integer m, we have $2\pi m\mathscr{V}_\omega \in \sum_{i=1}^r \mathbb{Z}\mathscr{Z}_i$. Since $\exp(2\pi m\mathscr{V}_\omega) = 1$ in T, we now conclude that $\exp(2\pi m\tilde{\mathscr{V}}_{\omega\mathbb{R}}) = \mathrm{id}_X$.

(3.4) *Proof of Corollary G.* In the above proof of Theorem F, let q be the smallest positive rational number such that $q\mathscr{V}_\omega \in \sum_{i=1}^r \mathbb{Z}\mathscr{Z}_i$. Therefore, the group $W_{\mathbb{C}} := \exp(\mathbb{C}q\tilde{\mathscr{V}}_\omega)$ is an algebraic torus ($\cong \mathbb{G}_{\mathrm{m}}$). We may write $W_{\mathbb{C}} = \{w; w \in \mathbb{C}^*\}$ in such a way that

$$q\tilde{\mathscr{V}}_\omega = \sqrt{-1}w\partial/\partial w\,.$$

Since the maximal compact subgroup $W(\cong S^1)$ of $W_{\mathbb{C}}$ is a subgroup of T above, W acts isometrically on (X,ω). Note also that

$$h^\omega_{q\mathscr{V}_\omega} = qh^\omega_{\mathscr{V}_\omega} = q\,\mathrm{pr}(\sigma_\omega)\,.$$

The basis $q\tilde{\mathscr{V}}_\omega(= \sqrt{-1}w\partial/\partial w)$ for the Lie algebra $\mathfrak{w}$ of W allows us to identify $\mathfrak{w}$ with $\mathbb{R}$, and therefore the moment map $\mu^W_\omega : X \to \mathfrak{w}^*$ associated with the W-action on X is expressible as

$$\mu : X \to \mathbb{R}, \quad x \mapsto h^\omega_{q\mathscr{V}_\omega}(x) = q\,\mathrm{pr}(\sigma_\omega)(x)\,.$$

Then a modification (see [14]) of a result of Guillemin and Sternberg [7] shows that the vertices of the image of the moment map μ^W_ω are $\mathbb{Q}$-rational points, i.e.,

$$\max_X\{q\,\mathrm{pr}(\sigma_\omega)\} \in \mathbb{Q} \text{ and } \min_X\{q\,\mathrm{pr}(\sigma_\omega)\} \in \mathbb{Q}\,.$$

Hence, both $\max_X \mathrm{pr}(\sigma_\omega)$ and $\min_X \mathrm{pr}(\sigma_\omega)$ are rational numbers, as required.

Remark. If $\omega \in \mathscr{M}_{c_1(X)}$ is an extremal Kähler metric in Corollary G, then by $\mathrm{pr}(\sigma_\omega) = \sigma_\omega - n$, Corollary G asserts that $\max_X \sigma_\omega \in \mathbb{Q}$ and $\min_X \sigma_\omega \in \mathbb{Q}$.

Appendix 1

Let $\mathscr{L}$ be the real line bundle as in Sect. 1, and τ be the nowhere vanishing section to $\mathscr{L}^*$ defined below. In this appendix, we shall show that the holomorphic $\tilde{K}_{\mathbb{C}}$-action on X lifts to a quasi-holomorphic (cf. [13]) $\tilde{K}_{\mathbb{C}}$-action on $\mathscr{L}$ such that the associated infinitesimal action of $\tilde{\mathfrak{k}}_{\mathbb{C}}$ on $\mathscr{L}^*$ satisfies

$$\tilde{\mathscr{Y}}\tau = \sqrt{-1}\,h^\omega_{\mathscr{Y}}\tau, \quad \mathscr{Y} \in \mathfrak{k}_{\mathbb{C}}\,.$$

More generally, such a quasi-holomorphic lifting can be obtained also for the G-action, though we do not go into details. Take a sufficiently fine Stein cover $X = \bigcup_{i\in I} U_i$ such that $\mathscr{L}$ admits a collection of local bases e_i over U_i, $i \in I$, with positive real-valued transition functions

$$\theta_{ij} := e_i/e_j \in H^0(U_i \cap U_j, |\mathscr{O}^*|^2), \quad i,j \in I\,.$$

Let e_i^* be the basis for $\mathscr{L}^*_{|U_i}$ dual to e_i. We then have a smooth section τ for $\mathscr{L}^*$ on X, unique up to positive constant multiple, such that τ is written as $\tau_i e_i^*$ on each U_i with a positive smooth function τ_i and that the first Chern form $c_1(\mathscr{L};\tau)$ for $\mathscr{L}$ with respect to τ coincides with ω (cf. [13]). We now define a smooth function $\psi_{\mathscr{Y},i}$ on U_i by

$$\psi_{\mathscr{Y},i} := \tilde{\mathscr{Y}} \log \tau_i - \sqrt{-1} h^{\omega}_{\mathscr{Y}} .$$

Then $\bar\partial \psi_{\mathscr{Y},i} = \bar\partial\{i_{\tilde{\mathscr{Y}}}(\partial \log \tau_i)\} - \sqrt{-1}\, i_{\tilde{\mathscr{Y}}}(2\pi\omega) = -i_{\tilde{\mathscr{Y}}}(\bar\partial\partial \log \tau_i) - \sqrt{-1}\, i_{\tilde{\mathscr{Y}}}(2\pi\omega) = 0$. Hence, $\psi_{\mathscr{Y},i}$ is holomorphic. Define an infinitesimal action of $\tilde{\mathscr{Y}}$ on $\mathscr{L}_{|U_i}$ by $\tilde{\mathscr{Y}} e_i := \psi_{\mathscr{Y},i} e_i$, $i \in I$. In view of $e_i = \theta_{ij} e_j$ and $\tau_i = \theta_{ij}\tau_j$, we have

$$\begin{aligned}(\tilde{\mathscr{Y}} e_i)_{|U_i\cap U_j} &= \psi_{\mathscr{Y},i} e_{i\,|U_i\cap U_j} = \{\tilde{\mathscr{Y}} \log(\theta_{ij}\tau_j) - \sqrt{-1}\, h^{\omega}_{\mathscr{Y}}\}\theta_{ij} e_{j\,|U_i\cap U_j} \\ &= \{\theta_{ij}\psi_{\mathscr{Y},j} + \tilde{\mathscr{Y}}\theta_{ij}\} e_{j\,|U_i\cap U_j} = \theta_{ij}(\tilde{\mathscr{Y}} e_j)_{|U_i\cap U_j} + (\tilde{\mathscr{Y}}\theta_{ij}) e_{j\,|U_i\cap U_j} .\end{aligned}$$

Hence, the infinitesimal actions of $\tilde{\mathscr{Y}}$ on $\mathscr{L}_{|U_i}$, $i \in I$, glue together to form a global infinitesimal action of $\tilde{\mathscr{Y}}$ on $\mathscr{L}$. Then $\tilde{\mathscr{Y}}\tau = \sqrt{-1}\, h^{\omega}_{\mathscr{Y}}\tau$ follows from

$$\begin{aligned}(\sqrt{-1}\tau_i e_i^*)^{-1}\tilde{\mathscr{Y}}(\tau_i e_i^*) &= -\sqrt{-1}\tilde{\mathscr{Y}} \log \tau_i + (\sqrt{-1} e_i^*)^{-1}\tilde{\mathscr{Y}} e_i^* \\ &= -\sqrt{-1}\tilde{\mathscr{Y}} \log \tau_i + \sqrt{-1}\,\psi_{\mathscr{Y},i} = h^{\omega}_{\mathscr{Y}} .\end{aligned}$$

Let J be the complex structure of X, and put $\tilde{\mathfrak{k}}^{\mathrm{real}}_{\mathbb{C}} := \{\mathscr{Y}_{\mathbb{R}};\mathscr{Y} \in \mathfrak{k}_{\mathbb{C}}\}$, where $\mathscr{Y}_{\mathbb{R}}$ is as in the introduction. Then by sending $\tilde{\mathscr{Y}} \in \tilde{\mathfrak{k}}_{\mathbb{C}}$ to $\tilde{\mathscr{Y}}_{\mathbb{R}} \in \tilde{\mathfrak{k}}^{\mathrm{real}}_{\mathbb{C}}$, we have the complex Lie algebra isomorphism

$$(\tilde{\mathfrak{k}}_{\mathbb{C}}, \sqrt{-1}) \cong (\tilde{\mathfrak{k}}^{\mathrm{real}}_{\mathbb{C}}, J)$$

with $\tilde{\mathscr{Y}} = (\tilde{\mathscr{Y}}_{\mathbb{R}} - \sqrt{-1} J \cdot \tilde{\mathscr{Y}}_{\mathbb{R}})/2$. Now, from the action of $\tilde{\mathscr{Y}}$ on $\mathscr{L}$, we can globally define an infinitesimal action of $\tilde{\mathscr{Y}}_{\mathbb{R}}$ on $\mathscr{L}$ by $\tilde{\mathscr{Y}}_{\mathbb{R}} e_i := (\psi_{\mathscr{Y},i} + \bar\psi_{\mathscr{Y},i}) e_i$. If $\mathscr{Y} \in \mathfrak{k}$, then $h^{\omega}_{\mathscr{Y}}$ is real-valued, and in particular

$$\begin{aligned}\tilde{\mathscr{Y}}_{\mathbb{R}}\tau = \tilde{\mathscr{Y}}_{\mathbb{R}}(\tau_i e_i^*) &= (\tilde{\mathscr{Y}}_{\mathbb{R}}\tau_i) e_i^* - \tau_i(\psi_{\mathscr{Y},i} + \bar\psi_{\mathscr{Y},i}) e_i^* \\ &= \tilde{\mathscr{Y}}_{\mathbb{R}}\tau_i - \tau_i(\tilde{\mathscr{Y}}_{\mathbb{R}} \log \tau_i - \sqrt{-1}\, h^{\omega}_{\mathscr{Y}} + \sqrt{-1}\, \bar h^{\omega}_{\mathscr{Y}}) = 0 .\end{aligned}$$

Therefore, we can lift the $\tilde K$-action on X naturally to a $\tilde K$-action on $\mathscr{L}$ in such a way that it leaves the section τ of $\mathscr{L}^*$ invariant. Now, regarding $\tilde{\mathfrak{k}}^{\mathrm{real}}_{\mathbb{C}}$ as the Lie algebra of $\tilde K_{\mathbb{C}}$, we can further lift the $\tilde K_{\mathbb{C}}$-action on X to a global $\tilde K_{\mathbb{C}}$-action on $\mathscr{L}$ by setting

$$\exp(s\tilde{\mathscr{Y}}_{\mathbb{R}}) \cdot e_i = |\exp(s\psi_{\mathscr{Y},i})|^2\, e_i, \quad i \in I ,$$

on $U_i \cap \{\exp(s\tilde{\mathscr{Y}}_{\mathbb{R}})\}(U_i)$, for all $s \in \mathbb{R}$ and $\mathscr{Y} \in \mathfrak{k}_{\mathbb{C}}$. Then this $\tilde K_{\mathbb{C}}$-action on $\mathscr{L}$ is quasi-holomorphic in the sense of [13].

Appendix 2

Let $\hat{\mathscr{H}}_\omega$ be the space of all complex smooth functions φ on X such that $\mathrm{grad}_\omega^{(1,0)}\varphi$ is holomorphic and that $\int_X \varphi\omega^n/n! = 0$. Then we have the complex Lie algebra isomorphism

$$\hat{\mathscr{H}}_\omega \cong \mathfrak{g}, \quad \varphi \leftrightarrow \mathrm{grad}_\omega^{(1,0)}\varphi ,$$

where $\hat{\mathscr{H}}_\omega$ has a Lie algebra structure by $[\varphi_1, \varphi_2] := \sqrt{-1}\sum_{\alpha,\beta} g^{\bar{\beta}\alpha}(\partial_\alpha\varphi_1\partial_{\bar{\beta}}\varphi_2 - \partial_{\bar{\beta}}\varphi_1\partial_\alpha\varphi_2)$. We then define a symmetric $\mathbb{C}$-bilinear form $\hat{B}_\omega : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ by

$$\hat{B}_\omega(\mathscr{W}_1, \mathscr{W}_2) := \int_X \varphi_1\varphi_2\,\omega^n/n!, \quad \varphi_1, \varphi_2 \in \hat{\mathscr{H}}_\omega ,$$

where $\mathscr{W}_1 := \mathrm{grad}_\omega^{(1,0)}\varphi_1$ and $\mathscr{W}_2 := \mathrm{grad}_\omega^{(1,0)}\varphi_2$. We shall now show the following:

Theorem H. *The $\mathbb{C}$-bilinear form $\hat{B}_\omega$ on $\mathfrak{g}$ is independent of the choice of ω in γ.*

Proof. We choose another Kähler metric ω' on X cohomologous to the original ω. Then there exists a real-valued smooth function ψ on X such that $\omega' = \omega_1$, where we set

$$\omega_t := \omega + \frac{\sqrt{-1}}{2\pi}t\partial\bar{\partial}\psi , \quad 0 \leqq t \leqq 1 .$$

Let $\mathscr{W}_\nu := \mathrm{grad}_\omega^{(1,0)}\varphi_\nu$, $\nu = 1, 2$, be arbitrary elements in $\mathfrak{g}$ with $\varphi_\nu \in \hat{\mathscr{H}}_\omega$. We can then find $\varphi_{\nu,t} \in \hat{\mathscr{H}}_{\omega_t}$ such that $\mathscr{W}_\nu := \mathrm{grad}_{\omega_t}^{(1,0)}\varphi_{\nu,t}$. It now suffices to show

$$\int_X \varphi_1\varphi_2\,\omega^n/n! = \int_X \varphi_{1,t}\varphi_{2,t}\omega_t^n/n!, \quad 0 \leqq t \leqq 1 .$$

Write $\omega_t = (\sqrt{-1}/2\pi)\sum_{\alpha,\beta} g_{t\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}}$ in terms of a system $(z^1, z^2, \ldots, z^n)$ of holomorphic local coordinates on X. By $i_{\mathscr{W}_\nu}(2\pi\omega) = \bar{\partial}\varphi_\nu$ and $i_{\mathscr{W}_\nu}(2\pi\omega_t) = \bar{\partial}\varphi_{\nu,t}$, we have

$$\begin{aligned} i_{\mathscr{W}_\nu}(2\pi\omega_t) &= i_{\mathscr{W}_\nu}\{2\pi\omega + \sqrt{-1}t\bar{\partial}(-\partial\psi)\} = \bar{\partial}\varphi_\nu + \sqrt{-1}\bar{\partial}\{t\, i_{\mathscr{W}_\nu}(\partial\psi)\} \\ &= \bar{\partial}\{\varphi_\nu + \sqrt{-1}t\mathscr{W}_\nu\psi\} = \bar{\partial}\left\{\varphi_\nu + t\sum_{\alpha,\beta} g^{\bar{\beta}\alpha}\partial_{\bar{\beta}}\varphi_\nu\partial_\alpha\psi\right\} , \end{aligned}$$

i.e., $\varphi_{\nu,t}$ coincides with $\eta_{\nu,t} := \varphi_\nu + t\sum_{\alpha,\beta} g^{\bar{\beta}\alpha}\partial_{\bar{\beta}}\varphi_\nu\partial_\alpha\psi$ on X up to an additive constant. Since $\sum_{\alpha,\beta} g^{\bar{\beta}\alpha}\partial_{\bar{\beta}}\varphi_\nu\partial_\alpha\psi = \sqrt{-1}\mathscr{W}_\nu\psi = \sum_{\alpha,\beta} g_t^{\bar{\beta}\alpha}\partial_{\bar{\beta}}\varphi_{\nu,t}\partial_\alpha\psi$, we obtain

$$\begin{aligned} \frac{d}{dt}\int_X \eta_{\nu,t}\omega_t^n/n! &= \int_X \left\{\left(\sum_{\alpha,\beta} g^{\bar{\beta}\alpha}\partial_{\bar{\beta}}\varphi_\nu\partial_\alpha\psi\right) + \varphi_{\nu,t}\square_{\omega_t}\psi\right\}\omega_t^n/n! \\ &= \int_X \left\{\left(\sum_{\alpha,\beta} g_t^{\bar{\beta}\alpha}\partial_{\bar{\beta}}\varphi_{\nu,t}\partial_\alpha\psi\right) + \varphi_{\nu,t}\square_{\omega_t}\psi\right\}\omega_t^n/n! \\ &= (\bar{\partial}\varphi_{\nu,t}, \bar{\partial}\bar{\psi})_{L^2(X,\omega)} + (\varphi_{\nu,t}, \square_{\omega_t}\bar{\psi})_{L^2(X,\omega)} = 0 , \end{aligned}$$

where $(\,,\,)_{L^2(X,\omega)}$ denotes the Hermitian L^2 inner product, relative to ω, for smooth functions or differential forms on X. Therefore, $\int_X \eta_{v,0}\omega_0^n/n! = \int_X \varphi_v \omega^n/n! = 0$ implies $\int_X \eta_{v,t}\omega_t^n/n! = 0$ for all $0 \leqq t \leqq 1$, hence $\varphi_{v,t} = \eta_{v,t}$ for all v and t. We now obtain the required identity by

$$\begin{aligned}
&\frac{d}{dt}\int_X \varphi_{1,t}\varphi_{2,t}\omega_t^n/n! = \frac{d}{dt}\int_X \eta_{1,t}\eta_{2,t}\omega_t^n/n! \\
&= \int_X \left\{ \varphi_{1,t}\left(\sum g^{\bar\beta\alpha}\partial_{\bar\beta}\varphi_2\partial_\alpha\psi\right) + \left(\sum g^{\bar\beta\alpha}\partial_{\bar\beta}\varphi_1\partial_\alpha\psi\right)\varphi_{2,t} + \varphi_{1,t}\varphi_{2,t}\square_{\omega_t}\psi \right\}\omega_t^n/n! \\
&= \int_X \left\{ \varphi_{1,t}\left(\sum g_t^{\bar\beta\alpha}\partial_{\bar\beta}\varphi_{2,t}\partial_\alpha\psi\right) + \left(\sum g_t^{\bar\beta\alpha}\partial_{\bar\beta}\varphi_{1,t}\partial_\alpha\psi\right)\varphi_{2,t} + \varphi_{1,t}\varphi_{2,t}\square_{\omega_t}\psi \right\}\omega_t^n/n! \\
&= \int_X \left\{ \left(\sum g_t^{\bar\beta\alpha}\partial_{\bar\beta}(\varphi_{1,t}\varphi_{2,t})\partial_\alpha\psi\right) + \varphi_{1,t}\varphi_{2,t}\square_{\omega_t}\psi \right\}\omega_t^n/n! \\
&= (\bar\partial(\varphi_{1,t}\varphi_{2,t}), \bar\partial\bar\psi)_{L^2(X,\omega)} + (\varphi_{1,t}\varphi_{2,t}, \square_{\omega_t}\psi)_{L^2(X,\omega)} = 0\,.
\end{aligned}$$

Remark. By Theorem H, we write the bilinear form $\hat{B}_\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ just as $\hat{B}_\gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. Then by the same argument as in (1.3), we obtain $\hat{B}_\gamma([\mathscr{W}_1, \mathscr{W}_2], \mathscr{W}_3) = \hat{B}_\gamma(\mathscr{W}_1, [\mathscr{W}_2, \mathscr{W}_3])$ for all $\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3$ in $\mathfrak{g}$. Moreover, the bilinear forms $B_\gamma : \mathfrak{k}_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}} \to \mathbb{C}$ and $\hat{B}_\gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ are related by

$$B_\gamma(\mathscr{Y}_1, \mathscr{Y}_2) = \hat{B}_\gamma(\iota_*\mathscr{Y}_1, \iota_*\mathscr{Y}_2), \quad \mathscr{Y}_1, \mathscr{Y}_2 \in \mathfrak{k}_{\mathbb{C}}\,,$$

where $\iota : G/R_u(= K_{\mathbb{C}}) \cong \iota(G/R_u) \subset G$ is an isomorphism as in the introduction. It is easily checked that the independence of $B_{\tilde{K},\omega}$ on ω can be proved also by Theorem H.

Remark. Let $0 < m \in \mathbb{Z}$. Define a symmetric $\mathbb{C}$-multilinear form $\hat{B}_\omega^{(m)} : \mathfrak{g}^m \to \mathbb{C}$ by setting $\hat{B}_\omega^{(m)}(\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_m) := \int_X \varphi_1\varphi_2\ldots\varphi_m\omega^n/n!$, $\varphi_j \in \hat{\mathscr{H}}_\omega$, where $\mathscr{W}_j := \operatorname{grad}_\omega^{(1,0)}\varphi_j$. Then a slight modification of the above arguments shows that $\hat{B}_\omega^{(m)}$ is also independent of the choice of ω in γ. As in the Remark just above, this induces a multilinear form on $\mathfrak{k}_{\mathbb{C}}$ depending only on the class γ, and when restricted to $\mathfrak{k}$, it coincides with the multilinear form in remark of (1.1).

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