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The generalized corona theorem

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Introduction

Let $H^\infty = H^\infty(\mathbb{D})$ be the Banach algebra of all bounded analytic functions in the open disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Its spectrum or maximal ideal space is denoted by $M(H^\infty)$. Carleson's famous corona theorem says that $\mathbb{D}$ is dense in $M(H^\infty)$. An equivalent algebraic formulation tells us that the ideal

$$I = I(f_1, \dots, f_N) = \left\{ \sum_{j=1}^{N} h_j f_j : h_j \in H^\infty \right\}$$

generated by the functions $f_j \in H^\infty$ equals the whole algebra if and only if $\sum_{j=1}^N |f_j| \geqq \delta > 0$ in $\mathbb{D}$. Let

$$J = J(f_1, \dots, f_N) = \left\{ f \in H^\infty : \exists C = C(f) \text{ with } |f| \leqq C \sum_{j=1}^{N} |f_j| \text{ in } \mathbb{D} \right\} .$$

It is obvious that J is an ideal containing I.

Carleson's theorem implies that whenever $J = H^\infty$, then $I = J$. However, a well known example due to Rao (see below or p. 365 of [5]) shows that, in general, the inclusion is proper. For example, one can take the functions $(1-z)^2$ and $\left(e^{-\frac{1+z}{1-z}}\right)^2$ as generators and let f be the function $(1-z)e^{-\frac{1+z}{1-z}}$.

Von Renteln [17] showed that there exist finitely generated ideals $I \neq H^\infty$ for which $I = J$. In fact, if I contains an interpolating Blaschke product B, then $I = J$. This result was later extended by Tolokonnikov [18], who proved

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that $I = J$ provided that J contains an interpolating Blaschke product. An easy proof of this latter result can be found in [15].

It is therefore a natural question to ask for a necessary and sufficient condition on the generators $f_1, \ldots, f_N$ in order that $I = J$. In [15] it was conjectured that this holds if and only if I contains an interpolating Blaschke product (provided, of course, that the generators have no common factor). This conjecture may also be rephrased as an analytic condition on the generators (see Theorem 1.10). It is the aim of this paper to confirm this conjecture for the case of two generators. To our surprise, however, this does not remain true for more than two generators (see Proposition 1.11).

In the second section of this paper we solve Wolff's f^2-problem ([21] and [5, p. 329]) under the additional hypothesis that the generators do not all vanish on any point φ in the spectrum of H^∞ where the Gleason part of φ is trivial. This hypothesis may also be rephrased as an analytic condition on the generators (see Corollary 2.7). For related material see [1, 2, 3, 4, 12, 13].

We assume that the reader is familiar with the theory of bounded analytic functions which is nicely presented in Garnett's book [5].

0 Preliminaries

A sequence $\{z_n\}$ in $\mathbb{D}$ is said to be an **interpolating sequence** if for every bounded sequence (w_n) of complex numbers there exists $f \in H^\infty$ with $f(z_n) = w_n$ for every $n \in \mathbb{N}$. A Blaschke product

$$B(z) = \prod_{j=1}^{\infty} \frac{\bar{a}_j}{|a_j|} \cdot \frac{a_j - z}{1 - \bar{a}_j z}$$

whose zero sequence is an interpolating sequence is called an **interpolating Blaschke product.**

By Carleson's theorem $\{z_n\}$ is interpolating if and only if

$$\inf_{n \in \mathbb{N}} \prod_{\substack{j=1 \\ j \neq n}}^{\infty} \rho(z_j, z_n) \geqq \delta > 0 ,$$

where $\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|$ denotes the pseudohyperbolic distance in $\mathbb{D}$. As usual, the extension of ρ to the whole spectrum of H^∞ is defined by

$$\rho(x, y) = \sup\{|\hat{f}(x)| : f \in H^\infty, \|f\|_\infty \leqq 1, \hat{f}(y) = 0\} .$$

Here $\hat{f}$ denotes the Gelfand transform of f defined by $\hat{f}(m) = m(f)$ ($m \in M(H^\infty)$). We shall always identify $\hat{f}$ with f.

If $f \in H^\infty$, then $Z(f) = \{m \in M(H^\infty) : f(m) = 0\}$ denotes its zero set; $Z_{\mathbb{D}}(f) = Z(f) \cap \mathbb{D}$. If I is an ideal, then $Z(I) = \bigcap_{f \in I} Z(f)$ is the hull or zero set of the ideal I.

Following Hoffman, define for every $m \in M(H^\infty)$ and $f \in H^\infty$ with $f(m) = 0$ the order of the zero m of f by

$$\operatorname{ord}(f,m) = \sup\{n \in \mathbb{N} : f = f_1 \dots f_n,\ f_j \in H^\infty,\ f_j(m) = 0\ (j = 1,\dots,n)\}\,.$$

If $f(m) \neq 0$, then $\operatorname{ord}(f,m) = 0$.

Using the fact that for every $m \in M(H^\infty)$ there exists an analytic map L_m of $\mathbb{D}$ onto the Gleason part $P(m) = \{x \in M(H^\infty) : \rho(m,x) < 1\}$ of m with $L_m(0) = m$, we see that $\operatorname{ord}(f,m)$ is the usual multiplicity of the zero of the analytic function $f \circ L_m$ at the origin. Note that the latter is infinite if f vanishes identically on $P(m)$.

If $f, g \in H^\infty$ then gcd (f,g) denotes a greatest common divisor of the functions f and g. It is well known that in contrast to the disk algebra, a greatest common divisor always exists and is uniquely determined modulo invertible functions. It is easy to prove that whenever $\gcd(f,h) = 1$ and h divides $f \cdot g$, then h divides g.

The following results will be used throughout this paper.

Hoffman's lemma 0.1 ([9, p. 86], and [5, p. 404]). *Let* $0 < \delta < 1$, $0 < \eta < (1 - \sqrt{1-\delta^2})/\delta$, *i.e.*, $0 < \eta < \rho(\delta,\eta)$, *and let*

$$0 < \varepsilon \leqq \varepsilon(\delta) := \frac{\delta - \eta}{1 - \delta\eta}\,\eta\,.$$

If b is any interpolating Blaschke product with zeros $\{z_n\}$ such that

$$\delta(b) = \inf_{n\in\mathbb{N}} (1 - |z_n|^2)|b'(z_n)| \geqq \delta\,,$$

then

$$\begin{aligned}\{z \in \mathbb{D} : |b(z)| < \varepsilon\} &\subseteq \{z \in \mathbb{D} : \rho(z, Z(b)) < \eta\}\\ &\subseteq \{z \in \mathbb{D} : |b(z)| < \eta\}\,. \end{aligned} \tag{1}$$

It is easily shown that $(1 - \sqrt{1-\delta^2})/\delta$ is a monotone increasing function of $\delta \in (0,1)$, that $\varepsilon < \eta < \delta$ and that $0 < (1 - \sqrt{1-\delta^2})/\delta < \delta$. We shall also use the fact that $\eta < 2\eta/(1+\eta^2) < \delta$ is equivalent to $0 < \eta < (1-\sqrt{1-\delta^2})/\delta$.

Using (1) and Schwarz's lemma [5, Exercise 1, p. 41] we have

$$\frac{\delta - \eta}{1 - \eta\delta}\,\rho(z, z_n) \leqq |b(z)| \leqq \rho(z, z_n) \tag{2}$$

whenever $\rho(z,z_n) < \eta$ and b is an interpolating Blaschke product with $\delta(b) \geqq \delta$ and zeros z_n. Finally, we note that the pseudohyperbolic disks $D(z_n,\eta) = \{z \in \mathbb{D} : \rho(z,z_n) < \eta\}$ are pairwise disjoint.

Lemma 0.2 [5, p. 310]. *Let $\{z_n\}$ be an interpolating sequence in $\mathbb{D}$ with* $\inf_n \prod_{j\neq n} \rho(z_j,z_n) \geqq \delta > 0$. *Let* $0 < \eta < (1 - \sqrt{1-\delta^2})/\delta$, *and let* $w_n \in \mathbb{D}$

satisfy $\rho(z_n, w_n) < \eta$ *for every* $n \in \mathbb{N}$. *Then* $\{w_n\}$ *is an interpolating sequence with*

$$\inf_n \prod_{j \neq n} \rho(w_j, w_n) \geqq \frac{\delta - \frac{2\eta}{1+\eta^2}}{1 - \delta \frac{2\eta}{1+\eta^2}} .$$

1 Necessary and sufficient conditions for $I = J$

Our first objective is to give necessary conditions in terms of the order of the zeros of the generators which guarantee that $I = J$. Using these results together with the fact that an ideal $I \neq H^\infty$ is generated by interpolating Blaschke products if and only if $\mathrm{ord}(I, m) = 1$ for every $m \in Z(I)$ (see Proposition 1.8), we will prove that, for two generators, $I = J$ if and only if I contains an interpolating Blaschke product times the greatest common divisor of the generators.

The proof of the main result will be divided into two major steps. First we shall be concerned with finitely generated ideals whose zero sets contain a point m with $2 \leqq \mathrm{ord}(I, m) < \infty$; the second case deals with ideals for which $\mathrm{ord}(I, m) = \infty$ for some $m \in Z(I)$.

Lemma 1.1 *Let* $f_j, g_j \in H^\infty$ $(j = 1, 2)$. *Assume that the functions* $f_1 g_1$ *and* $f_2 g_2$ *have no common factors. Then*

$$I(f_1 g_1, f_2 g_2) = J(f_1 g_1, f_2 g_2)$$

implies that

$$I(f_1, f_2) = J(f_1, f_2) .$$

Proof. Let $f \in J(f_1, f_2)$. Then $|g_1 g_2 f| \leqq C(|g_1 f_1| + |g_2 f_2|)$ for some constant C. By hypothesis, there exist $x_1, x_2 \in H^\infty$ such that

$$g_1 g_2 f = x_1 f_1 g_1 + x_2 f_2 g_2 .$$

Thus $g_1(g_2 f - x_1 f_1) = x_2 f_2 g_2$. Since $\gcd(g_1, f_2 g_2) = 1$, dividing by g_1 yields $g_2 f - x_1 f_1 = y_2 f_2 g_2$, for some $y_2 \in H^\infty$. Similarly, since $\gcd(g_2, f_1 g_1) = 1$, dividing by g_2 yields $f = \tilde{x}_1 f_1 + \tilde{x}_2 f_2$ for some $\tilde{x}_1, \tilde{x}_2 \in H^\infty$. Hence $J(f_1, f_2) \subseteq I(f_1, f_2)$, from which we get our assertion. □

Lemma 1.2 [9, p. 100]. *Let* $f \in H^\infty$ *and let* $m \in M(H^\infty)$ *be a point with* $f(m) = 0$. *Then either* f *has a zero of infinite order at* m *or there exists an interpolating subsequence of the zero sequence of* f *in* $\mathbb{D}$ *which captures* m *in its closure.*

Proposition 1.3 *Let* f_1, f_2 *be two functions in* H^∞ *having no common factor, and let* $I = I(f_1, f_2)$. *Assume that* $\mathrm{ord}(I, m) = N$ *with* $2 \leqq N < \infty$ *for some* $m \in Z(I)$. *Then* I *is properly contained in* $J = J(f_1, f_2)$.

Proof. Step 1 Because $\operatorname{ord}(I,m) = \min_{j=1,2} \operatorname{ord}(f_j, m)$, we may assume without loss of generality that $\operatorname{ord}(f_1,m) = N \leqq \operatorname{ord}(f_2,m) \leqq \infty$. In this step we construct generators F_j such that $\operatorname{ord}(F_j,m) < \infty$ for all $j = 1,2$. If $\operatorname{ord}(f_2,m) < \infty$, we are done. Otherwise let $F_1 = f_1$ and $F_2 = f_2 + \varepsilon f_1$ for some $\varepsilon > 0$. Then $\operatorname{ord}(F_2,m) = N < \infty$ and $I = I(F_1,F_2)$.

So we may assume from now on that

$$N = \operatorname{ord}(f_1,m) \leqq \operatorname{ord}(f_2,m) < \infty . \tag{3}$$

Step 2 By Lemma 1.2, (3) implies that we have $f_j = b_j c_j g_j$, where the b_j and c_j are interpolating Blaschke products with $b_j(m) = c_j(m) = 0$ and $g_j \in H^\infty$ $(j = 1,2)$.

Since $I(c_1,c_2,b_1,b_2)$ is a proper ideal, there exists by the corona theorem an interpolating sequence $\{\xi_n\}$ in $\mathbb{D}$ such that

$$\varepsilon_n := \{|c_1| + |c_2| + |b_1| + |b_2|\}(\xi_n) \to 0 \quad \text{as } n \to \infty . \tag{4}$$

Let b be the interpolating Blaschke product associated with $\{\xi_n\}$ and let $\delta = \min\{\delta(c_1), \delta(c_2), \delta(b_1), \delta(b_2), \delta(b)\}$. Choose, according to Hoffman's lemma, the Hoffman constant η so small that $0 < \varepsilon_n < \eta(\delta - \eta)/(1 - \eta\delta)$ for $n \geqq n_0$ and $0 < \eta < (1 - \sqrt{1-\delta^2})/\delta$. Without loss of generality let $n_0 = 1$. By Hoffman's lemma, we may conclude from (4) that each pseudohyperbolic disk $D(\xi_n, \eta)$ contains exactly one zero of c_1, c_2, b_1 and b_2 for all n.

Let c_1^*, c_2^*, b_1^*, b_2^* be the associated subproducts. Thus $f_j = b_j^* c_j^* h_j$ for some $h_j \in H^\infty (j = 1,2)$. In order to prove our proposition, it is, by Lemma 1.1, sufficient to show that

$$I(b_1^*c_1^*, b_2^*c_2^*) \neq J(b_1^*c_1^*, b_2^*c_2^*) . \tag{5}$$

Step 3 We claim that there exist interpolating Blaschke products B, B^* and C, C^* so that

$$BB^* = b_1^*c_1^*, \quad CC^* = b_2^*c_2^* , \tag{6}$$

$$\begin{aligned} &\frac{B}{B^*} \text{ is bounded on the zero set of } C^* \text{ in } \mathbb{D} \text{ and} \\ &\frac{C}{C^*} \text{ is bounded on the zero set of } B^* \text{ in } \mathbb{D} . \end{aligned} \tag{7}$$

Fix n and look at the distribution of the zeros of b_1^*, c_1^* resp. b_2^*, c_2^* in $D(\xi_n, \eta)$. Let

$$Z(b_1^*c_1^*) \cap D(\xi_n, \eta) = \{\beta_n, \beta_n^*\}$$

and

$$Z(b_2^*c_2^*) \cap D(\xi_n, \eta) = \{\gamma_n, \gamma_n^*\} .$$

Now there exists among the numbers $\rho(\beta_n, \gamma_n)$, $\rho(\beta_n, \gamma_n^*)$, $\rho(\beta_n^*, \gamma_n)$, $\rho(\beta_n^*, \gamma_n^*)$ a biggest one. Without loss of generality let $\rho(\beta_n^*, \gamma_n^*)$ be this number. Then we put the zero β_n^* to a Blaschke product called B^*, the zero γ_n^* to a Blaschke product called C^*. The other zeros β_n, resp. γ_n, are put to Blaschke products called B resp. C. By construction we have formula (6). Moreover,

$$\frac{\rho(\beta_n,\gamma_n^*)}{\rho(\beta_n^*,\gamma_n^*)} \leqq 1 \quad \text{and} \quad \frac{\rho(\gamma_n,\beta_n^*)}{\rho(\gamma_n^*,\beta_n^*)} \leqq 1 . \tag{8}$$

By Lemma 0.2, the sequences (β_n), (β_n^*), (γ_n) and (γ_n^*) are interpolating sequences whose associated interpolating Blaschke products B, B^*, C and C^* satisfy

$$\delta(B),\delta(B^*),\delta(C),\delta(C^*) \geqq \frac{\delta - \frac{2\eta}{1+\eta^2}}{1-\delta\frac{2\eta}{1+\eta^2}} =: \delta^* . \tag{9}$$

Moreover, we have that $\tilde{\varepsilon}_n = |B^*(\xi_n)| + |C^*(\xi_n)| + |B(\xi_n)| + |C(\xi_n)| \to 0$ as $n \to \infty$. In fact, by (2),

$$\begin{aligned} \tilde{\varepsilon}_n &\leqq \rho(\beta_n^*,\xi_n) + \rho(\gamma_n^*,\xi_n) + \rho(\beta_n,\xi_n) + \rho(\gamma_n,\xi_n) \\ &= \rho(\xi_n,Z(b_1)) + \rho(\xi_n,Z(b_2)) + \rho(\xi_n,Z(c_1)) + \rho(\xi_n,Z(c_2)) \\ &\to 0 \quad \text{as } n \to \infty . \end{aligned}$$

If η is so small that we also have

$$0 < \eta < \frac{2\eta}{1+\eta^2} =: \eta^* < \frac{1-\sqrt{1-\delta^{*2}}}{\delta^*} ,$$

we can use Lemma 0.2 and (2), (8), (9) to obtain the following estimates

$$\left(\text{note that} \quad \rho(\gamma_n^*,\beta_n^*) \leqq \frac{\rho(\gamma_n^*,\xi_n)+\rho(\xi_n,\beta_n^*)}{1+\rho(\gamma_n^*,\xi_n)\rho(\xi_n,\beta_n^*)} \leqq \frac{2\eta}{1+\eta^2} = \eta^* \right) :$$

$$\frac{B(\gamma_n^*)}{B^*(\gamma_n^*)} \leqq \frac{\rho(\beta_n,\gamma_n^*)}{\rho(\beta_n^*,\gamma_n^*)\rho(\delta^*,\eta^*)} \leqq \frac{1}{\rho(\delta^*,\eta^*)} = M$$

and similarly

$$\frac{C(\beta_n^*)}{C^*(\beta_n^*)} \leqq M \quad \text{for every } n .$$

This proves (7).

Step 4 Relation (7) now implies that we can solve the interpolation problems

$$\frac{B}{B^*}(\gamma_n^*) = H_1(\gamma_n^*) \quad \text{and} \quad \frac{C}{C^*}(\beta_n^*) = H_2(\beta_n^*)$$

for bounded analytic functions H_1 and H_2. Hence there exist $K_1, K_2 \in H^\infty$ so that

$$B = H_1B^* + K_1C^* \quad \text{and} \quad C = H_2C^* + K_2B^* .$$

Let $z \in \mathbb{D}$ be so that $|C(z)| \leqq |B(z)|$. Then, using $|B| \leqq \text{const}(|B^*| + |C^*|)$ in $\mathbb{D}$, one gets

$$\begin{aligned} |BC(z)| &\leqq \text{const} \cdot [|B^*(z)||C(z)| + |C^*(z)||C(z)|] \\ &\leqq \text{const} \cdot [|B^*B(z)| + |C^*C(z)|] \,. \end{aligned}$$

Changing the role of B and C we see that

$$BC \in J(B^*B, C^*C) \,.$$

But $BC \notin I(B^*B, C^*C)$, for otherwise we would have

$$BC = xB^*B + yC^*C \quad \text{for some } x, y \in H^\infty \,.$$

Since there are no common factors, we can divide by BC to obtain

$$1 = \tilde{x}B^* + \tilde{y}C^* \quad \text{for some } \tilde{x}, \tilde{y} \in H^\infty \,.$$

This contradicts the fact that

$$B^*(\xi_n) \to 0 \quad \text{and} \quad C^*(\xi_n) \to 0 \,.$$

Hence $I(BB^*, CC^*) \subsetneqq J(BB^*, CC^*)$. In view of (6), this proves (5) and hence Proposition 1.3. □

Proposition 1.4 [16] *Let* $\{z_n\}$ *be a finite union of interpolating sequences in* $\mathbb{D}$. *If* $w_n \in \mathbb{C}$ *satisfies*

$$|w_n| \leqq \prod_{j \neq n} \rho(z_j, z_n) \,,$$

then there exists a function $f \in H^\infty$ *such that*

$$f(z_n) = w_n \quad \textit{for all } n \in \mathbb{N} \,.$$

Remark. Nakazi's proof of this result in [16] is not constructive. However, the explicit solution formula of [20] of the interpolation problem of Carleson also works in this case. In fact,

$$f(z) = \sum_{n=1}^{\infty} w_n \frac{1}{B_n(z_n)} \left(\frac{1 - |z_n|^2}{1 - \bar{z}_n z} \right)^2 B_n(z) \exp(\alpha_n(z_n) - \alpha_n(z)) \,,$$

where

$$\alpha_n(z) = \sum_{k \geqq n} \frac{1 + \bar{z}_k z}{1 - \bar{z}_k z} (1 - |z_n|^2) \quad \text{and} \quad B_n(z) = B(z) \Big/ \frac{|z_n|}{z_n} \cdot \frac{z_n - z}{1 - \bar{z}_n z} \,,$$

is a solution.

For a function $f \in H^\infty$, $f \not\equiv 0$, let $f = BF$ be the Riesz factorization of f, where B is a Blaschke product and F is a function which does not vanish in $\mathbb{D}$.

Proposition 1.5 *Let $f_j = B_jF_j \in H^\infty$ have no common factors. Assume that $I(f_1, f_2) = J(f_1, f_2)$. Then*

$$Z(F_1) \cap Z(F_2) = \emptyset .$$

Proof. The Cauchy-Schwarz inequality implies that $B_1B_2\sqrt{F_1}\sqrt{F_2} \in J(f_1, f_2)$. Hence, by our hypothesis, there exist $x, y \in H^\infty$ so that

$$B_1B_2\sqrt{F_1}\sqrt{F_2} = xB_1F_1 + yB_2F_2 .$$

Division by $B_1B_2\sqrt{F_1}\sqrt{F_2}$ yields functions $\tilde{x}$, $\tilde{y} \in H^\infty$ such that

$$1 = \tilde{x}\sqrt{F_1} + \tilde{y}\sqrt{F_2} .$$

Hence $Z(F_1) \cap Z(F_2) = \emptyset$. □

Remark. We do not know whether the assertion of Proposition 1.5 holds for ideals $I = I(f_1, \dots, f_N)$ with more than two generators and satisfying $I(f_1, \dots, f_N) = J(f_1, \dots, f_N)$.

In view of Proposition 1.3, we must now consider the case where the generators B_1 and B_2 generate a proper ideal I such that $\operatorname{ord}(I, m) = \infty$ for some $m \in Z(I)$. The main difficulty is that the zeros of I need not lie in the closure of an interpolating subsequence of B_1 or B_2. Hence we cannot use the factorization argument given in the proof of Step 2 of Proposition 1.3. Using the following proposition, however, we shall be able to reduce our new situation to that of Proposition 1.3. Our proof is based on the following factorization of Hoffman ([9, p. 95], and [5, p. 411]) and Izuchi.

Factorization Theorem 1.6 (Hoffman-Izuchi [10, p. 55]) *Let $B \in H^\infty$ be a function with $Z_\infty(B) := \{m \in M(H^\infty) : \operatorname{ord}(B, m) = \infty\} \neq \emptyset$. Then B admits a factorization $B = B_1B_2$ such that $Z_\infty(B_1) = Z_\infty(B_2) = Z_\infty(B)$.*

Remark. Although this factorization theorem has only been stated for Blaschke products in [10], it remains true for any function $f \in H^\infty$, because the zeros of singular inner and outer functions are always of infinite order.

Proposition 1.7 *Let B_1, B_2 be two functions having no common factors, and let $I = I(B_1, B_2)$. Suppose that $\operatorname{ord}(B_j, m) = \infty$ for some $m \in Z(I)$ ($j = 1, 2$). Then there exist $\tilde{m} \in Z(I)$, interpolating Blaschke products b_1, b_2 and functions C_2 and D_2 such that*

(1) $B_2 = C_2D_2$;
(2) $b_1(\tilde{m}) = b_2(\tilde{m}) = C_2(\tilde{m}) = D_2(\tilde{m}) = 0$;
(3) $B_1 \in I(b_1b_2, C_2)$.

Proof. Step 1 First we factor the functions B_j according to Theorem 1.6 as a product $B_j = C_jD_j$ of two functions so that $\operatorname{ord}(C_j, m) = \operatorname{ord}(D_j, m) = \infty$. Then $I(C_1, C_2, D_1, D_2)$ is a proper ideal, and there exists, by the corona theorem, an interpolating sequence $\{\zeta_n\}$ in $\mathbb{D}$ so that

$$\{|C_1| + |C_2| + |D_1| + |D_2|\}(\zeta_n) \to 0 \quad \text{as } n \to \infty . \tag{10}$$

Note that every cluster point of the ζ_j lies in $Z(I)$.

We shall now construct the interpolating Blaschke products b_1 and b_2. To this end, let

$$E_1 = \{\zeta_n : |D_1(\zeta_n)| \leqq |C_1(\zeta_n)|\} . \tag{11}$$

Without loss of generality, E_1 is infinite (otherwise we can rename the functions). Let

$$E_2 = \{\zeta_n \in E_1 : |D_2(\zeta_n)| \leqq |C_2(\zeta_n)|\} . \tag{12}$$

Without loss of generality, we may assume that E_2 is infinite. Let

$$E_3 = \left\{\zeta_n \in E_2 : \left|\frac{C_1}{C_2}(\zeta_n)\right| \leqq 1\right\} . \tag{13}$$

Also here we may assume that E_3 is infinite, for otherwise we would look at the quotient C_2/C_1.

Since we will consider subsequences of $\{\zeta_n\}$, we may assume that the ζ_n satisfy Eqs. (10)–(13). Note that

$$\left|\frac{D_1}{C_2}(\zeta_n)\right| \leqq 1 . \tag{14}$$

Let $\eta_n := D_1(\zeta_n)$. Then $\eta_n \to 0$. Hence, by (14) and (11),

$$|C_2(\zeta_n)| \geqq \eta_n \quad \text{and} \quad |C_1(\zeta_n)| \geqq \eta_n . \tag{15}$$

Let b be the interpolating Blaschke product with zeros $\{\zeta_n\}$ and let $\delta = \delta(b)$. Because $\eta_n \to 0$ we can choose for n big enough, say $n \geqq n_0$, the Hoffman constants ε_n and η as follows:

$$0 < \eta < \frac{1-\sqrt{1-\delta^2}}{\delta}, \quad 0 < \varepsilon_n < \eta_n < \eta < \delta \text{ with } \varepsilon_n = \eta_n \frac{\delta - \eta_n}{1-\eta_n\delta} .$$

Without loss of generality let $n_0 = 1$.

For every n choose two different points v_n and w_n from $\partial D\left(\zeta_n, \frac{\eta_n}{4}\right)$ satisfying $\arg v_n = \arg w_n = \arg \zeta_n$. This implies that $\rho(v_n, w_n) \geqq \eta_n/4$. Note that the pseudohyperbolic disks $D(\zeta_n, \eta)$ are pairwise disjoint. By Lemma 0.2, both $\{v_n\}$ and $\{w_n\}$ are interpolating sequences satisfying

$$\frac{\eta_n}{4} \leqq \rho(v_n, w_n) = \frac{\rho(v_n,\zeta_n)+\rho(\zeta_n,w_n)}{1+\rho(v_n,\zeta_n)\rho(\zeta_n,w_n)} = \frac{\frac{\eta_n}{2}}{1+\frac{\eta_n^2}{16}} \leqq \frac{\eta_n}{2} . \tag{16}$$

Let b_1 respectively b_2 be the interpolating Blaschke products associated with $\{v_n\}$ respectively $\{w_n\}$. Since $\rho(v_n,\zeta_n) = \rho(w_n,\zeta_n) = \eta_n/4 \to 0$, we see that $b_1(m) = b_2(m) = 0$ for every $m \in [\mathrm{cl}\{\zeta_n : n \in \mathbb{N}\}]\backslash\{\zeta_n : n \in \mathbb{N}\} \subseteq Z(I)$. By Lemma 0.2, we have

$$\delta(b_j) \geqq \frac{\delta - \frac{2\eta}{1+\eta^2}}{1-\delta\frac{2\eta}{1+\eta^2}} =: \delta^* . \tag{17}$$

We are now going to prove that the quotient C_1/C_2 is bounded away from zero on both interpolating sequences $\{v_n\}$ and $\{w_n\}$.

Step 2 By Schwarz-Pick's lemma $\rho(f(\zeta_n), f(v_n)) \leqq \rho(\zeta_n, v_n)$ for every $f \in H^\infty$ with $\|f\|_\infty \leqq 1$. Hence we obtain

$$|f(\zeta_n)| - \frac{\eta_n}{2} \leqq |f(v_n)| \leqq |f(\zeta_n)| + \frac{\eta_n}{2}\,. \tag{18}$$

Replacing f by C_1, respectively C_2, and by using (15) and (13), we obtain

$$\left|\frac{C_1}{C_2}(v_n)\right| \leqq \frac{|C_1(\zeta_n)| + \frac{\eta_n}{2}}{|C_2(\zeta_n)| - \frac{\eta_n}{2}} \leqq \frac{\frac{3}{2}|C_1(\zeta_n)|}{\frac{1}{2}|C_2(\zeta_n)|} \leqq 3 \tag{19}$$

for every $n \in \mathbb{N}$.

Similarly, we get $\left|\frac{C_1}{C_2}(w_n)\right| \leqq 3$ for every $n \in \mathbb{N}$.

Step 3 Next we shall solve an interpolation problem simultaneously for both interpolating sequences $\{v_n\}$ and $\{w_n\}$. The existence of a solution will be guaranteed by Proposition 1.4. To this end, let

$$\omega_n = \prod_{k \neq n} \rho(v_k, v_n) \prod_{k=1}^{\infty} \rho(w_k, v_n) = (1 - |v_n|^2)|b_1'(v_n)||b_2(v_n)|$$

and let $\omega_n' = (1 - |w_n|^2)|b_2'(w_n)||b_1(w_n)|$. Finally, let $\delta_j = \delta(b_j)$ $(j = 1,2)$. By (17), $\delta_j \geqq \delta^*$.

Let $0 < \eta^* < (1 - \sqrt{1 - \delta^{*2}})/\delta^*$. Then (16) and the fact that $\eta_n \to 0$ imply that, for every $n \geqq n_0$, we have $\rho(v_n, Z_{\mathbb{D}}(b_2)) = \rho(v_n, w_n) \leqq \eta_n/2 < \eta^*$. Hence, by (2) of Hoffman's lemma, we obtain that

$$|b_2(v_n)| \geqq \frac{\delta^* - \eta^*}{1 - \eta^*\delta^*}\,\rho(v_n, w_n)\,. \tag{20}$$

The same holds for b_1.

We claim that both $|D_1(v_n)|/\omega_n$ and $|D_1(w_n)|/\omega_n'$ are bounded. In fact, by (18), (20) and (16), we have for $n \geqq n_0$

$$\begin{aligned}\left|\frac{D_1(v_n)}{\omega_n}\right| &\leqq \frac{|D_1(\zeta_n)| + \frac{\eta_n}{2}}{\omega_n} = \frac{\frac{3}{2}\eta_n}{\omega_n} \leqq \frac{\frac{3}{2}\eta_n}{\delta_1|b_2(v_n)|} \\ &\leqq \frac{\frac{3}{2}\eta_n}{\delta^* \frac{\delta^* - \eta^*}{1 - \eta^*\delta^*}\rho(v_n, w_n)} \leqq \frac{3}{2} \cdot 4[\delta^*\rho(\delta^*, \eta^*)]^{-1} =: M\,.\end{aligned} \tag{21}$$

Similarly, $D_1(w_n)/\omega_n'$ is bounded.

Thus, by (19), for $n \geqq n_0$ we have

$$\begin{aligned}\frac{1}{\omega_n}\left|\frac{B_1}{C_2}(v_n)\right| &= \left|\frac{C_1}{C_2}(v_n)\right| \cdot \left|\frac{D_1(v_n)}{\omega_n}\right| \leqq 3M, \\ \frac{1}{\omega_n'}\left|\frac{B_1}{C_2}(w_n)\right| &= \left|\frac{C_1}{C_2}(w_n)\right| \cdot \left|\frac{D_1(w_n)}{\omega_n'}\right| \leqq 3M\,.\end{aligned} \tag{22}$$

By Proposition 1.4, we can now solve the interpolation problem

$$h(y_j) = \frac{B_1}{C_2}(y_j), \quad h \in H^\infty,$$

where $\{y_j\}$ is the union of $\{v_n\}$ and $\{w_n\}$. Thus there exist $h \in H^\infty$ and $k \in H^\infty$ so that

$$B_1 = hC_2 + kb_1b_2 .$$

This proves Proposition 1.7. □

Before we can prove our main theorem, we need the following result of [15].

Proposition 1.8 [15] *Let $I \subsetneq H^\infty$ be a proper ideal. Then* $\mathrm{ord}(I,m) = 1$ *for every $m \in Z(I)$ if and only if I is generated by interpolating Blaschke products.*

Theorem 1.9 *Let $f_1, f_2 \in H^\infty$ have no common factors. Then $I(f_1,f_2) = J(f_1,f_2)$ if and only if* $\min_{j=1,2} \mathrm{ord}(f_j,m) \leqq 1$ *for every $m \in M(H^\infty)$.*

Proof. Let $I = I(f_1,f_2)$. Assume that $I(f_1,f_2) = J(f_1,f_2)$. If there exists $m \in Z(I) = Z(f_1) \cap Z(f_2)$ with $2 \leqq \mathrm{ord}(I,m) < \infty$, then Proposition 1.4 tells us that $I(f_1,f_2) \subsetneq J(f_1,f_2)$. So let $\mathrm{ord}(I,m) = \infty$ for some $m \in Z(I)$.

Now we apply Proposition 1.7. Let $f_2 = C_2D_2$ be the factorization obtained there. Then, by Lemma 1.1, we have $I(f_1,C_2) = J(f_1,C_2)$. Since $f_1 \in I(b_1b_2,C_2)$ for some interpolating Blaschke products b_1, b_2, there exist $x, y \in H^\infty$ so that $f_1 = xb_1b_2 + yC_2$. Hence $I(f_1,C_2) = I(xb_1b_2,C_2)$ and $J(f_1,C_2) = J(xb_1b_2,C_2)$. Applying Lemma 1.1 once again, we obtain that $\tilde{I} = I(b_1b_2,C_2) = J(b_1b_2,C_2)$.

Note that by Proposition 1.7 there exists $\tilde{m}$ with $b_1(\tilde{m}) = b_2(\tilde{m}) = C_2(\tilde{m}) = 0$. Hence $\mathrm{ord}(\tilde{I},\tilde{m}) = 2$. Proposition 1.3 now implies that $I(b_1b_2,C_2) \subsetneq J(b_1b_2,C_2)$, which is a contradiction. Thus we see that, whenever $2 \leqq \mathrm{ord}(I,m) \leqq \infty$ for some $m \in Z(I)$, then $I = I(f_1,f_2) \subsetneq J(f_1,f_2)$. Hence $I(f_1,f_2) = J(f_1,f_2)$ implies that

$$\min_{j=1,2} \mathrm{ord}(f_j,m) \leqq 1 \text{ for every } m \in Z(I) .$$

To prove the converse, let $\mathrm{ord}(I,m) = 1$ for every $m \in Z(I)$. Then, by Proposition 1.8, I is generated by interpolating Blaschke products. Hence, by [17] or [15], $I(f_1,f_2) = J(f_1,f_2)$. For the reader's convenience we briefly present the proof.

Let $|f| \leqq |f_1|+|f_2|$ and let $\{z_n\}$ be the zeros of any interpolating Blaschke product b belonging to I. By the definition of an interpolating sequence there exist $g_1, g_2 \in H^\infty$ satisfying

$$g_j(z_n) = \frac{f(z_n)\bar{f}_j(z_n)}{|f_1(z_n)|^2 + |f_2(z_n)|^2} \quad (j = 1,2), \quad n \in \mathbb{N} .$$

Hence $f - (g_1f_1 + g_2f_2) = gb$ for some $g \in H^\infty$. This shows that $f \in I(f_1,f_2)$. □

The following theorem now proves a conjecture in [15] for the case of two generators.

Theorem 1.10 *Let $f_1, f_2 \in H^\infty$ have no common factors. Let $I = I(f_1, f_2)$ and $J = J(f_1, f_2)$. Then the following assertions are equivalent*:

(1) $I = J$;

(2) $\operatorname{ord}(I, m) = 1$ *for every* $m \in Z(I)$;

(3) *I contains an interpolating Blaschke product*;

(4) *J contains an interpolating Blaschke product*;

(5) $|f_1(z)|^2 + (1-|z|^2)|f_1'(z)| + |f_2(z)|^2 + (1-|z|^2)|f_2'(z)| \geqq \delta > 0$ *for every* $z \in \mathbb{D}$.

Remark. The equivalences (3), (4), (5) have been proven in a more general setting by Tolokonnikov [18] by very deep operator-theoretic techniques. We present another proof here. For (4) $\Rightarrow$ (3) see also [12] and [15].

Proof. (1) $\Leftrightarrow$ (2) is Theorem 1.9.

(2) $\Rightarrow$ (3) is a special case of Proposition 1.8.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (2) is trivial.

$\rceil(5) \Rightarrow \rceil(2)$: Assume that there exists a net (z_α) in $\mathbb{D}$ converging to some point $m \in M(H^\infty)$, so that the expression in (5) tends to zero. Then $m \in Z(I)$. Because $(f \circ L_{z_\alpha})'(0) = (1-|z_\alpha|^2)|f'(z_\alpha)|$, the compact convergence of $f \circ L_{z_\alpha}$ to $f \circ L_m$ shows that $(f_j \circ L_m)'(0) = 0$, hence $\operatorname{ord}(f_j, m) \geqq 2$. Thus $\operatorname{ord}(I, m) \geqq 2$. Since all steps are reversible, we also get that $\rceil(2) \Rightarrow \rceil(5)$. □

Remark. If we do not assume that the generators are relatively prime, then we get the result that $I = I(f_1, f_2) = J(f_1, f_2)$ if and only if I contains an interpolating Blaschke product times the greatest common divisor $\gcd(f_1, f_2)$ of the generators.

In [15], it is shown that (4) $\Rightarrow$ (1). Proposition 1.8 shows that (3) follows from (2) for every ideal. The equivalences of (3) to (5) are in [18]. In view of this, it is reasonable to ask (see [15]):

> Does $I = I(f_1, \ldots, f_N) = J(f_1, \ldots, f_N)$ imply that I contains an interpolating Blaschke product?

To our surprise this is *not* the case, as the following easy example shows.

Proposition 1.11 *Let B and C be two interpolating Blaschke products. Then $I(B^2, C^2, BC) = J(B^2, C^2, BC) = J(B^2, C^2)$.*

Proof. Clearly we have $J(B^2, C^2, BC) = J(B^2, C^2)$. Now let $|f| \leqq |B|^2 + |C|^2$. We denote the zeros of B by b_n and those of C by c_n. Without loss of generality, assume $Z_{\mathbb{D}}(B) \cap Z_{\mathbb{D}}(C) = \emptyset$. The assumption on f implies that $|f(b_n)| \leqq |C(b_n)|^2$. Because $\{b_n\}$ is an interpolating sequence, there exists a function $H_1 \in H^\infty$ such that

$$\frac{f}{C^2}(b_n) = H_1(b_n) \text{ for every } n\,.$$

Hence $f = H_1C^2 + H_2B$ for some $H_2 \in H^\infty$. This implies that

$$|H_2(c_n)| = \left|\frac{f(c_n)}{B(c_n)}\right| \leqq |B(c_n)| .$$

Again, by solving an interpolation problem, there exists $H_3 \in H^\infty$ so that

$$\frac{H_2}{B}(c_n) = H_3(c_n) \text{ for every } n .$$

Therefore $H_2 = H_3B + H_4C$ for some $H_4 \in H^\infty$. Hence $f = H_1C^2 + (H_3B + H_4C)B \in I(B^2, C^2, BC)$. □

Remark. If B and C are interpolating Blaschke products and if $I = I(B^2, C^2)$ is a proper ideal, then $I(B^2, C^2) \subsetneqq J(B^2, C^2) = I(B^2, C^2, BC)$.

2 Wolff's f^2-problem

Let $f_1, \ldots, f_N \in H^\infty$. Wolff proved that $|f| \leqq \sum_{j=1}^N |f_j|$ implies that $f^3 \in I(f_1, \ldots, f_N)$ (see [5, p. 329]). The question of whether $|f| \leqq \sum_{j=1}^N |f_j|$ implies that $f^2 \in I(f_1, \ldots, f_N)$ remains open. In the following theorem we shall give a positive answer under the additional hypothesis that $\min_{1 \leqq j \leqq N} \operatorname{ord}(f_j, m) < \infty$ for every $m \in M(H^\infty)$.

The methods of our proof are the standard ones developed by Wolff to prove the corona theorem. First let us recall that a positive Borel measure μ on $\mathbb{D}$ is called a **Carleson measure** if there exists a constant C so that

$$\int_{\mathbb{D}} |f|\, d\mu \leqq C\|f\|_1$$

for every f in the Hardy space

$$H^1 = \left\{ f : f \text{ analytic in } \mathbb{D}, \|f\|_1 = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta < \infty \right\} .$$

It is well known that Carleson measures are those positive measures μ for which there exists a constant K such that

$$\mu(Q) \leqq Kl(Q)$$

for every Carleson cube Q defined by

$$Q = \{re^{i\theta} \in \mathbb{D} : 1 - r < l(Q), \theta_0 - l(Q) < \theta < \theta_0 + l(Q)\} .$$

Another equivalent condition is that

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \bar{z}w|^2}\, d\mu(w) < +\infty \tag{23}$$

(see [5, p. 31 and p. 238]).

Let $N(\mu) = \sup\left\{\frac{\mu(Q)}{l(Q)} : Q \text{ Carleson cube}\right\}$ denote the Carleson norm of μ.

We will use two results, due to Carleson and Wolff, on the existence of bounded solutions of $\bar{\partial}$-equations.

Theorem [5, p. 320–322]. (a) *Let* $G \in C(\mathbb{D})$ *induce the Carleson measure* μ *on* $\mathbb{D}$. *If* G *is bounded then the* $\bar{\partial}$*-equation* $\bar{\partial}h = G$ *admits a solution* $h \in C(\bar{\mathbb{D}}) \cap C^1(\mathbb{D})$ *with* $\|h\|_{L^\infty(\partial\mathbb{D})} \leqq C_1$, *where* C_1 *only depends on the Carleson norm of* μ.

(b) *Let* G *be a bounded* C^1 *function on* $\mathbb{D}$ *and assume that the two measures*

$$\mu_1 = |G(z)|^2(1-|z|)dxdy, \quad \mu_2 = \left|\frac{\partial G}{\partial z}(z)\right|(1-|z|)dxdy$$

are Carleson measures. Then the $\bar{\partial}$*-equation* $\bar{\partial}h = G$ *admits a solution* $h \in C(\bar{\mathbb{D}}) \cap C^1(\mathbb{D})$ *with* $\|h\|_{L^\infty(\partial\mathbb{D})} \leqq C_2$ *for some constant* C_2 *depending only on the Carleson norms of* μ_1 *and* μ_2.

For later reference we note that whenever G induces a Carleson measure μ, then the dilation G_r defined by $G_r(z) = G(rz)$ induces a Carleson measure μ_r for which $N(\mu_r) \leqq N(\mu)$.

According to [19], a Blaschke product B is said to be of Carleson-Newman type (a **CN-Blaschke product**, for short) if B is a finite product of interpolating Blaschke products.

A well known theorem (for example, see [14]) tells us that a Blaschke product B with zero sequence (z_n) is a CN-Blaschke product if and only if the measure $\mu = \sum_{n=1}^{\infty}(1-|z_n|^2)\delta_{z_n}$ associated with B is a Carleson measure. Any such sequence will be called a **CN-sequence**.

We shall also need the following results.

Proposition 2.1 [6, 8] *Let* $f \in H^\infty$ *be a function satisfying* $\operatorname{ord}(f, m) < \infty$ *for every* $m \in Z(f)$. *Then* f *has the form* $f = FB$, *where* F *is an invertible outer function and* B *is a* CN-*Blaschke product.*

Lemma 2.2 *Let* $D = D(z_0, \eta)$ *be the pseudohyperbolic disk* $\{z \in \mathbb{D} : \rho(z, z_0) < \eta\}$. *Then*:

(1) *The area of* D *is* $\pi\eta^2 \left[\frac{1-|z_0|^2}{1-\eta^2|z_0|^2}\right]^2$.

(2) *For every* $z \in D$ *we have*

$$\frac{1}{4}(1-\eta^2) < \frac{1-|z|^2}{1-|z_0|^2} < \frac{4}{1-\eta^2}.$$

(3) $\{w \in \mathbb{C} : |w - z_0| < \frac{\eta}{2}(1-|z_0|^2)\} \subseteq D \subseteq \{w \in \mathbb{C} : |w - z_0| < \frac{\eta}{1-\eta}(1-|z_0|^2)\}$.

(4) *Let* $0 < h < 1/4$ *and let* $z_0 \in \mathbb{D}$ *satisfy* $1 - |z_0| < h$, $|\arg z_0| < h$. *Then for every* $z_1 \in \mathbb{D}$ *with* $|z_1 - z_0| < h$ *we have* $1 - |z_0| < 3h$ *and* $|\arg z_1| < 3h$.

Proof. Elementary calculation. See also [5, p. 3]. □

Lemma 2.3 *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic with $f(0) = 0$, $|f'(0)| \geqq \delta$. Then f is schlicht in $|z| < \tau^* = (1 - \sqrt{1 - \delta^2})/\delta < \delta$. Moreover, τ^* is best possible.*

Proof. Use Exercise 1 of [5, Sect. 1], and note that $\tau^* = (\delta - \tau^*)/(1 - \tau^*\delta)$. To show that τ^* is best possible, just look at the function $f(z) = z(z - \tau^*)/(1 - \tau^* z)$. □

Lemma 2.4 *Let (z_n) be a CN-sequence. Fix an arbitrary η, $0 < \eta < 1$. For every $n \in \mathbb{N}$ let $\xi_n \in D(z_n, \eta)$. Then (ξ_n) is a CN-sequence.*

Proof. Using (2) of Lemma 2.2 one gets

$$1 - |\xi_n| \leqq \frac{4}{1 - \eta^2}(1 - |z_n|) .$$

Also $\xi_n \in Q = \{re^{i\theta} \in \mathbb{D} : 1 - r < l(Q),\ \theta_0 - l(Q) < \theta < \theta_0 + l(Q)\}$ implies that there exists a constant $C = C(\eta)$, depending only on η, such that $z_n \in CQ = \{re^{i\theta} \in \mathbb{D} : 1 - r < Cl(Q),\ \theta_0 - Cl(Q) < \theta < \theta_0 + Cl(Q)\}$. Hence,

$$\sum_{\xi_n \in Q} (1 - |\xi_n|) \leqq \frac{4}{1 - \eta^2} \sum_{z_n \in Q} (1 - |z_n|) \leqq K(\eta) l(Q) .$$

So, $\sum_n (1 - |\xi_n|)\delta_{\xi_n}$ is a Carleson measure and $\{\xi_n\}$ is a CN-sequence. □

We are now able to prove the following key lemma.

Proposition 2.5 *Let (z_n) be a CN-sequence of distinct points and let*

$$U = \bigcup_{n=1}^{\infty} D(z_n, \eta)$$

for some η, $0 < \eta < 1$. Let Ψ_U denote the characteristic function of U. Then, for any interpolating Blaschke product B, the measure

$$\mu = \frac{|B'|}{|B|} \Psi_U dxdy \tag{24}$$

is a Carleson measure.

Proof. The additive property of the logarithmic derivative yields

$$\frac{B'(z)}{B(z)} = \sum_j \frac{1 - |\xi_j|^2}{(z - \xi_j)(1 - \bar{\xi}_j z)} ,$$

where $\{\xi_j\}$ are the zeros of B in $\mathbb{D}$.

From now on, $c_i = c_i(\eta)$ will denote different constants depending only on η. Let us point out that we may assume without loss of generality that η is small, because otherwise one can cover each $D(z_n, \eta)$ by a finite number N of pseudohyperbolic disks of smaller radius, N being independent of n. The new centers z_n again form a CN-sequence.

Let

$$Q = \{re^{i\theta} \in \mathbb{D} : 1 - r < l(Q),\ \theta_0 - l(Q) < \theta < \theta_0 + l(Q)\}$$

and $A = \{z_n : D(z_n, \eta) \cap Q \neq \emptyset\}$. By Lemma 2.2, there exists $c_1 = c_1(\eta)$ such that $z_n \in A$ implies $D(z_n, \eta) \subset c_1 Q$. Let $\eta_1 = (1+\eta)/2$. From (24) and the definition of U we have

$$\begin{aligned}
&\int_Q \left|\frac{B'(z)}{B(z)}\right| \Psi_U(z)\,dxdy \\
&\leqq \sum_{z_n \in A} \sum_j (1 - |\xi_j|^2) \int_{D(z_n,\eta)} \frac{dxdy}{|z - \xi_j|\,|1 - \bar{\xi}_j z|} \\
&= \sum_{z_n \in A} \left[\sum_{j:\rho(\xi_j, z_n) \leqq \eta_1} (1 - |\xi_j|^2) \int_{D(z_n,\eta)} \frac{dxdy}{|z - \xi_j|\,|1 - \bar{\xi}_j z|} \right. \\
&\quad \left. + \sum_{j:\rho(\xi_j, z_n) > \eta_1} (1 - |\xi_j|^2) \int_{D(z_n,\eta)} \frac{dxdy}{|z - \xi_j|\,|1 - \bar{\xi}_j z|} \right] \\
&= \sum_{z_n \in A} [(I)_n + (II)_n]\,.
\end{aligned}$$

Concerning $(II)_n$, observe that if $\rho(\xi_j, z_n) > \eta_1$ and $\rho(z, z_n) < \eta$, we have (using the monotonicity of the functions $(a-x)/(1-ax)$ and $(x-a)/(1-ax)$):

$$\left|\frac{z - \xi_j}{1 - \bar{\xi}_j z}\right| = \rho(z, \xi_j) \geqq \frac{\rho(z_n, \xi_j) - \rho(z, z_n)}{1 - \rho(z, z_n)\rho(z_n, \xi_j)} \geqq \frac{\eta_1 - \eta}{1 - \eta_1 \eta} \geqq \frac{1}{2} \cdot \frac{1}{1+\eta}\,.$$

If $z \in D(z_n, \eta)$ and $\rho(\xi_j, z_n) > \eta_1$, applying (3) of Lemma 2.2, one has

$$\left|\frac{z_n - z}{1 - \bar{\xi}_j z_n}\right| \leqq \frac{|z_n - z|}{|\xi_j - z_n|} \leqq \frac{\eta(1-\eta)^{-1}(1 - |z_n|^2)}{2^{-1}\eta_1(1 - |z_n|^2)} = \frac{\eta(1-\eta)^{-1}}{2^{-1}\eta_1} \leqq \frac{1}{2}\,,$$

if η is sufficiently small. Hence

$$|1 - \bar{\xi}_j z| \geqq |1 - \bar{\xi}_j z_n| - |z_n - z| \geqq \tfrac{1}{2}|1 - \bar{\xi}_j z_n|\,.$$

So

$$\begin{aligned}
\int_{D(z_n,\eta)} \frac{dxdy}{|1 - \bar{\xi}_j z|^2} &\leqq \frac{4}{|1 - \bar{\xi}_j z_n|^2} \operatorname{Area}(D(z_n, \eta)) \\
&\leqq \frac{4\pi\eta^2}{(1 - \eta^2)^2} \cdot \frac{(1 - |z_n|^2)^2}{|1 - \bar{\xi}_j z_n|^2}\,.
\end{aligned}$$

So, using (23) with the Carleson measure $\mu = \sum(1-|\xi_j|^2)\delta_{\xi_j}$, we have

$$\begin{aligned}(II)_n &\leqq 2(1+\eta) \sum_{j:\rho(\xi_j,z_n)>\eta_1} (1-|\xi_j|^2) \int_{D(z_n,\eta)} \frac{dxdy}{|1-\bar{\xi}_j z|^2} \\ &\leqq c_2 \sum_j \frac{(1-|\xi_j|^2)(1-|z_n|^2)^2}{|1-\bar{\xi}_j z_n|^2} \\ &\leqq c_3(1-|z_n|)\,.\end{aligned}$$

Concerning $(I)_n$, observe that if $\rho(\xi_j,z_n) \leqq \eta_1$, there exists $c_4 = c_4(\eta)$ such that $D(z_n,\eta) \subset D(\xi_j,c_4)$. Let $K = \{z \in \mathbb{C} : |z-\xi_j| \leqq \frac{c_4}{1-c_4}(1-|\xi_j|^2)\}$. By Lemma 2.2 (3) we have

$$\int_{D(\xi_j,c_4)} \frac{dxdy}{|z-\xi_j|} \leqq \int_K \frac{dxdy}{|z-\xi_j|} = \frac{c_4\, 2\pi(1-|\xi_j|^2)}{1-c_4}\,.$$

Moreover, if $\rho(\xi_j,z_n) \leqq \eta$, then by (3), (4) in Lemma 2.2, $\xi_j \in c_5 Q(z_n)$, where

$$Q(z_n) = \{w \in \mathbb{D} : 1-|w| \leqq 1-|z_n|,\ |\arg w - \arg z_n| < 1-|z_n|\}\,.$$

So, using the fact that $\sum(1-|\xi_j|^2)\delta_{\xi_j}$ is a Carleson measure once again, one gets

$$\begin{aligned}(I)_n &\leqq 2 \sum_{j:\rho(\xi_j,z_n)\leqq\eta_1} \int_{D(\xi_j,c_4)} \frac{dxdy}{|z-\xi_j|} \\ &\leqq c_5 \sum_{j:\rho(\xi_j,z_n)\leqq\eta_1} (1-|\xi_j|) \leqq c_6 \sum_{\xi_j\in c_5 Q(z_n)} (1-|\xi_j|) \\ &\leqq c^* c_6(1-|z_n|)\,,\end{aligned}$$

where c^* is the Carleson norm of the measure $\sum(1-|\xi_j|)\delta_{\xi_j}$. Hence

$$\begin{aligned}\int_Q \left|\frac{B'(z)}{B(z)}\right| \Psi_U(z)dxdy &\leqq c_7 \sum_{z_n\in A}(1-|z_n|) \leqq c_7 \sum_{z_n\in c_1 Q}(1-|z_n|) \\ &\leqq c_8 l(Q)\,,\end{aligned}$$

and this finishes the proof. □

We are now ready to prove the main theorem of this section.

Theorem 2.6 *Let $f_j \in H^\infty$ $(j=1,\ldots,N)$ and let $I = I(f_1,\ldots,f_N)$. Assume that* $\operatorname{ord}(I,m) < \infty$ *for every $m \in Z(I)$. Then $|f| \leqq \sum_{j=1}^N |f_j|$ in $\mathbb{D}$ implies that*

$$f^2 \in I(f_1,\ldots,f_N)\,.$$

Proof. Step 1 First we show that I can be generated by $N+1$ Blaschke products satisfying the Carleson-Newman condition. In fact, the hypothesis $\mathrm{ord}(I,m) < \infty$ for every $m \in Z(I)$ implies that $Z(I)$ does not meet the set S of trivial Gleason parts of $M(H^\infty)$. Hence, by [18] (see also [7, Corollary 2.4], for an easy proof) I actually contains a CN-Blaschke product b_0.

Let $g_j = b_0 + \varepsilon f_j$. Since interpolating Blaschke products do not vanish on S, we see that $|g_j| \geqq \delta > 0$ on S for ε small. By a result of Guillory et al. [8], resp. Gorkin [6], (see also Proposition 2.1) it follows that there exist CN-Blaschke products b_j and outer functions F_j invertible in H^∞ so that $g_j = b_j F_j$ $(j = 1,\dots,N)$.

It is now clear that $I = I(b_0, g_1, \dots, g_N) = I(b_0, b_1, \dots, b_N)$.

Step 2 Let $|g| \leqq \sum_{j=0}^N |b_j|$. As in [5, Sect. 8], we may assume without loss of generality that $\bigcap_{j=0}^N Z_{\mathbb{D}}(b_j) = \emptyset$. Let

$$\varphi_j = \frac{\bar{b}_j}{\sum_{k=0}^N |b_k|^2}, \quad G_{jk} = \varphi_j \frac{\partial \varphi_k}{\partial \bar{z}} \quad (j,k = 0,\dots,N).$$

As usual we now use a normal families argument. For technical reasons, functions and their dilations are represented by the same symbol. Suppose we can solve the $\bar{\partial}$-equations

$$\frac{\partial h_{jk}}{\partial \bar{z}} = g^2 G_{jk} \quad (j,k = 0,\dots,N)$$

with $\|h_{jk}\|_{L^\infty(\partial\mathbb{D})} \leqq M$. Then $g_j = g^2\varphi_j + \sum_{k=0}^N (h_{jk} - h_{kj}) b_k$ satisfy

$$\sum_{j=0}^N g_j b_j = g^2 \quad \text{and} \quad \frac{\partial g_j}{\partial \bar{z}} = 0 \quad (j = 0,\dots,N)$$

(see [5, p. 329]). Thus we have only to show that these $\bar{\partial}$-equations admit bounded solutions. Fix j and k and write $G = G_{jk}$. Let $\{z_n\}$ denote the zeros of the CN-Blaschke product b_0 of Step 1. Take $a \in C^\infty(\mathbb{D})$, $0 \leqq a \leqq 1$ such that $a \equiv 1$ on $\{z : \rho(z,\{z_n\}) < \varepsilon\}$, $a \equiv 0$ on $\{z : \rho(z,\{z_n\}) \geqq 2\varepsilon\}$ and $(1-|z|)|\nabla a(z)|$ is bounded.★

Let $U = \bigcup_{n=1}^\infty D(z_n, 2\varepsilon)$. Now, $G = G_1 + G_2$, where $G_1 = Ga$ and $G_2 = G(1-a)$. An elementary calculation (see [5, p. 330]) yields that

$$|g^2 G_1|^2 \leqq c_0 a^2 \frac{\sum_{l=0}^N |b_l'|^2}{\sum_{l=0}^N |b_l|^2}.$$

Hence,

$$|g^2 G_1| \leqq c_1 a \frac{\sum_{l=0}^N |b_l'|}{\sum_{l=0}^N |b_l|} \leqq c_1 a \sum_{l=0}^N \frac{|b_l'|}{|b_l|}.$$

Since the b_j are CN-Blaschke products, there exists a finite number of interpolating Blaschke products B_l $(l = 0,\dots,L)$ so that

★ Note that, in general, the disks $D(z_n,\varepsilon)$ are not pairwise disjoint.

$$|g^2 G_1| \leqq c_1 a \sum_{l=0}^{L} \left| \frac{B_l'}{B_l} \right| .$$

Since a vanishes outside U, we obtain from Proposition 2.5 that

$$|g^2 G_1(z)|\, dxdy$$

is a Carleson measure and hence the $\bar{\partial}$-equation $\bar{\partial} h = g^2 G_1$ admits bounded solutions.

On the other hand, since b_0 is a CN-Blaschke product with zeros $\{z_n\}$, one has $|b_0(z)| \geqq c(\varepsilon)$ if $\rho(z, \{z_n\}) \geqq \varepsilon$. Now an elementary calculation (as in [5, p. 330]), yields

$$|g^2 G_2(z)|^2 (1 - |z|) \leqq \frac{c_0}{c(\varepsilon)^2} \sum_{l=0}^{N} |b_l'(z)|^2 (1 - |z|) ,$$

which is a Carleson measure (see [5, p. 330]). Also

$$|\partial(g^2 G_2)(z)|(1 - |z|)$$

is a Carleson measure. Hence the $\bar{\partial}$-equation $\bar{\partial} h = g^2 G_2$ admits bounded solutions. This finishes the proof. □

We shall finish our paper by giving an analytic condition on the generators which guarantee a positive solution to Wolff's f^2-problem. To this end, let

$$D^k(f)(z) = \frac{d^k}{(d\xi)^k} f\left(\frac{z + \xi}{1 + \xi \bar{z}}\right)\Bigg|_{\xi=0}$$

be the k-th pseudohyperbolic derivative of a function $f \in H^\infty$ ($k = 0, 1, 2, \ldots$). For $k = 1$, e.g., we obtain $D^1(f)(z) = (1 - |z|^2) f'(z)$. If $f = (f_1, \ldots, f_N) \in (H^\infty)^N$, then we put

$$|D^k f| := \sqrt{\sum_{j=1}^{N} |D^k f_j|^2} .$$

As a corollary we now obtain

Corollary 2.7 *Let* $f = (f_1, \ldots, f_N) \in (H^\infty)^N$. *Assume that for some* $n \in \mathbb{N} \cup \{0\}$ *we have* $\sum_{l=0}^{n} |D^l f| \frac{1}{l!} \geqq \delta > 0$. *Then* $|g| \leqq \sum_{j=1}^{N} |f_j|$ *implies that*

$$g^2 \in I(f_1, \ldots, f_N) .$$

Proof. By a result of Tolokonnikov [19] the hypothesis $\sum_{l=0}^{n} |D^l f| \frac{1}{l!} \geqq \delta > 0$ is equivalent to the fact that the ideal $I = I(f_1, \ldots, f_N)$ contains a CN-Blaschke product. Hence $\operatorname{ord}(I, m) < \infty$ for every $m \in Z(I)$. By Theorem 2.6, the result follows. □

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