Math. Ann. 301, 69–104 (1995)

Mathematische Annalen

The direct image theorem in formal and rigid geometry

Peter Ullrich

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstrasse 62, D-48149 Münster, Germany

Received: 9 February 1994

Mathematics Subject Classification (1991): 11Gxx, 14Fxx, 14Gxx, 32Cxx, 32Jxx, 32P05

In order to give the foundations for analytic geometry over non-archimedean valued fields Tate, influenced by ideas of Grothendieck, introduced "rigid spaces" in his seminar held at Harvard in 1961 [T]. In the middle of the sixties Grauert and Remmert transferred the Weierstraß theory of local complex analysis to the non-archimedean situation and, by this, provided the means for a successful treatment of foundational problems of rigid analytic geometry (cf., [GR1, GR2, GG], also the monograph [BGR]).

Besides this approach to rigid spaces which is orientated towards complex analysis there is the concept, as presented by Raynaud in 1970 in [R], to treat these by means of the theory of formal schemes developed by Grothendieck and to deduce the fundamental theorems from the results in [EGA]. This approach has recently been taken up again by Bosch and Lütkebohmert who succeeded in proving results not accessible to the classical analytical methods [BL1, BL2, L].

Alas, in [EGA $\mathrm{I}_{\mathrm{new}}$, par. 10] the theory of formal schemes is mainly and in [EGA III, pars. 3–5] exclusively presented for the noetherian case. Therefore, in the situation over an absolute base ring R (which is separated and complete with respect to an ideal of definition $\mathfrak{I}$), these results only apply to the case that R is noetherian, e.g., a discrete valuation ring and not to the "geometric" case of R the valuation ring of an algebraically closed and (non-trivially) valued complete field, e.g., of $\mathbb{C}_p$. Already F. Mehlmann has transferred the greatest part of [EGA $\mathrm{I}_{\mathrm{new}}$, par. 10] to the case of formal schemes locally of topologically finite presentation over an arbitrary valuation ring (for a height 1 valuation) in his doctoral dissertation [Me]. The theorem on the coherence of the higher direct images of coherent sheaves with respect to proper morphisms of formal

---

The present article is a revised version of the author's "Habilitationsschrift" presented to the Fachbereich Mathematik of the Westfälische Wilhelms-Universität at Münster

schemes has, however, up to now only been proved for the noetherian situation [EGA III, 3.4.2].

So the main purpose of the present article is to deduce this direct image theorem for formal schemes locally of topologically finite presentation over an arbitrary valuation ring R for a height 1 valuation. (For sake of completeness the case of R noetherian will always be taken into consideration, too.)

In the first two sections one finds fundamental facts on modules and topologically finitely presented algebras over R resp. on formal R-schemes locally of topologically finite presentation, e.g., a substitute for the Artin-Rees lemma in the non-noetherian situation (Lemma 1.3), the coherence of (topologically) finitely presented R-algebras (Corollary 1.8), and Theorem A for coherent module sheaves over formal R-schemes of topologically finite presentation (Proposition 2.3).

Sections 3 to 5 serve to the proof of

Direct image theorem for formal schemes 5.3 *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism of formal R-schemes which are locally of topologically finite presentation and $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Then $R^q\mathfrak{f}_*\mathcal{M}$ is a coherent $\mathcal{O}_{\mathfrak{Y}}$-module for each $q \in \mathbb{Z}$.*

Here the morphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is called ***proper*** if for any – resp. each (Lemma 2.1b)) – natural number λ the morphism $\mathfrak{f}_\lambda : \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda$ of schemes over $R/\mathfrak{I}^{\lambda+1}$ induced by reduction modulo $\mathfrak{I}^{\lambda+1}$ is proper in the sense of algebraic geometry.

In its overall structure the proof coincides with the one given in [EGA III, 3.4] for the noetherian situation: At first, for $\lambda \in \mathbb{N}$ arbitrary one considers the proper morphism $\mathfrak{f}_\lambda : \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda$ and the coherent $\mathcal{O}_{\mathfrak{X}_\lambda}$-module $\mathcal{M}_\lambda := \mathcal{M}/\mathfrak{I}^{\lambda+1}\mathcal{M}$. However, if R is not noetherian then $\mathfrak{X}_\lambda$ and $\mathfrak{Y}_\lambda$ are no noetherian schemes so that the direct image theorem for noetherian schemes cannot be applied. In Sect. 3 we draw on a coherence theorem of Kiehl for relatively pseudo-coherent sheaves instead [K3, Theorem 2.9′a], which furnishes the coherence of $R^q(\mathfrak{f}_\lambda)_*(\mathcal{M}_\lambda)$.

After this step [EGA III, 3.4] uses results from [EGA 0_{III}, par. 13] in order to interchange cohomology with the projective limit with respect to λ and by this gets the coherence of the direct image sheaves. There, alas, the noetherian property explicitly comes in again, in particular in the proof of [EGA 0_{III}, 13.7.7]. In Sect. 4 we will instead investigate for the case of a principal ideal $\mathfrak{I}$ – e.g., for R a valuation ring – how the cohomology of complexes behaves with respect to lifting modulo powers of $\mathfrak{I}$. As the cohomology on locally topologically finitely presented formal R-schemes with values in coherent sheaves can be computed by means of Čech complexes (Lemma 5.2), Proposition 4.3 gives the coherence of the cohomology groups, whereas Proposition 4.4 implies the compatibility of cohomology with complete localization. From Proposition 4.5 one then gets a posteriori that also in the situation over an arbitrary valuation ring cohomology and the projective limit with respect to λ commute.

These results are used in Sect. 6 in order to compare the cohomology of a usual proper scheme over a topologically finitely presented R-algebra A and the cohomology of its completion along the closed subscheme defined by $\mathfrak{I}$. As it is not known for the non-noetherian case whether the cohomology of a proper A-scheme with values in a coherent sheaf always is coherent, we again take resort to the notion of relative pseudo-coherence for the 1st and the 2nd GAGA Theorem (6.4 resp. 6.5). In the situation of a smooth proper A-scheme, however, the structure sheaf automatically is relatively pseudo-coherent, by which one gets the 3rd GAGA Theorem 6.8 ("Chow's theorem") in its usual formulation. Using this result one can then generalize Grothendieck's algebraization criterion [EGA III, 5.4.5] to the situation over an arbitrary valuation ring (Proposition 6.9).

In the last section we follow Raynaud's suggestion in [R] and deduce the fundamental results on the cohomology of rigid spaces from the results on formal schemes, e.g., Tate's acyclicity theorem ([T] resp. [GG]) on the structure sheaf of a rigid space (Proposition 7.1) and Kiehl's results [K2] on coherent module sheaves on rigid spaces (Proposition 7.3).

1 Preliminaries

Throughout this article R will always denote a noetherian ring or a valuation ring for a valuation of height 1 and $\mathfrak{I}$ a *finitely* generated ideal in R such that R is separated and complete with respect to the $\mathfrak{I}$-adic topology.

For M an R-module let

$$T_{\mathfrak{I}}(M) := \{m \in M;\ \text{There is a } \nu \in \mathbb{N} \text{ with } \mathfrak{I}^{\nu}m = \{0\}\ .\}$$

denote the ***$\mathfrak{I}$-torsion submodule of M***. The module M is called ***without (non-trivial) $\mathfrak{I}$-torsion*** if $T_{\mathfrak{I}}(M) = \{0\}$ holds and a ***(mere) $\mathfrak{I}$-torsion module*** if $T_{\mathfrak{I}}(M) = M$ holds.

Directly from the definition it follows that for each R-module M the module $M/T_{\mathfrak{I}}(M)$ has no (non-trivial) $\mathfrak{I}$-torsion. Furthermore, one has

Remark 1.1 Let $(a_i)_{i\in I} \subset \mathfrak{I}$ be a generating system of $\mathfrak{I}$ over R. Then an R-module M is without $\mathfrak{I}$-torsion if and only if the R-module homomorphism $M \to \prod_{i\in I} M,\ m \mapsto (a_i m)_{i\in I}$ is injective.

From now on we will assume in addition that the ring R is without $\mathfrak{I}$-torsion.

In the classical rigid case of a valuation ring R the ideal $\mathfrak{I}$ is generated by *one* element $t \in R$, which necessarily is a non-zero element of the maximal ideal of R. Furthermore, in this situation the $\mathfrak{I}$-torsion submodule $T_{\mathfrak{I}}(M)$ of an R-module M equals the usual torsion submodule of all elements of M which are annihilated by a non-zero element of R, and one has that M is without $\mathfrak{I}$-torsion if and only if it is flat over R.

For the more general case of $\mathfrak{I}$ a *principal* ideal one easily verifies:

Remark 1.2 Let $\mathfrak{I} = tR$ for a $t \in R$ and M be an R-module. Let $T := T_{\mathfrak{I}}(M)$ be the $\mathfrak{I}$-torsion submodule of M. Then $T \cap t^n M = t^n T$ holds for each $n \in \mathbb{N}$.

By this remark one gets as a substitute for the Artin-Rees lemma:

Lemma 1.3 *Let $\mathfrak{I} = tR$ for a $t \in R$.*

a) *Let $\psi : M \to N$ be an R-module homomorphism. Assume that there is an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{I}}(N) = \{0\}$. Then for each $\nu \in \mathbb{N}$ one has*

$$\ker\psi \cap t^{\nu+n}M = t^\nu(\ker\psi \cap t^n M)\,.$$

b) *Let $\varphi : L \to M$ be an R-module homomorphism. Assume that there is an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{I}}\big(M/\varphi(L)\big) = \{0\}$. Then φ is strict with respect to the respective $\mathfrak{I}$-adic topologies.*

Proof. a) Obviously, one has $t^\nu(\ker\psi \cap t^n M) \subset \ker\psi \cap t^{\nu+n}M$. Let conversely $m \in \ker\psi \cap t^{\nu+n}M$ be arbitrary. Then there is an $m' \in t^n M$ with $m = t^\nu m'$ where $\psi(m')$ lies in $t^n\psi(M) \subset t^n N$ and, because of $0 = \psi(m) = t^\nu\psi(m')$, is an $\mathfrak{I}$-torsion element of N. By Remark 1.2 and the assumption one has $T_{\mathfrak{I}}(N) \cap t^n N = t^n T_{\mathfrak{I}}(N) = \{0\}$, hence $\psi(m') = 0$ and $m = t^\nu m' \in t^\nu(\ker\psi \cap t^n M)$.

b) The continuity of φ is clear. Applying part a) to the residue class projection $M \to M/\varphi(L)$ gives $\varphi(L) \cap t^{\nu+n}M = t^\nu(\varphi(L) \cap t^n M) \subset t^\nu\varphi(L) = \varphi(t^\nu L)$ for $\nu \in \mathbb{N}$ arbitrary, hence the strictness of φ. □

In general one has:

Lemma 1.4 a) *Let $\psi : M \to N$ be a surjective R-module homomorphism. Then ψ is strict with respect to the $\mathfrak{I}$-adic topologies, hence the homomorphism $\hat{\psi} : \hat{M} \to \hat{N}$ induced by $\mathfrak{I}$-adic (separated) completion also surjective. Together with M also N is complete with respect to the $\mathfrak{I}$-adic topology.*

b) *Let $L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ be an exact sequence of R-modules. Then also the sequence*

$$\hat{L} \xrightarrow{\hat{\varphi}} \hat{M} \xrightarrow{\hat{\psi}} \hat{N} \longrightarrow 0$$

induced by $\mathfrak{I}$-adic (separated) completion is exact if $\hat{M}/\hat{\varphi}(\hat{L})$ is separated with respect to the $\mathfrak{I}$-adic topology.

c) *Let $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ be an exact sequence of R-modules such that there exists an $n \in \mathbb{N}$ with $\mathfrak{I}^n T_{\mathfrak{I}}(N) = \{0\}$. If $\mathfrak{I}$ is not a principal ideal assume furthermore that M is a finitely generated module over a noetherian R-algebra. Then also the sequence*

$$0 \longrightarrow \hat{L} \xrightarrow{\hat{\varphi}} \hat{M} \xrightarrow{\hat{\psi}} \hat{N} \longrightarrow 0$$

induced by $\mathfrak{I}$-adic (separated) completion is exact.

Proof. Let $\iota_M : M \to \hat{M}$ denote the canonical isometric homomorphism, analogously $\iota_L : L \to \hat{L}$ and $\iota_N : N \to \hat{N}$.

a) The first claim is clear and implies the last one since M is $\mathfrak{I}$-adically complete if and only if ι_M is surjective, analogously for N.

b) As $\iota_M(\varphi(L)) \subset \hat{\varphi}(\hat{L})$ holds, the map $M \xrightarrow{\iota_M} \hat{M} \xrightarrow{\psi'} \hat{M}/\hat{\varphi}(\hat{L}) =: N'$ factors over $\psi : M \to M/\varphi(L) = N$ by an R-module homomorphism $\iota : N \to N'$. By the right-exactness of tensorization one hence has a commutative diagram of R-modules

$$\begin{array}{ccccccc}
L \otimes_R R_\lambda & \xrightarrow{\varphi\otimes \mathrm{id}_{R_\lambda}} & M \otimes_R R_\lambda & \xrightarrow{\psi\otimes \mathrm{id}_{R_\lambda}} & N \otimes_R R_\lambda & \longrightarrow & 0 \\
\downarrow{\scriptstyle \iota_L\otimes \mathrm{id}_{R_\lambda}} & & \downarrow{\scriptstyle \iota_M\otimes \mathrm{id}_{R_\lambda}} & & \downarrow{\scriptstyle \iota\otimes \mathrm{id}_{R_\lambda}} & & \\
\hat{L} \otimes_R R_\lambda & \xrightarrow{\hat{\varphi}\otimes \mathrm{id}_{R_\lambda}} & \hat{M} \otimes_R R_\lambda & \xrightarrow{\psi'\otimes \mathrm{id}_{R_\lambda}} & N' \otimes_R R_\lambda & \longrightarrow & 0
\end{array}$$

with exact rows and isomorphisms in the first two columns for each $\lambda \in \mathbb{N}$. But then also $\iota\otimes\mathrm{id}_{R_\lambda}$ is an R-module isomorphism. So the system of the $\iota\otimes\mathrm{id}_{R_\lambda}$, $\lambda \in \mathbb{N}$, induces an R-module isomorphism between $\hat{N}$ and the $\mathfrak{I}$-adic completion of N'. However, $N' = \hat{M}/\hat{\varphi}(\hat{L})$ is complete by a), furthermore separated with respect to the $\mathfrak{I}$-adic topology by assumption, and hence coincides with its $\mathfrak{I}$-adic completion.

c) It suffices to show that φ and ψ are strict with respect to the $\mathfrak{I}$-adic topologies: For ψ the strictness holds by a), for φ it follows from Lemma 1.3b) if $\mathfrak{I}$ is a principal ideal and, by use of the additional assumption, from [AC, chap. III, par. 3, n°2, corollaire to théorème 2], else. □

For $r \in \mathbb{N}$ arbitrary the R-algebra $R\langle\zeta_1, \ldots, \zeta_r\rangle$ of strictly convergent power series (cf. [BGR, Sect. 1.4]) – resp. of restricted formal power series (cf. [AC, chap. III, par. 4, n°2]) – consists of those formal power series $\sum_{\nu\in\mathbb{N}^r} \alpha_\nu\zeta^\nu$ with coefficients $\alpha_\nu \in R$, $\nu \in \mathbb{N}^r$, that fulfil $\lim_{|\nu|\to\infty} \alpha_\nu = 0$ with respect to the $\mathfrak{I}$-adic topology.

An R-algebra A is called ***topologically finitely generated over*** R if there is an R-algebra epimorphism $\sigma : R\langle\zeta_1, \ldots, \zeta_r\rangle \to A$ for a convenient $r \in \mathbb{N}$ and ***topologically finitely presented over*** R if there is such a σ whose kernel is a finitely generated ideal in $R\langle\zeta_1, \ldots, \zeta_r\rangle$.

As in [BL1] a topologically finitely presented R-algebra A is called ***admissible*** if it has no $\mathfrak{I}$-torsion, i.e., if $T_{\mathfrak{I}}(A) = \{0\}$ holds.

By [EGA $0_{\mathrm{I\,new}}$, 7.5.2(ii)] (resp. [AC, chap. III, par. 2, n°11, corollaire 2 to proposition 14]) one has:

Lemma 1.5 *Each algebra that is topologically finitely generated over a noetherian ring again is a noetherian ring.*

We remind of the notion of coherence: A module M over a ring A is called ***coherent over*** A if it is finitely generated over A and each finitely generated A-submodule of M is finitely presented over A. The ring A itself is called ***coherent*** if it is coherent as a module over itself (cf. [AC, chap. I, par. 2, exerc. 11) and 12)], also [EGA $0_{\mathrm{I\,new}}$, 5.3]).

Directly by definition, finitely generated A-submodules of coherent A-modules are coherent A-modules, too. Obviously, each noetherian ring is coherent.

In analogy to [EGA I_{new}, 1.4.3] and in generalization of [EGA I_{new}, 1.5.1] the results on associated module sheaves from [EGA I_{new}, 1.3 and 1.4] imply that for an affine scheme $X = \operatorname{Spec} A$ an $\mathcal{O}_X$-module $\mathcal{M}$ is coherent if and only if it is associated to a coherent A-module (i.e., if and only if it is quasi-coherent and $\Gamma(X, \mathcal{M})$ is coherent over A), in particular, the structure sheaf $\mathcal{O}_X$ is coherent if and only if A is a coherent ring.

Fundamental for the sequel is (cf. also [BL1, Proposition 1.3]):

Proposition 1.6 *Let A be a finitely generated or a topologically finitely generated R-algebra. Then each finitely generated A-module without $\mathfrak{I}$-torsion is coherent over A.*

Proof. By Lemma 1.5 only the case of R a valuation ring has to be considered. Then it suffices to show that each finitely generated A-module N which has no torsion over R is finitely presented over A.

In any case, there exists a presentation $0 \to M \to F \to N \to 0$ of N with F free and finitely generated over A. In order to show that M is finitely generated over A one can assume without restriction that $A = R[\zeta_1, \dots, \zeta_r]$ resp. $A = R\langle\zeta_1, \dots, \zeta_r\rangle$ holds for a convenient $r \in \mathbb{N}$. In the "algebraic" situation $A = R[\zeta_1, \dots, \zeta_r]$ the claim follows directly from the devissage result [RG, I, théorème 3.4.6]. In the "analytic" situation $A = R\langle\zeta_1, \dots, \zeta_r\rangle$ one can deduce the claim by tensoring the situation over R with its field of fractions $Q(R)$ and using the theory of orthonormal bases [B, Satz 2.1] or Weierstraß theory [BGR, Theorem 5.2.7/7]. A third possibility is to reduce modulo $\mathfrak{I}$ and apply again [RG, I, théorème 3.4.6]. □

Using [AC, chap. I, par. 2, n° 8, lemme 9] resp. [EGA $0_{\mathrm{I}\,\text{new}}$, 5.3.13] this proposition implies the following two corollaries:

Corollary 1.7 *Let A be a (topologically) finitely generated R-algebra, M a finitely generated A-module and $T := T_{\mathfrak{I}}(M)$. Then M/T is coherent and T finitely generated over A. In particular, there is an $n \in \mathbb{N}$ with $\mathfrak{I}^n T = \{0\}$.*

Corollary 1.8 *Each (topologically) finitely presented R-algebra is a coherent ring.*

Lemma 1.9 *Each finitely presented module M over a topologically finitely presented R-algebra A is separated and complete with respect to the $\mathfrak{I}$-adic topology.*

Proof. As a topologically finitely presented R-algebra A is separated and complete with respect to the $\mathfrak{I}$-adic topology, cf. [BL1, Proposition 1.1(b)]. Then the same properties are valid for each free A-module of finite rank. Using Lemma 1.4a) this implies that each finitely generated A-module is $\mathfrak{I}$-adically complete. Furthermore, each A-submodule of a free A-module of finite rank is $\mathfrak{I}$-adically separated.

Now, let $0 \to K \to F \to M \to 0$ be a finite presentation of M over A with F free and of finite rank and K finitely generated over A. By Corollary 1.7 there is an $n \in \mathbb{N}$ with $\mathfrak{J}^n T_{\mathfrak{J}}(M) = \{0\}$. If $\mathfrak{J}$ is no principal ideal, hence R noetherian, then A is a noetherian R-algebra by Lemma 1.5.

Therefore, the hypotheses of Lemma 1.4c) are fulfilled and the sequence

$$0 \to \hat{K} \to \hat{F} \to \hat{M} \to 0$$

that one gets by $\mathfrak{J}$-adic (separated) completion is exact, too. But, as the canonical maps $F \to \hat{F}$ and $K \to \hat{K}$ are isomorphisms of R-modules then also the canonical map $M \to \hat{M}$ is an isomorphism of R-modules. □

2 Formal schemes and coherent module sheaves

For the general definition of formal schemes the reader is referred to [EGA $\mathrm{I}_{\mathrm{new}}$, par. 10]. In the sequel we will restrict to the situation described in [BL1, Sect. 1] of formal schemes locally of topologically finite presentation over a formal scheme S which is noetherian and without torsion with respect to an ideal of definition or admissible over a valuation ring (for a height 1 valuation); for the second case also confer to [Me, par. 2].

The structure sheaf $\mathcal{O}_{\mathfrak{X}}$ of such a formal S-scheme $\mathfrak{X}$ is coherent ([EGA $\mathrm{I}_{\mathrm{new}}$, 10.11.2] for the noetherian case, [Me, 2.2.8(a)] for the situation over a valuation ring). If $\mathfrak{J}$ is a coherent $\mathcal{O}_S$-ideal which defines the topology of S then $\mathfrak{J}\mathcal{O}_{\mathfrak{X}}$ is a coherent ideal of definition of $\mathfrak{X}$.

For $\lambda \in \mathbb{N}$ consider the closed subspace S_λ of S which is defined by $\mathfrak{J}^{\lambda+1}$. Then $\mathfrak{X}_\lambda := \mathfrak{X} \times_S S_\lambda$ has the structure sheaf $\mathcal{O}_{\mathfrak{X}_\lambda} = \mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_S} (\mathcal{O}_S/\mathfrak{J}^{\lambda+1}) \cong \mathcal{O}_{\mathfrak{X}}/\mathfrak{J}^{\lambda+1}\mathcal{O}_{\mathfrak{X}}$. By [EGA $\mathrm{I}_{\mathrm{new}}$, 10.5.3 and 10.6] one has $\mathfrak{X} = \varinjlim_\lambda \mathfrak{X}_\lambda$ as an inductive limit in the category of formal schemes in this situation. In particular, by [EGA $\mathrm{I}_{\mathrm{new}}$, 10.6.1] the topological space underlying $\mathfrak{X}$ coincides with the one underlying $\mathfrak{X}_\lambda$ for $\lambda \in \mathbb{N}$ arbitrary so that, e.g., the existence of any $\mu \in \mathbb{N}$ with $\mathfrak{X}_\mu$ quasi-compact implies that $\mathfrak{X}$ and all $\mathfrak{X}_\lambda$, $\lambda \in \mathbb{N}$, are quasi-compact.

Lemma 2.1 *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal S-schemes which are locally of topologically finite presentation.*

a) *If there is a $\mu \in \mathbb{N}$ such that the morphism $\mathfrak{f}_\mu : \mathfrak{X}_\mu \to \mathfrak{Y}_\mu$ induced from $\mathfrak{f}$ by base change with S_μ is separated then all $\mathfrak{f}_\lambda : \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda$, $\lambda \in \mathbb{N}$, and $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ are separated.*

b) *If there is a $\mu \in \mathbb{N}$ such that $\mathfrak{f}_\mu : \mathfrak{X}_\mu \to \mathfrak{Y}_\mu$ is proper then all $\mathfrak{f}_\lambda : \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda$, $\lambda \in \mathbb{N}$, are proper.*

Proof. For $\lambda, \lambda' \in \mathbb{N}$ arbitrary with $\lambda' \geqq \lambda$ the nilreduction $(\mathfrak{X}_\lambda)_{\mathrm{red}}$ of $\mathfrak{X}_\lambda$ coincides with the nilreduction $(\mathfrak{X}_{\lambda'})_{\mathrm{red}}$ of $\mathfrak{X}_{\lambda'}$ since $\mathcal{O}_{\mathfrak{X}_\lambda}$ is the quotient of $\mathcal{O}_{\mathfrak{X}_{\lambda'}}$ by a nilpotent ideal. Analogously, the morphisms $\mathfrak{f}_\lambda : \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda$ and $\mathfrak{f}_{\lambda'} : \mathfrak{X}_{\lambda'} \to \mathfrak{Y}_{\lambda'}$ have the same nilreduction $(\mathfrak{f}_\lambda)_{\mathrm{red}} = (\mathfrak{f}_{\lambda'})_{\mathrm{red}} : (\mathfrak{X}_\lambda)_{\mathrm{red}} = (\mathfrak{X}_{\lambda'})_{\mathrm{red}} \to (\mathfrak{Y}_\lambda)_{\mathrm{red}} = (\mathfrak{Y}_{\lambda'})_{\mathrm{red}}$.

Hence part a) follows from [EGA $\mathrm{I}_{\mathrm{new}}$, 5.3.1(vi) and 10.15.2], and part b) follows from part a) and [EGA II, 5.4.6]. □

In the sequel of this section we will consider formal schemes over the absolute basis Spf R with $\mathfrak{I}$ as an ideal of definition. Set $R_\lambda := R/\mathfrak{I}^{\lambda+1}$ for $\lambda \in \mathbb{N}$.

For an R-algebra A and $g \in A$ arbitrary let

$$A\langle g^{-1}\rangle := \varprojlim_{\lambda} \left((A \otimes_R R_\lambda)[g^{-1}]\right) \cong A\langle\zeta\rangle/(1 - g\zeta)$$

denote the complete localization of A with respect to g. Then Spf $A\langle g^{-1}\rangle$ is an open formal R-subscheme of Spf A and the open subschemes of this type form a basis of the topology of Spf A. Together with A also $A\langle g^{-1}\rangle$ is topologically finitely presented over R.

Combining the proof of [EGA $\mathrm{I}_{\mathrm{new}}$, 10.6.3] and [BL 1, Proposition 1.7] one gets:

Lemma 2.2 *Let $\mathfrak{X}$ be a formal R-scheme locally of topologically finite presentation. Assume that there is a $\mu \in \mathbb{N}$ such that $\mathfrak{X}_\mu$ is an affine scheme.*

Then $\mathfrak{X}$ is an affine formal scheme, $\mathfrak{X} = \mathrm{Spf}A$ with a topologically finitely presented R-algebra A.

For A a topologically finitely presented R-algebra, $\mathfrak{X} := \mathrm{Spf}\,A$ the corresponding affine formal R-scheme, M an A-module, and $\lambda \in \mathbb{N}$ let $(M \otimes_R R_\lambda)^\sim$ denote the $\mathcal{O}_{\mathfrak{X}_\lambda}$-module associated to $M \otimes_R R_\lambda$ in the usual sense of algebraic geometry. Then the ***$\mathcal{O}_{\mathfrak{X}}$-module M^Δ associated to M*** is defined as $\varprojlim_{\lambda}(M \otimes_R R_\lambda)^\sim$, where the projective limit has to be perfected in the category of sheaves of topological groups on $\mathfrak{X}$ (cf. [EGA $\mathrm{I}_{\mathrm{new}}$, 10.10.1], also [Me, 2.2.1]).

By [EGA $\mathrm{I}_{\mathrm{new}}$, 10.10.2.9] (for the noetherian case) resp. [Me, 2.2.5 (a)$\Leftrightarrow$(c)] (for the case of a valuation ring) one has:

Proposition 2.3 (Theorem A for formal schemes) *Let $\mathfrak{X} = \mathrm{Spf}A$ be an affine formal scheme of topologically finite presentation over R and $\mathcal{M}$ an $\mathcal{O}_{\mathfrak{X}}$-module. Then $\mathcal{M}$ is coherent if and only if there is a coherent A-module M such that as an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is isomorphic to the $\mathcal{O}_{\mathfrak{X}}$-module M^Δ associated to M.*

The A-module M in Proposition 2.3 is uniquely determined by $\mathcal{M}$ up to A-module isomorphism. Namely from the algebraic analogue [EGA $\mathrm{I}_{\mathrm{new}}$, 1.3.7] over R_λ one gets in the projective limit over $\lambda \in \mathbb{N}$ by use of Lemma 1.9:

Lemma 2.4 *Let $\mathfrak{X} = \mathrm{Spf}A$ be an affine formal scheme of topologically finite presentation over R. Let M be a coherent A-module and M^Δ the $\mathcal{O}_{\mathfrak{X}}$-module associated to M. Then for each $g \in A$ one has a canonical isomorphism*

$$\Gamma(\mathrm{Spf}A\langle g^{-1}\rangle, M^\Delta) \cong M \otimes_A A\langle g^{-1}\rangle\ .$$

(For the case of a valuation ring confer also to [Me, 2.2.1]. The noetherian case for $g = 1$ is found in [EGA $\mathrm{I}_{\mathrm{new}}$, 10.10.2.1].)

As over an affine formal R-scheme $\mathfrak{U} = \mathrm{Spf}A$ of topologically finite presentation the functor $M \rightsquigarrow M^\Delta$ from the category of coherent A-modules to the category of coherent $\mathcal{O}_{\mathfrak{U}}$-modules is exact ([EGA $\mathrm{I}_{\mathrm{new}}$, 10.10.2.1] resp. [Me, 2.2.1]), Proposition 2.3 and Lemma 2.4 imply as in the algebraic situation [EGA $\mathrm{I}_{\mathrm{new}}$, 1.3.11] that for each *affine* open formal R-subscheme $\mathfrak{U} = \mathrm{Spf}A$ of a formal R-scheme $\mathfrak{X}$ locally of topologically finite presentation the "section functor" $\mathcal{M} \rightsquigarrow \mathcal{M}(\mathfrak{U})$ from the category of coherent $\mathcal{O}_{\mathfrak{X}}$-modules to the category of coherent A-modules is exact, too.

Let $\mathfrak{X} = \mathrm{Spf}A$ be an affine formal scheme of topologically finite presentation over R and $\mathcal{M}$ a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Set $M := \mathcal{M}(\mathfrak{X})$, $T := T_{\mathfrak{I}}(M)$, and $F := M/T$. Then M is a coherent A-module by Proposition 2.3 and Lemma 2.4 so that, by Corollary 1.7, the A-modules T and F are coherent, too. Especially, there is an $n \in \mathbb{N}$ with $\mathfrak{I}^n T = \{0\}$.

By the exactness of the functor "Δ" the exact sequence $0 \to T \to M \to F \to 0$ of A-modules induces an exact sequence

$$0 \to T^{\Delta} =: \mathcal{T} \to M^{\Delta} \cong \mathcal{M} \to F^{\Delta} =: \mathcal{F} \to 0$$

of $\mathcal{O}_{\mathfrak{X}}$-modules.

Consider an arbitrary $g \in A$. Then Lemma 2.4 implies $\mathcal{T}(\mathrm{Spf}A\langle g^{-1}\rangle) \cong T \otimes_A A\langle g^{-1}\rangle$, hence $\mathfrak{I}^n\mathcal{T}(\mathrm{Spf}A\langle g^{-1}\rangle) = \{0\}$ so that, in particular, $\mathcal{T}(\mathrm{Spf}A\langle g^{-1}\rangle)$ is an $\mathfrak{I}$-torsion module. As the system $\{\mathrm{Spf}A\langle g^{-1}\rangle; g \in A\}$ is a basis of the topology of $\mathfrak{X}$ then $\mathcal{T}$ is an *$\mathfrak{I}$-torsion sheaf*, i.e., each stalk $\mathcal{T}_x$, $x \in \mathfrak{X}$, is an $\mathfrak{I}$-torsion module, and, furthermore, $\mathfrak{I}^n\mathcal{T}_x = \{0\}$ holds for each $x \in \mathfrak{X}$. So $\mathcal{T}(\mathfrak{U})$ is an $\mathfrak{I}$-torsion module for each open formal R-subscheme $\mathfrak{U}$ of $\mathfrak{X}$ and fulfills $\mathfrak{I}^n\mathcal{T}(\mathfrak{U}) = \{0\}$.

Similarly, for $g \in A$ arbitrary, Remark 1.1 implies that $\mathcal{F}(\mathrm{Spf}\,A\langle g^{-1}\rangle) \cong F \otimes_A A\langle g^{-1}\rangle$ has no $\mathfrak{I}$-torsion since $A\langle g^{-1}\rangle$ is a flat A-module ([BL1, Proposition 1.7]; for the noetherian case also [EGA $0_{\mathrm{I\,new}}$, 7.6.13] and for the case of a valuation ring [Me, 1.11.4]). Hence $\mathcal{F}$ is a sheaf *without $\mathfrak{I}$-torsion*, i.e., each stalk $\mathcal{F}_x$, $x \in \mathfrak{X}$, is an R-module without $\mathfrak{I}$-torsion. So for each open formal R-subscheme $\mathfrak{U}$ of $\mathfrak{X}$ the R-module $\mathcal{F}(\mathfrak{U})$ has no $\mathfrak{I}$-torsion, too.

Hence for each open formal subscheme $\mathfrak{U}$ of $\mathfrak{X}$ one concludes by the left exactness of the functor "taking sections on $\mathfrak{U}$" that $\mathcal{T}(\mathfrak{U})$ is not only an $\mathfrak{I}$-torsion module but, in fact, the whole $\mathfrak{I}$-torsion submodule $T_{\mathfrak{I}}(\mathcal{M}(\mathfrak{U}))$ of $\mathcal{M}(\mathfrak{U})$.

Therefore it is clear how to globalize the affine situation discussed above and to get

Proposition 2.5 *Let $\mathfrak{X}$ be a formal R-scheme locally of topologically finite presentation and $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Then there is an exact sequence of coherent $\mathcal{O}_{\mathfrak{X}}$-modules*

$$0 \to \mathcal{T} \to \mathcal{M} \to \mathcal{F} \to 0$$

such that $\mathcal{T}$ is an $\mathfrak{I}$-torsion sheaf and $\mathcal{F}$ is without $\mathfrak{I}$-torsion.

If $\mathfrak{X}$ is quasi-compact then there is an $n \in \mathbb{N}$ such that one has $\mathfrak{I}^n\mathcal{T}(\mathfrak{U}) = \{0\}$ and $\mathcal{T}(\mathfrak{U}) = T_{\mathfrak{I}}(\mathcal{M}(\mathfrak{U}))$ for each open formal R-subscheme $\mathfrak{U}$ of $\mathfrak{X}$.

An analogous decomposition as in the above proposition holds if $\mathcal{M}$ is a quasi-coherent module sheaf over a (usual) R-scheme X. In both cases the $\mathfrak{I}$-torsion sheaf $\mathcal{T}$ is uniquely determined by $\mathcal{M}$ and will be denoted by $\mathcal{T}_{\mathfrak{I}}(\mathcal{M})$ in the sequel.

For further reference we collect the information on the modules of q-cochains with values in a coherent sheaf over a formal scheme gained so far:

Remark 2.6 Let $\mathfrak{X}$ be a formal R-scheme locally of topologically finite presentation and $\mathfrak{U}$ a covering of $\mathfrak{X}$ by affine open formal R-subschemes such that each finite intersection of sets from $\mathfrak{U}$ again is formal affine (e.g., assume that $\mathfrak{X}_0$ is separated, cf. Lemma 2.2 and [EGA $\mathrm{I}_{\mathrm{new}}$, 5.3.6]). Let $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathfrak{X}}$.-module. For $q \in \mathbb{Z}$ let $C^q(\mathfrak{U}, \mathcal{M})$ denote the R-module of q-cochains on $\mathfrak{U}$ with values in $\mathcal{M}$.

Then for $\lambda \in \mathbb{N}$ arbitrary one has $C^q(\mathfrak{U}, \mathfrak{I}^{\lambda+1}\mathcal{M}) = \mathfrak{I}^{\lambda+1}C^q(\mathfrak{U}, \mathcal{M})$ and

$$C^q\left(\mathfrak{U}, \mathcal{M}/(\mathfrak{I}^{\lambda+1}\mathcal{M})\right) \cong C^q(\mathfrak{U}, \mathcal{M})/(\mathfrak{I}^{\lambda+1}C^q(\mathfrak{U}, \mathcal{M})) \cong C^q(\mathfrak{U}, \mathcal{M}) \otimes_R R_\lambda\,,$$

and the module $C^q(\mathfrak{U}, \mathcal{M})$ is separated and complete with respect to the $\mathfrak{I}$-adic topology.

If $\mathfrak{X}$ is quasi-compact there is an $n \in \mathbb{N}$ with $\mathfrak{I}^n T_{\mathfrak{I}}\left(C^q(\mathfrak{U}, \mathcal{M})\right) = \{0\}$ for all $q \in \mathbb{Z}$.

Proof. For $q < 0$ the claims are clear, so assume without restriction $q \geqq 0$. Because of the hypotheses on $\mathfrak{U}$ the module $C^q(\mathfrak{U}, \mathcal{M})$ is of the form $\prod_{j \in J_q} \mathcal{M}(\mathfrak{V}_j)$ with some affine open formal R-subschemes $\mathfrak{V}_j = \mathrm{Spf} A_j$ of $\mathfrak{X}$.

By the exactness of the functor "taking sections on $\mathfrak{V}_j$ in coherent $\mathcal{O}_{\mathfrak{X}}$-modules" one has for each $j \in J_q$ that $(\mathfrak{I}^{\lambda+1}\mathcal{M})(\mathfrak{V}_j) = \mathfrak{I}^{\lambda+1} \cdot \mathcal{M}(\mathfrak{V}_j)$ and

$$(\mathcal{M}/\mathfrak{I}^{\lambda+1}\mathcal{M})(\mathfrak{V}_j) \cong \mathcal{M}(\mathfrak{V}_j)/(\mathfrak{I}^{\lambda+1} \cdot \mathcal{M}(\mathfrak{V}_j)) \cong \mathcal{M}(\mathfrak{V}_j) \otimes_R R_\lambda$$

holds for each $\lambda \in \mathbb{N}$, hence also the analogous statements for the modules of q-cochains.

Furthermore, for each $j \in J_q$ the A_j-module $\mathcal{M}(\mathfrak{V}_j)$ is finitely presented by Proposition 2.3 and Lemma 2.4, hence $\mathfrak{I}$-adically separated and complete by Lemma 1.9. These properties transfer to $C^q(\mathfrak{U}, \mathcal{M})$ where for the $\mathfrak{I}$-adic completeness – if $\mathfrak{U}$ is not finite – one may use [BGR, Proposition 2.1.5/6] since the $\mathfrak{I}$-adic norm on the $\mathcal{M}(\mathfrak{V}_j)$ is bounded by 1 from above.

The last claim follows from Proposition 2.5. □

3 The direct image theorem for – not necessarily noetherian – schemes

For the proof of the direct image theorem for formal schemes at first the corresponding statement "modulo powers of $\mathfrak{I}$" is needed. As, however, in the case of a non-discrete valuation ring R also the rings $R_\lambda = R/\mathfrak{I}^{\lambda+1}$, $\lambda \in \mathbb{N}$, are not noetherian, one cannot draw on the direct image theorem for locally noetherian schemes [EGA III, 3.2.1] for this purpose. Instead of this, we will use a result of Kiehl (Theorem 3.1), but at first remind of some notions from [SGA 6] in order to formulate it (cf. [SGA 6, exp. I, introduction]):

Let $(X, \mathcal{O}_X)$ be a ringed space. Then an $\mathcal{O}_X$-module $\mathcal{M}$ is called ***pseudo-coherent*** if for each $n \in \mathbb{N}$ locally on X there is a finite presentation of $\mathcal{M}$ of length n, i.e., if there exists an open covering $\mathfrak{U}$ of X such that for each $U \in \mathfrak{U}$ there is an exact sequence

$$\mathcal{L}_n \to \mathcal{L}_{n-1} \to \cdots \to \mathcal{L}_0 \to \mathcal{M}|_U \to 0$$

with $\mathcal{L}_i$ a free and finitely generated $\mathcal{O}_U$-module for $i = 0, \ldots, n$. – Note that this notion of "pseudo-coherence" is *not* in agreement with the one introduced in [AC, chap. I, par. 2, exerc. 11)], which is named "précohérent" in [SGA 6, exp. I, definition 3.1.] –

A complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules – resp. the element of the derived category $D(\mathrm{Mod}\, X)$ of the category of $\mathcal{O}_X$-modules represented by this complex – is called ***pseudo-coherent*** if for each $n \in \mathbb{Z}$ locally (in the above sense) there is a quasi-isomorphism $\mathcal{L}^\bullet \to \mathcal{K}^\bullet$ with $\mathcal{L}^\bullet$ a complex that is bounded from above and whose components of degree greater or equal n are free and finitely generated $\mathcal{O}_X$-modules [SGA 6, exp. I, definition 2.3]. Obviously, an $\mathcal{O}_X$-module $\mathcal{M}$ is pseudo-coherent if and only if the complex $\bar{\mathcal{M}}$ that equals $\mathcal{M}$ at the zeroth place and vanishes else is pseudo-coherent.

Let $f : X \to Y$ be a morphism of schemes which is locally of finite type and $\mathcal{K}^\bullet$ a complex of $\mathcal{O}_X$-modules. Then $\mathcal{K}^\bullet$ is called ***relatively pseudo-coherent with respect to*** f if there is an open covering $\mathfrak{U}$ of X and for each $U \in \mathfrak{U}$ a factorization of $f|_U$ over a closed immersion $\iota_U : U \to Z$ into a smooth Y-scheme Z such that $(\iota_U)_*(\mathcal{K}^\bullet)$ is a pseudo-coherent complex of $\mathcal{O}_Z$-modules [SGA 6, exp. III, théorème 1.1 and definition 1.2]. An $\mathcal{O}_X$-module $\mathcal{M}$ is called ***relatively pseudo-coherent with respect to*** f if the complex $\bar{\mathcal{M}}$ is relatively pseudo-coherent with respect to f.

With these notions one has the following version of the direct image theorem:

Theorem 3.1 [K3, Theorem 2.9′a)] *Let $f : X \to Y$ be a proper morphism of schemes and $\mathcal{K}^\bullet$ a complex of $\mathcal{O}_X$-modules which is bounded from below and relatively pseudo-coherent with respect to f. Then $\mathcal{R}^\bullet f_* \mathcal{K}^\bullet$ is a pseudo-coherent complex of $\mathcal{O}_Y$-modules.*

Now we are going to adapt this result to the situation of interest in the present article. At first, for the standard situation of a coherent structure sheaf one has by [EGA $0_{\mathrm{I\,new}}$, 5.3.3 and 5.3.4] resp. [SGA 6, exp. I, corollaire 3.5b)]:

Lemma 3.2 *Let $(X, \mathcal{O}_X)$ be a ringed space with a coherent structure sheaf $\mathcal{O}_X$. Then an $\mathcal{O}_X$-module $\mathcal{M}$ is pseudo-coherent if and only if it is coherent, and a complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules that is quasi-isomorphic to a complex which is bounded from above is pseudo-coherent if and only if its cohomology $H^q(\mathcal{K}^\bullet)$ is coherent for all $q \in \mathbb{Z}$.*

In order to get a sufficient criterion for relative pseudo-coherence we have to recourse on the coherence of the structure sheaf not only of the scheme X itself but also of all affine spaces over X. The property of coherence, however, is in general not inherited by polynomial rings (for an example see [Sou, proposition 18]) so that one has to define explicitly:

A ring is called ***stably*** (or ***universally***) ***coherent*** if each polynomial ring in finitely many variables over this ring is coherent.

Obviously, noetherian rings are stably coherent. Important for the following is

Example 3.3 Let R and $\mathfrak{I}$ be as in Sect. 1. Then, as each polynomial ring in finitely many variables over a finitely presented R-algebra again is a finitely presented R-algebra, Corollary 1.8 implies that each finitely presented R-algebra, in particular each finitely presented algebra over $R/\mathfrak{I}^{\lambda+1}$ for $\lambda \in \mathbb{N}$, is stably coherent.

One has the following criterion for relative pseudo-coherence:

Lemma 3.4 *Let $f : X \to Y$ be a morphism of schemes which is locally of finite presentation, and let $Y = \operatorname{Spec} A$ be affine with a stably coherent ring A. Then each coherent $\mathcal{O}_X$-module $\mathcal{M}$ is relatively pseudo-coherent with respect to f.*

Proof. As relative pseudo-coherence is a local property on X, one can exchange X and assume it to be affine. Then it follows from [EGA $\mathrm{I}_{\mathrm{new}}$, 6.2.8 and 6.2.9] that, for a convenient $r \in \mathbb{N}$, there is a closed immersion $\iota : X \to \operatorname{Spec} A[\zeta_1, \ldots, \zeta_r] = \mathbb{A}_Y^r$ of Y-schemes and $\iota(X)$ is defined in $\mathbb{A}_Y^r$ by a finitely generated and quasi-coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_{\mathbb{A}_Y^r}$. Because of the stable coherence of A the structure sheaf of $\mathbb{A}_Y^r$ is coherent so that $\mathcal{J}$ is a coherent ideal. As $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, the module $\iota_*\mathcal{M}$ is coherent over $\iota_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{A}_Y^r}/\mathcal{J}$ by [EGA $0_{\mathrm{I\,new}}$, 5.3.15]. Now [EGA $0_{\mathrm{I\,new}}$, 5.3.13] implies the coherence of $\iota_*\mathcal{M}$ also over $\mathcal{O}_{\mathbb{A}_Y^r}$. So by Lemma 3.2 the module $\iota_*\mathcal{M}$ resp. the complex $\overline{\iota_*\mathcal{M}} = \iota_*(\bar{\mathcal{M}})$ is pseudo-coherent. □

From Kiehl's result Theorem 3.1 one now gets:

Direct image theorem for schemes 3.5 *Let $f : X \to Y$ be a proper morphism of schemes which is (locally) of finite presentation and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module. Assume that Y has an open covering by spectra of stably coherent rings. Then $R^q f_*\mathcal{M}$ is a coherent $\mathcal{O}_Y$-module for each $q \in \mathbb{Z}$.*

Proof. As coherence, properness and (local) finite presentability are local properties on Y, one can assume $Y = \operatorname{Spec} A$ with a stably coherent ring A. Then the coherent $\mathcal{O}_X$-module $\mathcal{M}$ is relatively pseudo-coherent with respect to the proper morphism f by Lemma 3.4. Therefore, Theorem 3.1 implies the pseudo-coherence of $\mathcal{R}^{\cdot} f_*(\bar{\mathcal{M}})$.

By [EGA III, 1.4.12] the direct image $R^q f_*\mathcal{M} = H^q(\mathcal{R}^{\cdot} f_*(\bar{\mathcal{M}}))$ vanishes for q sufficiently large. Hence $\mathcal{R}^{\cdot} f_*(\bar{\mathcal{M}})$ is quasi-isomorphic to a complex that is bounded from above. As A is a (stably) coherent ring, the structure sheaf $\mathcal{O}_Y$ is coherent. Lemma 3.2 now implies that $H^q(\mathscr{R}^{\cdot}f_*(\bar{\mathscr{M}})) = R^q f_*\mathscr{M}$ is coherent over $\mathcal{O}_Y$ for all $q \in \mathbb{Z}$. □

Besides the direct image theorem also Serre's results on the higher direct image sheaves for projective morphisms [Se, n° 66] can be transferred to the stably coherent case. So [SGA 6, exp. III, théorème 2.2.2 or corollaire 2.3] together with Lemma 3.4 and [EGA II, 4.4.6] imply

Proposition 3.6 *Let Y be a scheme which has a finite open covering by spectra of stably coherent rings, $f : X \to Y$ a projective morphism of schemes which is (locally) of finite presentation, and $\mathscr{L}$ an invertible $\mathcal{O}_X$-module which is ample with respect to f. For each $\mathcal{O}_X$-module $\mathscr{M}$ and each $n \in \mathbb{Z}$ set $\mathscr{M}(n) := \mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}$.*

Then for each coherent $\mathcal{O}_X$-module $\mathscr{M}$ there is an $N \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$ with $n \geqq N$ one has $R^q f_*(\mathscr{M}(n)) = 0$ for all $q > 0$.

Note. The direct image theorem 3.5 for *projective* morphisms f and this proposition can also be proved directly, without any recourse on Theorem 3.1 resp. [SGA 6, exp. III, 2] and the theory of pseudo-coherent complexes since one can transfer the arguments in [EGA III, 2.2.1 and 2.2.2] in a rather canonical way from the noetherian to the stably coherent situation.

4 Lifting of the cohomology of complexes

Let R and $\mathfrak{J}$ be as above. Throughout this section assume in addition that $\mathfrak{J} = tR$ is a *principal* ideal. Let A be an R-algebra and set $A_\lambda := A/\mathfrak{J}^{\lambda+1}A = A/t^{\lambda+1}A$ for $\lambda \in \mathbb{N}$.

For a complex $C^{\cdot} = (C^q, \partial^q)$ of A-modules let $C^{\cdot}_\lambda$ denote the complex of A_λ-modules that arises from $C^{\cdot}$ by tensoring all modules with A_λ and all homomorphisms with id_{A_λ} over A. Then the exact sequence of complexes

$$0 \longrightarrow t^{\lambda+1}C^{\cdot} \xrightarrow{\iota_\lambda} C^{\cdot} \xrightarrow{\sigma_\lambda} C^{\cdot}_\lambda \longrightarrow 0$$

induces a long exact cohomology sequence

$$\dots \longrightarrow H^q(t^{\lambda+1}C^{\cdot}) \xrightarrow{\iota^q_\lambda} H^q(C^{\cdot}) \xrightarrow{\sigma^q_\lambda} H^q(C^{\cdot}_\lambda) \xrightarrow{\delta^q_\lambda} H^{q+1}(t^{\lambda+1}C^{\cdot}) \longrightarrow \dots$$

Lemma 4.1 *Let $C^{\cdot}$ be a complex of A-modules and $q \in \mathbb{Z}$ such that there is an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{J}}(C^{q+1}) = \{0\}$. Then for each $\lambda \geqq n$ one has:*

a) $t^{\lambda+1}H^q(C^{\cdot}) \subset \iota^q_\lambda(H^q(t^{\lambda+1}C^{\cdot})) \subset t^{\lambda-n+1}H^q(C^{\cdot})$.

b) *Let A and C^{q-1} be complete and C^q be separated with respect to the respective $\mathfrak{J}$-adic topology. If $\beta_1, \dots, \beta_m$ are elements of $H^q(C^{\cdot})$ such that $\sigma^q_\lambda(\beta_1), \dots, \sigma^q_\lambda(\beta_m)$ generate $\sigma^q_\lambda(H^q(C^{\cdot}))$ as an A-module then $\beta_1, \dots, \beta_m$ already generate $H^q(C^{\cdot})$ as an A-module.*

Proof. a) By Lemma 1.3a), applied to the R-module homomorphism $\partial^q : C^q \to C^{q+1}$, one has $\ker \partial^q \cap t^{\lambda+1} C^q = t^{\lambda-n+1}(\ker \partial^q \cap t^n C^q)$, hence

$$t^{\lambda+1} \ker \partial^q \subset \ker \partial^q \cap t^{\lambda+1} C^q \subset t^{\lambda-n+1} \ker \partial^q .$$

Now the claim follows by applying the residue class map $p : \ker \partial^q \to H^q(C^{\cdot})$ to this chain of inclusions.

b) For $\mu \in \{1, \dots, m\}$ let $\alpha_\mu \in \ker \partial^q$ be an inverse image of $\beta_\mu \in H^q(C^{\cdot})$ with respect to p. Furthemore set $F := A^m$ and define an A-module homomorphism $\Phi : F \to \ker \partial^q$ by mapping the μ-th element of the canonical basis of F to α_μ for $\mu = 1, \dots, m$. Then the A-module homomorphism

$$F \xrightarrow{\Phi} \ker \partial^q \xrightarrow{p} \ker \partial^q / \operatorname{im} \partial^{q-1} = H^q(C^{\cdot}) \xrightarrow{\sigma^q_\lambda} \operatorname{im} \sigma^q_\lambda$$

is surjective by assumption. So for $\varphi := p \circ \Phi$ one has $H^q(C^{\cdot}) = \varphi(F) + \ker \sigma^q_\lambda = \varphi(F) + \operatorname{im} \iota^q_\lambda = \varphi(F) + tH^q(C^{\cdot})$, where the last equality holds by part a). This implies

$$\ker \partial^q = \Phi(F) + \operatorname{im} \partial^{q-1} + t \ker \partial^q .$$

Hence for any $a = a_0 \in \ker \partial^q$ there is a $b_1 \in F$, a $c_1 \in C^{q-1}$, and an $a_1 \in \ker \partial^q$ with $a_0 = \Phi(b_1) + \partial^{q-1}(c_1) + ta_1$. As Φ and ∂^{q-1} are homomorphisms of A-modules, by iteration one gets three sequences $(b_\nu) \subset F$, $(c_\nu) \subset C^{q-1}$, and $(a_\nu) \subset \ker \partial^q$ from this with

$$a = a_0 = \Phi\left(\sum_{\nu=1}^{\nu_0} t^{\nu-1} b_\nu\right) + \partial^{q-1}\left(\sum_{\nu=1}^{\nu_0} t^{\nu-1} c_\nu\right) + t^{\nu_0} a_{\nu_0}$$

for each $\nu_0 \in \mathbb{N}$. As A, hence also F, and C^{q-1} are complete with respect to the respective $\mathfrak{J}$-adic topology, one infers the existence of $b := \sum_{\nu=1}^{\infty} t^{\nu-1} b_\nu \in F$ and $c := \sum_{\nu=1}^{\infty} t^{\nu-1} c_\nu \in C^{q-1}$. Then, by the $\mathfrak{J}$-adic separatedness of C^q, the above equalities give in the limit $\nu_0 \to \infty$ that $a = \Phi(b) + \partial^{q-1}(c)$ holds.

As $a \in \ker \partial^q$ was arbitrary, this implies $\ker \partial^q = \Phi(F) + \operatorname{im} \partial^{q-1}$, hence $H^q(C^{\cdot}) = (p \circ \Phi)(F) = \sum_{\mu=1}^{m} A\, p(\alpha_\mu) = \sum_{\mu=1}^{m} A\, \beta_\mu$. □

From part b) of the above lemma one gets the following consequence, which generalizes part (1) of the (homologically clothed) Theorem 8.5 in [Mo]:

Consequence 4.2 *Let A be complete with respect to the $\mathfrak{J}$-adic topology and $C^{\cdot}$ be a complex of A-modules. Let $q \in \mathbb{Z}$ such that C^{q-1} is complete and C^q is separated with respect to the respective $\mathfrak{J}$-adic topology and such that there is an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{J}}(C^{q+1}) = \{0\}$. Then $H^q(C^{\cdot}_\lambda) = 0$ for a $\lambda \in \mathbb{N}$ with $\lambda \geqq n$ already implies $H^q(C^{\cdot}) = 0$.*

Proposition 4.3 *Let $C^{\cdot}$ be a complex of A-modules and $q \in \mathbb{Z}$ such that there is an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{J}}(C^{q+1}) = \{0\}$.*

a) *Let A and C^{q-1} be complete and C^q be separated with respect to the respective $\mathfrak{J}$-adic topology. If $H^q(C^{\cdot}_\lambda)$ is a finitely generated A-module and*

$H^{q+1}(t^{\lambda+1}C^{\boldsymbol\cdot})$ a coherent A-module for a $\lambda \in \mathbb{N}$ with $\lambda \geqq n$ then $H^q(C^{\boldsymbol\cdot})$ is a finitely generated A-module.

b) *Let A be topologically finitely generated over R. If $H^q(C^{\boldsymbol\cdot})$ is a finitely generated A-module and $H^q(C^{\boldsymbol\cdot}_\lambda)$ is a coherent A-module for each $\lambda \in \mathbb{N}$ with $\lambda \geqq n$ then $H^q(C^{\boldsymbol\cdot})$ is a coherent A-module.*

Proof. One again considers the long exact cohomology sequence

$$\dots \longrightarrow H^q(t^{\lambda+1}C^{\boldsymbol\cdot}) \xrightarrow{\iota^q_\lambda} H^q(C^{\boldsymbol\cdot}) \xrightarrow{\sigma^q_\lambda} H^q(C^{\boldsymbol\cdot}_\lambda) \xrightarrow{\delta^q_\lambda} H^{q+1}(t^{\lambda+1}C^{\boldsymbol\cdot}) \longrightarrow \dots$$

a) Together with $H^q(C^{\boldsymbol\cdot}_\lambda)$ also $\delta^q_\lambda\big(H^q(C^{\boldsymbol\cdot}_\lambda)\big)$ is finitely generated over A. As a submodule of the coherent A-module $H^{q+1}(t^{\lambda+1}C^{\boldsymbol\cdot})$ then $\delta^q_\lambda\big(H^q(C^{\boldsymbol\cdot}_\lambda)\big)$ is even finitely presented over A. By [AC, chap. I, par. 2, n° 8, lemme 9] this implies that $\ker \delta^q_\lambda = \operatorname{im} \sigma^q_\lambda$ is finitely generated over A since $H^q(C^{\boldsymbol\cdot}_\lambda)$ is finitely generated over A. Because of $\lambda \geqq n$ then by Lemma 4.1b) also $H^q(C^{\boldsymbol\cdot})$ is a finitely generated A-module.

b) Let $T := T_{\mathfrak{I}}\big(H^q(C^{\boldsymbol\cdot})\big)$ be the $\mathfrak{I}$-torsion submodule of the finitely generated A-module $H^q(C^{\boldsymbol\cdot})$. Then, by Corollary 1.7, the A-module $H^q(C^{\boldsymbol\cdot})/T$ is coherent and the A-module T is finitely generated. In particular, there is a $\lambda' \in \mathbb{N}$ with $\{0\} = t^{\lambda'+1}T = T \cap t^{\lambda'+1}H^q(C^{\boldsymbol\cdot})$, using Remark 1.2. As $\ker \sigma^q_\lambda = \operatorname{im} \iota^q_\lambda \subset t^{\lambda'+1}H^q(C^{\boldsymbol\cdot})$ holds for $\lambda := \lambda' + n$ by Lemma 4.1a), then $\sigma^q_\lambda|_T : T \to H^q(C^{\boldsymbol\cdot}_\lambda)$ is injective, hence T isomorphic to an A-submodule of $H^q(C^{\boldsymbol\cdot}_\lambda)$. By assumption the A-module $H^q(C^{\boldsymbol\cdot}_\lambda)$ is coherent, hence also each finitely generated A-submodule of $H^q(C^{\boldsymbol\cdot}_\lambda)$, therefore T, too. Now, by [AC, chap. I, par. 2, exerc. 11) a)] resp. [EGA $0_{\mathrm{I\,new}}$, 5.3.2], together with $H^q(C^{\boldsymbol\cdot})/T$ and T also $H^q(C^{\boldsymbol\cdot})$ itself is a coherent A-module. □

Let $g \in A$ be arbitrary, $C^{\boldsymbol\cdot}$ a complex of A-modules, $\tilde{C}^{\boldsymbol\cdot}$ one of $A\langle g^{-1}\rangle$-modules, and $\Psi : C^{\boldsymbol\cdot} \to \tilde{C}^{\boldsymbol\cdot}$ a homomorphism of complexes of A-modules. Then for each $\lambda \in \mathbb{N}$ one has a commutative diagram of A-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t^{\lambda+1}C^{\boldsymbol\cdot} & \xrightarrow{\iota_\lambda} & C^{\boldsymbol\cdot} & \xrightarrow{\sigma_\lambda} & C^{\boldsymbol\cdot}_\lambda & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t^{\lambda+1}\tilde{C}^{\boldsymbol\cdot} & \xrightarrow{\tilde\iota_\lambda} & \tilde{C}^{\boldsymbol\cdot} & \xrightarrow{\tilde\sigma_\lambda} & \tilde{C}^{\boldsymbol\cdot}_\lambda & \longrightarrow & 0
\end{array}$$

with columns induced by Ψ and exact rows, which induces a commutative diagram of A-modules

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H^q(t^{\lambda+1}C^{\boldsymbol\cdot}) & \xrightarrow{\iota^q_\lambda} & H^q(C^{\boldsymbol\cdot}) & \xrightarrow{\sigma^q_\lambda} & H^q(C^{\boldsymbol\cdot}_\lambda) & \xrightarrow{\delta^q_\lambda} & H^{q+1}(t^{\lambda+1}C^{\boldsymbol\cdot}) & \longrightarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \longrightarrow & H^q(t^{\lambda+1}\tilde{C}^{\boldsymbol\cdot}) & \xrightarrow{\tilde\iota^q_\lambda} & H^q(\tilde{C}^{\boldsymbol\cdot}) & \xrightarrow{\tilde\sigma^q_\lambda} & H^q(\tilde{C}^{\boldsymbol\cdot}_\lambda) & \xrightarrow{\tilde\delta^q_\lambda} & H^{q+1}(t^{\lambda+1}\tilde{C}^{\boldsymbol\cdot}) & \longrightarrow \dots
\end{array}$$

with exact rows. As the cohomology groups of the lower row are $A\langle g^{-1}\rangle$-modules, by tensoring the upper row with $A\langle g^{-1}\rangle$ over A one gets the following commutative diagram of $A\langle g^{-1}\rangle$-modules

$$\begin{tikzcd}
H^q(t^{\lambda+1}C^\cdot)\otimes_A A\langle g^{-1}\rangle \arrow[r, "\iota^q_\lambda\otimes\mathrm{id}_{A\langle g^{-1}\rangle}"] \arrow[d, "\varphi^q_\lambda"] & H^q(C^\cdot)\otimes_A A\langle g^{-1}\rangle \arrow[r, "\sigma^q_\lambda\otimes\mathrm{id}_{A\langle g^{-1}\rangle}"] \arrow[d, "\varphi^q"] & {} \\
H^q(t^{\lambda+1}\tilde{C}^\cdot) \arrow[r, "\tilde{\iota}^q_\lambda"] & H^q(\tilde{C}^\cdot) \arrow[r, "\tilde{\sigma}^q_\lambda"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
H^q(C^\cdot_\lambda)\otimes_A A\langle g^{-1}\rangle \arrow[r, "\delta^q_\lambda\otimes\mathrm{id}_{A\langle g^{-1}\rangle}"] \arrow[d, "\Phi^q_\lambda"] & H^{q+1}(t^{\lambda+1}C^\cdot)\otimes_A A\langle g^{-1}\rangle \arrow[d, "\varphi^{q+1}_\lambda"] \\
H^q(\tilde{C}^\cdot_\lambda) \arrow[r, "\tilde{\delta}^q_\lambda"] & H^{q+1}(t^{\lambda+1}\tilde{C}^\cdot)
\end{tikzcd}$$

whose upper row is also exact if A is topologically finitely presented over R because then $A\langle g^{-1}\rangle$ is flat over A by [BL1, Proposition 1.7].

Proposition 4.4 *Let A be a topologically finitely presented R-algebra, $g \in A$ arbitrary, $C^\cdot$ a complex of A-modules, $\tilde{C}^\cdot$ one of $A\langle g^{-1}\rangle$-modules, and $\Psi : C^\cdot \to \tilde{C}^\cdot$ a homomorphism of complexes of A-modules. Let $q \in \mathbb{Z}$ such that $H^q(C^\cdot)$ is a coherent A-module. Then one has:*

a) *If there exists an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{J}}(C^{q+1}) = \{0\}$ and if Φ^q_λ is injective for each $\lambda \in \mathbb{N}$ with $\lambda \geqq n$ then also φ^q is injective.*

b) *Let $\tilde{C}^{q-1}$ be complete and $\tilde{C}^q$ be separated with respect to the respective $\mathfrak{J}$-adic topology, and let there be an $\tilde{n} \in \mathbb{N}$ with $t^{\tilde{n}} T_{\mathfrak{J}}(\tilde{C}^{q+1}) = \{0\}$. If Φ^q_λ is surjective and φ^{q+1}_λ is injective for a $\lambda \in \mathbb{N}$ with $\lambda \geqq \tilde{n}$ then φ^q is surjective.*

Hence, in particular, φ^q is an isomorphism if φ^{q+1}_λ and Φ^q_λ are isomorphisms for all $\lambda \in \mathbb{N}$, $\tilde{C}^{q-1}$ is complete and $\tilde{C}^q$ separated with respect to the respective $\mathfrak{J}$-adic topology, and there is an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{J}}(C^{q+1}) = \{0\}$ and $t^n T_{\mathfrak{J}}(\tilde{C}^{q+1}) = \{0\}$.

Proof. a) Let $\lambda \in \mathbb{N}$ with $\lambda \geqq n$ be arbitrary. Then, by the injectivity of Φ^q_λ and the exactness of the first row of the above diagram, one has

$$\begin{aligned}
\ker\varphi^q \subset \ker(\tilde{\sigma}^q_\lambda \circ \varphi^q) &= \ker\left(\Phi^q_\lambda \circ (\sigma^q_\lambda \otimes \mathrm{id}_{A\langle g^{-1}\rangle})\right) \\
&= \ker(\sigma^q_\lambda \otimes \mathrm{id}_{A\langle g^{-1}\rangle}) = \mathrm{im}\,(\iota^q_\lambda \otimes \mathrm{id}_{A\langle g^{-1}\rangle}) \\
&= (\mathrm{im}\,\iota^q_\lambda) \otimes_A A\langle g^{-1}\rangle\,.
\end{aligned}$$

Using Lemma 4.1a) this implies $\ker\varphi^q \subset t^{\lambda-n+1}\left(H^q(C^\cdot)\otimes_A A\langle g^{-1}\rangle\right)$.

By assumption $H^q(C^\cdot)$ is coherent over A, hence $H^q(C^\cdot)\otimes_A A\langle g^{-1}\rangle$ finitely presented over $A\langle g^{-1}\rangle$ and therefore separated with respect to the $\mathfrak{J}$-adic topology by Lemma 1.9. As $\lambda \in \mathbb{N}$ with $\lambda \geqq n$ was arbitrary, one hence concludes $\ker\varphi^q = \{0\}$.

b) By some diagram chasing (cf. [AC, chap. I, par. 1, n°4, proposition 1(i)]) the surjectivity of Φ^q_λ, the injectivity of φ^{q+1}_λ, and the exactness of the rows of the above diagram imply

$$\tilde{\sigma}^q_\lambda\left(\varphi^q(H^q(C^\cdot)\otimes_A A\langle g^{-1}\rangle)\right) = \tilde{\sigma}^q_\lambda(H^q(\tilde{C}^\cdot))\,.$$

Let $H^q(C^\cdot)$ be generated over A by the finitely many elements $\alpha_1, \dots \alpha_m$. Then the images of $\beta_1 := \varphi^q(\alpha_1 \otimes 1_{A\langle g^{-1}\rangle}), \dots, \beta_m := \varphi^q(\alpha_m \otimes 1_{A\langle g^{-1}\rangle})$ with respect to $\tilde{\sigma}^q_\lambda$ generate $\tilde{\sigma}^q_\lambda(H^q(\tilde{C}^\cdot))$ over $A\langle g^{-1}\rangle$. As $\lambda \geqq \tilde{n}$ holds, then by Lemma 4.1b) the elements $\beta_1, \dots, \beta_m \in \operatorname{im} \varphi^q$ generate $H^q(\tilde{C}^\cdot)$ over $A\langle g^{-1}\rangle$, which implies the surjectivity of φ^q. □

Proposition 4.5 *Let $C^\cdot$ be a complex of A-modules and $q \in \mathbb{Z}$ such that there is an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{I}}(C^i) = \{0\}$ for $i = q, q+1, q+2$ and a $\nu \in \mathbb{N}$ with $t^\nu T_{\mathfrak{I}}\left(H^{q+1}(t^n C^\cdot)\right) = \{0\}$. Then $\varprojlim_\lambda H^q(C^\cdot_\lambda)$ and the $\mathfrak{I}$-adic (separated) completion of $H^q(C^\cdot)$ are canonically isomorphic.*

If, furthermore, A is topologically finitely presented over R and $H^q(C^\cdot)$ is coherent over A then even $H^q(C^\cdot)$ itself and $\varprojlim_\lambda H^q(C^\cdot_\lambda)$ are canonically isomorphic.

Proof. For $\lambda, \mu \in \mathbb{N}$ with $\mu \geqq \lambda$ the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t^{\mu+1}C^\cdot & \xrightarrow{\iota_\mu} & C^\cdot & \xrightarrow{\sigma_\mu} & C^\cdot_\mu & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & t^{\lambda+1}C^\cdot & \xrightarrow{\iota_\lambda} & C^\cdot & \xrightarrow{\sigma_\lambda} & C^\cdot_\lambda & \longrightarrow & 0
\end{array}$$

induces the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
H^q(t^{\mu+1}C^\cdot) & \xrightarrow{\iota^q_\mu} & H^q(C^\cdot) & \xrightarrow{\sigma^q_\mu} & H^q(C^\cdot_\mu) & \xrightarrow{\delta^q_\mu} & H^{q+1}(t^{\mu+1}C^\cdot) & \xrightarrow{\iota^{q+1}_\mu} & H^{q+1}(C^\cdot) \\
\downarrow \iota^q_{\mu,\lambda} & & \| & & \downarrow & & \downarrow \iota^{q+1}_{\mu,\lambda} & & \| \\
H^q(t^{\lambda+1}C^\cdot) & \xrightarrow{\iota^q_\lambda} & H^q(C^\cdot) & \xrightarrow{\sigma^q_\lambda} & H^q(C^\cdot_\lambda) & \xrightarrow{\delta^q_\lambda} & H^{q+1}(t^{\lambda+1}C^\cdot) & \xrightarrow{\iota^{q+1}_\lambda} & H^{q+1}(C^\cdot)
\end{array}.$$

From this one gets the following exact sequence of projective systems

$$(*) \qquad 0 \to H^q(C^\cdot)/\iota^q_\lambda\big(H^q(t^{\lambda+1}C^\cdot)\big) \to H^q(C^\cdot_\lambda) \to \operatorname{im} \delta^q_\lambda \to 0\,.$$

In order to show that $\varprojlim_\lambda \operatorname{im} \delta^q_\lambda = \{0\}$ holds, let $\lambda \in \mathbb{N}$ with $\lambda \geqq n$ be arbitrary. As for $i \in \{q, q+1, q+2\}$ all $\mathfrak{I}$-torsion elements of C^i are annihilated by t^n, one has by Remark 1.2 that $t^n C^i$ is without $\mathfrak{I}$-torsion, hence isomorphic over A to $t^{\lambda+1}C^i$. Therefore also $H^{q+1}(t^n C^\cdot)$ and $H^{q+1}(t^{\lambda+1}C^\cdot)$ are isomorphic over A, by which t^ν annihilates each $\mathfrak{I}$-torsion element of $H^{q+1}(t^{\lambda+1}C^\cdot)$, too.

For $\mu := \lambda + \nu$ let $x \in \operatorname{im} \delta^q_\mu$ be arbitrary. As $t^{\mu+1}H^q(C^\cdot_\mu) = \{0\}$ holds, then x is an $\mathfrak{I}$-torsion element of $H^{q+1}(t^{\mu+1}C^\cdot)$, hence also the image $\iota^{q+1}_{\mu,\lambda}(x)$ of x with respect to the canonical homomorphism $\iota^{q+1}_{\mu,\lambda} : H^{q+1}(t^{\mu+1}C^\cdot) \to H^{q+1}(t^{\lambda+1}C^\cdot)$ an element of the $\mathfrak{I}$-torsion submodule $T := T_{\mathfrak{I}}\left(H^{q+1}(t^{\lambda+1}C^\cdot)\right)$ of $H^{q+1}(t^{\lambda+1}C^\cdot)$. On the other hand, by the above, $t^{\lambda+1}C^{q+2}$ is without $\mathfrak{I}$-torsion, which implies by Lemma 4.1a), applied to the complex $t^{\lambda+1}C^\cdot$

at the place $q+1$, that $\iota_{\mu,\lambda}^{q+1}(x) \in t^\nu H^{q+1}(t^{\lambda+1}C^\cdot)$ since $\mu = \lambda+\nu$. By Remark 1.2 one hence concludes $\iota_{\mu,\lambda}^{q+1}(x) \in T \cap t^\nu H^{q+1}(t^{\lambda+1}C^\cdot) = t^\nu T = \{0\}$. As $x \in \operatorname{im}\delta_\mu^q$ was arbitrary, one therefore has $\iota_{\mu,\lambda}^{q+1}(\operatorname{im}\delta_\mu^q) = \{0\} \subset \operatorname{im}\delta_\lambda^q$, hence, since also $\lambda \in \mathbb{N}$ with $\lambda \geqq n$ was arbitrary, $\varprojlim_\lambda \operatorname{im}\delta_\lambda^q = \{0\}$.

So the short exact sequence $(*)$ induces an exact sequence (e.g., by [EGA $0_{\mathrm{I\,new}}$, 7.2.8])

$$0 \to \varprojlim_\lambda \left(H^q(C^\cdot)/\,\iota_\lambda^q(H^q(t^{\lambda+1}C^\cdot))\right) \to \varprojlim_\lambda H^q(C_\lambda^\cdot) \to \varprojlim_\lambda \operatorname{im}\delta_\lambda^q = 0\,.$$

As the systems $\left(\iota_\lambda^q(H^q(t^{\lambda+1}C^\cdot))\right)_\lambda$ and $\left(t^{\lambda+1}H^q(C^\cdot)\right)_\lambda$ define the same topology on $H^q(C^\cdot)$ by Lemma 4.1a), then $\varprojlim_\lambda H^q(C_\lambda^\cdot)$ is always isomorphic to the $\mathfrak{I}$-adic completion of $H^q(C^\cdot)$, and the addendum follows from Lemma 1.9. □

Note. If one assumes that $C^\cdot$ consists of $\mathfrak{I}$-adically separated and complete A-modules then one can deduce the addendum even without a condition on $H^{q+1}(C^\cdot)$ as follows:

Now $C^\cdot \cong \varprojlim_\lambda C_\lambda^\cdot$ holds. As the system $(C_\lambda^\cdot)_\lambda$ is surjective, then [EGA 0_{III}, 13.2.3] implies that the canonical homomorphism $H^q(C^\cdot) \cong H^q\left(\varprojlim_\lambda C_\lambda^\cdot\right) \to \varprojlim_\lambda H^q(C_\lambda^\cdot)$ is surjective. The injectivity can be seen by the same arguments as above.

5 The direct image theorem for formal schemes

As in Sect. 2 let S denote a formal scheme which is noetherian and without torsion with respect to an ideal of definition or admissible over a valuation ring (for a height 1 valuation). For S noetherian each formal S-scheme locally of topologically finite presentation is locally noetherian (Lemma 1.5) so that the direct image theorem for formal S-schemes locally of topologically finite presentation holds by [EGA III, 3.4.2] in this case. Hence it only remains to be proved if S is admissible over an – in particular, non-discrete – valuation ring R.

Let R and $\mathfrak{I}$ be as above and set $R_\lambda := R/\mathfrak{I}^{\lambda+1}$ for $\lambda \in \mathbb{N}$. Theorem A for coherent modules over formal R-schemes locally of topologically finite presentation is already documented in the literature (see Proposition 2.3). Furthermore, one has

Proposition 5.1 (Theorem B for formal schemes) *Let $\mathfrak{X}$ be an affine formal R-scheme of topologically finite presentation and $\mathcal{M}$ a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Then $H^q(\mathfrak{X}, \mathcal{M}) = 0$ holds for each $q > 0$.*

At first, we will give a proof for this proposition by means of Consequence 4.2 that, however, needs the additional assumption that $\mathfrak{I}$ is generated by *one* element $t \in R$ – which, in fact, is the case that interests in the sequel –. For sake of completeness we will then sketch a proof for the general case which draws on [EGA 0_{III}, 13.3].

First proof. So, assume additionally that $\mathfrak{I} = tR$.

Because of Cartan's comparison theorem one only has to show that for each affine open formal R-subscheme $\mathfrak{Z}$ of $\mathfrak{X}$ and each covering $\mathfrak{U}$ of $\mathfrak{Z}$ by affine open formal R-subschemes the cohomology $H^q(\mathfrak{U}, \mathcal{M})$ of the Čech complex $C^\bullet := C^\bullet(\mathfrak{U}, \mathcal{M})$ of cochains on $\mathfrak{U}$ with values in $\mathcal{M}$ vanishes for each $q > 0$.

By Remark 2.6 the complex $C^\bullet$ consists of $\mathfrak{I}$-adically separated and complete R-modules and there is an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{I}}(C^q) = \{0\}$ for all $q \in \mathbb{Z}$. Then $\mathfrak{U}_n := \{U \times_R R_n; U \in \mathfrak{U}\}$ is an open affine covering of the affine R_n-scheme $\mathfrak{Z}_n = \mathfrak{Z} \times_R R_n$ and $\mathcal{M}_n := \mathcal{M}/t^{n+1}\mathcal{M} \cong \mathcal{M} \otimes_R R_n$ is a coherent $\mathcal{O}_{\mathfrak{Z}_n}$-module. By Remark 2.6 one has $C^\bullet(\mathfrak{U}_n, \mathcal{M}_n) = C^\bullet(\mathfrak{U}, \mathcal{M}_n) \cong C^\bullet \otimes_R R_n$, hence $H^q(C^\bullet \otimes_R R_n) \cong H^q(\mathfrak{U}_n, \mathcal{M}_n) = 0$ for $q > 0$ arbitrary, e.g., by [EGA III, 1.3.1]. From Consequence 4.2 one therefore gets $H^q(\mathfrak{U}, \mathcal{M}) = H^q(C^\bullet) = 0$ for each $q > 0$. □

Second proof. Let again $\mathfrak{Z}$ be an arbitrary affine open formal R-subscheme of $\mathfrak{X}$ and set $\mathcal{M}_\lambda := \mathcal{M}/\mathfrak{I}^{\lambda+1}\mathcal{M}$ for $\lambda \in \mathbb{N}$ arbitrary. Then $H^q(\mathfrak{Z}, \mathcal{M}_\lambda) = H^q(\mathfrak{Z}_\lambda, \mathcal{M}_\lambda) = 0$ holds for all $q > 0$. Furthermore, for $\mu \in \mathbb{N}$ with $\mu \leqq \lambda$ arbitrary the canonical map $\mathcal{M}_\lambda \to \mathcal{M}_\mu$ is a surjective morphism of coherent $\mathcal{O}_{\mathfrak{Z}}$-modules so that by the exactness of the functor "taking sections on $\mathfrak{Z}$" the projective system $\left(H^q(\mathfrak{Z}, \mathcal{M}_\lambda)\right)_\lambda$ fulfills the Mittag-Leffler condition of [EGA 0_{III}, 13.1.2] for each $q \geqq 0$. As by [EGA $\mathrm{I}_{\mathrm{new}}$, 10.11.3] (for the noetherian case) resp. [Me, 2.2.8(b)] (for the case of a valuation ring) one has $\mathcal{M} \cong \varprojlim_\lambda \mathcal{M}_\lambda$, then one gets the claim from [EGA 0_{III}, 13.3.1 and 13.3.2(ii)]. □

By Proposition 5.1 and Leray's theorem one can, as usual, compute cohomology by means of affine coverings:

Lemma 5.2 *Let $\mathfrak{X}$ be a formal R-scheme locally of topologically finite presentation and $\mathfrak{U}$ a covering of $\mathfrak{X}$ by affine open formal R-subschemes such that each finite intersection of sets from $\mathfrak{U}$ is formal affine again (e.g., assume that $\mathfrak{X}_0$ is separated). Let $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathfrak{X}}$-module.*

Then the canonical morphism $H^q(\mathfrak{U}, \mathcal{M}) \to H^q(\mathfrak{X}, \mathcal{M})$ is bijective for each $q \in \mathbb{Z}$.

A morphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ of formal S-schemes which are locally of topologically finite presentation is called *proper* if the morphism $\mathfrak{f}_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ of (ordinary) schemes induced by base change with S_0 over S is proper in the sense of algebraic geometry. – Note that in this situation $\mathfrak{X}_0$ is of finite type over S_0, hence also over $\mathfrak{Y}_0$ by which $\mathfrak{f}$ always is of finite type (cf. [EGA $\mathrm{I}_{\mathrm{new}}$, 10.13.3]). So the above definition is consistent with the one in [EGA III, 3.4.1.] –

By Lemma 2.1b) the morphism of formal schemes $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is proper if and only if for each $\lambda \in \mathbb{N}$ the morphism of (ordinary) schemes $\mathfrak{f}_\lambda : \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda$ is proper. In particular, the notion introduced above does not depend on the choice of the ideal of definition of S.

Direct image theorem for schemes 5.3 *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism of formal S-schemes which are locally of topologically finite presentation and $\mathscr{M}$ a coherent $\mathscr{O}_\mathfrak{X}$-module. Then $R^q\mathfrak{f}_*\mathscr{M}$ is a coherent $\mathscr{O}_\mathfrak{Y}$-module for each $q \in \mathbb{Z}$.*

Proof. By the introductory remark to this section only the case that an ideal of definition of S is generated by *one* element is left to be considered. As all properties in the statement of the theorem are local on $\mathfrak{Y}$, one can assume without restriction then that $S = \operatorname{Spf} R$ holds with a ring R as above, whose ideal of definition $\mathfrak{I}$ is generated by one element $t \in R$, and that $\mathfrak{Y} = \operatorname{Spf} A$ holds for a topologically finitely presented R-algebra A. For $g \in A$ arbitrary set $\mathfrak{X}^g := \mathfrak{f}^{-1}(\operatorname{Spf} A\langle g^{-1}\rangle)$.

By Proposition 2.3 and Lemma 2.4 it suffices for the proof of the direct image theorem to show that for each coherent $\mathscr{O}_\mathfrak{X}$-module $\mathscr{M}$ and each $q \in \mathbb{Z}$ one has

1) The A-module $H^q(\mathfrak{X}, \mathscr{M})$ is coherent.

2) For all $g \in A$ the canonical map $H^q(\mathfrak{X}, \mathscr{M}) \to H^q(\mathfrak{X}^g, \mathscr{M})$ induces an isomorphism $H^q(\mathfrak{X}, \mathscr{M}) \otimes_A A\langle g^{-1}\rangle \cong H^q(\mathfrak{X}^g, \mathscr{M})$.

As $\mathfrak{f}_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ is quasi-compact, there is a covering $\mathfrak{U}$ of $\mathfrak{X}$ that consists of finitely many, say $q_0 + 1$, affine open formal R-subschemes of $\mathfrak{X}$. Then for $g \in A$ arbitrary $\{U \cap \mathfrak{X}^g; U \in \mathfrak{U}\}$ is an open covering of $\mathfrak{X}^g$ by at most q_{0+1} affine formal R-subschemes. By Lemma 5.2 one hence has $H^q(\mathfrak{X}^g, \mathscr{M}) = 0$ for all $q > q_0$.

So 1) and 2) are trivially true for all $q > q_0$ and one can use a descending induction on q for their proof: Let $q \in \mathbb{Z}$ be fixed, $(0 \leqq)\ q \leqq q_0$, with the property that the above claims are true for $q + 1$ instead of q.

1) For the finite covering $\mathfrak{U}$ of $\mathfrak{X}$ chosen above set $C^\cdot := C^\cdot(\mathfrak{U}, \mathscr{M})$. Then, by Remark 2.6, the complex $C^\cdot$ consists of $\mathfrak{I}$-adically separated and complete A-modules and there is an $n \in \mathbb{N}$ with $t^n T_\mathfrak{I}(C^{q+1}) = \{0\}$ since $\mathfrak{X}$ is quasi-compact. Furthermore, by this remark one has $C^\cdot(\mathfrak{U}, t^{n+1}\mathscr{M}) = t^{n+1}C^\cdot$ and $C^\cdot(\mathfrak{U}, \mathscr{M}_\lambda) \cong C^\cdot \otimes_R R_\lambda =: C^\cdot_\lambda$ for each $\lambda \in \mathbb{N}$, where $\mathscr{M}_\lambda := \mathscr{M}/t^{\lambda+1}\mathscr{M} \cong \mathscr{M} \otimes_R R_\lambda$.

Together with $\mathscr{M}$ also $t^{n+1}\mathscr{M}$ is a coherent $\mathscr{O}_\mathfrak{X}$-module so that – by induction hypothesis – the A-module $H^{q+1}(\mathfrak{X}, t^{n+1}\mathscr{M}) \cong H^{q+1}(\mathfrak{U}, t^{n+1}\mathscr{M}) = H^{q+1}(t^{n+1}C^\cdot)$ is coherent.

Now $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is a proper morphism of formal R-schemes which are locally of topologically finite presentation so that for each $\lambda \in \mathbb{N}$ the proper morphism $\mathfrak{f}_\lambda : \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda$ is locally of finite presentation by [EGA $\mathrm{I}_{\mathrm{new}}$, 6.2.6 (v)]. As $A_\lambda := A \otimes_R R_\lambda$ is a finitely presented R_λ-algebra, hence stably coherent by Example 3.3, one can apply the direct image theorem for schemes 3.5 to $\mathfrak{f}_\lambda : \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda = \operatorname{Spec} A_\lambda$ and $\mathscr{M}_\lambda$ and finds that

$R^q(\mathfrak{f}_\lambda)_*(\mathcal{M}_\lambda)$ is a coherent $\mathcal{O}_{\mathfrak{Y}_\lambda}$-module. Then by [EGA III, 1.4.11] the A_λ-module $H^q(\mathfrak{X}_\lambda, \mathcal{M}_\lambda) \cong H^q(\mathfrak{U}, \mathcal{M}_\lambda) \cong H^q(C_\lambda^\bullet)$ is coherent. Hence $H^q(C_\lambda^\bullet)$ is also coherent over A by [EGA $0_{\mathrm{I\,new}}$, 5.3.13].

So the hypotheses of Proposition 4.3 are fulfilled for $C^\bullet = C^\bullet(\mathfrak{U}, \mathcal{M})$ and the coherence of $H^q(C^\bullet) = H^q(\mathfrak{U}, \mathcal{M}) \cong H^q(\mathfrak{X}, \mathcal{M})$ over A follows.

2) Let $g \in A$ be arbitrary, however fixed in the sequel. Again, consider $\mathfrak{U}^g := \{U \cap \mathfrak{X}^g; U \in \mathfrak{U}\}$ and $C^\bullet := C^\bullet(\mathfrak{U}, \mathcal{M})$. Furthermore, set $\tilde{C}^\bullet := C^\bullet(\mathfrak{U}^g, \mathcal{M})$.

Let $\Psi : C^\bullet \to \tilde{C}^\bullet$ denote the homomorphism of complexes of A-modules that is given by restricting cochains on $\mathfrak{U}$ to cochains on $\mathfrak{U}^g$. Using the notations introduced before Proposition 4.4 one has for $\lambda \in \mathbb{N}$ arbitrary that φ_λ^{q+1} : $H^{q+1}(t^{\lambda+1}C^\bullet) \otimes_A A\langle g^{-1}\rangle \to H^{q+1}(t^{\lambda+1}\tilde{C}^\bullet)$ is induced by the canonical map

$$H^{q+1}(t^{\lambda+1}C^\bullet) \cong H^{q+1}(\mathfrak{X}, t^{\lambda+1}\mathcal{M}) \to H^{q+1}(\mathfrak{X}^g, t^{\lambda+1}\mathcal{M}) \cong H^{q+1}(t^{\lambda+1}\tilde{C}^\bullet)$$

hence an $A\langle g^{-1}\rangle$-module isomorphism by the induction hypothesis, applied to the coherent $\mathcal{O}_{\mathfrak{X}}$-module $t^{\lambda+1}\mathcal{M}$. In an analogous way one gets from [EGA III, 1.4.11 and 1.4.13] that the canonical map $H^q(C_\lambda^\bullet) \cong H^q(\mathfrak{X}_\lambda, \mathcal{M}_\lambda) \to H^q(\mathfrak{X}_\lambda^g, \mathcal{M}_\lambda) \cong H^q(\tilde{C}_\lambda^\bullet)$ induces an $A\langle g^{-1}\rangle$-module isomorphism

$$\Phi_\lambda^q : H^q(C_\lambda^\bullet) \otimes_A A\langle g^{-1}\rangle \cong H^q(C_\lambda^\bullet) \otimes_A \left((A \otimes_R R_\lambda)[g^{-1}]\right) \to H^q(\tilde{C}_\lambda^\bullet) .$$

By Remark 2.6 and as 1) has already been proven for q, the assumptions of Proposition 4.4 are fulfilled and one now concludes from that proposition that the map

$$\begin{aligned} \varphi^q : H^q(\mathfrak{X}, \mathcal{M}) \otimes_A A\langle g^{-1}\rangle &\cong H^q(\mathfrak{U}, \mathcal{M}) \otimes_A A\langle g^{-1}\rangle \\ &\to H^q(\mathfrak{U}^g, \mathcal{M}) \cong H^q(\mathfrak{X}^g, \mathcal{M}) \end{aligned}$$

induced by $H^q(\mathfrak{X}, \mathcal{M}) \to H^q(\mathfrak{X}^g, \mathcal{M})$ is an $A\langle g^{-1}\rangle$-module isomorphism. □

Note. The proof of the analogue of 2) in the algebraic situation needs less expenses since there one only has to remark that for $g \in A$ tensorizing with $A[g^{-1}]$ over A commutes both with the formation of the complex of sections and with the formation of the cohomology because of the flatness of $A[g^{-1}]$ over A.

Indeed, in the above, formal situation $A\langle g^{-1}\rangle$ is a flat A-module, too. The formation of the complex of sections does, however, not commute with the usual but only with the $\mathfrak{I}$-*adically completed* tensor product with $A\langle g^{-1}\rangle$ which makes a limit argument as in the proof of Proposition 4.4 necessary.

In the above proof it was, in particular, shown for the situation over a valuation ring – for the noetherian situation see [EGA III, 3.4.6] –:

Lemma 5.4 *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y} = \mathrm{Spf}\, A$ be a proper morphism of formal R-schemes where $\mathfrak{X}$ is (locally) of topologically finite presentation over R and A a topologically finitely presented R-algebra, and $\mathcal{M}$ be a coherent*

$\mathcal{O}_{\mathfrak{X}}$-module. Then for each $g \in A$ and each $q \in \mathbb{Z}$ one has a canonical isomorphism

$$H^q\left(\mathfrak{f}^{-1}(\operatorname{Spf} A\langle g^{-1}\rangle), \mathcal{M}\right) \cong \Gamma\left(\operatorname{Spf} A\langle g^{-1}\rangle, R^q\mathfrak{f}_*\mathcal{M}\right) .$$

In particular, $H^q\left(\mathfrak{f}^{-1}(\operatorname{Spf} A\langle g^{-1}\rangle), \mathcal{M}\right)$ is a coherent $A\langle g^{-1}\rangle$-module.

The proof of the direct image theorem for noetherian formal schemes in [EGA III, 3.4.4] uses that cohomology and projective limit commute. The approach to the proof in the present article does not need this result, quite the contrary, it follows from the results and methods which are now at hand:

Proposition 5.5 *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism of formal S-schemes which are locally of topologically finite presentation and $\mathcal{M}$ a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Let $\mathfrak{J}$ be an ideal of definition of S and $\mathcal{M}_\lambda := \mathcal{M}/\mathfrak{J}^{\lambda+1}\mathcal{M}$ for $\lambda \in \mathbb{N}$. Then for each $q \in \mathbb{Z}$ there is a canonical isomorphism*

$$R^q\mathfrak{f}_*\mathcal{M} \xrightarrow{\sim} \varprojlim_{\lambda} R^q\mathfrak{f}_*(\mathcal{M}_\lambda) .$$

Proof. By [EGA III, 3.4.3] only the case remains to be considered that $\mathfrak{J}$ is generated by one element. As in the proof of the direct image theorem for formal schemes 5.3 one can restrict then to the case that $S = \operatorname{Spf} R$ holds with a ring R as above, whose ideal of definition $\mathfrak{J}$ is generated by one element $t \in R$, and that $\mathfrak{Y} = \operatorname{Spf} A$ for a topologically finitely presented R-algebra A.

Let $q \in \mathbb{Z}$ be arbitrary, however fixed. By Lemma 5.4 it suffices to show that for each $g \in A$ the canonical morphism $H^q(\mathfrak{X}^g, \mathcal{M}) \to \varprojlim_{\lambda} H^q(\mathfrak{X}^g, \mathcal{M}_\lambda)$ is an isomorphism where again $\mathfrak{X}^g := \mathfrak{f}^{-1}(\operatorname{Spf} A\langle g^{-1}\rangle)$. For this consider a covering $\mathfrak{U}^g$ of $\mathfrak{X}^g$ by affine open formal R-subschemes and set $C^\cdot := C^\cdot(\mathfrak{U}^g, \mathcal{M})$.

By Remark 2.6 there is an $n \in \mathbb{N}$ with $t^n T_{\mathfrak{J}}(C^i) = \{0\}$ for $i = q$, $q+1, q+2$ and one has $C^\cdot(\mathfrak{U}^g, \mathcal{M}_\lambda) \cong C^\cdot \otimes_R R_\lambda =: C^\cdot_\lambda$ for $\lambda \in \mathbb{N}$ arbitrary and $C^\cdot\,(\mathfrak{U}^g, t^n\mathcal{M}) = t^n C^\cdot$. As together with $\mathcal{M}$ also $t^n\mathcal{M}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module, then $H^{q+1}(t^nC^\cdot)$ and $H^q(C^\cdot)$ are coherent $A\langle g^{-1}\rangle$-modules by Lemmata 5.2 and 5.4. In particular, by Corollary 1.7 there is a $\nu \in \mathbb{N}$ with $t^\nu T_{\mathfrak{J}}\left(H^{q+1}(t^nC^\cdot)\right) = \{0\}$. Hence the hypotheses of Proposition 4.5 are fulfilled, with the topologically finitely presented R-algebra $A\langle g^{-1}\rangle$ instead of A, and that proposition gives the claim. □

By a technique well-known for the case of noetherian (usual) schemes (cf. [EGA III, 6.10.5] or [Mu, Chap. II, par. 5]) the direct image theorem for formal schemes 5.3 implies

Proposition 5.6 *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y} = \operatorname{Spf} A$ be a proper morphism of formal R-schemes where $\mathfrak{X}$ is (locally) of topologically finite presentation over R and A a topologically finitely presented R-algebra, and $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Assume that A and $\mathcal{M}$ are flat over R.*

Then there is a finite complex $L^\cdot$ of coherent and R-flat $\mathcal{O}_{\mathfrak{Y}}$-modules which is quasi-isomorphic to $\mathcal{R}^\cdot\,\mathfrak{f}_*\mathcal{M}$. Furthermore, for each $\lambda \in \mathbb{N}$ and each $q \in \mathbb{Z}$ this quasi-isomorphism induces an isomorphism

$$H^q(L^\cdot \otimes_R R_\lambda) \xrightarrow{\sim} R^q\mathfrak{f}_*(\mathcal{M}_\lambda)\,.$$

Proof. As before let $\mathfrak{U}$ be a covering of $\mathfrak{X}$ by affine open formal subschemes and let $C^\cdot := C^\cdot(\mathfrak{U}, \mathcal{M})$ denote the Čech complex of cochains on $\mathfrak{U}$ with values in $\mathcal{M}$. Then $H^\cdot(C^\cdot) = H^\cdot(\mathfrak{X}, \mathcal{M})$ is a finite complex of coherent A-modules by Lemma 5.4. Now apply [EGA 0_{III}, 11.9.1] with $\mathbf{C}$ the category of all A-modules, $\mathbf{K}'$ the set of all coherent A-modules, and $\mathbf{K}''$ the set of all free A-modules of finite rank: One gets that there is a complex $(D^\cdot, \partial^\cdot)$ of free A-modules of finite rank which is bounded from above and quasi-isomorphic to $C^\cdot$. If one substitutes D^0 by $H^0(C^\cdot) \oplus \operatorname{im} \partial^0$ and ∂^0 by the projection of this direct sum onto $\operatorname{im} \partial^0$ (followed by the inclusion of $\operatorname{im} \partial^0$ in D^1) and defines the modules and homomorphisms with negative index to be zero one gets a finite complex $K^\cdot$ of coherent A-modules from $D^\cdot$ which obviously is quasi-isomorphic to $C^\cdot$, too. As A is flat over R, all modules $K^q = D^q$ with $q > 0$ trivially are flat over R. The R-flatness of K^0 follows verbatim as in the last part of the proof of Lemma 1 in [Mu, Chap. II, par. 5] from the fact that the $C^q = C^q(\mathfrak{U}, \mathcal{M})$ are flat R-modules. (For R a valuation ring one can verify this immediately from the definition of K^0 since in this case a module is flat over R if and only if it has no $\mathfrak{I}$-torsion.)

Now define $L^\cdot$ to be the complex of $\mathcal{O}_\mathfrak{Y}$-modules associated to $K^\cdot$, i.e., set $L^q := (K^q)^\Delta$. Then $L^\cdot$ is a finite complex of coherent and R-flat $\mathcal{O}_\mathfrak{Y}$-modules and by the exactness of the functor "Δ" and Lemma 5.4 one has

$$H^q(L^\cdot) \cong \left(H^q(K^\cdot)\right)^\Delta \cong \left(H^q(\mathfrak{X}, \mathcal{M})\right)^\Delta \cong R^q\mathfrak{f}_*\mathcal{M}$$

for each $q \in \mathbb{Z}$, hence the first claim. As $C^\cdot$ and $K^\cdot$ consist of flat R-modules this implies by [Mu, Chap. II, par. 5, Lemma 2] that for each R-algebra B the quasi-isomorphism induces an isomorphism $H^q(L^\cdot \otimes_R B) \xrightarrow{\sim} H^q\big((\mathcal{R}^\cdot \mathfrak{f}_*\mathcal{M}) \otimes_R B\big)$ for each $q \in \mathbb{Z}$, so, in particular, the second claim. □

For the formulation of the proper mapping theorem we remind of the definition of a closed formal subscheme (cf. [EGA I_{new}, 10.14.2] for the noetherian case, [Me, 2.3.2] for the case of a valuation ring):

Let $\mathfrak{X}$ be a formal R-scheme locally of topologically finite presentation and $\mathcal{J}$ a coherent $\mathcal{O}_\mathfrak{X}$-ideal. Then the support $\mathfrak{Z} := \operatorname{Supp}(\mathcal{O}_\mathfrak{X}/\mathcal{J})$ of the quotient $\mathcal{O}_\mathfrak{X}/\mathcal{J}$ is a closed subspace of the topological space underlying $\mathfrak{X}$. So $\mathfrak{Z}$, endowed with the structure sheaf $(\mathcal{O}_\mathfrak{X}/\mathcal{J})|_\mathfrak{Z}$, becomes a formal R-scheme locally of topologically finite presentation, which is called the ***closed formal subscheme of*** $\mathfrak{X}$ ***defined by*** $\mathcal{J}$.

For example, for $\mathfrak{X}$ a formal R-scheme locally of topologically finite presentation and $\mathcal{M}$ a coherent $\mathcal{O}_\mathfrak{X}$-module let $\mathcal{A}nn_{\mathcal{O}_\mathfrak{X}}(\mathcal{M})$ denote the kernel of the canonical $\mathcal{O}_\mathfrak{X}$-module homomorphism $\mathcal{O}_\mathfrak{X} \to \mathcal{H}om_{\mathcal{O}_\mathfrak{X}}(\mathcal{M}, \mathcal{M})$ which assigns the homothety $m \mapsto am$ to each section a in $\mathcal{O}_\mathfrak{X}$. Then the support $\operatorname{Supp}(\mathcal{M})$ of $\mathcal{M}$ coincides with that of $\mathcal{O}_\mathfrak{X}/\mathcal{A}nn_{\mathcal{O}_\mathfrak{X}}(\mathcal{M})$. As $\mathcal{A}nn_{\mathcal{O}_\mathfrak{X}}(\mathcal{M})$ is a coherent $\mathcal{O}_\mathfrak{X}$-ideal by [EGA $0_{\text{I new}}$, 5.3.4 and 5.3.7] then in a canonical way $\operatorname{Supp}(\mathcal{M})$ bears the structure of a closed formal subscheme of $\mathfrak{X}$.

A morphism $\mathfrak{i} : \mathfrak{X} \to \mathfrak{Y}$ of formal R-schemes which are locally of topologically finite presentation is called a ***closed immersion*** if it factorizes into the form $\mathfrak{X} \xrightarrow{\mathfrak{g}} \mathfrak{Z} \xrightarrow{\mathfrak{j}} \mathfrak{Y}$ with a closed formal subscheme $\mathfrak{Z}$ of $\mathfrak{Y}$, an isomorphism of formal R-schemes $\mathfrak{g}$, and $\mathfrak{j}$ the canonical injection of $\mathfrak{Z}$ in $\mathfrak{Y}$ (cf. [EGA I_{new}, 10.14.2] resp. [Me, 2.3.2]).

By [EGA III, 4.8.10] (for R noetherian) resp. [Me, 2.3.4] (for R a valuation ring) one has:

Lemma 5.7 *A morphism $\mathfrak{i} : \mathfrak{X} \to \mathfrak{Y}$ of formal R-schemes which are locally of topologically finite presentation is a closed immersion if and only if $\mathfrak{i}_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ is a closed immersion of R_0-schemes.*

Now the means are at hand in order to derive as usual the mapping theorem for proper morphisms from the direct image theorem.

Proper mapping theorem 5.8 *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism of formal R-schemes which are locally of topologically finite presentation. Then for each closed formal subscheme $\mathfrak{Z}$ of $\mathfrak{X}$ the image $\mathfrak{f}(\mathfrak{Z})$ of $\mathfrak{Z}$ with respect to $\mathfrak{f}$ (defined in the sense of morphisms of topological spaces) bears the structure of a closed formal subscheme of $\mathfrak{Y}$.*

Proof. By Lemma 5.7 and [EGA II, 5.4.2(i) and (ii)] the composition of a closed immersion with a proper morphism of formal R-schemes locally of topologically finite presentation is proper again. So one can assume $\mathfrak{Z} = \mathfrak{X}$ without restriction.

The morphisms $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ and $\mathfrak{f}_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ coincide as maps of the underlying topological spaces. As $\mathfrak{f}_0$ is a proper morphism of ordinary schemes, hence a closed map, the image $\mathfrak{f}_0(\mathfrak{X}_0)$ is closed in $\mathfrak{Y}_0$. Therefore as a topological space $\mathfrak{f}(\mathfrak{X})$ is closed in $\mathfrak{Y}$ and hence coincides with $\operatorname{Supp}(\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}})$. On the other hand, $\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}}$ is a coherent $\mathcal{O}_{\mathfrak{Y}}$-module by the direct image theorem for formal schemes 5.3 so that its support $\operatorname{Supp}(\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}})$ can be considered as a closed formal subscheme of $\mathfrak{Y}$. □

6 GAGA theorems

For a (usual) R-scheme X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ the ***completion $\hat{\mathcal{M}}$ of $\mathcal{M}$ along the closed subscheme $X_0 = V(\mathfrak{I}\mathcal{O}_X)$ of X defined by $\mathfrak{I}$*** is defined as the restriction of the sheaf

$$\varprojlim_{\lambda} \mathcal{M}_\lambda \text{ with } \mathcal{M}_\lambda := \mathcal{M} \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathfrak{I}^{\lambda+1}\mathcal{O}_X) \text{ for } \lambda \in \mathbb{N}$$

to (the topological space) X_0 (cf. [EGA I_{new}, 10.8.2]). As $\mathfrak{I}$ is finitely generated, the ringed space $(X_0, \widehat{\mathcal{O}_X})$ is a formal scheme by [EGA I_{new}, 10.8.3], which will be denoted by $\hat{X}$. More precisely, if $X = \operatorname{Spec} B$ is an affine R-scheme then $\hat{X} = \operatorname{Spf} \hat{B}$ holds with $\hat{B}$ the $\mathfrak{I}$-adic (separated) completion of B.

Similarly, each morphism $f : X \to Y$ of (usual) R-schemes induces a morphism $f_\lambda := f \times \mathrm{id}_{\mathrm{Spec}\, R_\lambda} : X_\lambda := X \times_R R_\lambda \to Y_\lambda := Y \times_R R_\lambda$ of R_λ-schemes, $R_\lambda = R/\mathfrak{I}^{\lambda+1}$, for each $\lambda \in \mathbb{N}$, hence (cf. [EGA $\mathrm{I}_{\mathrm{new}}$, 10.9.1]) a morphism $\hat{f} : \hat{X} \to \hat{Y}$ of formal R-schemes, which is called the ***continuation of f to the completion of X and Y along X_0 and Y_0***.

In the sequel we will study the situation of schemes of finite type over a topologically finitely presented R-algebra A. By Lemmata 1.4b) and 1.9 the $\mathfrak{I}$-adic completion of a finitely presented A-algebra B is topologically finitely presented over R and by Remark 1.1, Lemma 1.4a), c), and Proposition 1.6 it is even admissible if B is a finitely generated A-algebra without $\mathfrak{I}$-torsion.

Let $\mathcal{M}$ be a quasi-coherent module sheaf over $X = \mathrm{Spec}\, B$, say the $\mathcal{O}_X$-module associated to the B-module M and assume that $\hat{B}$ is topologically finitely presented over R. Then, if M finitely presented over B or finitely generated over B and without $\mathfrak{I}$-torsion the $\hat{B}$-module $\hat{M}$ is coherent (use the same results from Sect. 1 as above) and $\hat{\mathcal{M}}$ equals the $\mathcal{O}_{\hat{X}}$-module $\hat{M}^{\Delta}$ associated to $\hat{M}$ by Lemma 2.4.

Hence one has

Lemma 6.1 *Let A be a topologically finitely presented R-algebra and X an A-scheme locally of finite type whose structure sheaf has no $\mathfrak{I}$-torsion or locally of finite presentation.*

Then $\hat{X}$ is a formal R-scheme locally of topologically finite presentation and for each quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ which is finitely generated and without $\mathfrak{I}$-torsion or finitely presented the $\mathcal{O}_{\hat{X}}$-module $\hat{\mathcal{M}}$ is coherent.

By taking sections on affine open formal subschemes Lemma 1.4b) (together with Lemma 1.9) implies

Lemma 6.2 *Let X be an R-scheme such that $\hat{X}$ is locally of topologically finite presentation over R and $\mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ be an exact sequence of quasi-coherent $\mathcal{O}_X$-modules with $\hat{\mathcal{L}}$ and $\hat{\mathcal{M}}$ coherent over $\mathcal{O}_{\hat{X}}$.*

Then also the sequence $\hat{\mathcal{L}} \to \hat{\mathcal{M}} \to \hat{\mathcal{N}} \to 0$ induced by completion is exact.

Having this result at hand one can copy the proof of [EGA $\mathrm{I}_{\mathrm{new}}$, 10.8.8(ii)] from the noetherian situation: One gets for each R-scheme X with $\hat{X}$ locally of topologically finite presentation over R that the functorial homomorphism $i^*\mathcal{M} \to \hat{\mathcal{M}}$ induced by the canonical homomorphism $i : \hat{X} \to X$ of ringed spaces is an isomorphism of $\mathcal{O}_{\hat{X}}$-modules for each finitely presented $\mathcal{O}_X$-module $\mathcal{M}$.

As the formation of the inverse image of module sheaves commutes with the tensor product this implies that over schemes of this type the formation of the tensor product of finitely presented module sheaves commutes with the completion (cf. [EGA $\mathrm{I}_{\mathrm{new}}$, 10.8.10.1] for the noetherian case). Furthermore, similarly as in [EGA $\mathrm{I}_{\mathrm{new}}$, 10.9.5], over these schemes the inverse image of finitely presented module sheaves commutes with completion resp. continuation of the morphism.

The remainder of this section is devoted to the study of the behaviour of the completion with respect to the direct image for proper morphisms:

First of all, if $f : X \to Y$ is a proper morphism of (usual) R-schemes with $\hat{X}$ and $\hat{Y}$ locally of topologically finite presentation over R then, by [EGA II, 5.4.2(iii)], also $f_0 = (\hat{f})_0 : X \times_R R_0 = X_0 = (\hat{X})_0 \to Y \times_R R_0 = Y_0 = (\hat{Y})_0$ is proper so that $\hat{f} : \hat{X} \to \hat{Y}$ is proper by definition. Therefore, if $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module with $\hat{\mathcal{M}}$ coherent over $\mathcal{O}_{\hat{X}}$ one knows by Proposition 5.5 that there is a canonical isomorphism

$$R^q(\hat{f})_*(\hat{\mathcal{M}}) \xrightarrow{\sim} \varprojlim_{\lambda} R^q(\hat{f})_*\mathcal{M}_\lambda = \varprojlim_{\lambda} R^q f_*\mathcal{M}_\lambda .$$

Applying Proposition 4.5 analogously as in the proof of Proposition 5.5, but now in the algebraic setting, hence gives

Proposition 6.3 *Let $f : X \to Y$ be a proper morphism of R-schemes with $\hat{X}$ and $\hat{Y}$ locally of topologically finite presentation over R. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module such that there is an $n \in \mathbb{N}$ with $\mathfrak{I}^n\mathcal{T}_{\mathfrak{I}}(\mathcal{M}) = 0$ and such that $\hat{\mathcal{M}}$ is a coherent $\mathcal{O}_{\hat{X}}$-module.*

If $q \in \mathbb{Z}$ has the property that there is a $\nu \in \mathbb{N}$ with $\mathfrak{I}^\nu\mathcal{T}_{\mathfrak{I}}\left(R^{q+1}f_*(\mathfrak{I}^n\mathcal{M})\right) = 0$ then one has a canonical isomorphism of $\mathcal{O}_{\hat{Y}}$-modules

$$(R^q f_*\mathcal{M})^\wedge \xrightarrow{\sim} R^q(\hat{f})_*(\hat{\mathcal{M}}) .$$

Now consider the case that in the above proposition $Y = \operatorname{Spec} A$ holds with a topologically finitely presented R-algebra A.

If R is noetherian then X and Y are noetherian schemes and the direct image theorem for noetherian schemes [EGA III, 3.2.1] gives that the $\mathcal{O}_Y$-module $R^{q+1}f_*(\mathfrak{I}^n\mathcal{M})$ is finitely generated for each coherent $\mathcal{O}_X$-module $\mathcal{M}$. Hence the $\mathfrak{I}$-torsion submodule both of $\mathcal{M}$ and of $R^{q+1}f_*(\mathfrak{I}^n\mathcal{M})$ is finitely generated so that the assumptions of Proposition 6.3 are fulfilled for each coherent $\mathcal{O}_X$-module $\mathcal{M}$ (cf. [EGA III, 4.1.5]).

If, on the other hand, R is not (necessarily) noetherian one really has to take the conditions on the $\mathfrak{I}$-torsion submodules into consideration: The one on $\mathcal{T}_{\mathfrak{I}}(\mathcal{M})$ is not restrictive for the applications to rigid analysis since there one can even assume that $\mathcal{M}$ has no $\mathfrak{I}$-torsion at all (cf. Sect. 7). Also the one on $R^{q+1}f_*(\mathfrak{I}^n\mathcal{M})$ causes no difficulties *supposed one knows that this sheaf is finitely generated over* $\mathcal{O}_Y$ since then Corollary 1.7 furnishes a $\nu \in \mathbb{N}$ with $\mathfrak{I}^\nu T_{\mathfrak{I}}\left(H^{q+1}(X, \mathfrak{I}^n\mathcal{M})\right) = 0$, hence $\mathfrak{I}^\nu\mathcal{T}_{\mathfrak{I}}\left(R^{q+1}f_*(\mathfrak{I}^n\mathcal{M})\right) = 0$ (cf. the proof of Proposition 2.5).

It is, however, not known – at least to the author of the present article – whether for $f : X \to Y = \operatorname{Spec} A$ a proper morphism and A a topologically finitely presented R-algebra the higher direct images of coherent $\mathcal{O}_X$-modules are always finitely generated over $\mathcal{O}_Y$. (By the direct image theorem for schemes 3.5 it would suffice to know that A is stably coherent. But the best result in the literature towards this direction seems to be [RG, théorème 3.4.6] which implies that the absolute base ring R is stably coherent, cf. Example 3.3.)

Therefore we resort to the notion of relative pseudo-coherence introduced in Sect. 3 which guarantees the coherence of the higher direct image sheaves by the result of Kiehl (Theorem 3.1). So by taking sections Proposition 6.3 implies the

1st GAGA Theorem 6.4 *Let A be a topologically finitely presented R-algebra and $f : X \to Y := \operatorname{Spec} A$ a proper morphism of R-schemes with $\hat{X}$ (locally) of topologically finite presentation over R. Furthermore, let $\mathcal{M}$ be an $\mathcal{O}_X$-module which is relatively pseudo-coherent with respect to f, fulfills $\mathfrak{J}^n\mathcal{T}_{\mathfrak{J}}(\mathcal{M}) = 0$ for an $n \in \mathbb{N}$ and whose completion $\hat{\mathcal{M}}$ is coherent over $\mathcal{O}_{\hat{X}}$.*

Then for each $q \in \mathbb{Z}$ one has a canonical isomorphism of A-modules

$$H^q(X, \mathcal{M}) \xrightarrow{\sim} H^q(\hat{X}, \hat{\mathcal{M}}) .$$

Note. The condition that $\hat{\mathcal{M}}$ is coherent over $\mathcal{O}_{\hat{X}}$ can be deduced from the other ones.

As for a free $\mathcal{O}_X$-module $\mathcal{O}_X^r$ of rank $r \in \mathbb{N}$ the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^r, \mathcal{M})$ is isomorphic to $\mathcal{M}^r$ one has that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{M})$ is a finitely presented $\mathcal{O}_X$-module and relatively pseudo-coherent with respect to f if $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of finite rank and $\mathcal{M}$ is a finitely presented $\mathcal{O}_X$-module and relatively pseudo-coherent with respect to f. Hence the above 1st GAGA Theorem implies in the usual way

2nd GAGA Theorem 6.5 *Let A be a topologically finitely presented R-algebra and $f : X \to Y := \operatorname{Spec} A$ a proper morphism of R-schemes with $\hat{X}$ (locally) of topologically finite presentation over R. Furthermore, let $\mathcal{M}$ be a finitely presented $\mathcal{O}_X$-module which is relatively pseudo-coherent with respect to f and fulfills $\mathfrak{J}^n\mathcal{T}_{\mathfrak{J}}(\mathcal{M}) = 0$ for an $n \in \mathbb{N}$, and $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank.*

Then one has a canonical isomorphism of A-modules

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{M}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_{\hat{X}}}(\hat{\mathcal{F}}, \hat{\mathcal{M}}) .$$

Though the above GAGA results are somewhat unsatisfactory, in particular because of the condition of relative pseudo-coherence, they suffice to deduce the 3rd GAGA Theorem 6.8 for projective morphisms ("Chow's theorem") without unpleasant technical conditions. To this aim, we first generalize [EGA III, 5.2.3] to the present situation:

Proposition 6.6 *Let A be a topologically finitely presented R-algebra and $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y} := \operatorname{Spf} A$ a proper morphism of formal R-schemes where $\mathfrak{X}$ is (locally) of topologically finite presentation over R.*

Furthermore, let $\mathcal{L}$ be an invertible $\mathcal{O}_{\mathfrak{X}}$-module such that $\mathcal{L}_0 = \mathcal{L}/\mathfrak{J}\mathcal{L} \cong \mathcal{L} \otimes_R R_0$ is an ample $\mathcal{O}_{\mathfrak{X}_0}$-module. For each $\mathcal{O}_{\mathfrak{X}}$-module and each $n \in \mathbb{Z}$ set $\mathcal{M}(n) := \mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}^{\otimes n}$.

Then for each coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ there is an $N \in \mathbb{Z}$ such that for all $n \geqq N$ the following holds:

(i) *One has $H^q(\mathfrak{X}, \mathcal{M}(n)) = 0$ for all $q > 0$.*

(ii) *The canonical homomorphism $H^0(\mathfrak{X}, \mathcal{M}(n)) \to H^0(\mathfrak{X}, \mathcal{M}_\lambda(n))$ is surjective for all $\lambda \in \mathbb{N}$.*

Proof. The noetherian case is treated in [EGA III, 5.2.3] so that one can restrict to the situation $\mathfrak{I} = tR$ for a $t \in R$.

By Proposition 2.5 one finds an $m \in \mathbb{N}$ with $t^m \mathcal{T}_{\mathfrak{I}}(\mathcal{M}) = 0$. As the $\mathcal{O}_{\mathfrak{X}_0}$-module $\mathcal{L}_0 \cong \mathcal{L}_m / t\mathcal{L}_m$ is ample, $\mathcal{L}_m$ is ample with respect to the proper morphism $\mathfrak{f}_m : \mathfrak{X}_m \to \mathfrak{Y}_m$ induced by $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ by [EGA II, 4.5.13 and 4.6.6]. So there is a $d > 0$ with $\mathcal{L}_m^{\otimes d}$ very ample with respect to $\mathfrak{f}_m$. By Corollary 1.8 the structure sheaf of $\mathfrak{X}_m$ is coherent, hence also the invertible $\mathcal{O}_{\mathfrak{X}_m}$-module $\mathcal{L}_m^{\otimes d}$. As $A \otimes_R R_m$ is stably coherent (Example 3.3) the direct image theorem for schemes 3.5 now implies that $(\mathfrak{f}_m)_*(\mathcal{L}_m^{\otimes d})$ is a coherent, in particular finitely generated $\mathcal{O}_{\mathfrak{Y}_m}$-module. Now [EGA II, 5.5.4] gives that $\mathfrak{f}_m : \mathfrak{X}_m \to \mathfrak{Y}_m$ is projective.

So Proposition 3.6 furnishes an $N' \in \mathbb{Z}$ with $R^q(\mathfrak{f}_m)_*(\mathcal{M}_m \otimes_{\mathcal{O}_{\mathfrak{X}_m}} \mathcal{L}_m^{\otimes n}) = 0$, hence $H^q(\mathfrak{X}, \mathcal{M}(n)_m) = 0$ for all $n \geqq N'$ and all $q > 0$. Similarly as in the first proof of Theorem B for formal schemes (Proposition 5.1) one can "lift" this statement by means of Consequence 4.2 and hence gets claim (i) (with N' in place of N).

In order to prove claim (ii), apply the fact just proved to the coherent $\mathcal{O}_{\mathfrak{X}}$-modules $t\mathcal{M}, t^2\mathcal{M}, \ldots, t^m\mathcal{M}$. For each $\lambda \in \mathbb{N}$ with $0 \leqq \lambda \leqq m-1$ this provides an $N_\lambda \in \mathbb{Z}$ with $H^1(\mathfrak{X}, (t^{\lambda+1}\mathcal{M})(n)) = 0$ for all $n \geqq N_\lambda$. Set $N'' := \max_{0 \leqq \lambda \leqq m-1} N_\lambda$. By the choice of m one has $t^m \mathcal{T}_{\mathfrak{I}}(\mathcal{M}) = 0$, which implies by Remark 1.2 that the $\mathcal{O}_{\mathfrak{X}}$-module $t^m\mathcal{M}$ is without $\mathfrak{I}$-torsion, hence isomorphic to $t^\lambda \mathcal{M}$ for each $\lambda \geqq m$. Therefore one has $H^1(\mathfrak{X}, (t^{\lambda+1}\mathcal{M})(n)) = 0$ for all $\lambda \in \mathbb{N}$ and all $n \geqq N''$. An application of the long exact cohomology sequence induced by $0 \to (t^{\lambda+1}\mathcal{M})(n) \to \mathcal{M}(n) \to \mathcal{M}_\lambda(n) \to 0$ now gives the surjectivity of $H^0(\mathfrak{X}, \mathcal{M}(n)) \to H^0(\mathfrak{X}, \mathcal{M}_\lambda(n))$ for all $\lambda \in \mathbb{N}$ and all $n \geqq N''$ so that the claims (i) and (ii) hold for $N := \max\{N', N''\}$. □

Corollary 6.7 *Under the assumptions of Proposition* 6.6 *for each coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ there is an $N \in \mathbb{Z}$ such that $\mathcal{M}(n)$ is generated by global sections on $\mathfrak{X}$ for all $n \geqq N$, i.e., $\mathcal{M}$ is an epimorphic image of an $\mathcal{O}_{\mathfrak{X}}$-module of the form $(\mathcal{O}_{\mathfrak{X}}(-n))^m$.*

Proof. One can argue as in the proof of [EGA III, 5.2.4] since for the application of [EGA II, 4.5.5] it is not necessary that $\mathfrak{X}_0$ – there: X_0 – is noetherian, but sufficient that $\mathfrak{X}_0$ is separated and quasi-compact or that the space underlying $\mathfrak{X}_0$ is noetherian which both is the case in the present situation. □

3rd GAGA Theorem 6.8 *Let A be a topologically finitely presented R-algebra and X a projective A-scheme such that the completion $\mathfrak{X} := \hat{X}$ of X is a formal R-scheme (locally) of topologically finite presentation.*

Then for each coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ there is a finitely presented $\mathcal{O}_X$-module $\mathcal{M}'$ whose completion $\widehat{\mathcal{M}'}$ is isomorphic to $\mathcal{M}$. In particular, for each closed formal subscheme $\mathfrak{Z}$ of $\mathfrak{X}$ there is a closed subscheme Z of X of finite presentation over A whose completion $\hat{Z}$ is isomorphic to $\mathfrak{Z}$.

Proof. Let $\imath : X \to \mathbb{P}^r_A =: P$ be a closed immersion of A-schemes for $r \in \mathbb{N}$ convenient. Then, by Lemma 5.7, also its continuation $\hat{\imath} : \mathfrak{X} \to \hat{P} =: \mathfrak{P}$ is a closed immersion, in particular, the kernel of the canonical $\mathcal{O}_{\mathfrak{P}}$-module homomorphism $\mathcal{O}_{\mathfrak{P}} \to (\hat{\imath})_* \mathcal{O}_{\mathfrak{X}}$ is a coherent, hence finitely generated $\mathcal{O}_{\mathfrak{P}}$-ideal.

Now let $\mathcal{M}$ be an arbitrary coherent $\mathcal{O}_{\mathfrak{X}}$-module. Then $(\hat{\imath})_* \mathcal{M}$ is coherent over $(\hat{\imath})_* \mathcal{O}_{\mathfrak{X}}$, hence by the above remark also over $\mathcal{O}_{\mathfrak{P}}$. Setting $\mathcal{L} := \mathcal{O}_P(1)$, the $\mathcal{O}_{\mathfrak{P}}$-module $\hat{\mathcal{L}}$ is invertible and the $\mathcal{O}_{\mathfrak{P}_0}$-module $(\hat{\mathcal{L}})_0 = \mathcal{L}_0 \cong \mathcal{O}_{\mathbb{P}^r_{A \otimes_R R_0}}(1)$ is ample. Furthermore, together with $f : P \to \operatorname{Spec} A$ also $\hat{f} : \mathfrak{P} \to \operatorname{Spf} A$ is proper. Hence two applications of Corollary 6.7 give an exact sequence

$$(\hat{\mathcal{L}}^{\otimes n'})^{m'} \cong \left((\mathcal{L}^{\otimes n'})^{m'}\right)^{\wedge} \xrightarrow{\hat{\beta}} (\hat{\mathcal{L}}^{\otimes n})^{m} \cong \left((\mathcal{L}^{\otimes n})^{m}\right)^{\wedge} \xrightarrow{\hat{\alpha}} (\hat{\imath})_* \mathcal{M} \longrightarrow 0$$

of $\mathcal{O}_{\mathfrak{P}}$-modules with $n, n' \in \mathbb{Z}$, $m, m' \in \mathbb{N}$.

As $P = \mathbb{P}^r_A$ is smooth over $\operatorname{Spec} A$ and, trivially, $\mathcal{O}_P$ a pseudo-coherent $\mathcal{O}_P$-module, $\mathcal{O}_P$ is relatively pseudo-coherent with respect to f, hence also the locally free $\mathcal{O}_P$-module $(\mathcal{L}^{\otimes n})^m$. Furthermore, by Proposition 2.5, there is a $\nu \in \mathbb{N}$ with $\mathfrak{I}^{\nu} \mathcal{T}_{\mathfrak{I}}\left((\mathcal{L}^{\otimes n})^m\right) = 0$. Hence the assumptions of the 2nd GAGA Theorem 6.5 are fulfilled and there exists an $\mathcal{O}_P$-module homomorphism $\beta : (\mathcal{L}^{\otimes n'})^{m'} \to (\mathcal{L}^{\otimes n})^m$ whose completion is $\hat{\beta}$.

As the inverse image of finitely presented module sheaves is compatible with the completion of modules, the completion of $\imath^* \beta : \imath^*\left((\mathcal{L}^{\otimes n'})^{m'}\right) \to \imath^*\left((\mathcal{L}^{\otimes n})^m\right)$ is

$$\begin{aligned}(\hat{\imath})^*(\hat{\beta}) : \left(\imath^*((\mathcal{L}^{\otimes n'})^{m'})\right)^{\wedge} &\cong (\hat{\imath})^*\left((\widehat{\mathcal{L}}^{\otimes n'})^{m'}\right) \\ &\to \left(\imath^*((\mathcal{L}^{\otimes n})^{m})\right)^{\wedge} \cong (\hat{\imath})^*\left((\widehat{\mathcal{L}}^{\otimes n})^{m}\right)\end{aligned}$$

where all the involved $\mathcal{O}_X$- and $\mathcal{O}_{\mathfrak{X}}$-modules are locally free of finite rank, in particular, $\left(\imath^*((\mathcal{L}^{\otimes n'})^{m'})\right)^{\wedge}$ and $\left(\imath^*((\mathcal{L}^{\otimes n})^{m})\right)^{\wedge}$ are coherent over $\mathcal{O}_{\mathfrak{X}}$.

Now $\mathcal{M}' := \operatorname{coker}(\imath^* \beta)$ is a finitely presented $\mathcal{O}_X$-module. By Lemma 6.2 the completion of $\operatorname{coker}(\imath^* \beta)$, i.e., $\widehat{\mathcal{M}'}$, and the cokernel of $(\hat{\imath})^*(\hat{\beta})$ are isomorphic as $\mathcal{O}_{\mathfrak{X}}$-modules. But because of the right-exactness of the functor $(\hat{\imath})^*$ the cokernel of $(\hat{\imath})^*(\hat{\beta})$ coincides with $(\hat{\imath})^*\left(\operatorname{coker}(\hat{\beta})\right) = (\hat{\imath})^*\left((\hat{\imath})_* \mathcal{M}\right)$, and, as $\hat{\imath}$ is a closed immersion, $(\hat{\imath})^*\left((\hat{\imath})_* \mathcal{M}\right)$ and $\mathcal{M}$ are canonically isomorphic (cf. [EGA I_{new}, 10.14.6] for R noetherian resp. [Me, 2.4.9 and 2.4.7] for R a valuation ring). □

By use of this theorem one can also transfer the algebraization criterion [EGA III, 5.4.5] to the non-noetherian situation:

Proposition 6.9 *Let A be a topologically finitely presented R-algebra and $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y} := \operatorname{Spf} A$ a proper morphism of formal R-schemes where $\mathfrak{X}$ is (locally) of topologically finite presentation over R.*

If there is an invertible $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{L}$ such that $\mathcal{L}_0 = \mathcal{L}/\mathfrak{I}\mathcal{L} \cong \mathcal{L} \otimes_R R_0$ is an ample $\mathcal{O}_{\mathfrak{X}_0}$-module then there exists a projective A-scheme X of finite presentation whose completion $\hat{X}$ is isomorphic to $\mathfrak{X}$.

Proof. From Proposition 6.6(ii) one gets an $N \in \mathbb{Z}$ such that the canonical homomorphism $H^0(\mathfrak{X}, \mathcal{L}^{\otimes n}) \to H^0(\mathfrak{X}, \mathcal{L}_0^{\otimes n}) = H^0(\mathfrak{X}_0, \mathcal{L}_0^{\otimes n})$ is surjective for all $n \geqq N$. By [EGA II, 4.5.10(ii)], one can choose an $n \geqq N$ such that $\mathcal{L}_0^{\otimes n}$ is very ample with respect to the proper morphism $\mathfrak{f}_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0 = \operatorname{Spec}(A \otimes_R R_0)$. The ring $A \otimes_R R_0$ is stably coherent (Example 3.3), the invertible $\mathcal{O}_{\mathfrak{X}_0}$-module $\mathcal{L}_0^{\otimes n}$ is coherent and $\mathfrak{f}_0$ is (locally) of finite presentation. Hence $R^0(\mathfrak{f}_0)_*(\mathcal{L}_0^{\otimes n})$ is a coherent $\mathcal{O}_{\mathfrak{X}_0}$-module by the direct image theorem for schemes 3.5 so that the module $H^0(\mathfrak{X}_0, \mathcal{L}_0^{\otimes n})$ is coherent over $A \otimes_R R_0$ and therefore also over A.

Choose a finitely generated A-submodule E of $H^0(\mathfrak{X}, \mathcal{L}^{\otimes n})$ such that the image of E in $H^0(\mathfrak{X}_0, \mathcal{L}_0^{\otimes n})$ generates this module over A. Exactly as in the proof of [EGA III, 5.4.5] one now constructs a morphism $g : \mathfrak{X} \to (\mathbf{P}(E))\hat{}$ of formal R-schemes from $\mathfrak{X}$ to the completion of the projective bundle $\mathbf{P}(E)$ such that $g_0 : \mathfrak{X}_0 \to \big((\mathbf{P}(E))\hat{}\big)_0 = \mathbf{P}(E/\mathfrak{I}E)$ is a closed immersion: One simply has to note that for none of the auxiliary results ([EGA I, 9.1.5]=[EGA $\mathrm{I}_{\mathrm{new}}$, 3.3.4] and [EGA II, 4.1.3, 4.2.2, 4.2.10 and 4.4.4[!]]) cited there one needs a noethericity condition and that each morphism of formal R-schemes automatically is adic in the sense of [EGA $\mathrm{I}_{\mathrm{new}}$, 10.12.1] by which the results of [EGA $\mathrm{I}_{\mathrm{new}}$, 10.12] transfer to the case of a valuation ring (e.g., by means of [EGA $\mathrm{I}_{\mathrm{new}}$, 10.6.9]).

As E is finitely generated over A there is an A-module epimorphism $u : F \to E$ with a free A-module F of finite rank $r+1$, $r \in \mathbb{N}$, hence a closed immersion $j := \mathbf{P}(u) : \mathbf{P}(E) \to \mathbf{P}(F) = \mathbb{P}_A^r$. Denoting by $\hat{\jmath} : (\mathbf{P}(E))\hat{} \to (\mathbb{P}_A^r)\hat{}$ the continuation of j then $\hat{\jmath} \circ g : \mathfrak{X} \to (\mathbb{P}_A^r)\hat{}$ is a morphism of formal R-schemes which are locally of topologically finite presentation. As the reduction $(\hat{\jmath} \circ g)_0 = j_0 \circ g_0 : \mathfrak{X}_0 \to \big((\mathbb{P}_A^r)\hat{}\big)_0 = \mathbf{P}(F/\mathfrak{I}F)$ is a closed immersion, Lemma 5.7 now implies that $\hat{\jmath} \circ g$ is a closed immersion of formal R-schemes and the 3rd GAGA Theorem 6.8 gives the claim. □

7 Rigid spaces

As in Sect. 2 let S be a formal scheme which is noetherian and without torsion with respect to an ideal of definition or admissible over a valuation ring (for a height 1 valuation) and $\mathfrak{I}$ be a coherent $\mathcal{O}_S$-ideal which defines the topology of S. A morphism $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ of admissible formal S-schemes is called an ***admissible formal blowing-up of an open coherent ideal*** $\mathcal{A} \subset \mathcal{O}_{\mathfrak{X}}$ if its restriction to each affine open formal subscheme $\mathfrak{U} = \operatorname{Spf} A$ of $\mathfrak{X}$ equals the completion of the usual blowing-up of the ideal $\mathcal{A}(\mathfrak{U})$ in $\operatorname{Spec} A$ along the subscheme defined by $\mathfrak{J}(\mathfrak{U})$ (cf. [BL 1, Sect. 2]; in particular, a formal S-scheme is called ***admissible*** if it is locally of topologically finite presentation over S and its structure sheaf is without $\mathfrak{J}$-torsion).

In the case that S equals the formal spectrum of a valuation ring R one knows by Raynaud that the category of quasi-compact and quasi-separated rigid analytic spaces over the quotient field of R coincides with the localization of the category of quasi-compact admissible formal R-schemes by the admissible formal blowing-ups ([R, p. 326]; for a proof see [Me, 4.3] resp. [BL 1, Theorem 4.1]).

For an arbitrary base scheme S as above one turns the statement of Raynaud's result into a definition and defines the category of (quasi-compact) rigid S-spaces as the localization of the category of quasi-compact admissible formal S-schemes by the admissible formal blowing-ups. In particular, one has a canonical functor which assigns a rigid S-space $\mathfrak{X}_{\mathrm{rig}}$ to each quasi-compact admissible formal S-scheme $\mathfrak{X}$.

In order to define a structure sheaf $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$ on $\mathfrak{X}_{\mathrm{rig}}$, one proceeds as follows (cf. [BL 1, Sect. 5]): For each quasi-compact admissible formal S-scheme $\mathfrak{X}$ one defines the sheaf $\mathcal{O}'_{\mathfrak{X}}$ of the sections on $\mathfrak{X}$ which are defined on the complement of the special fibre by setting $\mathcal{O}'_{\mathfrak{X}}(\mathrm{Spf}\,A) := \mathcal{O}_{\mathrm{Spec}\,A}(\mathrm{Spec}\,A - V(\mathfrak{J}))$ for each affine open formal subscheme $\mathrm{Spf}\,A$ of $\mathfrak{X}$ and then continuing this as a sheaf to all open formal subschemes of $\mathfrak{X}$. Each open rigid S-subspace $\mathfrak{U}_{\mathrm{rig}}$ of $\mathfrak{X}_{\mathrm{rig}}$ has an open immersion $\mathfrak{U} \hookrightarrow \mathfrak{X}$ of formal S-schemes as a representative and one sets $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}(\mathfrak{U}_{\mathrm{rig}}) := \mathcal{O}'_{\mathfrak{U}}(\mathfrak{U})$.

Similarly, the coherent $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$-modules can be described by assigning the sheaf $\mathcal{M}'$ of sections defined on the complement of the special fibre of $\mathfrak{X}$ to each coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ and defining an $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$-module $\mathcal{M}_{\mathrm{rig}}$ in the above situation by $\mathcal{M}_{\mathrm{rig}}(\mathfrak{U}_{\mathrm{rig}}) := \mathcal{M}'(\mathfrak{U})$.

One has to verify, of course, that this functor $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$ is well-defined and, in fact, a sheaf on $\mathfrak{X}_{\mathrm{rig}}$ and that each coherent $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$-module $\mathcal{M}_{\mathrm{rig}}$ has a formal $\mathcal{O}_{\mathfrak{X}}$-model over each S-model $\mathfrak{X}$ of $\mathfrak{X}_{\mathrm{rig}}$. For a noetherian base scheme S this is proved in [BL 1, Sect. 5] by use of results from [EGA III, pars. 4–5] on noetherian formal schemes. For the proof in the situation over an – in particular, non-discrete – valuation ring, however, [BL 1] takes resort to Tate's acyclicity theorem ([T, Theorem 8.2] resp. [GG, Satz 1.2], also [BGR, Theorem 8.2.1/1]) and Theorem A for coherent modules over rigid spaces [K 2, Theorem 1.2], which are results from classical rigid analytic theory.

In the sequel we show how to deduce these theorems from the results on formal schemes proved within the last two sections, following the lines of [L, Sect. 2] for the discrete case.

To this purpose let again S equal the formal spectrum $\mathrm{Spf}\,R$ of a ring R of the usual kind. By definition two quasi-compact admissible formal S-schemes that differ only by an admissible formal blowing-up represent the same rigid S-space. Hence one can blow up the ideal $\mathfrak{J}$ in R, if necessary, and therefore assume even in the noetherian situation that $\mathfrak{J} = tR$ is a *principal* ideal. Let K denote the localization of R by the multiplicative system $\{t^{\nu}; \nu \in \mathbb{N}\}$. As R

is without $\mathfrak{I}$-torsion, the canonical map $R \to K$ is injective. In the noetherian case, however, K need not be a field.

Then for a quasi-compact admissible formal R-scheme $\mathfrak{X}$ and a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ one has by the above definitions

$$\mathcal{O}'_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}} \otimes_R K \quad \text{and} \quad \mathcal{M}' = \mathcal{M} \otimes_R K .$$

In order to prove that $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$ is a well-defined sheaf on $\mathfrak{X}_{\mathrm{rig}}$, it hence suffices to show

Proposition 7.1 (Tate's acyclicity theorem) *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be an admissible formal blowing-up of admissible formal R-schemes. Then the canonical map $\mathcal{O}_{\mathfrak{Y}} \to \mathfrak{f}_* \mathcal{O}_{\mathfrak{X}}$ induces an isomorphism*

$$\mathcal{O}_{\mathfrak{Y}} \otimes_R K \xrightarrow{\sim} (\mathfrak{f}_* \mathcal{O}_{\mathfrak{X}}) \otimes_R K ,$$

and $(R^q \mathfrak{f}_* \mathcal{O}_{\mathfrak{X}}) \otimes_R K$ vanishes for all $q > 0$.

The basic idea of the following proof is simple enough: Of course, one can assume $\mathfrak{Y}$ to be formal affine, say $\mathfrak{Y} = \operatorname{Spf} A$ with an admissible R-algebra A. Then, by definition, $\mathfrak{f}$ is the completion of the blowing-up $f : X \to Y := \operatorname{Spec} A$ of an open coherent ideal $\mathfrak{a}$ in A. As f is an isomorphism outside the closed subscheme of Y defined by $\mathfrak{I} = tR$, one has an isomorphism $\mathcal{O}_Y \otimes_R K \xrightarrow{\sim} f_* \mathcal{O}_X \otimes_R K$ and has $(R^q f_* \mathcal{O}_X) \otimes_R K = 0$ for $q > 0$. A GAGA argument should now give the claim of Proposition 7.1.

The technical difficulties in applying the GAGA result 6.3 in the non-noetherian situation, which were discussed after that proposition in Sect. 6, make the realization of this idea a little bit involved. Relatively straightforward is the

Proof of Proposition 7.1 for the special case $A = R\langle \zeta_1 \ldots, \zeta_r \rangle$. Let $\mathfrak{f}$ be the completion of the blowing-up $f : X \to \operatorname{Spec} A$ of the open coherent ideal $\mathfrak{a}$ in A. As $\mathfrak{a}$ is open, one can assume a set of generating elements $a_1, \ldots, a_N$ of $\mathfrak{a}$ to be in $R[\zeta_1, \ldots, \zeta_r]$ and, say, a_1 in $R - \{0\}$.

Let $\mathfrak{a}'$ denote the ideal generated in $R[\zeta_1, \ldots, \zeta_r]$ by $a_1, \ldots, a_N$. Then $\mathfrak{a}'$ is coherent, since $R[\zeta_1, \ldots, \zeta_r]$ is coherent by Example 3.3, and open. For the blowing-up $f' : X' \to Y' := \operatorname{Spec} R[\zeta_1, \ldots, \zeta_r]$ of $\mathfrak{a}'$ in $R[\zeta_1, \ldots, \zeta_r]$ one has $f' \times \mathrm{id}_{\operatorname{Spec} R_\lambda} = f \times \mathrm{id}_{\operatorname{Spec} R_\lambda} = \mathfrak{f} \times \mathrm{id}_{\operatorname{Spec} R_\lambda}$ for each $\lambda \in \mathbb{N}$, which implies that $\mathfrak{f}$ can also be considered as the completion of f'. But, as R is stably coherent by Example 3.3, the direct image theorem for schemes 3.5 implies that $R^q f'_* \mathcal{O}_{X'}$ is a coherent $\mathcal{O}_{Y'}$-module for each $q \in \mathbb{N}$. As the blown-up ideal $\mathfrak{a}'$ is open, $\mathcal{O}_{X'}$ has no $\mathfrak{I}$-torsion and Corollary 1.7 implies that for each $q \in \mathbb{N}$ there is a $v_q \in \mathbb{N}$ with $t^{v_q} \mathcal{T}_{\mathfrak{I}}(R^q f'_* \mathcal{O}_{X'}) = 0$.

So Proposition 6.3 gives an isomorphism $(R^q f'_* \mathcal{O}_{X'})^{\wedge} \xrightarrow{\sim} R^q \mathfrak{f}_* \mathcal{O}'_{\mathfrak{X}}$. For $q > 0$ one has $(R^q f'_* \mathcal{O}_{X'}) \otimes_R K = 0$, hence $t^{v_q} R^q f'_* \mathcal{O}_{X'} = 0$. Therefore $R^q f'_* \mathcal{O}_{X'}$ and $R^q \mathfrak{f}_* \mathcal{O}_{\mathfrak{X}}$ are isomorphic and $(R^q \mathfrak{f}_* \mathcal{O}_{\mathfrak{X}}) \otimes_R K = 0$. A similar argument, applied to kernel and cokernel of the canonical map $\mathcal{O}_{Y'} \to f'_* \mathcal{O}_{X'}$ resp. $\mathcal{O}_{\mathfrak{Y}}$

$\to \mathfrak{f}_*\mathcal{O}_{\mathfrak{X}}$, gives the isomorphism $\mathcal{O}_{\mathfrak{Y}} \otimes_R K \xrightarrow{\sim} \mathfrak{f}_*\mathcal{O}_{\mathfrak{X}} \otimes_R K$ from $\mathcal{O}_{Y'} \otimes_R K \xrightarrow{\sim} f'_*\mathcal{O}_{X'} \otimes_R K$. □

The reader will have noted that the same kind of argument given above for $A = R\langle\zeta_1,\dots,\zeta_r\rangle$ works for each admissible R-algebra A which is the $\mathfrak{I}$-adic (separated) completion of a finitely presented R-algebra. The special case $A = R\langle\zeta_1,\dots,\zeta_r\rangle$, however, suffices to attack the situation of a general A by showing first

Lemma 7.2 *Let A be an admissible R-algebra and $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y} := \operatorname{Spf} A$ an admissible formal blowing-up of admissible formal R-schemes.*

Assume that the statement of Proposition 7.1 holds for this particular blowing-up. Then one has:

a) *For each $q > 0$ and each coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ the module $(R^q\mathfrak{f}_*\mathcal{M}) \otimes_R K$ vanishes.*

b) *For each coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ the canonical map $\mathfrak{f}^*\mathfrak{f}_*\mathcal{M} \to \mathcal{M}$ induces an isomorphism $(\mathfrak{f}^*\mathfrak{f}_*\mathcal{M}) \otimes_R K \xrightarrow{\sim} \mathcal{M} \otimes_R K$.*

c) *For each coherent $\mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{N}$ the canonical map $\mathcal{N} \to \mathfrak{f}_*\mathfrak{f}^*\mathcal{N}$ induces an isomorphism $\mathcal{N} \otimes_R K \xrightarrow{\sim} (\mathfrak{f}_*\mathfrak{f}^*\mathcal{N}) \otimes_R K$.*

Proof. Let $\mathfrak{f}$ be the completion of the blowing-up $f : X \to Y := \operatorname{Spec} A$ of the open coherent ideal $\mathfrak{a}$ in A. Then there is a closed immersion $\imath$ of X into a projective space $\mathbb{P}^r_A$ with $r \in \mathbb{N}$ convenient over which f factors and for which $\imath^*(\mathcal{O}_{\mathbb{P}^r_A}(1))$ equals the invertible $\mathcal{O}_X$-ideal $\mathfrak{a}\mathcal{O}_X$. By the explicit description of admissible formal blowing-ups in [BL 1, Lemma 2.2] the completion $\mathcal{L} := \big(\imath^*(\mathcal{O}_{\mathbb{P}^r_A}(1))\big)^\wedge \cong (\hat{\imath})^*\big((\mathcal{O}_{\mathbb{P}^r_A}(1))^\wedge\big)$ fulfills $\mathcal{L} = \mathfrak{a}\mathcal{O}_{\mathfrak{X}}$ and one has a canonical isomorphism $\mathcal{L}^{\otimes n} \cong \mathfrak{a}^n\mathcal{O}_{\mathfrak{X}}$ for each $n \in \mathbb{N}$. In particular, the $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{L}$ is invertible and $\mathcal{L}_0 = \mathcal{L}/\mathfrak{I}\mathcal{L} \cong (\imath \times \mathrm{id}_{\operatorname{Spec} R_0})^*\big(\mathcal{O}_{\mathbb{P}^r_{A\otimes_R R_0}}(1)\big)$ is an ample $\mathcal{O}_{\mathfrak{X}_0}$-module.

As $\mathfrak{a}$ is open in A, there is a $\nu \in \mathbb{N}$ with $t^\nu \in \mathfrak{a}$, hence $t^{n\nu}\mathcal{O}_{\mathfrak{X}} \hookrightarrow \mathcal{L}^{\otimes n} \hookrightarrow \mathcal{O}_{\mathfrak{X}}$ for each $n \in \mathbb{Z}$ with $n \geqq 0$. So one has a canonical isomorphism $\mathcal{L}^{\otimes n} \otimes_R K \cong \mathcal{O}_{\mathfrak{X}} \otimes_R K$, at first only for those $n \in \mathbb{Z}$ with $n \geqq 0$, but then, by the exactness of tensorization with $\mathcal{L}^{\otimes n}$ over $\mathcal{O}_{\mathfrak{X}}$, also for $n < 0$.

Hence the assumption that Proposition 7.1 is true for the present $\mathfrak{f}$ implies that the above claims a) and b) hold for each $\mathcal{O}_{\mathfrak{X}}$-module of the form $(\mathcal{L}^{\otimes n})^m$ with $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Furthermore, by this assumption, claim c) holds for each $\mathcal{O}_{\mathfrak{Y}}$-module of the form $\mathcal{O}_{\mathfrak{Y}}^m$ with $m \in \mathbb{N}$.

But for an arbitrary coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ two applications of Corollary 6.7 give an exact sequence

$$(\mathcal{L}^{\otimes n'})^{m'} \xrightarrow{\beta} (\mathcal{L}^{\otimes n})^m \xrightarrow{\alpha} \mathcal{M} \longrightarrow 0$$

of $\mathcal{O}_{\mathfrak{X}}$-modules with $n, n' \in \mathbb{Z}$ and $m, m' \in \mathbb{N}$. And for an arbitrary $\mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{N}$ there is an exact sequence

$$\mathcal{O}_{\mathfrak{Y}}^{m'} \to \mathcal{O}_{\mathfrak{Y}}^m \to \mathcal{N} \to 0$$

of $\mathcal{O}_{\mathfrak{Y}}$-modules with $m', m \in \mathbb{N}$ by Theorem A for formal schemes (Proposition 2.3).

So claim a) can be shown simultaneously for all coherent $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{M}$ by looking at the long cohomology sequence induced by the short exact sequence

$$0 \longrightarrow \ker \alpha \longrightarrow (\mathcal{L}^{\otimes n})^m \xrightarrow{\alpha} \mathcal{M} \longrightarrow 0$$

of coherent $\mathcal{O}_{\mathfrak{X}}$-modules and using descending induction on q. Then the claims b) resp. c) follow from the above presentations of $\mathcal{M}$ resp. $\mathcal{N}$ by use of part a) for $q = 1$ and the right-exactness of the functor $\mathfrak{f}^*$. □

Proof of Proposition 7.1 for an arbitrary admissible A: As A is topologically of finite presentation, there is an R-algebra epimorphism $\sigma : R\langle\zeta_1, \ldots, \zeta_r\rangle \to A$ with finitely generated kernel for a convenient $r \in \mathbb{N}$. It induces a closed immersion of formal schemes $i : \mathfrak{Y} = \operatorname{Spf} A \to \operatorname{Spf} R\langle\zeta_1, \ldots, \zeta_r\rangle =: \mathbb{D}_R^r$.

Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y} = \operatorname{Spf} A$ be the admissible formal blowing-up of the open coherent ideal $\mathfrak{a}$ in A. For a convenient set of generators of $\mathfrak{a}$ in A one can choose inverse images with respect to σ which generate an open coherent ideal $\mathfrak{b}$ in $R\langle\zeta_1, \ldots, \zeta_r\rangle$. Denote the admissible formal blowing-up of $\mathfrak{b}$ in $R\langle\zeta_1, \ldots, \zeta_r\rangle$ by $\mathfrak{g} : \mathfrak{Z} \to \mathbb{D}_R^r$. Then one has a commutative diagram of formal schemes

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mathfrak{j}} & \mathfrak{Z} \\ \downarrow{\scriptstyle \mathfrak{f}} & & \downarrow{\scriptstyle \mathfrak{g}} \\ \mathfrak{Y} & \xrightarrow{\mathfrak{i}} & \mathbb{D}_R^r \end{array}$$

with a closed immersion $\mathfrak{j} : \mathfrak{X} \to \mathfrak{Z}$, and, furthermore, $\mathfrak{j}_*\mathcal{O}_{\mathfrak{X}} \otimes_R K = \mathfrak{g}^*(\mathfrak{i}_*\mathcal{O}_{\mathfrak{Y}}) \otimes_R K$ holds, cf. the explicit description of admissible formal blowing-ups in [BL 1, Lemma 2.2].

As $\mathfrak{j}$ is a closed immersion of formal schemes, $\mathfrak{j}_*\mathcal{O}_{\mathfrak{X}}$ is a coherent $\mathcal{O}_{\mathfrak{Z}}$-module so that Lemma 7.2a) in combination with the proved special case of Proposition 7.1 implies $\left(R^q\mathfrak{g}_*(\mathfrak{j}_*\mathcal{O}_{\mathfrak{X}})\right) \otimes_R K = 0$ for $q > 0$.

By Theorem B for formal schemes (Proposition 5.1) the higher direct images ($q > 0$) of coherent module sheaves with respect to the closed immersions both $\mathfrak{i}$ and $\mathfrak{j}$ vanish. Hence two applications of Leray's spectral sequence and $\mathfrak{g} \circ \mathfrak{j} = \mathfrak{i} \circ \mathfrak{f}$ give

$$R^q\mathfrak{g}_*(\mathfrak{j}_*\mathcal{O}_{\mathfrak{X}}) = R^q(\mathfrak{g} \circ \mathfrak{j})_*\mathcal{O}_{\mathfrak{X}} = R^q(\mathfrak{i} \circ \mathfrak{f})_*\mathcal{O}_{\mathfrak{X}} = \mathfrak{i}_*(R^q\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}})$$

for $q \in \mathbb{Z}$. So $\mathfrak{i}_*(R^q\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}}) \otimes_R K$ vanishes for $q > 0$ and one has

$$\mathfrak{i}_*(\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}}) \otimes_R K = \mathfrak{g}_*(\mathfrak{j}_*\mathcal{O}_{\mathfrak{X}}) \otimes_R K = \mathfrak{g}_*\mathfrak{g}^*(\mathfrak{i}_*\mathcal{O}_{\mathfrak{Y}}) \otimes_R K \cong \mathfrak{i}_*\mathcal{O}_{\mathfrak{Y}} \otimes_R K$$

where the isomorphy holds by Lemma 7.2c).

As $\mathfrak{i}_*$ is a closed immersion, this implies Tate's acyclicity theorem in the general case. □

By means of an argument which is independent from the fact whether the base ring R is noetherian or not ([L, proof of Lemma 2.2] resp. [BL1, Lemma 5.7]) one shows that coherent module sheaves over open rigid subspaces of a rigid R-space $\mathfrak{X}_{\mathrm{rig}}$ can be glued together if they are induced by coherent module sheaves on open formal subschemes of the *same* formal model $\mathfrak{X}$ of $\mathfrak{X}_{\mathrm{rig}}$.

As Proposition 7.1 has been proved for each admissible blowing-up, Lemma 7.2b), c) now implies that a coherent $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$-module has a formal model on *each* formal model of $\mathfrak{X}$ (cf. [L, Lemma 2.2] and [BL 1, Proposition 5.6]). In particular, one can generalize the reasoning from [L, Theorem 2.3] to arbitrary valuation rings and hence gets Kiehl's results on coherent modules [K 2] in full generality:

Proposition 7.3 (Theorems A and B for rigid spaces) *Let $\mathfrak{X}_{\mathrm{rig}}$ be a rigid R-space which has an affine formal model. Then for each coherent $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$-module $\mathcal{M}_{\mathrm{rig}}$ one has:*

(A) *The module $\mathcal{M}_{\mathrm{rig}}$ is generated by global sections.*

(B) *For each $q > 0$ one has $H^q(\mathfrak{X}_{\mathrm{rig}}, \mathcal{M}_{\mathrm{rig}}) = 0$.*

For the *proof* of (A) one uses Theorem A for the formal situation (Proposition 2.3), for that of (B) Lemma 5.4 and Proposition 7.2a) or Theorem B for the formal situation (Proposition 5.1).

In an analogous way one can deduce the finiteness theorem of Kiehl for proper rigid morphisms [K 1, Theorem 3.3] for the situation over an arbitrary valuation ring by means of formal methods, too. One only has to remark that each morphism of rigid spaces which is proper in the sense introduced by Kiehl [K 1, Definition 2.3] has a formal model which is proper in the sense defined in Sect. 5 (cf. [R, p. 326], also [L, Lemma 2.6]) and can argue as in the proof of [L, Theorem 2.7] then, just with the exception that instead of [EGA III, 3.4.2] one has to use the version of the direct image theorem which has been proved in the present article.

Acknowledgement. I would like to express my thanks to Prof. S. Bosch, University of Münster, for his suggestion to work at the formal algebraic approach to rigid analytic geometry. He and Prof. W. Lütkebohmert, University of Ulm, have essentially advanced this work by their interest, in particular, to them and Prof. H.-J. Nastold, University of Münster, I owe the hint on Kiehl's article [K 3]. My special thanks go to Prof. R. Remmert, University of Münster, for his support during the whole of my academic studies.

References

[AC] Bourbaki, N.: Algèbre commutative. Chap. I–IX, Paris: Hermann 1961–65, Paris: Masson 1980–85

[B] Bosch, S.: Orthonormalbasen in der nichtarchimedischen Funktionentheorie. Manuscr. Math. **1**, 35–57 (1969)

[BGR] Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean analysis. (Grundl. Math. Wiss. 261) Berlin Heidelberg New York: Springer 1984

[BL 1] Bosch, S., Lütkebohmert, W.: Formal and rigid geometry. I. Rigid spaces. Math. Ann. **295**, 291–317 (1993)

[BL 2] Bosch, S., Lütkebohmert, W.: Formal and rigid geometry. II. Flattening techniques. Math. Ann. **296**, 403–429 (1993)

[EGA] Grothendieck, A., Dieudonné, J.: Eléments de géométrie algébrique. Publ. Math., Inst. Hautes Etud. Sci. **4, 8, 11, 17, 20, 24, 28, 32** (1960–67)

[EGA_{new}] Grothendieck, A., Dieudonné, J.A.: Eléments de Géométrie Algébrique. (Grundl. Math. Wiss. 166) Berlin Heidelberg New York: Springer 1971

[GG] Gerritzen, L., Grauert, H.: Die Azyklizität der affinoiden Überdeckungen. In: Global Analysis. Papers in Honor of K. Kodaira, 159–184. Tokyo: University of Tokyo Press, Princeton: Princeton University Press 1969

[GR 1] Grauert, H., Remmert, R.: Nichtarchimedische Funktionentheorie. In: Festschrift zur Gedächtnisfeier für Karl Weierstraß1815–1965, 393–476. (Wissenschaftl. Abh. Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen 33) Köln, Opladen: Westdeutscher Verlag 1966

[GR 2] Grauert, H., Remmert, R.: Über die Methode der diskret bewerteten Ringe in der nichtarchimedischen Analysis. Invent. Math. **2**, 87–133 (1966)

[K 1] Kiehl, R.: Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie. Invent. Math. **2**, 191–214 (1967)

[K 2] Kiehl, R.: Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie. Invent. Math. **2**, 256–273 (1967)

[K 3] Kiehl, R.: Ein „Descente"-Lemma und Grothendiecks Projektionssatz für nichtnoethersche Schemata. Math. Ann. **198**, 287–316 (1972)

[L] Lütkebohmert, W.: Formal-algebraic and rigid-analytic geometry. Math. Ann. **286**, 341–371 (1990)

[Me] Mehlmann, F.: Ein Beweis für einen Satz von Raynaud über flache Homomorphismen affinoider Algebren. Schriftenr. Math. Inst. Univ. Münster, 2. Ser. **19** (1981)

[Mo] Monsky, P.: p-adic analysis and zeta functions. (Lect. Math. 4) Tokyo: Kinokuniya 1970

[Mu] Mumford, D.: Abelian Varieties. Oxford University Press 1974

[R] Raynaud, M.: Géométrie analytique rigide d'aprés Tate, Kiehl, ... Table ronde d'analyse non archimedienne. Bull. Soc. Math. Fr., **Mém. 39-40**, 319–327 (1974)

[RG] Raynaud, M., Gruson, L.: Critères de platitude et de projectivité. Invent. Math. **13**, 1–89 (1971)

[Se] Serre, J-P.: Faisceaux algébriques cohérents. Ann. Math., II. Ser. **61**, 197–278 (1955)

[SGA 6] Théorie des Intersections et Théorème de Riemann-Roch. Séminaire de Géométrie Algébrique du Bois Marie 1966/67. (Lect. Notes Math. 225) Berlin Heidelberg New York: Springer 1971

[Sou] Soublin, J-P.: Anneaux et modules cohérents. J. Algebra **15**, 455–472 (1970)

[T] Tate, J.T.: Rigid analytic spaces. Private Notes (1962). Reprinted in Invent. Math. **12**, 257–289 (1971)

Note added in proof

Recently K. Fujiwara, Nagoya University, has informed me that in July 1994 O. Gabber, IHES, has shown (notations as in the present paper): Let B denote a finitely generated algebra over a topologically finitely generated R-algebra. Then each finitely generated B-module without $\mathcal{J}$-torsion is finitely presented over B. – The proof of this fact will be contained in Fujiwara's forthcoming article "Theory of tubular neighborhood in etale topology". –

Using this result it follows that each topologically finitely presented R-algebra A is stably coherent. So far each proper morphism $f : X \longrightarrow Y = \operatorname{Spec} A$ the higher direct images of coherent $\mathcal{O}_X$-modules with respect to f are coherent $\mathcal{O}_Y$-modules by the direct image theorem for schemes 3.5. Hence one can apply the GAGA result Proposition 6.3 also in the situation over a non-discrete valuations ring R in the way one is used to from the noetherian situation (cf. the considerations following that Proposition) and therefore gets the 1st and 2nd GAGA Theorem for all coherent $\mathcal{O}_Y$-modules $\mathcal{M}$ and $\mathcal{F}$. Furthermore, the proof given in Sect. 7 for Tate's acyclicity theorem can considerably be simplified.