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Geometric invariant theory on Stein spaces

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Actions of complex reductive groups of holomorphic transformations on complex spaces can often be studied by means which are very close to those of geometric invariant theory. For example, if the complex reductive group G is acting on a Stein space, then the fibers of the categorical quotient are affine algebraic and G acts on them in an algebraic fashion [S].

Our goal here is to present results on actions of compact groups of holomorphic transformations on Stein spaces which also can be applied in situations where there are no actions of complex groups, e.g. bounded domains. The essential ingredient is the

Complexification Theorem. *Let K be a compact Lie group and $K^{\mathbb{C}}$ a complexification of K. If K acts on a reduced Stein space X, then there exists a complex space $X^{\mathbb{C}}$ with a holomorphic action $K^{\mathbb{C}} \times X^{\mathbb{C}} \to X^{\mathbb{C}}$ and a K-equivariant holomorphic map $\imath: X \to X^{\mathbb{C}}$ with the following properties:*

(i) *$\imath: X \to X^{\mathbb{C}}$ is an open embedding and $\imath(X)$ is a Runge subset of $X^{\mathbb{C}}$ such that $K^{\mathbb{C}} \cdot \imath(X) = X^{\mathbb{C}}$.*

(ii) *$X^{\mathbb{C}}$ is a Stein space.*

(iii) *If ϕ is a K-equivariant holomorphic map from X into a complex space Y on which $K^{\mathbb{C}}$ acts holomorphically, then there exists a unique $K^{\mathbb{C}}$-equivariant holomorphic map $\phi^{\mathbb{C}}: X^{\mathbb{C}} \to Y$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\imath} & X^{\mathbb{C}} \\ & \phi \searrow \quad \swarrow \phi^{\mathbb{C}} & \\ & Y & \end{array}$$

commutes.

The complexification $X^{\mathbb{C}}$ of a Stein space with a fixed K-action is uniquely determinated up to $K^{\mathbb{C}}$-equivariant biholomorphisms. The property (iii) can be viewed in a slightly different way. Since K is a compact Lie group, every holomorphic function on X can be expanded in a Fourier series with respect to the action of K. The summands are K-finite holomorphic functions on X which automatically extend to $K^{\mathbb{C}}$-finite holomorphic functions on $X^{\mathbb{C}}$. Thus $X^{\mathbb{C}}$ is a natural domain of definition of the K-finite holomorphic functions on X. The simplest K-finite functions are the invariant ones. They form a subalgebra $\mathcal{O}(X)^K$ of the algebra $\mathscr{F}_K(X)$ of K-finite holomorphic functions on X. Associated to $\mathcal{O}(X)^K$ is the categorical quotient $X//K$, i.e. the quotient of X with respect to the equivalence relation

$$R=\{(x,y)\in X\times X;\ f(x)=f(y) \text{ for all } f\in\mathcal{O}(X)^K\}.$$

Let π_X denote the quotient map. The correspondence $Q\to\mathcal{O}_X(\pi_X^{-1}(Q))^K$ defines a sheaf $\mathcal{O}_X^K$ on $X//K$ so that $(X//K,\mathcal{O}_X^K)$ is a $\mathbb{C}$-ringed space. In fact, more is true. In Sect. 6.5 we prove the following

Quotient Theorem. *Let X be a reduced Stein space equipped with an action of a compact Lie group K. Then the $\mathbb{C}$-ringed space $(X//K,\mathcal{O}_X^K)$ is a Stein space.*

This is a generalization of the result in [H 2], where it is proved for normal Stein spaces. The main point here is that the open embedding ι induces an isomorphism of the categorical quotients $X//K$ and $X^{\mathbb{C}}//K$. Moreover, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^{\mathbb{C}} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_X^{\mathbb{C}}} \\ X//K & \xrightarrow[\cong]{} & X^{\mathbb{C}}//K \end{array}$$

commutes.

In fact, the complexification $X^{\mathbb{C}}$ is constructed as a sort of twisted fiber space over $X//K$. In order to do that, a local version of the Complexification Theorem is proved in Sect. 6.3. This result can be thought of as a Slice Theorem for an action of a compact Lie group on a Stein space. A simplified version is the following

Linearization Theorem. *Let K be a compact Lie group which acts on a Stein manifold Y and fix a point p in $Y//K$. Then there exist a point $a\in\pi_Y^{-1}(p)$, an open neighbourhood Q of p and an equivariant open embedding ϕ of $\pi_Y^{-1}(Q)$ into a homogeneous vector bundle $N=K^{\mathbb{C}}\times_{K_a^{\mathbb{C}}}\mathbb{C}^m$ such that ϕ induces an open embedding $\phi//K$ of Q into $N//K\cong\mathbb{C}^m//K_a$.*

One important step in the proof of the Linearization Theorem is to find such a distinguished point a in the special case where Y is locally $K^{\mathbb{C}}$-homogeneous. For applications of the Complexification Theorem see the last section.

1 Actions of real Lie groups and complexifications

1.1 Spaces with a group action

Let G be a Lie group. A topological space X together with a group homomorphism ϱ from G into the group of topological self maps of X is called a *G-space* if the action $G\times X\to X$, $(g,x)\to\varrho(g)(x)$ is continuous. We often write $g\cdot x$ for $\varrho(g)(x)$. For a subset U in a G-space we set $G\cdot U=\{g\cdot u;\ g\in G,\ u\in U\}$.

A subset U of a G-space is called a *G-subset* or *G-invariant* if $G\cdot U=U$ holds. If U is a G-subset of a G-space X, then also the topological closure $\bar{U}$ of U in X is a

G-subset. For a point x in X the G-set $G \cdot x = G \cdot \{x\}$ is called the *G-orbit* through x. The *isotropy group* G_x at x is the subgroup $G_x = \{g \in G;\ g \cdot x = x\}$ of G.

The set of G-fixed points of a G-space X will be denoted by X^G. Note that $X^G = \{x \in X;\ G \cdot x = \{x\}\} = \{x \in X;\ G_x = G\}$.

Let H be a Lie group and $\tau: H \to G$ a continuous group homomorphism. Then every G-space X is viewed as a H-space. The H-action is given by $(h, x) \to \varrho(\tau(h))(x)$. The group G itself is a G-space with respect to the group multiplication $G \times G \to G$, $(g, x) \to gx$. In this case we speak about the G-space G without specifying the action. With respect to any subgroup H of G the group G is a H-space.

The *quotient of a G-space* X is the set $X/G = \{G \cdot x;\ x \in X\}$ endowed with the quotient topology. The G-space X is called *G-connected* if the quotient X/G is connected. An open G-connected G-subset U of X will be called *G-domain*. If the group G is connected then an open G-subset in X is a G-domain if and only if it is connected.

Remark. If we write G/H where H is a subgroup of G, then, unless otherwise mentioned, G/H will be the space $\{gH;\ g \in G\}$. So we let H act on G by $H \times G \to G$, $(h, g) \to gh^{-1}$.

A map $\phi: X \to Y$ between two G-spaces is called a *G-map* or *equivariant*, if $\phi(g \cdot x) = g \cdot \phi(x)$ for all $g \in G$ and $x \in X$.

1.2 Complex spaces with a group action

Let G be a Lie group. A *complex G-space X* is a reduced complex space X with countable topology which is a G-space such that for every $g \in G$ the map $X \to X$, $x \to g \cdot x$ is holomorphic. The action $G \times X \to X$ is in this case a real analytic map (see [K 1]). A *complex G-space X* is called *holomorphic* if G is a complex Lie group and if the action $G \times X \to X$ is holomorphic. In this case the isotropy groups are closed complex subgroups of G.

A *locally analytic G-subset* of a complex G space X is a G-subset of X which is a locally analytic subset of X. If the locally analytic G-subset is closed in X, then it will be called an *analytic G-set*. An analytic G-set A in X is called *G-irreducible* if there exists an irreducible analytic component A_0 of A such that $G \cdot A_0 = A$. For a connected group G the notions G-irreducible and irreducible coincide.

Complex *G-manifolds* are by definition complex G-spaces without singular points. Note that the set of singular points of a complex G-space is always an analytic G-set.

The set of holomorphic maps from a complex G-space X into a complex G-space Y is denoted by $\mathrm{Hol}(X, Y)$. Endowed with the compact-open topology, the set $\mathrm{Hol}(X, Y)$ is a G-space. The action is defined by $(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x)$. The set of G-fixed points is in this case denoted by $\mathrm{Hol}_G(X, Y)$. Thus an element in $\mathrm{Hol}_G(X, Y)$ is a holomorphic G-map from X into Y.

Every complex space Y can be viewed as a G-space with the *trivial action* which is given by $g \cdot y = y$ for all $g \in G$ and $y \in Y$. The set of G-invariant holomorphic maps from X into Y is then $\mathrm{Hol}_G(X, Y)$. For the *algebra $\mathcal{O}(X)^G$ of invariant holomorphic functions* on a complex G-space X we have $\mathcal{O}(X)^G = \mathrm{Hol}_G(X, \mathbb{C})$, where $\mathbb{C}$ is endowed with the trivial action.

A *linearly equivariant map* on a G-space X is a G-map from X into some complex vector space V of finite dimension, where the G-action on V is given by a continuous representation of G into the general linear group $GL(V)$.

A Stein space which is an open subspace of a complex space Y is called an *open Stein subset of* Y. The meaning of notions like Stein G-domain is evident.

A holomorphic map $\phi: X \to Y$ of complex spaces is called *immersive along a subset* S *of* X if ϕ is an immersion at every point $x \in S$, i.e. to every point $x \in S$ there exists an open neighbourhood which is mapped biholomorphically by ϕ onto a locally analytic set in Y. For a G-map ϕ of G-spaces X and Y the set of points in X where ϕ is an immersion is a complement of an analytic G-subset in X.

We shall make repeated use of plurisubharmonic functions on complex spaces. In order to avoid technical difficulties these are always assumed to be differentiable.

1.3 Complexification of a Lie group

Every real Lie algebra $\mathfrak{g}$ determines a complex Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. A complexification can also be constructed on the group level, cf. [Ho]:

Let G be a real Lie group. A complex Lie group $G^{\mathbb{C}}$ together with a continuous group homomorphism $\iota: G \to G^{\mathbb{C}}$ is called a *complexification* of G if for a given continuous group homomorphism ϕ from G into a complex Lie group H, there exists one and only one holomorphic group homomorphism $\phi^{\mathbb{C}}$ from $G^{\mathbb{C}}$ into H such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\iota} & G^{\mathbb{C}} \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi^{\mathbb{C}}} & \\ & H & \end{array}$$

commutes.

A complexification is unique up to biholomorphisms. Every Lie group is already real analytic and in the above definition the maps ι and ϕ can be assumed to be analytic.

The construction of $G^{\mathbb{C}}$ can be found in [Ho], at least for connected groups. If the group G is not connected then G can be identified as a G-space with $G \times_{G_1} G_1$, where G_1 denotes the connected component of the identity of G. Finally, $G^{\mathbb{C}}$ can be identified in a natural way with the G-space $G \times_{G_1} G_1^{\mathbb{C}}$. It is straightforward to define a complex Lie group structure on $G^{\mathbb{C}}$ and to check the above universality condition.

Remark. In general the map $\iota: G \to G^{\mathbb{C}}$ is not injective. For the universal covering group G of $SL(\mathbb{R}^2)$, the complexification $G^{\mathbb{C}}$ is equal to $SL(\mathbb{C}^2)$ and the kernel of $\iota: G \to G^{\mathbb{C}}$ is isomorphic to $\mathbb{Z}$.

We often make use of the following identity principle.

Identity Theorem. *Let H be a closed complex subgroup of $G^{\mathbb{C}}$ and let U denote an open subset in $G^{\mathbb{C}}/H$. Let x be a point in $G^{\mathbb{C}}/H$ such that the G-orbit $G \cdot x$ intersects every connected component of U. Then*

(i) *every holomorphic function f on U such that $f | G \cdot x \cap U = 0$ is identically zero on U, and*

(ii) *the only analytic subset of U which contains $G \cdot x \cap U$ is U.*

Proof. The construction of $G^{\mathbb{C}}$ implies that the real tangent space of a G-orbit in $G^{\mathbb{C}}$ generates the complex tangent space of $G^{\mathbb{C}}$. Since $G^{\mathbb{C}} = G \cdot G_1^{\mathbb{C}}$, the Identity Theorem holds for $H=\{1\}$. Applying this to $\pi^{-1}(U)$, where $\pi: G^{\mathbb{C}} \to G^{\mathbb{C}}/H$ is the canonical projection, we obtain the desired result. □

Corollary. *Every holomorphic G-map of holomorphic $G^{\mathbb{C}}$-spaces is a $G^{\mathbb{C}}$-map. In particular, $\mathcal{O}(G^{\mathbb{C}})^G = \mathcal{O}(G^{\mathbb{C}})^{G^{\mathbb{C}}} = \mathbb{C}$.* □

The G-orbit of a point x in a $G^{\mathbb{C}}$-homogeneous complex space X is in general not a totally real submanifold. However, $\dim_{\mathbb{R}} G \cdot x \geqq \dim_{\mathbb{C}} G^{\mathbb{C}} \cdot x = \dim_{\mathbb{C}} X$.

Example 1. The manifold $\mathbb{C}^n \setminus \{0\}$ is homogeneous with respect to the linear action of $GL(\mathbb{C}^n)$, which is the complexification of the unitary group $U(\mathbb{C}^n)$. The orbits of $U(\mathbb{C}^n)$ are the $(2n-1)$-dimensional spheres.

Example 2. The manifold $SL(\mathbb{C}^2)/H$ where $H = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix};\ t \in \mathbb{C} \setminus \{0\} \right\}$ is a complex $SU(\mathbb{C}^2)$-space. The $SU(\mathbb{C}^2)$-orbit of $p=H$ in the manifold $SL(\mathbb{C}^2)/H$ is the totally real submanifold $SU(\mathbb{C}^2)/S^1$. Any other $SU(\mathbb{C}^2)$-orbit in $SL(\mathbb{C}^2)/H$ is a real hypersurface.

1.4 Complexification of a group action

Let G be a Lie group and X a complex G-space. Let $G^{\mathbb{C}}$ be a complexification of G.

A holomorphic $G^{\mathbb{C}}$-space $X^{\mathbb{C}}$ together with a G-map ι from X into $X^{\mathbb{C}}$ is called a *G-complexification of the G-space X* if to every holomorphic G-map ϕ from X into a holomorphic $G^{\mathbb{C}}$-space Y there exists one and only one holomorphic $G^{\mathbb{C}}$-map $\phi^{\mathbb{C}}$ from $X^{\mathbb{C}}$ into Y such that the diagram.

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^{\mathbb{C}} \\ & {}^{\phi}\searrow \quad \swarrow^{\phi^{\mathbb{C}}} & \\ & Y & \end{array}$$

commutes.

A G-complexification $X^{\mathbb{C}}$ of a complex G-space is unique up to biholomorphic $G^{\mathbb{C}}$-maps. Therefore, provided it exists, we shall refer to the G-complexification of X.

Example 1. Let X be a holomorphic $G^{\mathbb{C}}$-space. If it is viewed as a complex G-space, then the G-complexification of X is X itself (Identity Theorem). A special case of this fact is a compact complex G-space X. Then the group $\mathrm{Aut}(X)$ of biholomorphic automorphisms of X is a complex Lie group and consequently X is its own complexification.

In contrast to this one can also complexify a compact complex torus X of dimension n also as a real Lie group. In this case one has $X^{\mathbb{C}} = (\mathbb{C}^*)^n$.

Example 2. The disc $\Delta = \{z \in \mathbb{C};\ |z| < 1\}$ is a complex S^1-space with respect to the linear S^1-action on $\mathbb{C}$. The S^1-complexification of Δ is $\mathbb{C}$.

More generally, one can show that the $(S^1)^n$-complexification of a holomorphically convex Reinhardt domain U in $\mathbb{C}^n$ is $U^{\mathbb{C}} = (\mathbb{C}^*)^n \cdot U = (\mathbb{C}^*)^r \times \mathbb{C}^{n-r}$.

Let G be a Lie group. As an immediate consequence of the Identity Theorem (1.3) one obtains the

Lifting Lemma. *Let U be an open G-subspace of a complex G-space X. Assume that U intersects every irreducible component of X and furthermore that there exists a G-complexification $U^{\mathbb{C}}$ of U such that the corresponding map $\iota_U: U \to U^{\mathbb{C}}$ can be extended to a holomorphic map $\iota: X \to U^{\mathbb{C}}$. Then $U^{\mathbb{C}}$ with the map $\iota: X \to U^{\mathbb{C}}$ is a G-complexification of X.* □

1.5 Extension of equivariant maps

Let X be a $G^{\mathbb{C}}$-space. For a given point $x \in X$ let $b_x: G^{\mathbb{C}} \to X$, $b_x(g) = g \cdot x$ denote the orbit map. A G-subset U of X is called *orbit-connected* if $b_x^{-1}(U)$ is a G-connected subset of $G^{\mathbb{C}}$ for all $x \in X$.

Note that we have $b_{g\cdot x}^{-1}(U) = b_x^{-1}(U) \cdot g^{-1}$ for all $g \in G^{\mathbb{C}}$ and $x \in X$. In particular, a G-subset of the $G^{\mathbb{C}}$-space $G^{\mathbb{C}}$ is orbit-connected if and only if it is G-connected.

Extension Lemma. *If U is an orbit-connected open G-subset of a holomorphic $G^{\mathbb{C}}$-space X, then $G^{\mathbb{C}} \cdot U$ with the inclusion $\iota: U \to G^{\mathbb{C}} \cdot U$ is a G-complexification of U.*

Proof. Let Y be a holomorphic $G^{\mathbb{C}}$-space and ϕ a holomorphic G-map from U into Y. We define the extension $\phi^{\mathbb{C}}$ by the rule $\phi^{\mathbb{C}}(g \cdot x) = g \cdot \phi(x)$. If this defines a map $\phi^{\mathbb{C}}: G^{\mathbb{C}} \cdot U \to Y$, then it is automatically a holomorphic $G^{\mathbb{C}}$-map.

For a fixed point $x \in U$ we set $U_x = b_x^{-1}(U)$, where b_x denotes the orbit map. It follows from the Identity Theorem (1.3), that the holomorphic G-maps

$$\psi_1: G^{\mathbb{C}} \to Y, \quad g \to g \cdot \phi(x),$$
$$\psi_2: U_x \to Y, \quad g \to \phi(g \cdot x)$$

are equal on U_x.

Let now y be a point in $G^{\mathbb{C}} \cdot U$ such that $y = g_1 \cdot x_1 = g_2 \cdot x_2$ for some $g_j \in G^{\mathbb{C}}$ and $x_j \in U$. Then $g_2^{-1} \cdot g_1 \in U_{x_1}$ and from the above it follows that

$$g_1 \cdot \phi(x_1) = g_2 \cdot (g_2^{-1} \cdot g_1) \cdot \phi(x_1) = g_2 \cdot \phi(g_2^{-1} \cdot g_1 \cdot x_1) = g_2 \cdot \phi(x_2). \quad \square$$

Future tube example. Let $\mathbb{C}^{2\times 2}$ denote the vector space of complex 2×2-matrices. Let $\langle\,,\,\rangle$ be the usual Hermitian inner product on $\mathbb{C}^2$. For a matrix $Z \in \mathbb{C}^{2\times 2}$ define the adjoint $Z^* \in \mathbb{C}^{2\times 2}$ by $\langle Z \cdot v, w\rangle = \langle v, Z^* \cdot w\rangle$ for all $v, w \in \mathbb{C}^2$.

The domain $U = \left\{ Z \in \mathbb{C}^{2\times 2};\ \left\langle \frac{1}{2i}(Z^* - Z)\cdot v, v\right\rangle > 0 \text{ for all } v \in \mathbb{C}^2\backslash\{0\} \right\}$ in a $SL(\mathbb{C}^2)$-domain in $\mathbb{C}^{2\times 2}$ where the $SL(\mathbb{C}^2)$-action on $\mathbb{C}^{2\times 2}$ is given by the map $SL(\mathbb{C}^2) \times \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$, $(g, Z) \to gZg^*$. The complexification of the real Lie group $G = SL(\mathbb{C}^2)$ is the complex Lie group $G^{\mathbb{C}} = SL(\mathbb{C}^2) \times SL(\mathbb{C}^2)$ and $\iota: G \to G^{\mathbb{C}}$, $\iota(g) = (g, g^*)$ is the corresponding map. Consequently, the G-action on $\mathbb{C}^{2\times 2}$ is a restriction of the holomorphic $G^{\mathbb{C}}$-action $G^{\mathbb{C}} \times \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$, $(g, h, Z) \to gZh^{-1}$. The so called m-fold future tube is the m-fold product U^m of the domain U equipped with the diagonal G-action, i.e. $g \cdot (Z_1, \ldots, Z_m) = (gZ_1g^*, \ldots, gZ_mg^*)$. The G-domain U^m is an orbit-connected subset of $(\mathbb{C}^{2\times 2})^m$, see [S, W, p. 91]. Hence $G^{\mathbb{C}} \cdot U^m$ is a G-complexification of U^m. It should be noted that it is not known if $G^{\mathbb{C}} \cdot U^m$ is a domain of holomorphy.

Remark. There exist domains of holomorphy in $\mathbb{C}^2$ invariant under the linear action of $SL(\mathbb{R}^2)$ such that the $SL(\mathbb{R}^2)$-complexification is $\mathbb{C}^2\backslash\{0\}$.

2 Linearly equivariant maps

2.1 Vector spaces of equivariant maps

Let G be a Lie group and X a complex G-space. The group G acts linearly on the vector space $\mathcal{O}(X)$ by $g \cdot f = f \circ g^{-1}$. A holomorphic function f on X is called *G-finite* if the orbit $G \cdot f = \{f \circ g^{-1};\, g \in G\}$ is contained in a G-invariant linear subspace of $\mathcal{O}(X)$ which is finitely-dimensional.

Let V be a finite-dimensional linear G-subspace of $\mathcal{O}(X)$ and V' the dual vector space. By duality, G-acts linearly on V', $(g \cdot \lambda)(f) = \lambda(g^{-1} \cdot f)$. The holomorphic map $A: X \to V'$, $A(x)(f) = f(x)$ is linearly equivariant. With respect to a basis of V' which is dual to a basis $\{f_1, \ldots, f_n\}$ of V the map A is given by $A(x) = (f_1(x), \ldots, f_n(x))$. On the other hand, the components of a linearly equivariant holomorphic map from X to $\mathbb{C}^n$ are G-finite holomorphic functions on X.

2.2 Fourier series

Let K be a compact Lie group and X a complex K-space. Every continuous representation $\varrho: K \to GL(V)$ of K in a finite-dimensional complex vector space V is unitary with respect to a K-invariant Hermitian inner product on V. Let $\hat{K}$ denote a complete system of irreducible unitary representations of K. For $\varrho: K \to GL(V)$ in $\hat{K}$ we consider $GL(V)$ as an open subset of $E(V) = \mathrm{Hom}_{\mathbb{C}}(V, V)$ The formula $k \cdot A = A \circ \varrho(k^{-1})$ defines a linear K action on $E(V)$.

Let μ be a Haar measure on K which is normalized by $\mu(K) = 1$. We write $dk = d\mu(k)$. The composition of the linear map $P(\varrho): \mathcal{O}(X) \to \mathrm{Hol}_K(X, E(V))$,

$$P(\varrho)(f) = f_\varrho = d(\varrho) \int_K (k \cdot f)\varrho(k^{-1})dk,$$

where $d(\varrho)$ denotes the dimension of V, with the trace function $\mathrm{Tr}: E(V) \to \mathbb{C}$ is a continuous projection

$$p_\varrho: \mathcal{O}(X) \to \mathcal{O}(X), \qquad p_\varrho(f) = \mathrm{Tr}\, f_\varrho,$$

which depends only on the equivalence class of the representation ϱ, see [W, p. 260].

The series $\sum_{\varrho \in \hat{K}} \mathrm{Tr}\, f_\varrho$ is called the *Fourier series of f*. A general result of Harish-Chandra states that the Fourier series of $f \in \mathcal{O}(X)$ converges in the topology of $\mathcal{O}(X)$ to the function f [W, p. 260].

For later applications we need a more precise statement in our special situation. For this let $\| \; \|_\varrho$ denote the K-invariant norm on $E(V)$ which is associated to the Hermitian product $\langle A, B \rangle_\varrho = \mathrm{Tr}(A \cdot B^*)$ on $E(V)$, where B^* denotes the adjoint of $B \in E(V)$. For a subset C_α of X and $g \in \mathcal{O}(X)$ set $|g|_\alpha = \sup_{x \in C_\alpha} |g(x)|$ and $\|g\|_{\varrho,\alpha} = \sup_{x \in C_\alpha} \|g_\varrho(x)\|_\varrho$.

Fourier Theorem. *Let K be a compact Lie group and X a complex K-space. Let $\{C_\alpha\}$ be a covering of X with relatively compact open subsets. Then for every $f \in \mathcal{O}(X)$ and α the series $\sum_{\varrho \in \hat{K}} d(\varrho) \|f\|_{\varrho,\alpha}$ is a convergent majorant of the series $\sum_{\varrho \in \hat{K}} |\mathrm{Tr} f_\varrho|_\alpha$. Furthermore, $f = \sum_{\varrho \in \hat{K}} \mathrm{Tr}\, f_\varrho$.*

Proof. The proof of the theorem is the same as the corresponding proof in [W, p. 260], if one use the estimate

$$\begin{aligned}|\mathrm{Tr} f_\varrho|_\alpha &\leq d(\varrho)\,\|f\|_{\varrho,\alpha}\\ &= d(\varrho)c(\varrho)^{-m}\|\Omega^m(f)\|_{\varrho,\alpha}\\ &\leq d(\varrho)^2 c(\varrho)^{-m}|\Omega^m(f)|_\alpha,\end{aligned}$$

where Ω and $c(\varrho)$ are defined as in [W]. □

Example. Let X be a Reinhardt domain in $\mathbb{C}^n$, i.e. a domain in $\mathbb{C}^n$ which is invariant under the linear $(S^1)^n$-action on $\mathbb{C}^n$ which is given by the representation

$$(S^1)^n \to GL(\mathbb{C}^n), \qquad (t_1, \ldots, t_n) \to \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}, \qquad t_j \in S^1.$$

In this case the Fourier series of $f \in \mathcal{O}(X)$ coincides with the usual Laurent series

$$f = \sum_{m \in \mathbb{Z}^n} f_m, \qquad f_m(z) = \int_{(S^1)^m} t^m \cdot f(t^{-1} \cdot z)\,dt.$$

It follows that $f_m(t \cdot z) = t^m \cdot f_m(z)$ for all $t \in (S^1)^n$, $z \in X$, $m \in \mathbb{Z}^n$. Hence, f_m is an eigenvector of the linear map $e_j : \mathcal{O}(X) \to \mathcal{O}(X)$, $e_j = z_j \dfrac{\partial}{\partial z_j}$, $j = 1, \ldots, j_n$ with the eigenvalue m_j, $m = (m_1, \ldots, m_n)$. Consequently, for $\Omega : \mathcal{O}(X) \to \mathcal{O}(X)$, $\Omega = 1 + \sum_{j=1}^n e_j^2$, one has $\Omega(f_m) = (1 + \|m\|^2) f_m$, where we set $\|m\|^2 = \sum_{j=1}^n m_j^2$. For $C_\alpha \Subset X$ it follows from $\Omega(f_m)(z) = \int_{(S^1)^n} t^m \cdot \Omega(f)(t^{-1} \cdot z)\,dt$ that

$$(*) \qquad |f_m|_\alpha = \frac{1}{1 + \|m\|^2}\,|\Omega(f_m)|_\alpha \leq \frac{1}{1 + \|m\|^2}\,|\Omega(f)|_\alpha.$$

The estimate $(*)$ implies the convergence of the series $\sum_{m \in \mathbb{Z}^n} f_m$.

Let K be a compact Lie group and X a complex K-space. By $\mathcal{F}_K(X)$ we denote the algebra of K-finite holomorphic functions on X. Since every summand $\mathrm{Tr} f_\varrho$ of the Fourier series is a K-finite holomorphic function on X, the Fourier Theorem implies the following

Corollary. *Let X be a complex K-space.*

(i) *If $\mathcal{O}(X)$ separates the points of X, then $\mathcal{F}_K(X)$ also separates the points of X.*

(ii) *If $\mathcal{O}(X)$ defines local coordinates at a point $x \in X$, then $\mathcal{F}_K(X)$ also defines local coordinates at x.* □

In particular, for a Stein K-space X, the linearly equivariant holomorphic maps on X separate points and for a given point $x \in X$ there exists a linearly equivariant map on X which is an immersion at x.

2.3 Invariant functions

Let K be a compact Lie group and X a complex K-space. As we have explained, the Fourier series of a holomorphic function has summands which are given by linearly equivariant maps. The simplest equivariant maps are the invariant functions. Associated to the algebra $\mathcal{O}(X)^K$ of invariant holomorphic functions is the equivalence relation

$$R=\{(x,y)\in X\times X,\ f(x)=f(y) \text{ for all } f\in\mathcal{O}(X)^K\}.$$

The topological quotient of X with respect to R will be called the *categorical quotient of the K-space X* and is denoted by $X//K$. The space $X//K$ is a Hausdorff topological space. The quotient map from X onto $X//K$ is denoted by π_X.

For the further study of the quotient $X//K$ we need a simple property of the invariant functions. We formulate this more generally for linearly equivariant maps.

Let the compact Lie group K act linearly on $\mathbb{C}^n$, i.e. there is given a continuous representation of K into $GL(\mathbb{C}^n)$. A holomorphic map $\phi: Y\to X$, where X, Y are complex K-spaces, induces a linear K-map $\phi^*:\mathrm{Hol}(X,\mathbb{C}^n)\to\mathrm{Hol}(Y,\mathbb{C}^n)$, where the K-action on $\mathrm{Hol}(X,\mathbb{C}^n)$ is defined by $(k\cdot f)(x)=k\cdot(f(k^{-1}\cdot x))$. The K-action on $\mathrm{Hol}(Y,\mathbb{C}^n)$ is defined analogously. In particular, ϕ^* maps the vector space $\mathrm{Hol}_K(X,\mathbb{C}^n)$ into $\mathrm{Hol}_K(Y,\mathbb{C}^n)$. Hence it preserves the linearly equivariant maps. With this notations one has the following

Lemma. *If* $\phi^*:\mathrm{Hol}(X,\mathbb{C}^n)\to\mathrm{Hol}(Y,\mathbb{C}^n)$ *is surjective, then* $\phi(\mathrm{Hol}_K(X,\mathbb{C}^n))=\mathrm{Hol}_K(Y,\mathbb{C}^n)$.

Proof. For $g\in\mathrm{Hol}_K(Y,\mathbb{C}^n)$ and $f\in\mathrm{Hol}(X,\mathbb{C}^n)$ such that $g=\phi^*(f)$ one has $g=\phi^*(\hat{f})$, where $\hat{f}\in\mathrm{Hol}_K(X,\mathbb{C}^n)$ is defined by integration over the compact group, $\hat{f}(x)=\int_K (k\cdot f)(x)dk$. □

This lemma reflects the special nature of compact transformation group.

Example. The representation $\mathbb{C}\to GL(\mathbb{C}^2)$, $t\to\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$ defines a linear action on $\mathbb{C}^2$. The restriction $r:\mathcal{O}(\mathbb{C}^2)\to\mathcal{O}(Y)$, for $Y=\{(0,y);\ y\in\mathbb{C}\}$ is surjective. But one has $r(\mathcal{O}(\mathbb{C}^2)^{\mathbb{C}})=\mathbb{C}\neq\mathcal{O}(Y)=\mathcal{O}(Y)^{\mathbb{C}}$.

In the following corollaries (cf. [H 2]) we denote by K a compact Lie group and by X a Stein K-space. Let $\pi_X: X\to X//K$ be the quotient map.

Corollary 1. *If $p\in X//K$, then $\mathcal{O}(Y)^K=\mathbb{C}$ for all analytic K-subsets $Y\subset\pi_X^{-1}(p)$.*

Proof. This follows from Theorem B for Stein spaces and the Lemma. □

For a point $p\in X//K$ let $E_X(p)$ denote the intersection over all non-empty analytic K-subsets which are contained in $\pi_X^{-1}(p)$.

Corollary 2. *The analytic K-subset $E_X(p)$ is non-empty for all $p\in X//K$. Moreover, if Y is a K-irreducible analytic K-subset of X, such that $Y\cap E_X(p)\neq\emptyset$, then either $Y=E_X(p)$ or $\dim_{\mathbb{C}} Y>\dim_{\mathbb{C}} E_X(p)$.*

Proof. It follows from Corollary 1 that an analytic K-subset $D(p)$ of $\pi_X^{-1}(p)$ of minimal dimension is K-connected and smooth. If Y is a non-empty analytic K-subset of $\pi_X^{-1}(p)$, then the assumption $Y\cap D(p)=\emptyset$ leads to the contradiction $\mathcal{O}(Y\cup D(p))^K\neq\mathbb{C}$. Hence, one obtains $D(p)=E_X(p)$. □

For $x\in X$ let $B_X(x)$ denote the smallest analytic K-subset of X which contains x.

Corollary 3. *For $x, y \in X$ it follows that $\pi_X(x) = \pi_X(y)$ if and only if $B_X(x) \cap B_X(y) \neq \emptyset$.* □

Let $\mathcal{O}_X^K$ be the sheaf on $X//K$ which is defined by the correspondence $Q \to \mathcal{O}_X(\pi_X^{-1}(Q))^K$.

Corollary 4. *The pair $(X//K, \mathcal{O}_X^K)$ is a $\mathbb{C}$-ringed space. The map $\pi_X: X \to X//K$ is a morphism of the $\mathbb{C}$-ringed spaces $(X, \mathcal{O}_X)$ and $(X//K, \mathcal{O}_X^K)$. Furthermore, to every K-invariant holomorphic map from X into a complex space Y there exists one and only one morphism $\phi//K: (X//K, \mathcal{O}_X^K) \to (Y, \mathcal{O}_Y)$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\pi_X} & X//K \\ & \phi \searrow \quad \swarrow \phi_{X//K} & \\ & Y & \end{array}$$

commutes.

Proof. From Corollary 3 it follows, that every K-invariant holomorphic map on X is constant on the fibers of π_X. In particular, the sheaf $\mathcal{O}_X^K$ can be identified with the sheaf of germs of continuous functions f on $X//K$ such that $f \circ \pi_X$ is holomorphic on X. □

Corollary 5. *Let ϕ be a holomorphic K-map from X into a complex K-space which is an immersion at $x \in E_X(p)$. Then ϕ is an immersion along $\pi_X^{-1}(p)$.* □

We close this section by remarking that the analytic decomposition of X which is defined by the fibers of $\pi_X: X \to X//K$ is the coarsest analytic decomposition of the Stein space X into analytic K-subsets F with $\mathcal{O}(F)^K = \mathbb{C}$.

3 Orbit convexity

3.1 Polar decomposition

An invertible complex matrix $g \in GL(\mathbb{C}^n)$ can be written as a product of the form $g = k \cdot \exp v$, where k is an unitary matrix and v is a Hermitian matrix. This decomposition of matrices can be used to introduce polar coordinates on a closed subgroup G of $GL(\mathbb{C}^n)$ which is stable under the involutive Cartan isomorphism $\Theta: GL(\mathbb{C}^n) \to GL(\mathbb{C}^n)$, $\Theta(g) = (g^*)^{-1}$, where g^* denotes the adjoint matrix of g with respect to the usual Hermitian product on $\mathbb{C}^n$.

The set $K = \{g \in G;\ \Theta(g) = g\}$ of fixed points is contained in the unitary group, and is therefore a compact Lie group. If we denote by $\mathrm{gl}(\mathbb{C}^n)$ the Lie algebra of $GL(\mathbb{C}^n)$ and by $\theta: \mathrm{gl}(\mathbb{C}^n) \to \mathrm{gl}(\mathbb{C}^n)$ the derivative of Θ at the identity, i.e. $\theta(v) = -v$, then the eigenspace $\mathfrak{k} = \{v \in \mathfrak{g};\ \theta(v) = v\}$ is the Lie algebra of K. Since $i\mathfrak{k} = \{v \in \mathfrak{g};\ \theta(v) = -v\}$, the sum $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ is direct and $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C}$ is the Lie algebra of G.

Let K be a compact Lie group. As a consequence of the Peter and Weyl Theorem, there is an embedding ϱ_0 of K into some $GL(\mathbb{C}^n)$ such that $G = \varrho_0^{\mathbb{C}}(K^{\mathbb{C}})$ is stable with respect to Θ and $\varrho_0(K) = \{g \in G;\ \Theta(g) = g\}$ (cf. [Ch]). It can be shown that G together with the homomorphism $\varrho_0: K \to G$ is a complexification of K (cf. [Ho, p. 207]).

The complexification $K^{\mathbb{C}}$ of a compact Lie group K is an affine algebraic group which has the following defining property:

To every continuous representation $\varrho: K \to GL(\mathbb{C}^m)$ there exists a regular representation $\varrho^{\mathbb{C}}: K^{\mathbb{C}} \to GL(\mathbb{C}^m)$ such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\iota} & K^{\mathbb{C}} \\ & {}_{\varrho}\searrow \quad \swarrow_{\varrho^{\mathbb{C}}} & \\ & GL(\mathbb{C}^m) & \end{array}$$

commutes.

Since $\varrho^{\mathbb{C}}$ is a regular map, the orbits of $K^{\mathbb{C}}$ on $\mathbb{C}^m$ are Zariski open in its closure, i.e. the topological closure of a $K^{\mathbb{C}}$-orbit in $\mathbb{C}^m$ is an algebraic $K^{\mathbb{C}}$-subset of $\mathbb{C}^m$, which only contains orbits of smaller dimension in its boundary. Furthermore, one has $\varrho(K)^{\mathbb{C}} = \varrho^{\mathbb{C}}(K^{\mathbb{C}})$ and $\varrho^{\mathbb{C}}(K^{\mathbb{C}})$ is a closed subgroup of $GL(\mathbb{C}^m)$. The representation ϱ is injective if and only if $\varrho^{\mathbb{C}}$ is injective [Ho].

As a consequence of Theorems 3 and 4 in [Mo 1], cf. [Mo 2], the existence of the special embedding $\varrho_0: K \to GL(\mathbb{C}^m)$ implies the following

Decomposition Theorem of Mostow. *Let L be a closed subgroup of the compact Lie group K with Lie algebra $\mathfrak{l}$. Let the group L act on the Lie algebra $\mathfrak{k}^{\mathbb{C}}$ of $K^{\mathbb{C}}$ by the adjoint representation* $\mathrm{Ad}: L \to GL(\mathfrak{k}^{\mathbb{C}})$. *Then there exists a L-invariant linear subspace $\mathfrak{m}$ of $\mathfrak{k}$, such that the map $K \times_L i\mathfrak{m} \to K^{\mathbb{C}}/L^{\mathbb{C}}$, $[k, v] \to k \cdot \exp v L^{\mathbb{C}}$ is an isomorphism of topological K-spaces.* □

Remark. Since $\mathfrak{m}$ is a L-invariant subspace, the linear subspace $i\mathfrak{m}$ of $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} + i\mathfrak{k}$ is also L-invariant. The K-space $K \times_L i\mathfrak{m}$ is by definition the quotient of $K \times i\mathfrak{m}$ with respect to the L-action on $K \times i\mathfrak{m}$ which is defined by the formula $h \cdot (k, v) = (kh^{-1}, h \cdot v)$ (see also Sect. 4.3).

The particular case $L = \{1\}$ in the Decomposition Theorem is called the Polar Decomposition of the complex reductive group G.

3.2 Convexity of invariant sets

If $\mathbb{C}^n$ is viewed as a holomorphic $(\mathbb{C}^*)^n$-space with $(\mathbb{C}^*)^n$-action which is given by $(t_1, \ldots, t_n) \to \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}$, then the $(S^1)^n$-domains in $\mathbb{C}^n$ are called Reinhardt domains. It can be shown that a holomorphically convex Reinhardt domain U possesses the following property:

For $z = (z_1, \ldots, z_n) \in U$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ such that $(e^{v_1} z_1, \ldots, e^{v_n} z_n) \in U$ it follows that $\exp tv \cdot z = (e^{tv_1} z_1, \ldots, e^{tv_n} z_n) \in U$ for $t \in [0, 1]$.

This notion of convexity, which refers only to the orbits, can be carried over to more general situations. We restrict ourselves to compact Lie groups and their complexifications.

Let K be a compact Lie group with Lie algebra $\mathfrak{k}$ and $K^{\mathbb{C}}$ a complexification of K. A K-subset U of a $K^{\mathbb{C}}$-space X is called *orbit-convex* if for every $x \in U$ and $v \in i\mathfrak{k}$ such that $\exp v \cdot x \in U$, it follows that $\exp tv \cdot x \in U$ for $t \in [0, 1]$.

For later use we note some properties of orbit-convex sets.

Properties of orbit-convex sets. *Let K be a compact Lie group and X a $K^{\mathbb{C}}$-space. Then the following hold:*

(i) *Every intersection of orbit-convex K-subsets of X is orbit convex.*

(ii) *If ϕ is a $K^{\mathbb{C}}$-map from X into a $K^{\mathbb{C}}$-space Y, then the preimage of an orbit-convex K-subset under ϕ is orbit convex.*

(iii) *If U is an orbit-convex K-subset of X and L a closed subgroup of K, then the L-subset U of the $L^{\mathbb{C}}$-space X is also orbit-convex.*

(iv) *A K-subset U of X is orbit-convex if and only if for all compact tori T in K the T-subset U is an orbit-convex subset of the $T^{\mathbb{C}}$-space X.*

Proof. The properties (i), (ii), and (iii) are direct consequences of the definitions, so we only give the proof of (iv). On the one hand, this follows from the inclusion $i\mathfrak{t} \subset i\mathfrak{k}$ for every compact torus T of K with Lie algebra $\mathfrak{t}$. On the other hand, the topological closure of the group $\{\exp tw;\ t\in\mathbb{R}\}$ for $w\in\mathfrak{k}$ is a compact torus T in K and iw is contained in $i\mathfrak{t}$. □

3.3 Complexification of orbit-convex subsets

A first application of the notion of convexity is the following

Proposition. *Let X be a holomorphic $K^{\mathbb{C}}$-space and U an orbit-convex open K-subset of X. Then every analytic K-subset A of U is an open orbit-convex subset of the $K^{\mathbb{C}}$-space $K^{\mathbb{C}}\cdot A$. Moreover, $K^{\mathbb{C}}\cdot A$ is an analytic subset of the open $K^{\mathbb{C}}$-subset $K^{\mathbb{C}}\cdot U$ of X and with respect to the inclusion $\iota: A\to K^{\mathbb{C}}\cdot A$ the holomorphic $K^{\mathbb{C}}$-space $K^{\mathbb{C}}\cdot A$ is a K-complexification of the K-space A.*

Proof. Let $\mathfrak{k}$ denote the Lie algebra of K. From the assumption that U is orbit-convex and A is a K-invariant analytic subset of U, it follows that

$$(*)\qquad \text{If}\quad a\in A \quad\text{and}\quad v\in i\mathfrak{k}\quad \text{are such that}\quad \exp v\cdot a\in U,\quad \text{then}\quad \exp tv\cdot a\in A \quad \text{for}\quad t\in[0,1].$$

In particular, the K-subset A of X is orbit-convex.

The sets $g\cdot U$, $g\in K^{\mathbb{C}}$ form an open covering of $K^{\mathbb{C}}\cdot U$. So from

$$(**)\qquad K^{\mathbb{C}}\cdot A\cap g\cdot U=g\cdot A \quad\text{for all}\quad g\in K^{\mathbb{C}}$$

it follows that $K^{\mathbb{C}}\cdot A$ is an analytic subset of $K^{\mathbb{C}}\cdot U$.

To prove (**) it is sufficient to verify $K^{\mathbb{C}}\cdot A\cap U=A$. For this, suppose that $x\in K^{\mathbb{C}}\cdot A\cap U\supset A$. There exists $k\in K$, $v\in i\mathfrak{k}$ (3.1 Polar Decomposition) and $a\in A$ such that $x=k\cdot\exp v\cdot a$. Hence $\exp v\cdot a\in U$ and by (*) it follows that $\exp v\cdot a\in A$. Consequently, $x=k\cdot\exp v\cdot a\in k\cdot A=A$.

Since every orbit-convex K-subset of X is orbit-connected (3.1 Polar Decomposition), $K^{\mathbb{C}}\cdot A\cap U=A$ implies that $K^{\mathbb{C}}\cdot A$ is a K-complexification of A (1.5 Extension Lemma). □

Corollary. *For an analytic K-subset A of an open orbit-convex K-subset U of a holomorphic $K^{\mathbb{C}}$-space X one has $K^{\mathbb{C}}\cdot A\cap U=A$.* □

By definition, the *orbit-convex hull* of a K-subset U of a $K^{\mathbb{C}}$-space Y is the smallest orbit-convex K-subset $\mathrm{Conv}_K(U)$ which contains U.

The orbit-convex hull of a K-subset U of a $K^{\mathbb{C}}$-space Y can be constructed inductively as follows.

Let $\mathfrak{k}$ be the Lie algebra of the compact Lie group K and set $U=U_0$. If U_k is constructed, then define $U_{k+1}=\{y\in Y;$ there exist $y_0\in U_k$ and $v\in i\mathfrak{k}$, such that $\exp v\cdot y_0\in U_k$ and $y=\exp t_0 v\cdot y_0$ for some $t_0\in[0,1]\}$. It follows that $\mathrm{Conv}_K(U)=\bigcup_{k=0}^{\infty} U_k$.

Note that if U is open in Y, then $\mathrm{Conv}_K(U)$ is also open in Y.

3.4 *Invariant domains of holomorphy*

There is a connection between orbit-convex domains and invariant plurisubharmonic functions which is at least implicitly known, cf. [Ro; L]. The simplest case is that where the compact group is a real torus $T \cong (S^1)^r$. The exponential map $\exp: \mathfrak{t}^{\mathbb{C}} \to T^{\mathbb{C}}$ from the Lie algebra $\mathfrak{t}^{\mathbb{C}}$ of $T^{\mathbb{C}}$ onto $T^{\mathbb{C}}$ is in this case the universal covering map of $T^{\mathbb{C}} \cong (\mathbb{C}^*)^r$. The decomposition $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t}$ defines a real structure on $\mathfrak{t}^{\mathbb{C}}$. A computation of the Hessian matrix in corresponding real coordinates proves the following

Lemma. *Let U be an orbit-convex T-domain in $T^{\mathbb{C}}$ which contains the identity* 1. *Then, for a plurisubharmonic K-invariant function $\phi: U \to \mathbb{R}$, the function $\hat{\phi}: \exp^{-1}(U) \to \mathbb{R}$, $\hat{\phi} = \phi \circ \exp$ is convex. It follows that*

$$\phi(\exp tv) = \hat{\phi}(tv) \leqq (1-t)\hat{\phi}(0) + t\hat{\phi}(v)$$
$$= (1-t)\phi(1) + t\phi(\exp v).$$

for all $t \in [0,1]$. If ϕ is strictly plurisubharmonic, then $\hat{\phi} \mid i\mathfrak{t} \cap \exp^{-1}(U): i\mathfrak{t} \cap \exp^{-1}(U) \to \mathbb{R}$ is strictly convex. □

Now we give an application of this Lemma to complex spaces. For this let $\mathfrak{k}$ be the Lie algebra of the compact Lie group K.

Proposition. *Let Y be a holomorphic $K^{\mathbb{C}}$-space and X an orbit-convex open K-subset of Y. For every plurisubharmonic K-function $\phi: X \to \mathbb{R}$, the K-set $D = \{x \in X;\ \phi(x) < 1\}$ is an orbit-convex subset of Y.*

Proof. Let x be a point in D and v an element in $i\mathfrak{k}$ such that $\exp v \cdot x$ is contained in D. There exists a torus T in K such that $v \in i\mathfrak{t}$, where $\mathfrak{t}$ denotes the Lie algebra of T. Since the orbit map $b: T^{\mathbb{C}} \to Y$, $t \to t \cdot x$ is holomorphic, the K-function $\phi \circ b: b^{-1}(x) \to \mathbb{R}$ is plurisubharmonic. The Lemma implies that $\phi(\exp tv \cdot x) < 1$ for $t \in [0,1]$. Thus D is orbit-convex. □

In the special case $X = Y$ the Proposition will be applied in the proof of the next theorem, which is a partial converse of the Proposition in Sect. 3.3. Remember that $\mathscr{F}_K(X)$ denotes the algebra of K-finite holomorphic functions on a complex K-space X.

Theorem. *Let K be a compact Lie group with complexification $K^{\mathbb{C}}$. Let X be an open K-subset of a holomorphic $K^{\mathbb{C}}$-space Y such that $K^{\mathbb{C}} \cdot X$ together with the inclusion $\imath: X \to K^{\mathbb{C}} \cdot X$ is a K-complexification of X. Then for $X^{\mathbb{C}} = K^{\mathbb{C}} \cdot X$, the following hold:*

(i) *The restriction map $\mathcal{O}(X^{\mathbb{C}}) \to \mathcal{O}(X)$ induces an isomorphism $\mathscr{F}_K(X^{\mathbb{C}}) \to \mathscr{F}_K(X)$. In particular, $(X^{\mathbb{C}}, X)$ is a Runge pair.*

(ii) *If X is a Stein K-space, then X is an orbit-convex K-subset of $X^{\mathbb{C}}$.*

Proof. Every K-finite holomorphic function $f: X \to \mathbb{C}$ is a component of linearly equivariant holomorphic map on X (2.1). The definition of a K-complexification implies the surjectivity of the restriction $\mathscr{F}_K(X^{\mathbb{C}}) \to \mathscr{F}_K(X)$. Hence it is an isomorphism. The Runge property (i) follows from the Fourier Theorem (2.2).

In order to prove property (ii), let f be a holomorphic function on X and $F = \sum_{\varrho \in \hat{K}} \operatorname{Tr} f_\varrho$ the Fourier series of f. Here we use the same notation as in Sect. 2.2. By the assumptions, the linearly equivariant holomorphic maps f_ϱ extend to holomorphic maps on $X^{\mathbb{C}}$ which we also denote by f_ϱ. Consequently, the K-invariant functions which are defined by $x \to \|f_\varrho(x)\|_\varrho^2$ are plurisubharmonic functions on $X^{\mathbb{C}}$. Let $C_{\hat{\alpha}}$ denote the orbit-convex hull (3.3) of $C_\alpha \Subset X^{\mathbb{C}}$ and $\hat{X}$ the orbit-convex hull of X in $X^{\mathbb{C}}$. Let $\{C_\alpha\}$ be a covering of X by relatively compact open K-subsets C_α. Assume that $C_\alpha \subset C_{\alpha+1}$. Now $\hat{X} = \bigcup_{\alpha=1}^{\infty} C_{\hat{\alpha}}$ and furthermore, the K-set $\{z \in X^{\mathbb{C}};\ \|f_\varrho(z)\|_\varrho < \|f\|_{\varrho,\alpha}\}$ is orbit-convex (Proposition) and contains C_α. It follows that $\|f\|_{\varrho,\alpha} = \|f\|_{\varrho,\hat{\alpha}}$, because $\|f\|_{\varrho,\hat{\alpha}} > \|f\|_{\varrho,\alpha}$ would imply that there exists $x \in C_{\hat{\alpha}} \setminus \{z \in X^{\mathbb{C}};\ \|f_\varrho(z)\|_\varrho < \|f\|_{\varrho,\alpha}\}$. Hence $F = \sum_{\varrho \in \hat{K}} \operatorname{Tr} f_\varrho$ defines a holomorphic continuation of f to $\hat{X}$ (Fourier Theorem, 2.2). But X is a Stein space, and therefore, $X = \hat{X}$. □

4 Locally homogeneous spaces

4.1 Holomorphically separable spaces

Let K be a compact Lie group and X a Stein K-space. We already know that there are uniquely determined minimal analytic K-subsets E in the fibers of the quotient map $\pi_X: X \to X//K$ (2.3). Our analysis of actions of compact groups on Stein spaces will begin with the study these minimal analytic subsets. It is easy seen that $K^{\mathbb{C}}$ acts "locally transitively" on such sets and thus one expects $E^{\mathbb{C}}$ to be $K^{\mathbb{C}}$-homogeneous.

Proposition. *Let K be a compact Lie group and X a holomorphically separable complex K-space. If X does not contain proper analytic K-subsets, then there exists a K-complexification $X^{\mathbb{C}}$. The complex space $X^{\mathbb{C}}$ is $K^{\mathbb{C}}$-homogeneous and holomorphically separable. The corresponding map $\iota: X \to X^{\mathbb{C}}$ is an open embedding.*

Proof (cf. [H 2]). From the Identity Theorem (1.1) and the assumption that X does not contain proper analytic subsets we have the following fact:

If ψ is a holomorphic K-map from X into a complex K-space Y, then $\psi(X)$ is contained in a smooth locally analytic K-subset of Y. The rank of ψ is constant. In particular, if Y is $K^{\mathbb{C}}$-homogeneous, then $\psi(X)$ is open in Y.

Let $\{C_\alpha\}$ be a covering of X by relatively compact open K-subsets C_α. We assume $C_\alpha \subset C_{\alpha+1}$. For every α there exists a linearly equivariant holomorphic K-map $\psi_\alpha: X \to \mathbb{C}^{n_\alpha}$ such that the restriction $\psi_\alpha | C_\alpha$ is injective (2.2 Corollary). We fix a point $x_1 \in C_1$ and set $w_\alpha = \psi_\alpha(x_1)$. In order to compare the maps ψ_α, we define

$$\phi_1 = \psi_1, \quad m_1 = n_1, \quad v_1 = w_1$$

and, after ϕ_α, m_α, v_α are defined,

$$\phi_{\alpha+1} = \phi_\alpha \times \psi_{\alpha+1}, \quad m_{\alpha+1} = m_\alpha + n_{\alpha+1}, \quad v_{\alpha+1} = (v_\alpha, w_{\alpha+1}) \in \mathbb{C}^{m_{\alpha+1}}.$$

Since the linear action of K on $\mathbb{C}^{m_\alpha}$ extends to an algebraic action of $K^{\mathbb{C}}$ on $\mathbb{C}^{m_\alpha}$ (1.1), for all α one obtains

1. $\phi_\alpha(X) \subset K^{\mathbb{C}} \cdot v_\alpha$.
2. ϕ_α is an open immersion.
3. $\phi_\alpha | C_\alpha$ is an injection.
4. $K^{\mathbb{C}} \cdot v_\alpha$ is $K^{\mathbb{C}}$-biholomorphically equivalent to $K^{\mathbb{C}}/H_\alpha$, where H_α denotes the $K^{\mathbb{C}}$-isotropy group of $K^{\mathbb{C}}$ at v_α.

By construction, the isotropy group $H_{\alpha+1}$ is contained in H_α. The diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\phi_\alpha}\swarrow & & \searrow{\scriptstyle\phi_{\alpha+1}} \\ K^{\mathbb{C}}\cdot v_\alpha & \underset{p_\alpha}{\longleftarrow} & K^{\mathbb{C}}\cdot v_{\alpha+1} \end{array}$$

commutes for all $\alpha\in\mathbb{N}$ where p_α denotes the restriction of the $K^{\mathbb{C}}$-equivariant projection from $\mathbb{C}^{m_{\alpha+1}}=\mathbb{C}^{m_\alpha}\times\mathbb{C}^{n_{\alpha+1}}$ to $\mathbb{C}^{m_\alpha}$. Since the isotropy groups H_α are linear algebraic groups, each of them has only finitely many connected components. Moreover, from $\dim_{\mathbb{C}}H_\alpha=\dim_{\mathbb{C}}K^{\mathbb{C}}-\dim_{\mathbb{C}}X$ it follows, that the connected component of the identity is the same group for all α. Hence there exists an α_0 such that $H_\alpha=H_{\alpha_0}$ for all $\alpha\geqq\alpha_0$. For such an α, the map p_α is biholomorphic. In particular, ϕ_{α_0} is an open embedding.

Let H be the smallest and closed complex subgroup of H_{α_0} such that there exists an open $K^{\mathbb{C}}$-equivariant embedding $\imath: X\to K^{\mathbb{C}}/H$. Note that H is obtained from H_{α_0} by omitting, if necessary, some connected components.

We set $X^{\mathbb{C}}=K^{\mathbb{C}}/H=K^{\mathbb{C}}\cdot v$, where $v=\imath(x_1)=H$. In order to prove the universality property for equivariant holomorphic maps, let ψ be a holomorphic K-map from X into a holomorphic $K^{\mathbb{C}}$-space Y. The image of the K-map $\imath\times\psi: X\to K^{\mathbb{C}}\cdot v\times Y$, $x\to(\imath(x),\psi(x))$ is contained in the $K^{\mathbb{C}}$-orbit $K^{\mathbb{C}}\cdot(v,w)$ $=\{(g\cdot v, g\cdot w); g\in K^{\mathbb{C}}\}$ where we set $w=\psi(x_1)$. Furthermore, viewed as a map from X into $K^{\mathbb{C}}\cdot(v,w)\cong K^{\mathbb{C}}/H\cap(K^{\mathbb{C}})_w$, $\imath\times\psi$ is an open embedding. Let $p: K^{\mathbb{C}}\cdot(v,w)$ $\to K^{\mathbb{C}}\cdot v$ denote the restriction of the natural projection. It follows from the definition of $K^{\mathbb{C}}/H$ that p is biholomorphic. Moreover, viewed as a map from $K^{\mathbb{C}}\cdot v$ into $K^{\mathbb{C}}\cdot v\times Y$, the inverse map p^{-1} is holomorphic. If we denote by q the projection $K^{\mathbb{C}}\cdot v\times Y\to Y$, then the diagram

$$\begin{array}{ccc} X & \xrightarrow{\imath} & K^{\mathbb{C}}\cdot v \\ {\scriptstyle\psi}\searrow & & \swarrow{\scriptstyle\psi^{\mathbb{C}}} \\ & Y & \end{array}$$

commutes, where $\psi^{\mathbb{C}}=q\circ p^{-1}$. Finally, the $K^{\mathbb{C}}$-map $\psi^{\mathbb{C}}$ is uniquely determined by $\psi^{\mathbb{C}}(v)=w$. □

Corollary. *Let X be a holomorphically separable K-connected open K-subset of a $K^{\mathbb{C}}$-homogeneous space Y. Then the inclusion $j: X\to Y$ induces a $K^{\mathbb{C}}$-equivariant holomorphic covering map $j^{\mathbb{C}}: X^{\mathbb{C}}\to Y$.* □

4.2 Totally real points

Let Y be a $K^{\mathbb{C}}$-homogeneous space and U an open K-subset. In general, it is not true that Y is the K-complexification of U. For example, the S^1-complexification of an annulus in a compact complex torus is $\mathbb{C}^*$.

This example is, of course, not relevant for the study of K-actions on Stein spaces. However, the following examples reflect phenomen of central importance. Let U be a sufficiently small open orbit-connected K-subset in $K^{\mathbb{C}}$ which contains K. In general, there exist many finite subgroups Γ in $K^{\mathbb{C}}$, such that $K\cap g\Gamma g^{-1}=\{1\}$ for all $g\in U$. For such groups, the image X of U in $K^{\mathbb{C}}/\Gamma$ is biholomorphically equivalent to U and the K-action on X is free, i.e. $K_x=\{1\}$ for all $x\in X$. By construction, the complexification of X is $K^{\mathbb{C}}$.

On the other hand, there are also "positive" examples. The compact group $SU(\mathbb{C}^2)$ acts on the holomorphic $SL(\mathbb{C}^2)$-manifold $Y=SL(\mathbb{C}^2)/H$ where $H=\left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}; t\in\mathbb{C}^*\right\}$. The $SU(\mathbb{C}^2)$-orbit through $p=H\in Y$ is of real dimension two and isomorphic to $SU(\mathbb{C}^2)/S^1$. Any other $SU(\mathbb{C}^2)$-orbit has real dimension three, hence is a hypersurface. Every open Stein $SU(\mathbb{C}^2)$-subset X of Y contains the "minimal" $SU(\mathbb{C}^2)$-orbit of dimension two and $X^{\mathbb{C}}=Y$.

In the above examples the K-complexification always exists and is determined by some "minimal" K-orbit. This is a general observation which is explained by the last theorem of this section. For this we begin by proving two lemma.

Lemma 1. *Let T be a real compact torus with Lie algebra $\mathfrak{t}$ and S a closed subgroup of T with Lie algebra $\mathfrak{f}$. Let π denote the quotient map $T^{\mathbb{C}}\to T^{\mathbb{C}}/S^{\mathbb{C}}$. Then for an orbit-convex T-domain U in $T^{\mathbb{C}}$ the T-domain $\pi(U)$ is also orbit-convex. In particular, for $M=T/S$ the M-domain $\pi(U)$ is orbit-convex in the $M^{\mathbb{C}}$-space $M^{\mathbb{C}}\cong T^{\mathbb{C}}/S^{\mathbb{C}}$.*

Proof. Let $v\in i\mathfrak{t}$ and $x\in U$ be such that $\exp(v)\cdot\pi(x)\in\pi(U)$. There exist $h\in S^{\mathbb{C}}$, $w\in i\mathfrak{f}$ and $s\in S$ such that $h=s\cdot\exp w$ and $\exp v\cdot x\cdot h=s\cdot\exp(w+v)\cdot x\in U$. Since U is orbit-convex, it follows that $\exp tv\cdot\pi(x)=\pi(\exp tv\cdot x\cdot s\cdot\exp tw)\in\pi(U)$ for $t\in[0,1]$. □

Let K be a compact Lie group with Lie algebra $\mathfrak{k}$ and L a closed subgroup of K with Lie algebra $\mathfrak{l}$.

Lemma 2. *Let U be an orbit-convex K-domain in $K^{\mathbb{C}}/L^{\mathbb{C}}$ which contains the point $p=L^{\mathbb{C}}\in K^{\mathbb{C}}/L^{\mathbb{C}}$ and $\phi: U\to\mathbb{R}$ a strictly plurisubharmonic K-function with a local minimum at p. If $g\cdot p\in U$ for some $g\in K^{\mathbb{C}}$, then the inequality $\phi(g\cdot p)\leqq\phi(p)$ implies that $g\cdot p\in K\cdot p$. If in addition $g=\exp v$ for some $v\in i\mathfrak{k}$, then it follows that $v\in i\mathfrak{l}$.*

Proof. Let $g\in K^{\mathbb{C}}$ with $g\cdot p\in U$ and $\phi(g\cdot p)\leqq\phi(p)$ be given. We write g as a product $g=k\cdot\exp v$, where $k\in K$ and $v\in i\mathfrak{k}$ (Polar Decomposition, 3.1). Since ϕ is K-invariant, we have $\phi(\exp v\cdot p)\leqq\phi(p)$. Because K is a compact Lie group, there exists a compact torus T with Lie algebra $\mathfrak{t}$ such that $v\in i\mathfrak{t}$. The T-set $U_T=U\cap T^{\mathbb{C}}\cdot p$ is orbit-convex (3.2).

Let S denote the isotropy group T_p and M the compact torus T/S. Then $T^{\mathbb{C}}\cdot p$ is canonically isomorphic to $M^{\mathbb{C}}$ and U_T is an orbit-convex M-subset in U_T (Lemma 1). The M-invariant function $\phi_M: U_T\to\mathbb{R}$, $\phi_M=\phi\mid U_T$ is strictly plurisubharmonic. It follows that $v\in i\mathfrak{f}\subset i\mathfrak{l}$ (3.4 Lemma) where $\mathfrak{f}$ denotes the Lie algebra of S.

This proves $\exp v\in L^{\mathbb{C}}$ and consequently $g\cdot p=k\cdot\exp v\cdot p=k\cdot p\in K\cdot p$. □

Remark. Note that the statement of Lemma 2 remains true, with the same proof, if one replaces the assumption "ϕ obtains a minimal value at p" by "p is a critical point of ϕ".

Let K be a compact Lie group and Y a holomorphic $K^{\mathbb{C}}$-space. For a point $y\in Y$ the isotropy group K_y is contained in the isotropy group $(K^{\mathbb{C}})_y$. The map $\imath:(K_y)^{\mathbb{C}}\to(K^{\mathbb{C}})_y$ is injective (see 3.1).

Example. The isotropy group of the unitary group $U(\mathbb{C}^2)$ at $(1,0)\in\mathbb{C}^2$, where $U(\mathbb{C}^2)$ acts on $\mathbb{C}^2$ by matrix multiplication, is the group $\left\{\begin{pmatrix} 1 & 0 \\ 0 & t\end{pmatrix}; |t|=1\right\}$ which is isomorphic to S^1. The isotropy group of $GL(\mathbb{C}^2)=U(\mathbb{C}^2)^{\mathbb{C}}$ at $(1,0)$ is the group $\left\{\begin{pmatrix} 1 & z \\ 0 & t\end{pmatrix}; t\neq 0\right\}$.

A point y in a holomorphic $K^{\mathbb{C}}$-space Y is called a *totally real K-point*, if the inclusion $\iota: K_y \to (K^{\mathbb{C}})_y$ induces an isomorphism $\iota^{\mathbb{C}}: (K_y)^{\mathbb{C}} \to (K^{\mathbb{C}})_y$. Note that the K-orbit of a totally real K-point is a totally real submanifold. The converse of this statement is not necessarily true.

Example. All points in the K-orbit through the point $p = L^{\mathbb{C}} \in K^{\mathbb{C}}/L^{\mathbb{C}}$ are totally real K-points. Every $K^{\mathbb{C}}$-fixed point is a totally real K-point.

Theorem. *Let K be a compact Lie group and X a Stein K-space which does not contain proper analytic K-subsets. Then there exists a point x in the complexification $X^{\mathbb{C}}$ of X such that $\iota(x) \in X^{\mathbb{C}}$ is a totally real K-point.*

Proof. The complexification $X^{\mathbb{C}}$ of X is a $K^{\mathbb{C}}$-homogeneous holomorphically separable complex space (4.1 Proposition). Since X is a Stein K-space, there exists a strictly plurisubharmonic K-function $\psi: X \to \mathbb{R}$ which obtains a minimal value in some point $x \in X$. For every K-orbit one has that $\dim_{\mathbb{R}} K \cdot x \geqq \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} X^{\mathbb{C}}$ (1.3). Because the set of critical points of the strictly plurisubharmonic function ψ is contained in a totally real submanifold of X, cf. [RH; W], it follows that $\dim_{\mathbb{R}} K/K_x = \dim_{\mathbb{C}} K^{\mathbb{C}} \cdot x = \dim_{\mathbb{C}} X^{\mathbb{C}}$.

The K-space X is an orbit-convex K-domain in $X^{\mathbb{C}}$ (3.4 Theorem). Let H denote the isotropy group $(K^{\mathbb{C}})_x$ and L the compact isotropy group K_x. Since $\dim_{\mathbb{C}} L^{\mathbb{C}} = \dim_{\mathbb{R}} L = \dim_{\mathbb{C}} H$, the complexification $L^{\mathbb{C}}$ is an open subgroup of H. The group H is algebraic. Hence it has only finitely many connected components. Thus the $K^{\mathbb{C}}$-equivariant projection $p: K^{\mathbb{C}}/L^{\mathbb{C}} \to K^{\mathbb{C}}/H$, $g \cdot L^{\mathbb{C}} \to g \cdot x$ is a finite covering map. Recall, that the inverse image $\hat{X} = p^{-1}(X)$ is an orbit-convex K-domain in $K^{\mathbb{C}}/L^{\mathbb{C}}$ (3.2).

We now show that the covering map $\hat{p}: \hat{X} \to X$, $\hat{p} = p \mid X$ is injective. For this, first note that $\phi = \psi \circ \hat{p}$ is a strictly plurisubharmonic K-function on $\hat{X}$. A point $z \in \hat{p}^{-1}(x)$ is of the form $z = g \cdot \hat{x}$ for some $g \in K^{\mathbb{C}}$, where we set $\hat{x} = L^{\mathbb{C}} \in K^{\mathbb{C}}/L^{\mathbb{C}}$. It follows from the definition that $\phi(z) = \phi(g \cdot \hat{x}) = \phi(\hat{x})$. Consequently, $g = k \cdot h$ for some $k \in K$ and $h \in L^{\mathbb{C}}$ (Lemma 2). From $k \cdot x = k \cdot \hat{p}(\hat{x}) = \hat{p}(g \cdot \hat{x}) = g \cdot x = x$ we obtain that $k \in K_x = L$. Thus $g = k \cdot h \in L^{\mathbb{C}}$ and $z = g \cdot \hat{x} = \hat{x}$. This proves that $\hat{p}^{-1}(x) = \hat{x}$. Consequently, $\hat{p}: \hat{X} \to X$ is an isomorphism and $H = L^{\mathbb{C}}$. □

The above proof also yields the following result which goes back to Matsushima and Onisčik.

Corollary. *Let K be a compact Lie group and Y a $K^{\mathbb{C}}$-homogeneous holomorphic $K^{\mathbb{C}}$-space. If there exists a strictly plurisubharmonic K-function $\phi: Y \to \mathbb{R}$ which obtains a minimal value at some point $x \in Y$, then the isotropy group $(K^{\mathbb{C}})_x = (K_x)^{\mathbb{C}}$ is a complex reductive group.* □

4.3 Homogeneous vector bundles

For a closed subgroup H of a Lie group G and a H-space Y, let $G \times_H Y$ denote the bundle with typical fiber Y associated to the principal bundle $G \to G/H$. This bundle is by definition the quotient of the H-space $G \times Y$ with respect to H, where the H action on $G \times Y$ is defined by $h \cdot (g, y) = (gh^{-1}, h \cdot y)$. The corresponding quotient map $G \times Y \to G \times_H Y$ is open. We denote the image of a point $(x, y) \in G \times Y$ in $G \times_H Y$ by $[x, y]$. The map $G \times G \times_H Y \to G \times_H Y$, $(g, [x, y]) \to [gx, y]$ is a G-action on $G \times_H Y$. Note that the bundle projection $G \times_H Y \to G/H$, $[g, y] \to gH$ is a G-map.

4.4 Minimal compact orbits

Let G be a Lie group and H a closed subgroup of G. We denote by $\mathrm{Typ}_G(X)$ the G-isomorphism class of the G-homogeneous space $X=G/H$ and call it the *G-orbit type*. There is a partial ordering on the set of G-orbit types which is defined as follows.

If $X_1=G/H_1$ and $X_2=G/H_2$ represent two G-orbit types, then X_1 is called smaller as X_2 if there exists a G-map from X_2 into X_1. Note that such a map is automatically surjective. We set $\mathrm{Typ}_G(X_1)\leqq\mathrm{Typ}_G(X_2)$ if and only if X_1 is smaller as X_2. The G-orbit types of two G-homogeneous spaces are equal if and only if the corresponding isotropy groups are G-conjugate.

The *orbit type of a G-space X* is by definition the set $\{\mathrm{Typ}_G(G\cdot x);\ x\in X\}$. A G-homogeneous space which represents a minimal element in $\{\mathrm{Typ}_G(G\cdot x);\ x\in X\}$ is called a *minimal G-orbit* in X.

Let K be a compact Lie group and L a closed subgroup of K. As a K-space, the $K^{\mathbb{C}}$-homogeneous complex space $K^{\mathbb{C}}/L^{\mathbb{C}}$ is isomorphic to a real K-vector bundle over K/L of the form $K\times_L F$ (3.1 Decomposition Theorem). Consequently, for every $x\in K^{\mathbb{C}}/L^{\mathbb{C}}$ the bundle projection $K\times_L F\to K/L$ induces a K-map $K\cdot x\to K/L$. Thus every K-orbit in $K^{\mathbb{C}}/L^{\mathbb{C}}$ can be compared with K/L and K/L is a minimal K-orbit in $K^{\mathbb{C}}/L^{\mathbb{C}}$. Furthermore, it follows that the K-orbit through a point $x\in K^{\mathbb{C}}/L^{\mathbb{C}}$ is minimal if and only if $\mathrm{Typ}_K(K\cdot x)=\mathrm{Typ}_K(K/L)$. This is the case if and only if K_x is conjugate to L by an element of K. The following is a characterization of minimal K-orbits in $K^{\mathbb{C}}/L^{\mathbb{C}}$:

Theorem. *Let K be a compact Lie group and L a closed subgroup of K. For a point $x\in K^{\mathbb{C}}/L^{\mathbb{C}}$ the following statements are equivalent.*

(i) *The point x is a totally real K-point.*

(ii) *The K-orbit through x is minimal.*

(iii) *On $K^{\mathbb{C}}/L^{\mathbb{C}}$ there exists a strictly plurisubharmonic proper K-invariant function $p: K^{\mathbb{C}}/L^{\mathbb{C}}\to[0,\infty)$, such that $K\cdot x=p^{-1}(0)$.*

(iv) *The K-orbit through x has a basis of neighbourhoods which consists of orbit-convex Stein K-subsets.*

Proof. First, we show that (i) implies (ii).

Every point $x\in K^{\mathbb{C}}/L^{\mathbb{C}}$ is representable by $gL^{\mathbb{C}}$ for a suitable $g\in K^{\mathbb{C}}$. By assumption, $\iota^{\mathbb{C}}:(K_x)^{\mathbb{C}}\to(K^{\mathbb{C}})_x$ is an isomorphism. It follows that $\dim_{\mathbb{R}}K_x=\dim_{\mathbb{C}}(K_x)^{\mathbb{C}}=\dim_{\mathbb{C}}(K^{\mathbb{C}})_x=\dim_{\mathbb{C}}L^{\mathbb{C}}=\dim_{\mathbb{R}}L$. Since every K-orbit in $K^{\mathbb{C}}/L^{\mathbb{C}}$ is comparable with the minimal orbit K/L (3.1 Decomposition Theorem), there exists a K-map from $K\cdot x$ onto K/L. Thus the equality $\dim_{\mathbb{R}}K_x=\dim_{\mathbb{R}}L$ implies that K_x is an open subgroup of kLk^{-1} for a suitable $k\in K$. From $(K_x)^{\mathbb{C}}=kL^{\mathbb{C}}k^{-1}=(K^{\mathbb{C}})_x=gL^{\mathbb{C}}g^{-1}$, it follows that K_x and kLk^{-1} have the same number of connected components. This implies $K_x=kLk^{-1}$ and consequently $\mathrm{Typ}_K(K\cdot x)=\mathrm{Typ}_K(K/L)$.

We now prove that (ii) implies (iii).

Let $\hat{L}$ denote the isotropy group K_x. Since the orbit $K\cdot x$ is minimal, it follows that $\hat{L}=kLk^{-1}$ for some $k\in K$. Thus $K^{\mathbb{C}}/L^{\mathbb{C}}$ is $K^{\mathbb{C}}$-isomorphic to $K^{\mathbb{C}}/kL^{\mathbb{C}}k^{-1}$, where the isomorphism is induced by the map $K^{\mathbb{C}}\to K^{\mathbb{C}}$, $g\to gk^{-1}$. Consequently, without lost of generality we can assume that $L=K_x$ and $x=L^{\mathbb{C}}$. Now we make use of the following consequence of the Peter and Weyl theorem:

There exists a real vectorspace W of finite dimension with a linear action of K such that $L=K_w$ for some $w\in W$ (cf. for example [J]).

The complexification $K^{\mathbb{C}}$ acts holomorphically on the complexification $W \otimes \mathbb{C}$, and there exists a K-invariant Hermitian inner product $\langle\,,\rangle$ on $W \otimes \mathbb{C}$ whose restriction to W is a K-invariant scalar product on W.

Since $K \cdot w$ is perpendicular to the line $\mathbb{R}w$, w is a critical point of the strictly plurisubharmonic K-function $d: W \otimes \mathbb{C} \to \mathbb{R}$, $d(v) = \langle v, v\rangle$. From this follows (see [P, S] or 5.4 Corollary 1) that

(1) the orbit $K^{\mathbb{C}} \cdot w$ is closed in $W \otimes \mathbb{C}$,

(2) the $K^{\mathbb{C}}$-map $K^{\mathbb{C}}/L^{\mathbb{C}} \to K^{\mathbb{C}} \cdot w$, $g \cdot x \to g \cdot w$ is biholomorphic, i.e. $L^{\mathbb{C}} = (K^{\mathbb{C}})_w$,

(3) $d(w) \leqq d(g \cdot w)$ for all $g \in K^{\mathbb{C}}$.

As a consequence of (1) and (2), the strictly plurisubharmonic K-function $p: K^{\mathbb{C}}/L^{\mathbb{C}} \to [0, \infty)$, $g \cdot x \to d(g \cdot w) - \mathrm{d(w)}$ is proper. Thus it follows that $p^{-1}(0) = K \cdot x$ (4.2 Lemma 2).

To prove that (iii) implies (iv) one need only observe that the sets $p^{-1}([0, \varepsilon))$, $\varepsilon > 0$ have the desired property.

Finally, we prove that (iv) implies (i).

From the Slice Theorem for compact group actions, see [J], it follows that $\tilde{U} = \{y \in K^{\mathbb{C}}/L^{\mathbb{C}};\ \mathrm{Typ}_K(K \cdot x) \leqq \mathrm{Typ}_K(K \cdot y)\}$ is an open neighbourhood of the orbit $K \cdot x$. Let U be an orbit-convex Stein K-domain in $\tilde{U}$ which contains $K \cdot x$. Since $K^{\mathbb{C}}/L^{\mathbb{C}}$ is a K-complexification of U (3.3), there exists a totally real K-point $y \in U$, i.e. $(K_y)^{\mathbb{C}} = (K^{\mathbb{C}})_y$. But then, as we have proved, the orbit $K \cdot y$ is minimal. The definition of $\tilde{U}$ implies that $K \cdot x$ is also a minimal K-orbit. Note that $x = gL^{\mathbb{C}}$ for some $g \in K^{\mathbb{C}}$, i.e. $(K^{\mathbb{C}})_x = gL^{\mathbb{C}}g^{-1}$. Since $K_x = kLk^{-1}$ for a suitable $k \in K$, it follows that $(K_x)^{\mathbb{C}} = kL^{\mathbb{C}}k^{-1} \subset (K^{\mathbb{C}})_x = gL^{\mathbb{C}}g^{-1}$. Hence, we obtain $(K_x)^{\mathbb{C}} = (K^{\mathbb{C}})_x$ □

Note that the space $K^{\mathbb{C}}$ is homogeneous with respect to the $K^{\mathbb{C}} \times K^{\mathbb{C}}$-action given by $(g, h, x) \to gxh^{-1}$. The $K \times K$-orbit through $1 \in K^{\mathbb{C}}$ is the group K and K is a minimal $K \times K$-orbit in $K^{\mathbb{C}}$. This proves the following

Corollary 1. *Let K be a compact Lie group and $K^{\mathbb{C}}$ the complexification of $K^{\mathbb{C}}$. Then the $K \times K$-subset K of $K^{\mathbb{C}}$ has a basis of open orbit-convex $K \times K$-neighbourhoods.* □

Let L be a closed subgroup of the compact Lie group K and $\delta: L \to GL(V)$ a representation of L in a finite-dimensional complex vector space V. We always assume the representation to be unitary. The corresponding $K^{\mathbb{C}}$-bundle $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} V$ is a holomorphic vector bundle over $K^{\mathbb{C}}/L^{\mathbb{C}}$ with typical fiber V. Every closed (resp. open) $L^{\mathbb{C}}$-subset U of V can be identified with the closed (resp. open) $K^{\mathbb{C}}$-subset $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} U$ of $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} V$. The embedding $j: V \to K^{\mathbb{C}} \times_{L^{\mathbb{C}}} V$, $z \to [1, z]$ induces an isomorphism of the $\mathbb{C}$-ringed spaces $V//L$ and $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} V//K$.

In the following, K/L is identified with the K-orbit through $L^{\mathbb{C}} \in K^{\mathbb{C}}/L^{\mathbb{C}}$ and $K^{\mathbb{C}}/L^{\mathbb{C}}$ with the zero section $K^{\mathbb{C}} \cdot [1, 0] = K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \{0\}$ in $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} V$. Furthermore, the group $K^{\mathbb{C}}$ is viewed as a homogeneous $K^{\mathbb{C}} \times L^{\mathbb{C}}$-space. Here the action is given by the map $K^{\mathbb{C}} \times L^{\mathbb{C}} \times K^{\mathbb{C}} \to K^{\mathbb{C}}$, $(g, h, x) \to gxh^{-1}$. Note that the open orbit-convex $K \times L$-subsets which contains K form a basis of neighbourhoods of K in $K^{\mathbb{C}}$ (Corollary 1).

Corollary 2. *For every orbit-convex $K \times L$-subset N of $K^{\mathbb{C}}$ and every orbit-convex L-subset D of V the K-subset $[N, D] = \{[g, z];\ (g, z) \in N \times D\}$ of $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} V$ is orbit-convex.*

Proof. Let $\mathfrak{k}$ denote the Lie algebra of K. If $(u, x) \in N \times D$ and $v \in i\mathfrak{k}$ are given and if $\exp v \cdot [u, x] \in [N, D]$, then it is necessary to show that $\exp tv \cdot [u, x] \in [N, D]$ for $t \in [0, 1]$.

By the definition of $[N,D]$, $(\exp v\cdot u\cdot h^{-1}, h\cdot x)\in N\times D$ for an appropriate $h\in L^{\mathbb{C}}$. Let $\mathfrak{l}$ denote the Lie algebra of L. There exist $m\in L$ and $w\in i\mathfrak{l}$, such that $h=m\cdot\exp w$ (3.1). Since N is $K\times L$-invariant and D is L-invariant, it follows from the orbit-convexity assumption that $(\exp tv\cdot u\cdot\exp(-tw), \exp tw\cdot x)\in N\times D$ for $t\in[0,1]$. Thus $\exp tv\cdot[u,x]=[\exp tv\cdot u\cdot(\exp tw)^{-1}, \exp tw\cdot x]\in[N,D]$. □

5 Linear actions

5.1 Linearization

Let K be a compact Lie group and $K^{\mathbb{C}}$ a complexification of K. We note the following useful

Remark. Let Y_1 and Y_2 be holomorphic $K^{\mathbb{C}}$-spaces and $\phi: Y_1\to Y_2$ a holomorphic K-map. If ϕ is an open immersion at $a_1\in Y_1$ such that the restriction of ϕ to $K\cdot a_1$ is injective, then ϕ maps a K-neighbourhood of $K\cdot a_1$ biholomorphically onto a neighbourhood of $K\cdot\phi(a_1)$. If $a_2=\phi(a_1)$ possesses a basis of K-connected open neighbourhoods, then ϕ^{-1} is defined in an open K-neighbourhood U of $K\cdot a_2$. Since $K^{\mathbb{C}}\cdot U$ is a K-complexification of U, ϕ^{-1} extends to a holomorphic $K^{\mathbb{C}}$-map on $K^{\mathbb{C}}\cdot U$ (1.5). Hence ϕ maps a $K^{\mathbb{C}}$-neighbourhood of $K\cdot a_1$ biholomorphically onto a $K^{\mathbb{C}}$-neighbourhood of $K\cdot a_2$.

Let Y be a holomorphic $K^{\mathbb{C}}$-space and a a smooth totally real K-point in Y. By definition, if we denote by L the isotropy group K_a, we have $L^{\mathbb{C}}=(K^{\mathbb{C}})_a$. The differential of the maps $Y\to Y$, $y\to h\cdot y$, $h\in L$, defines a representation of L in $GL(T_aY)$, where T_aY denotes the tangent space of Y at a. The L-action on Y can be linearized in a neighbourhood of a, i.e. there exists an L-equivariant biholomorphic map μ from an L-neighbourhood of a in Y onto an L-neighbourhood of zero in T_aY with $\mu(a)=0$.

The tangent space $T_a(K^{\mathbb{C}}\cdot a)$ at a of the orbit $K^{\mathbb{C}}\cdot a$ is an L-invariant linear subspace of T_aY. We choose an L-invariant Hermitian inner product on T_aY and decompose T_aY in an orthogonal sum $T_aY=T_a(K^{\mathbb{C}}\cdot a)\oplus V$. We call V an *L-invariant normal space at a to the orbit $K^{\mathbb{C}}\cdot a$.*

For a sufficiently small open L-neighbourhood D of $0\in V$, there exists a holomorphic L-map $\lambda: D\to Y$, $\lambda(x)=\mu^{-1}$, which maps D biholomorphically onto a local submanifold of Y. Note that, if D is an orbit-connected neighbourhood of $0\in V$, for example, a ball around zero, then $D^{\mathbb{C}}=L^{\mathbb{C}}\cdot D$ is an L-complexification of D. Consequently, $\lambda: D\to Y$ extends to a holomorphic $L^{\mathbb{C}}$-map $\lambda^{\mathbb{C}}: D^{\mathbb{C}}\to Y$. With this notation we have the following

Linearization Lemma. *Let K be a compact group with complexification $K^{\mathbb{C}}$ and Y a holomorphic $K^{\mathbb{C}}$-space. Let $a\in Y$ be a smooth totally real K-point in Y and set $L=K_a$. Assume that there exists a basis of open orbit-connected neighbourhoods of the orbit $K\cdot a$ in Y. If D is a sufficiently small open L-neighbourhood of zero in a L-invariant normal space V at a to $K^{\mathbb{C}}\cdot a$, then there exists an injective holomorphic $L^{\mathbb{C}}$-map $\lambda^{\mathbb{C}}: K^{\mathbb{C}}\cdot D\to Y$ such that $\Lambda: K^{\mathbb{C}}\times_{L^{\mathbb{C}}} L^{\mathbb{C}}\cdot D\to Y$, $[g,v]\to g\cdot\lambda(v)$, is an open embedding.*

Proof. For small D, there exists a holomorphic $L^{\mathbb{C}}$-map $\lambda^{\mathbb{C}}: K^{\mathbb{C}}\cdot D\to Y$ which maps D onto a local L-submanifold of Y. We can assume that this submanifold is transversal to the orbit $K^{\mathbb{C}}\cdot a$ at a. Consequently, the map $\Lambda: K^{\mathbb{C}}\times_L L^{\mathbb{C}}\cdot D\to Y$ is an open immersion at $[1,0]$ with $A([1,0])=\lambda^{\mathbb{C}}(0)=a$. Note that A maps $K/L\cong K\cdot[1,0]$ diffeomorphically onto $K\cdot a\cong K/L$. Thus by the Remark A maps an open $K^{\mathbb{C}}$-neighbourhood of $K^{\mathbb{C}}\cdot[1,0]$ biholomorphically onto the image. Recall that the quotient map $K^{\mathbb{C}}\times V\to K^{\mathbb{C}}\times_{L^{\mathbb{C}}}V$ is open. Hence, after shrinking D if necessary, we obtain the desired result. □

Example. Every point in $\mathbb{C}/\mathbb{Z}+i\mathbb{Z}$ is a totally real $S^1\cong\mathbb{R}/\mathbb{Z}$ point. But there are no proper S^1-connected subsets in $\mathbb{C}/\mathbb{Z}+i\mathbb{Z}$.

Even if Y is a K-vector space with a linear K-action, then not every K-orbit of a totally real K-point possesses arbitrary small K-connected neighbourhoods (see [Lu 1]).

5.2 Hilbert's Finiteness Theorem

Let K be a compact Lie group which acts linearly on $\mathbb{C}^n$, i.e. via a unitary representation $\varrho: K\to GL(\mathbb{C}^n)$. The Finiteness Theorem can be stated as follows:

The algebra $\mathbb{C}[z_1,\ldots,z_n]^K$ of K-invariant polynomials on $\mathbb{C}^n$ is finitely generated.

For the proof of the Finiteness Theorem one needs Hilbert's Basisatz and the existence of a Haar measure on the compact group K, see [We].

A set of generators $p_1,\ldots,p_k$ of the algebra $\mathbb{C}[z_1,\ldots,z_n]^K$ gives rise to a polynomial map $P:\mathbb{C}^n\to\mathbb{C}^k$. For an ideal I in $\mathbb{C}[z_1,\ldots,z_n]^K$ one obtains by integration over K the identity $I\cdot\mathbb{C}[z_1,\ldots,z_n]\cap[z_1,\ldots,z_n]^K=I$. From this it follows that (cf. [Kr] or [K, S, S]):

(i) *The images of Zariski closed K-subsets of $\mathbb{C}^n$ of P are closed in $\mathbb{C}^k$. In particular, $Z=P(\mathbb{C}^n)$ is a Zariski closed subvariety of $\mathbb{C}^k$, whose algebra of regular functions is isomorphic to $\mathbb{C}[z_1,\ldots,z_n]^K$.*

(ii) *The map P separates the closed $K^{\mathbb{C}}$-orbits in $\mathbb{C}^n$.*

By 2.3 Corollary 3, we know that there exists a continuous bijection $\bar{P}:\mathbb{C}^n//K\to Z$ such that $\bar{P}\circ\pi_n=P$ where $\pi_n:\mathbb{C}^n\to\mathbb{C}^n//K$ denotes the quotient map. It is known that Z is isomorphic to $\mathbb{C}^n//K$ as a $\mathbb{C}$-ringed space. This will be proved in detail in 6.4.

5.3 Complete invariant subsets

Let K be a compact Lie group. For a point x in a complex K-space X let $B_X(x)$ denote the smallest analytic K-subset in X which contains x (cf. 2.3).

Remark. If X is a locally analytic K-subset of a holomorphic $K^{\mathbb{C}}$-space Y, then $K^{\mathbb{C}}\cdot B_X(x)\subset B_Y(x)$ for all $x\in X$. This follows from the Identity Theorem (1.3).

A locally analytic K-subset X of a holomorphic $K^{\mathbb{C}}$-space Y is called *complete* if $K^{\mathbb{C}}\cdot B_X(x)=B_Y(x)$ for all $x\in X$. The locally analytic K-subset X is called *complete with respect to a closed subgroup L of K* if X is a complete L-subset of the holomorphic $L^{\mathbb{C}}$-space Y.

Example. Let U be a Reinhardt domain in $\mathbb{C}^n$. After a change of indices we may assume $U\cap\mathbb{C}^m\times\{0\}\neq\emptyset$ and $U\cap\{0\}\times\mathbb{C}^{n-m}=\emptyset$ for some $m\in\{0,\ldots,n-1\}$. The $(S^1)^n$-complexification of U is $(\mathbb{C}^*)^n\cdot U\subset(\mathbb{C}^*)^m\times\mathbb{C}^{n-m}$. One can show that U is a domain of holomorphy if and only if it is orbit-convex and complete with respect to every closed subgroup of $(S^1)^n$.

In the following we often use the fact that a locally analytic K-subset X of a holomorphic $K^{\mathbb{C}}$-space Y which is a union of some analytic K-subsets A_j, $j \in J$ is complete in Y if and only if $X \cap A_j$ is complete in A_j for all j.

Example. If the compact Lie group K acts linearly on $\mathbb{C}^n$, then the action of the complexification $K^{\mathbb{C}}$ on $\mathbb{C}^n$ is algebraic. Hence the topological closure $\overline{K^{\mathbb{C}} \cdot z}$ of an orbit $K^{\mathbb{C}} \cdot z \subset \mathbb{C}^n$ is an analytic $K^{\mathbb{C}}$-subset of $\mathbb{C}^n$. Thus, we have $B_{\mathbb{C}^n}(z) = \overline{K^{\mathbb{C}} \cdot z}$ for all $z \in \mathbb{C}^n$. It follows that a locally analytic K-subspace X in $\mathbb{C}^n$ is complete if and only if $\overline{K^{\mathbb{C}} \cdot x} \cap X$ is complete in $\overline{K^{\mathbb{C}} \cdot x}$ for all $x \in X$.

Let the compact Lie group K act linearly on $\mathbb{C}^n$ and denote the quotient map $\mathbb{C}^n \to \mathbb{C}^n//K$ by π_n.

Lemma. *Let Y be a closed complex $K^{\mathbb{C}}$-subset of a π_n-saturated open subset of $\mathbb{C}^n$. Then an open orbit-convex K-subset X of Y is complete if and only if $K^{\mathbb{C}} \cdot X$ is saturated with respect to $\pi_Y : Y \to Y//K$.*

Proof. Note first, since Y is a complete analytic $K^{\mathbb{C}}$-subset of $\mathbb{C}^n$, we have $B_Y(y) = K^{\mathbb{C}} \cdot B_Y(y) = B_{\mathbb{C}^n}(y) = \overline{K^{\mathbb{C}} \cdot y}$ for all $y \in Y$.

Assume that the open K-subset X of Y is complete. It is necessary to show that if $z \in \pi_Y^{-1}(\pi_Y(K^{\mathbb{C}} \cdot X)) = \pi_Y^{-1}(\pi_Y(X))$, then $z \in K^{\mathbb{C}} \cdot X$. Suppose $x \in X$ is such that $\pi_Y(z) = \pi_Y(x)$. Then $\emptyset \neq B_Y(z) \cap B_Y(x) = B_Y(z) \cap K^{\mathbb{C}} \cdot B_X(x)$. But since X is open in Y, $\emptyset \neq B_Y(z) \cap X = K^{\mathbb{C}} \cdot z \cap X$ implies that $K^{\mathbb{C}} \cdot z \cap X \neq \emptyset$. Consequently, we obtain $z \in K^{\mathbb{C}} \cdot X$.

Now assume that the open subset $K^{\mathbb{C}} \cdot X$ of Y is π_Y-saturated. For all $x \in X$ we have $B_Y(x) = B_{\mathbb{C}^n}(x) = \overline{K^{\mathbb{C}} \cdot x}$. Thus $\pi_Y(x) = \pi_Y(B_Y(x))$. Since $K^{\mathbb{C}} \cdot X$ is π_Y-saturated, it follows that $B_Y(x) \subset K^{\mathbb{C}} \cdot X$. Now, $B_X(x)$ is an analytic K-subset of the orbit-convex subset X of Y. Hence $K^{\mathbb{C}} \cdot B_X(x)$ is an analytic K-subset of $K^{\mathbb{C}} \cdot X$ (Proposition 3.3). Of course, $K^{\mathbb{C}} \cdot B_X(x) \subset B_Y(x)$. In the situation under consideration as we have shown $B_Y(x) \subset K^{\mathbb{C}} \cdot X$. Thus it follows that $K^{\mathbb{C}} \cdot B_X(x)$ is an analytic subset of $B_Y(x)$. The definition of $B_Y(x)$ implies that $K^{\mathbb{C}} \cdot B_X(x) = B_Y(x)$. □

As we will see later the statement of our Lemma remains true if one replace the holomorphic $K^{\mathbb{C}}$-space $\mathbb{C}^n$ by a holomorphic Stein $K^{\mathbb{C}}$-space.

5.4 *Completeness and invariant plurisubharmonic functions*

Let the compact Lie group K act linearly on $\mathbb{C}^n$ and denote by π_n the quotient map $\mathbb{C}^n \to \mathbb{C}^n//K$.

Proposition. *Let Y be a closed $K^{\mathbb{C}}$-subspace of a π_n-saturated open subset of $\mathbb{C}^n$. Then for a plurisubharmonic K-function $\phi : Y \to \mathbb{C}^n$ the open K-subset $D(\phi) = \{x \in Y; \phi(x) < 1\}$ of Y is orbit-convex and complete.*

Proof. Since the K-subset $D(\phi)$ is orbit-convex (3.4 Proposition), it suffices to show that $K^{\mathbb{C}} \cdot D(\phi)$ is saturated with respect to the quotient map $\pi_Y : Y \to Y//K$ (5.3 Lemma). For this let $x \in D(\phi)$ and $y \in Y$ and suppose that $\pi_Y(x) = \pi_Y(y)$. We shall prove that $y \in K^{\mathbb{C}} \cdot D(\phi)$. Since Y is π_n-saturated $\pi_Y(x) = \pi_Y(y)$ implies that $B_Y(x) \cap B_Y(y) = \overline{K^{\mathbb{C}} \cdot x} \cap \overline{K^{\mathbb{C}} \cdot y} \neq \emptyset$. From the Hilbert Lemma (see, for example, [Kr, III.24]) it follows that there exists a holomorphic group homomorphism $\gamma : \mathbb{C}^* \to K^{\mathbb{C}}$ with $\gamma(S^1) \subset K$ such that

(a) $y_0 = \lim_{z \to 0} \gamma(z) \cdot x$ exists and

(b) $y_0 \in E_n(x) = E_Y(\pi_Y(x)) \subset \overline{K^{\mathbb{C}} \cdot x}$.

The S^1-map $u:\mathbb{C}\to\mathbb{C}^n$ defined by $u(z)=\gamma(z)\cdot x$ for $z\in\mathbb{C}^*$ and $u(0)=y_0$ is holomorphic. Hence the S^1-invariant function $\hat{\phi}:\mathbb{C}\to\mathbb{R}$, $\hat{\phi}=\phi\circ u$ is subharmonic and consequently

$$\hat{\phi}(0)\leqq \int_{S^1}\hat{\phi}(tz)dt=\hat{\phi}(z)\quad \text{for all}\quad z\in\mathbb{C}.$$

This proves that $y_0\in D(\phi)$. Thus $K^{\mathbb{C}}\cdot y_0\subset\overline{K^{\mathbb{C}}\cdot y}$ implies $K^{\mathbb{C}}\cdot y\cap D(\phi)\neq\emptyset$ and it follows that $y\in K^{\mathbb{C}}\cdot D(\phi)$. □

In the following corollaries we assume that K is acting linearly on $\mathbb{C}^n$ and that Y is a closed complex K-subspace of a π_n-saturated open subset of $\mathbb{C}^n$. For $p\in Y//K$ let $E_Y(p)$ be the minimal $K^{\mathbb{C}}$-orbit in $\pi_Y^{-1}(p)$ (2.3 Corollary 2). The following is a straightforward generalization of a result of Kempf and Ness (cf. [K, N; D, K; P, S]).

Corollary 1. *Let $\phi:Y\to\mathbb{R}$ be a K-invariant function such that for $p\in Y//K$ the restriction $\phi_p:\pi_Y^{-1}(p)\to\mathbb{R}$, $\phi_p=\phi\,|\,\pi_Y^{-1}(p)$, is proper and strictly plurisubharmonic. Then every extremal value of ϕ_p is a minimal value and the set of such points is exactly one K-orbit through a totally real K-point $a\in E_Y(p)$.*

Proof. Let $\hat{\phi}$ be the function introduced in the proof of the Proposition. Then, since ϕ_p is strictly plurisubharmonic, for $x\in\pi_Y^{-1}(p)\backslash E_Y(p)$, the function $\mathbb{R}\to\mathbb{R}$, $t\to\hat{\phi}(e^t)=\phi(\gamma(e^t)\cdot x)$ is convex and strictly increasing. Hence the critical points of ϕ_p are contained in $E_Y(p)=K^{\mathbb{C}}\cdot y_0$ and the desired result follows from 4.2 Lemma 2. □

Corollary 2. *In every closed $K^{\mathbb{C}}$-orbit E of $\mathbb{C}^n$ there exists a totally real K-point a such that every open neighbourhood of $K\cdot a$ contains an orbit-convex complete open K-neighbourhood of $K\cdot a$.*

Proof. Recall that there exists an invariant polynomial map $P:\mathbb{C}^n\to\mathbb{C}^k$ such that $P^{-1}(P(z))=\pi_n^{-1}(\pi_n(z))$ for all $z\in\mathbb{C}^n$ (5.2 Hilbert's Finiteness Theorem). We may assume that K is a subgroup of the unitary group of $\mathbb{C}^n$. The K-invariant function $d:\mathbb{C}^n\to\mathbb{R}$, $d(z)=\|z\|^2$ is proper and strictly plurisubharmonic. Hence the map $\phi=P\times d$ is proper. Furthermore, for a suitable $a\in E$ we have $\phi^{-1}(\phi(a))=K\cdot a$ (Corollary 1). If U is a neighbourhood of $K\cdot a$, then the properness of ϕ implies that there exist open neighbourhoods Q of $P(a)$ and I of $d(a)$ such that $\phi^{-1}(Q\times I)\subset U$.

Finally, since $P^{-1}(Q)$ is π_n-saturated and d is plurisubharmonic, the Proposition implies that $\phi^{-1}(Q\times I)=P^{-1}(Q)\cap d^{-1}(I)=(d\,|\,P^{-1}(Q))^{-1}(I)$ is orbit-convex and complete if Q is sufficiently small. □

5.5 *Luna's Slice Theorem*

Let K be a compact Lie group with complexification $K^{\mathbb{C}}$. We consider a linear K action on $\mathbb{C}^n$ and denote by π_n the quotient map $\mathbb{C}^n\to\mathbb{C}^n//K$. Let a be a totally real K-point in a closed $K^{\mathbb{C}}$-orbit E in $\mathbb{C}^n$. We set $L=K_a$ and denote by V a L-invariant normal space at a to the orbit $K^{\mathbb{C}}\cdot a$ (5.1).

We identify $T_a\mathbb{C}^n$ with $\mathbb{C}^n$. The holomorphic $K^{\mathbb{C}}$-map $\Lambda:K^{\mathbb{C}}\times_{L^{\mathbb{C}}}V\to\mathbb{C}^n$, $[g,v]\to g\cdot(a+v)$ maps a neighbourhood of the orbit $K/L\cong K\cdot[1,0]\subset K^{\mathbb{C}}\times_{L^{\mathbb{C}}}V$ biholomorphically onto a neighbourhood of $K\cdot a$ in $\mathbb{C}^n$. Let $\pi_V:V\to V//K$ be the quotient map. Using the notion of orbit-convexity and completeness, we now give a proof of Luna's

Slice Theorem. *There exists a π_V-saturated neighbourhood S of $0 \in V$, such that*
(i) *$U = K^{\mathbb{C}} \cdot (a+S)$ is π_n-saturated and*
(ii) *the $K^{\mathbb{C}}$-map $\Lambda: K^{\mathbb{C}} \times_{L^{\mathbb{C}}} V \to U$, $[g,v] \to g \cdot (a+v)$ is biholomorphic.*

Proof. By 5.4 Corollary 2 there exists a totally real K-point $\tilde{a}$ in $K^{\mathbb{C}} \cdot a$ such that $K \cdot \tilde{a}$ has a basis of orbit-convex complete open neighbourhoods. Note that $\tilde{a} = g_0 \cdot a$ for an appropriate $g_0 \in K^{\mathbb{C}}$.

The holomorphic $K^{\mathbb{C}}$-map Λ is an immersion at $[1,0]$. Thus Λ is an immersion along $K^{\mathbb{C}} \cdot [1,0]$. In particular, Λ is an immersion at $[g_0, 0]$. Since $K^{\mathbb{C}}_{[g_0,0]} = g_0 L^{\mathbb{C}} g_0^{-1} = (K^{\mathbb{C}})_{\tilde{a}}$, the map Λ restricted to $K^{\mathbb{C}} \cdot [g_0, 0]$ is injective. After we choose a sufficiently small orbit-convex open neighbourhood $\tilde{U}$ of $K \cdot \tilde{a}$ we see that Λ maps an open $K^{\mathbb{C}}$-neighbourhood of $K^{\mathbb{C}} \cdot [1,0]$ in $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} V$ onto a π_n-saturated open neighbourhood of $K^{\mathbb{C}} \cdot a$ in $\mathbb{C}^n$ (5.1 and 5.3).

For a sufficiently small ball D around zero in V the $L^{\mathbb{C}}$-set $S = L^{\mathbb{C}} \cdot D$ is π_V-saturated (5.4) and Λ maps $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} S$ biholomorphically onto the π_n-saturated open set $U = K^{\mathbb{C}} \cdot (a+S) = K^{\mathbb{C}} \cdot (a+D) \subset \tilde{U}$. □

Corollary. *For every totally real K-point a of a closed $K^{\mathbb{C}}$-orbit in $\mathbb{C}^n$ the orbit $K \cdot a$ has a basis of orbit-convex complete open Stein K-neighbourhoods.*

Proof. This follows from the Slice Theorem and 4.4 Corollary 2. □

The π_V-saturated $L^{\mathbb{C}}$-subset S of V in the Slice Theorem contains arbitrary small open L-invariant balls D_r around the origin. If D_r is contained in S, then we call $D = a + D_r$ *a local linear slice at a to the closed orbit* $K^{\mathbb{C}} \cdot a$.

In particular, a local linear slice D at a is a locally analytic L-subset of $\mathbb{C}^n$ and one has $K^{\mathbb{C}} \cdot (a + L^{\mathbb{C}} \cdot D_r) = K^{\mathbb{C}} \cdot D$. The holomorphic $L^{\mathbb{C}}$-space $L^{\mathbb{C}} \cdot D = a + L^{\mathbb{C}} \cdot D_r$ is an L-complexification of D and the natural map $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} L^{\mathbb{C}} \cdot D \to L^{\mathbb{C}} \cdot D$ is biholomorphic. Furthermore, $K^{\mathbb{C}} \cdot D$ is a π_n saturated open neighbourhood of $K^{\mathbb{C}} \cdot a$ in $\mathbb{C}^n$.

6 Complexifications

6.1 Maps with the slice property

Let K be a compact Lie group and X a complex K-space. We say that a linearly equivariant holomorphic map $\phi: X \to \mathbb{C}^n$ has the slice property at $x \in X$ if it satisfies the following conditions:
(i) ϕ is an immersion at x,
(ii) ϕ restricted to $K \cdot x$ is injective,
(iii) $\phi(x)$ is contained in the closed $K^{\mathbb{C}}$-orbit $E_{\mathbb{C}^n}(\phi(x))$, and
(iv) $\phi(x)$ is a totally real K-point in $\mathbb{C}^n$.

The following is a direct consequence of the Slice Theorem (5.5).

Lemma. *Let K be a compact Lie group and X a complex K-space. If $\phi: X \to \mathbb{C}^n$ is a linearly equivariant holomorphic K-map which has the slice property at $x \in X$, then there exists a local linear slice D at $\phi(x)$ and an open K-neighbourhood U of x such that:*

(i) *$\phi(U)$ is an orbit-convex and complete open K-subset of the analytic K-subset $A = K^{\mathbb{C}} \cdot \phi(U)$ of $K^{\mathbb{C}} \cdot D$.*

(ii) *$\phi_U : U \to \phi(U)$, $\phi_U(y) = \phi(y)$ is biholomorphic.* □

Let $\hat{U}$ denote the union of irreducible components of the complex K-space $\phi^{-1}(A)$ which intersect U. Note that $\hat{U}$ is a closed analytic K-subset of $\phi^{-1}(A)$ and that A with the map $\hat{\phi} : \hat{U} \to A$, $\hat{\phi} = \phi \mid \hat{U}$, is a K-complexification of $\hat{U}$ (3.3 Proposition and 1.4 Lifting Lemma).

6.2 *Existence of maps with the slice property*

Let K be a compact Lie group and X a Stein K-space. For a point $x \in X$ let $B_X(x)$ denote the smallest analytic K-subset of X which contains x. Let $E_X(x)$ be the smallest non-empty analytic K-subset of $B_X(x)$. One has $E_X(x) = E_X(\pi_X(x))$ (2.3 Corollary 2) and $E_X(x)$ is the unique analytic K-subset in $\pi_X^{-1}(\pi_X(x))$ of minimal dimension, where π_X denotes the quotient map $X \to X//K$. Note that in general x does not belong to $E_X(x)$.

Lemma. *Let x be a point in a Stein K-space X. Then, for every totally real K-point $x_0 \in E_X(x)$, there exists a linearly equivariant holomorphic map $\phi : X \to \mathbb{C}^n$ which has the slice property at x_0.*

Proof. Since the algebra of K-finite functions is dense in $\mathcal{O}(X)$ (2.2 Fourier Theorem) there is a linearly equivariant holomorphic map $\phi_1 : X \to \mathbb{C}^{n_1}$ which is an immersion at x_0.

The homogeneous Stein space $K^{\mathbb{C}}/L^{\mathbb{C}}$ where $L = K_{x_0}$ is a K-complexification of $E_X(p)$ (4.2 Theorem and 4.4 Theorem). We can identify $K^{\mathbb{C}}/L^{\mathbb{C}}$ with a closed $K^{\mathbb{C}}$-orbit in some $\mathbb{C}^{n_2}$ (cf. the proof of 4.4 Theorem) and obtain a linearly equivariant holomorphic embedding $\psi : E_X(x) \to \mathbb{C}^{n_2}$. Let $\phi_2 : X \to \mathbb{C}^{n_2}$ be a K-equivariant extension of ψ (2.3 Lemma). Note that $\phi_2(x_0)$ is a totally real K-point. The group $K^{\mathbb{C}}$ acts diagonally on $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$. By construction, the holomorphic K-map $\phi : X \to \mathbb{C}^n$, $\phi(y) = (\phi_1(y), \phi_2(y))$ is an immersion at x_0 whose restriction to $E_X(x)$ is injective.

The orbit $K^{\mathbb{C}} \cdot \phi_2(x_0)$ is a K-complexification of $E_X(x_0)$. Hence the equivariant projection $q : \mathbb{C}^n \to \mathbb{C}^{n_2}$ maps the orbit $K^{\mathbb{C}} \cdot \phi(x_0) = \{(g \cdot \phi_1(x_0), g \cdot \phi_2(x_0));\ g \in K^{\mathbb{C}}\}$ biholomorphically onto $K^{\mathbb{C}} \cdot \phi_2(x_0)$. Thus $\phi(x_0)$ is a totally real K-point.

It remains to show that $K^{\mathbb{C}} \cdot \phi(x_0)$ is closed in $\mathbb{C}^n$. Assume the contrary is true. Then there exists $a \in \overline{K^{\mathbb{C}} \cdot \phi(x_0)} \backslash K^{\mathbb{C}} \cdot \phi(x_0)$. Since $K^{\mathbb{C}}$ acts algebraically on $\mathbb{C}^n$ it follows that $\dim K^{\mathbb{C}} \cdot a < \dim K^{\mathbb{C}} \cdot \phi(x_0)$. On the other hand, from $q(K^{\mathbb{C}} \cdot a) \subset \overline{K^{\mathbb{C}} \cdot \phi_2(x_0)} = K^{\mathbb{C}} \cdot \phi_2(x_0)$, we deduce that $q(K^{\mathbb{C}}) = K^{\mathbb{C}} \cdot \phi_2(x_0)$. Thus $\dim K^{\mathbb{C}} \cdot a \geq \dim K^{\mathbb{C}} \cdot \phi_2(x_0) = \dim K^{\mathbb{C}} \cdot \phi(x_0)$. This contradicts $\dim K^{\mathbb{C}} \cdot a < K^{\mathbb{C}} \cdot \phi(x_0)$. □

Remark. Using 6.1 Lemma, 5.5 Slice Theorem, and 3.1 Decomposition Theorem, one sees that a minimal K-orbit in $\pi_X^{-1}(\pi_X(x))$ is automatically contained in $E_X(x)$.

6.3 *Local complexification*

Let K be a compact Lie group and X a Stein K-space. As in the previous section, $B_X(x)$ denotes the smallest analytic K-subset of X which contains the point $x \in X$ and $E_X(x)$ is the smallest analytic K-subset of $B_X(x)$.

Lemma. *Let U be a subset of X and $\phi: X \to \mathbb{C}^n$ a linearly equivariant holomorphic map which is an immersion along U. If for $x \in X$ there exists $x_0 \in U \cap B_X(x)$ such that $\phi(x_0) \in E_{\mathbb{C}^n}(\phi(x))$, then it follows that $E_X(x) = E_X(x_0) = B_X(x_0)$.*

Proof. Let $\phi: X \to \mathbb{C}^n$ be a linearly equivariant holomorphic map. Then, for $x \in X$,
(a) $\phi(B_X(x)) \subset B_{\mathbb{C}^n}(\phi(x)) = \overline{K^{\mathbb{C}} \cdot \phi(x)}$;
(b) $\dim B_X(x) \geqq \dim B_{\mathbb{C}^n}(\phi(x))$.

Note that $\phi(B_X(x_0))$ is an open subset of $E_{\mathbb{C}^n}(\phi(x)) = K^{\mathbb{C}} \cdot \phi(x_0)$. For all $y \in B_X(x)$, it follows that:

$$\dim B_X(y) \geqq \dim E_{\mathbb{C}^n}(\phi(x)) = \dim \phi(B_X(x_0)) = \dim B_X(x_0).$$

Hence $B_X(x_0)$ is of minimal dimension in $B_X(x)$. Consequently, $B_X(x_0) = E_X(x_0) = E_X(x)$. □

For a Stein K-space X let $\pi_X: X \to X//K$ be the quotient map. We fix a point $p \in X//K$ and a minimal K-point $x_0 \in E_X(p)$. Let $\phi: X \to \mathbb{C}^n$ be a linearly equivariant holomorphic map which has the slice property in x_0, and let U denote the open K-neighbourhood of x_0 of 6.1 Lemma. The K-subset $\phi(U)$ is then orbit-convex and complete, and $A = K^{\mathbb{C}} \cdot \phi(U)$ is an analytic $K^{\mathbb{C}}$-subset in a π_n-saturated open set in $\mathbb{C}^n$. We set $\hat{U} = B_X(U) = \bigcup_{y \in U} B_X(y)$ and denote by $\iota: \hat{U} \to A$ the restriction of ϕ to $\hat{U}$.

Theorem on the existence of local complexifications. *The K-subset $\hat{U}$ of X is open and π_X-saturated. The analytic $K^{\mathbb{C}}$-subset $A = K^{\mathbb{C}}\iota(U)$ is a K-complexification of $\hat{U}$ and $\iota: \hat{U} \to A$ is an open embedding.*

Proof. In order to prove that $\hat{U}$ is π_X-saturated, let $z \in X$ with $\pi_X(z) \in \pi_X(\hat{U})$ be given. We have to show that $z \in \hat{U}$.

If $B_X(z) \cap U$ is non-empty and contained in $B_X(u)$ for some $u \in B_X(z) \cap U$, then, since $B_X(u)$ is by definition a K-irreducible analytic K-set, the identity principle for analytic sets implies that $B_X(z) \subset B_X(u) \subset \hat{U}$. In particular, $z \in \hat{U}$.

We show that $B_X(z) \cap U$ is non-empty. For this note that $\pi_X(\hat{U}) = \pi_X(U)$ (2.3 Corollary). We choose a $x \in U$ such that $\pi_X(z) = \pi_X(x)$. Since $A = K^{\mathbb{C}} \cdot \phi(U)$ is an analytic set in a π_n-saturated open subset of $\mathbb{C}^n$, we have $E_{\mathbb{C}^n}(\phi(x)) \subset B_{\mathbb{C}^n}(\phi(x)) = B_A(\phi(x))$. But $x \in U$ and $\phi(U)$ is complete, so it follows that $E_{\mathbb{C}^n}(\phi(x)) \subset B_A(\phi(x)) = K^{\mathbb{C}} \cdot B_{\phi(U)}(\phi(x)) = K^{\mathbb{C}} \cdot \phi(B_U(x))$, since $\phi \mid U$ is biholomorphic onto $\phi(U)$. Consequently, $E_{\mathbb{C}^n}(\phi(x)) \cap \phi(B_U(x))$ is non-empty and there is a point $z_0 \in B_U(x) \subset B_X(x) \cap U$ such that $\phi(z_0) \in E_{\mathbb{C}^n}(\phi(x))$. The previous Lemma implies that $E_X(x) \cap U \neq \emptyset$. Thus $E_X(x) = E_X(z) \subset B_X(z)$ (2.3) implies that $B_X(z) \cap U$ is non-empty.

In order to complete the proof of the statement that $\hat{U}$ is π_X-saturated it remains to show that $B_X(z) \cap U \subset B_X(u)$ for some $u \in B_X(z) \cap U$. For this first note that since the boundary $R = \overline{K^{\mathbb{C}} \cdot \phi(z)} \setminus K^{\mathbb{C}}\phi(z)$ of $K^{\mathbb{C}} \cdot \phi(z)$ in $\overline{K^{\mathbb{C}} \cdot \phi(z)}$ is a (possible empty) analytic K-subset of $\mathbb{C}^n$, there exists $u \in (B_X(z) \setminus \phi^{-1}(R)) \cap U \subset B_X(z) \cap U$ such that $\phi(u) \in K^{\mathbb{C}} \cdot \phi(z) \cap \phi(U)$. In particular, it follows that $B_{\mathbb{C}^n}(\phi(z)) = B_{\mathbb{C}^n}(\phi(u)) = B_A(\phi(u))$. Together with 3.3 Corollary, this implies that $\phi(B_X(z) \cap U) \subset B_{\mathbb{C}^n}(\phi(z)) \cap \phi(U) = K^{\mathbb{C}} \cdot B_{\phi(U)}(\phi(u)) \cap \phi(U) = B_{\phi(U)}(\phi(u)) = \phi(B_U(u))$. From this it follows that $B_X(z) \cap U \subset B_U(u) \subset B_X(u)$.

To prove that $\hat{U}$ is open, let $z \in \hat{U}$ be given. Then $B_X^{\circ}(z) = B_X(z) \setminus \phi^{-1}(\overline{K^{\mathbb{C}} \cdot \phi(z)} \setminus K^{\mathbb{C}} \cdot \phi(z))$ is, as a complement of a proper analytic K-subset of $B_X(z)$, a K-connected open K-subset of the K-irreducible analytic set $B_X(z)$. Furthermore, $B_X^\circ(z) \cap U$ is non-empty. Hence there exist $y \in B_X^\circ(z) \cap U$ and $k \in K$ such that y and $k \cdot z$ are contained in the same connected component of $B_X^\circ(z)$. Let $u: [0,1] \to B_X^\circ(z)$ be a curve with $u(0) = y$ and $u(1) = k \cdot z$. Note that $K^{\mathbb{C}} \to K^{\mathbb{C}} \cdot \phi(y)$, $g \to g \cdot \phi(y)$ is a bundle map. Thus there exists a curve $\gamma: [0,1] \to K^{\mathbb{C}}$, such that $\gamma(0) = 1$ and $\phi(u(t)) = \gamma(t) \cdot \phi(y)$ for $t \in [0,1]$.

The K-action on X can be extended to a local $K^{\mathbb{C}}$-action, cf. [K 1]. Thus in a relatively compact neighbourhood U_γ of $u([0,1])$ the operation $t \to \gamma(t) \cdot w$, $t \in [0,1]$, $w \in U_\gamma$, is well defined, cf. [H 2]. If $U_0 \subset U$ is a small neighbourhood of y, then $\gamma(1) \cdot U_0$ is a neighbourhood of $k \cdot z$ which is contained in $B_X(U_0) \subset \hat{U}$. Thus $z \in k^{-1} \cdot \gamma(1) \cdot U_0 \subset \hat{U}$. This proves that $\hat{U}$ is open.

Recall that the holomorphic $K^{\mathbb{C}}$-space $A = K^{\mathbb{C}} \cdot \phi(U)$ is a K-complexification of $\phi(U)$. Since $\iota \mid U: U \to \phi(U)$ is biholomorphic, A is a K-complexification of $\hat{U}$ (1.4 Lifting Lemma). Since X is a Stein space, it follows, by the definition of a K-complexification, that ι is an open embedding. □

6.4 The categorical quotient for linear actions

Let K be a compact Lie group and $\varrho: K \to GL(\mathbb{C}^n)$ a continuous representation which we always assume to be unitary. Let $\pi_n: \mathbb{C}^n \to \mathbb{C}^n//K$ denote the corresponding quotient map. The finitely generated algebra $\mathbb{C}[z_1, \ldots, z_n]^K$ defines an affine algebraic variety which will be denoted by Z.

A set of generators of $\mathbb{C}[z_1, \ldots, z_n]^K$ defines a surjective map $P: \mathbb{C}^n \to Z$, such that $\pi_n^{-1}(\pi_n(z)) = P^{-1}(P(z))$ for all $z \in \mathbb{C}^n$. Since Z is a normal variety, the complex space associated to Z is a normal complex space [Z, S, p. 320]. Let $\mathcal{O}_Z$ be the sheaf of germs of holomorphic functions on Z. Since $P: \mathbb{C}^n \to Z$ is an invariant holomorphic map, there is a bijective continuous map $\bar{P}: \mathbb{C}^n//K \to Z$ such that the diagram

$$\begin{array}{ccc} & \mathbb{C}^n & \\ {\scriptstyle \pi_n} \swarrow & & \searrow {\scriptstyle P} \\ \mathbb{C}^n//K & \xrightarrow[\bar{P}]{} & Z \end{array}$$

commutes.

Lemma. *The map $\bar{P}$ is homeomorphic.*

Proof. It follows from 5.5 Slice Theorem that for all open subsets $Q \subset \mathbb{C}^n//K$ the map $\pi_Q: \pi_n^{-1}(Q) \to Q$, $\pi_Q = \pi_n \mid \pi_n^{-1}(Q)$ is quasi-proper. In particular, $\mathbb{C}^n//K$ is a locally compact Hausdorff space. Since $\bar{P}^{-1}(\bar{P}(p)) = \{p\}$ for $p \in \mathbb{C}^n//K$, it follows that there exists an open neighbourhood Q of p and a connected open neighbourhood W of $\bar{P}(p)$, such that $\bar{P}_Q: Q \to W$, $\bar{P}_Q = \bar{P} \mid Q$ is a finite map. Since $P_Q: \pi_n^{-1}(Q) \to W$, $P_Q = P \mid \pi_n^{-1}(Q)$ is quasi-proper, $P_Q(\pi_N^{-1}(Q)) = \bar{P}(Q)$ is an analytic subset in W (see, for example, [G 2]). But W is a connected normal space, so we obtain $\bar{P}_Q(Q) = W$. This proves that $\bar{P}$ is open at p. □

Remark. The map $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} V \to \mathbb{C}^n$, $[g, v] \to g \cdot (a + v)$ in 5.5 Slice Theorem induces an algebraic map $V//K \to \mathbb{C}^n//K$ which maps a neighbourhood of $\pi_V(0)$ topologically onto a neighbourhood of $\pi_n(p)$ (Slice Theorem and Lemma). Thus the map

$V//L \to \mathbb{C}^n//K$ is biholomorphic at $\pi_V(0)$. Hence one obtains the Slice Theorem of Luna as formulated in [Lu 1] for the ground field $\mathbb{C}$.

Theorem. *The categorical quotient* $(\mathbb{C}^n//K, \mathcal{O}^K_{\mathbb{C}^n})$ *is a Stein space.*

Proof. We identify $Z = P(\mathbb{C}^n)$ and $\mathbb{C}^n//K$ as topological spaces. Now we repeat the arguments of Luna in [Lu 3]. For an open subset Q of $\mathbb{C}^n//K$ let $\mathscr{C}(Q)$ denote the Frechét space of continuous functions on Q. One can identify $\mathscr{C}(Q)$ with the closed subset $\mathscr{C}(\pi_n^{-1}(Q))^K$ of $\mathscr{C}(\pi_n^{-1}(Q))$ (cf. 2.3 Corollary 3). Thus $\mathcal{O}_Z(Q)$ is a closed subset of $\mathcal{O}^K_{\mathbb{C}^n}(Q) \subset \mathscr{C}(Q)$. For a local linear slice D at a totally real K-point $a \in E_X(p)$ where $p \in \mathbb{C}^n//K$ one can identify the algebra of K_a-invariant holomorphic polynomials on D with a dense subspace of $\mathcal{O}_Z(\pi_n(D))$. Since $\mathcal{O}_{\mathbb{C}^n}(K^{\mathbb{C}} \cdot D)^K \cong \mathcal{O}_D(D)^{K_a}$, it follows that $\mathcal{O}_Z(\pi_n(D)) \cong \mathcal{O}^K_{\mathbb{C}^n}(\pi_n(D))$. Consequently, $\mathcal{O}_Z \cong \mathcal{O}^K_{\mathbb{C}^n}$. □

This theorem will now be applied to orbit-convex complete K-subsets of $\mathbb{C}^n$.

Proposition. *Let U be an open orbit-convex complete K-subset of $\mathbb{C}^n$. Then for an analytic K-subset X in U it follows that*

(i) *$\pi_n(X)$ is an analytic subset in the open subset $\pi_n(U)$ of $\mathbb{C}^n//K$.*

(ii) *The inclusion $X \to U$ induces an isomorphism of the $\mathbb{C}$-ringed spaces $(X//K, \mathcal{O}^K_X)$ and $(\pi_n(X), \mathcal{O}_{\pi_n(X)})$. In particular, $(X//K, \mathcal{O}^K_X)$ is a complex space.*

Proof. Recall that X is an open K-subset of the analytic K-subset $A = K^{\mathbb{C}} \cdot X$ of $\hat{U} = K^{\mathbb{C}} \cdot U$ (3.3 Proposition). Moreover, $\hat{U}$ is a π_n-saturated open subset in $\mathbb{C}^n$ (5.3 Lemma). Thus $Q = \pi_n(\hat{U}) = \pi_n(U)$ is an open subset in $\mathbb{C}^n//K$. We set $\hat{A} = \pi_n^{-1}(\pi_n(A))$. The remainder of the proof will be carried out in three steps.

1 $\pi_n(X)$ is an analytic subset in $\pi_n(U)$. Since $\mathbb{C}^n//K$ is a complex space, we can cover $Q = \pi_n(U)$ with open Stein subsets Q_α. Note that $\hat{U}_\alpha = \pi_n^{-1}(Q_\alpha)$ is holomorphically convex. Hence $\{\hat{U}_\alpha\}$ is a covering of $\hat{U}$ with open Stein subsets. Let $\mathscr{I}$ denote the sheaf of ideals of the analytic subset $A = K^{\mathbb{C}} \cdot X$ of $\hat{U}$. Since $\hat{U}_\alpha$ are open Stein sets, we have $A_\alpha = A \cap \hat{U}_\alpha = \{z \in \hat{U}_\alpha;\ f(z) = 0 \text{ for all } f \in \mathscr{I}(\hat{U}_\alpha)\}$.

Since every closed $K^{\mathbb{C}}$-orbit in $\hat{A}$ is contained in A (2.3 Corollary 3), it follows for $\hat{A}_\alpha = \hat{A} \cap U_\alpha = \pi_n^{-1}(\pi_n(A_\alpha))$ that $\hat{A}_\alpha = \{x \in \hat{U}_\alpha;\ f(x) = 0 \text{ for all } f \in \mathscr{I}(\hat{U}_\alpha)^K\}$ (2.3 Lemma). In particular, $\pi_n(X) = \pi_n(\hat{A}) = \bigcup_\alpha \pi_n(\hat{A}_\alpha)$ is an analytic subset in $\pi_n(U)$.

2 The inclusion $j: A \to \hat{A}$ induces an isomorphism $\bar{j}$ of the $\mathbb{C}$-ringed spaces $(A//K, \mathcal{O}^K_A)$ and $(\pi_n(\hat{A}), \mathcal{O}_{\pi_n(\hat{A})})$. Let p be a point in $A//K$ and x a totally real K-point in $E_A(p)$. Since A is closed in the π_n-saturated set $\hat{A}$, we have $E_A(p) = E_X(x) = E_{\mathbb{C}^n}(x)$. For every open neighbourhood Q of $p \in A//K$ one has $\pi_A^{-1}(Q) = A \cap \tilde{U}$ for an appropriate open subset $\tilde{U}$ in $\mathbb{C}^n$. There exists a local linear slice D at x which is contained in $\tilde{U}$ (see 5.5 Slice Theorem). Then, for such a D, $\pi_A(A \cap D)$ is an open subset in $A//K$. Since $\bar{j}(\pi_A(A \cap D)) = \pi_n(A \cap D) = \pi_n(\hat{A}) \cap \pi_n(D)$, it follows that $\bar{j}$ is a homeomorphism.

Let $\hat{Q}$ be an open Stein subset in $\pi_n(\hat{U}) \subset \mathbb{C}^n//K$. Since every closed $K^{\mathbb{C}}$-orbit in $\hat{A}$ is contained in A, the restriction $\mathcal{O}_{\hat{A}}(\hat{A} \cap \pi_n^{-1}(\hat{Q}))^K \to \mathcal{O}_A(A \cap \pi_n^{-1}(\hat{Q}))^K$ is injective and, because $\hat{Q}$ is a Stein subset of $\pi_n(\hat{U})$, it is also surjective (2.3 Lemma). From $\mathcal{O}_{\pi_n(\hat{A})}(\pi_n((\hat{A}) \cap \hat{Q}) = \mathcal{O}_{\hat{A}}(\hat{A} \cap \pi_n^{-1}(\hat{Q}))^K$ it follows $\mathcal{O}_{\pi_n(\hat{A})} \cong \mathcal{O}^K_A$.

3 The inclusion $\imath: X \to A$ induces an isomorphism $\bar{\imath}$ of the $\mathbb{C}$-ringed spaces $(X//K, \mathcal{O}^K_X)$ and $(A//K, \mathcal{O}^K_A)$. The map $\bar{\imath}$ is a homeomorphism (cf. second part of the proof). Let W be an arbitrary π_A-saturated open subset of A. Note that $W \cap X$ is orbit-convex (3.2). Since $K^{\mathbb{C}} \cdot (W \cap X) = W$ (3.3 Corollary) the map $\bar{\imath}: \mathcal{O}^K_A \to \bar{\imath}\mathcal{O}^K_X$ is an isomorphism. □

6.5 Hilbert's Theorem for Stein spaces

Let K be a compact Lie group and X a Stein K-space. Then the $\mathbb{C}$-ringed space $(X//K, \mathcal{O}_X^K)$ is a Stein space.

Proof. Since $(X//K, \mathcal{O}_X^K)$ is a complex space (6.4 Proposition and 6.3 Local Complexification), it remains to show that $X//K$ is a Stein space. But this is obvious, because, if (p_k) is a discrete sequence in $X//K$, then $(\pi_X^{-1}(p_k))$ is a "discrete" sequence of analytic K-subsets of X and consequently there exists a function $f \in \mathcal{O}(X)^K$ such that $f \mid \pi_X^{-1}(p_k) = k$ (2.3 Lemma). □

Remark. Using 6.3 and the coherence result of Roberts [R], one can also see that $(X//K, \mathcal{O}_X^K)$ is a complex space. Moreover, Roberts arguments along with 6.3 can be used to prove analogous coherence results for Stein K-spaces.

6.6 Complexification of Stein spaces

Let K be a compact Lie group and X a Stein K-space. As before, we denote by π_X the quotient map $X \to X//K$. We now show that the local complexifications constructed above can be patched together.

Complexification Theorem. *For every Stein K-space X there exists a K-complexification $X^{\mathbb{C}}$ which has the following properties:*

(i) *X is an open orbit-convex complete Runge K-subset of $X^{\mathbb{C}}$.*

(ii) *The inclusion $X \to X^{\mathbb{C}}$ induces an isomorphism of the quotients $X//K$ and $X^{\mathbb{C}}//K$. For the quotient maps one has $(\pi_X)^{\mathbb{C}} = \pi_{X^{\mathbb{C}}}$.*

(iii) *If Q is a locally analytic subset in $X//K$, then $K^{\mathbb{C}} \cdot \pi_{X^{\mathbb{C}}}^{-1}(Q) = \pi_{\mathbb{C}}^{-1}(Q)$ is a K-complexification of $\pi_X^{-1}(Q)$.*

(iv) *The K-complexification $X^{\mathbb{C}}$ is a Stein space.*

Proof. First, we summarize the consequences of the Theorem on the Existence of local Complexifications. There exist:

(1) a covering $\{U_\alpha\}$ of X where U_α are open π_X-saturated Stein subsets of X,

(2) linear K-actions on appropriate $\mathbb{C}^{n_\alpha}$ and analytic $K^{\mathbb{C}}$-subsets A_α in π_α-saturated open subsets of $\mathbb{C}^{n_\alpha}$, where π_α denotes the quotient map $\mathbb{C}^{n_\alpha} \to \mathbb{C}^{n_\alpha}//K$, and

(3) orbit-convex open K-subsets Y_α of A_α with $A_\alpha = K^{\mathbb{C}} \cdot Y_\alpha$ and biholomorphic K-maps $\phi_\alpha : U_\alpha \to Y_\alpha$.

We can define K-spaces $Y_{\alpha\beta} = Y_\beta \cap \phi_\beta(U_\beta \cap \phi_\alpha^{-1}(Y_\alpha))$ and $A_{\alpha\beta} = K^{\mathbb{C}} \cdot Y_{\alpha\beta}$. The K-spaces $Y_{\alpha\beta}$ are orbit-convex open K-subspaces of $A_{\alpha\beta}$, i.e. $A_{\alpha\beta}$ is a K-complexification of $Y_{\alpha\beta}$ (3.3). The map $\iota_{\alpha\beta} : Y_{\alpha\beta} \to Y_{\beta\alpha}$, $\iota_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} \mid Y_{\alpha\beta}$ extends uniquely to a holomorphic K-map $\iota_{\alpha\beta}^{\mathbb{C}} : A_{\alpha\beta} \to A_{\beta\alpha}$. Note that $\iota_{\beta\alpha}^{\mathbb{C}}$ is biholomorphic with inverse $\iota_{\alpha\beta}^{\mathbb{C}}$. The holomorphic $K^{\mathbb{C}}$-space $X^{\mathbb{C}} = \bigcup A_\alpha/(\iota_{\alpha\beta}^{\mathbb{C}})$ contains the open K-subset $\bigcup_\alpha Y_\alpha/(\iota_{\alpha\beta})$ which will be identified with X. Since the K-subsets Y_α of A_α are orbit-convex and complete, this is also the case for the K-subset X. From $K^{\mathbb{C}} \cdot X = X^{\mathbb{C}}$, it follows that $X^{\mathbb{C}}$ is a K-complexification of X (3.3).

We prove that $X^{\mathbb{C}}$ is a Stein space. For this, let $\{z_n \in X^{\mathbb{C}};\ n \in \mathbb{N}\}$ be a discrete subset in $X^{\mathbb{C}}$. Since $X//K = X^{\mathbb{C}}//K$ is a Stein space, we can assume that the sequence $(\pi_X^{\mathbb{C}}(z_n))$ converges to $p \in X//K$. There exist a linearly equivariant holomorphic map

$\phi: X \to \mathbb{C}^m$ and a π_X-saturated open neighbourhood U of $\pi_X^{-1}(p)$ such that $A = K^{\mathbb{C}} \cdot \phi(U)$ is an analytic subset in a $\pi_m: \mathbb{C}^m \to \mathbb{C}^m//K$ saturated open subset of $\mathbb{C}^m$. Furthermore, the extension $\phi^{\mathbb{C}}: X^{\mathbb{C}} \to \mathbb{C}^m$ of ϕ restricted to $K^{\mathbb{C}} \cdot U$ is biholomorphic onto A (6.2). Replacing (z_n) by a subsequence, we can assume that $\{\phi(z_n); n \in \mathbb{N}\}$ is a discrete subset of $\mathbb{C}^m$. Since $\mathbb{C}^m$ is a Stein space, there exists a $f \in \mathcal{O}(\mathbb{C}^m)$ such that $|(f \circ \phi^{\mathbb{C}})(z_n)| = n$ for all $n \in \mathbb{N}$. This proves that $X^{\mathbb{C}}$ is holomorphically convex. The same argument applied to a set consisting of two points of $X^{\mathbb{C}}$ shows that $X^{\mathbb{C}}$ is holomorphically separable. This implies that $X^{\mathbb{C}}$ is a Stein space.

The Runge property of X in $X^{\mathbb{C}}$ is a consequence of 3.4 Theorem. □

6.7 Applications

In the sequal K denotes a compact Lie group.

(a) Let X be a Stein K-space and assume that $\mathcal{O}(X)^K = \mathbb{C}$. Then, of course, $X//K = \{p\}$. If x is a totally real K-point in $E_X(p)$, then $X^{\mathbb{C}}$ is a closed analytic K-subset in the $K^{\mathbb{C}}$-vector bundle $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} T_x X$ where L is the isotropy group of K at x. Thus $X^{\mathbb{C}} = K^{\mathbb{C}} \times_{L^{\mathbb{C}}} Y$, where Y is a closed L-subset of the L-vector space $T_x X$ through $0 \in T_x X$. Note that $K \cdot x$ is identified with $K/L = K \cdot [1, 0] \subset X^{\mathbb{C}}$. If one assumes that X is a contractible topological space, then we have $H_*(K/L, \mathbb{Z}) = \mathbb{Z}$ for the homology ring. This implies $L = K$ and consequently X is an open subset in Y which contains 0. In particular, if X is smooth, then $Y = T_x X$ and X is a domain in $T_x X$ which is invariant under the linear K-action on $T_x X$.

(b) Let X be a connected Stein K-manifold and assume that the set X^K of K-fixed points is non-empty. If there exists $x \in X$ such that $\mathcal{O}(V_x)^K = \mathbb{C}$ for the K-invariant normal space at x to X^K, then $X^{\mathbb{C}}$ is the normal bundle to the submanifold X^K of X, cf. [H 3].

(c) Let X be a Stein S^1-manifold and assume $X^{S^1} = \emptyset$. Then locally X is an open subset of the vector bundle $\mathbb{C}^* \times_{\mathbb{Z}_m} V$ where $\mathbb{Z}_m = \{t \in S^1; t^m = 1\}$. Hence $X^{\mathbb{C}}$ is a Seifert $\mathbb{C}^*$-principal bundle over $X//S^1$. The singularities of $X//S^1$ are given by finite quotients with respect to cyclic groups.

(d) Let M be a compact subgroup of the unitary group $U(\mathbb{C}^n)$ which contains the transformations $S = \{t \cdot id_{\mathbb{C}^n}, t \in S^1\}$. If U is an M-invariant domain of holomorphy in $\mathbb{C}^n$ which contains the origin, then $\mathbb{C}^n = M^{\mathbb{C}} \cdot U$ is an M-complexification of U. But then U is orbit-convex with respect to any compact subgroup K of M (3.2). For such a group it follows that $K^{\mathbb{C}} \cdot U$ is a domain of holomorphy.

In order to give a concrete example, let U be the m-fold product of $U_0 = \left\{ Z \in \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}; \langle Z \cdot w, Z \cdot w \rangle < \|w\|^2 \text{ for all } w \in \mathbb{C}^2 \setminus \{0\} \right\}$ and M the m-fold product of $U(\mathbb{C}^2) \times U(\mathbb{C}^2)$. The group $K = SU(\mathbb{C}^2) \times SU(\mathbb{C}^2)$ can be embedded diagonally in M and with respect to the diagonal action it follows that $K^{\mathbb{C}} \cdot U$ is a domain of holomorphy. This is a "compact" version of the future tube example (1.5).

(e) Let X be a real analytic K-manifold, i.e. the action $K \times X \to X$ is a real analytic map. From the result of Grauert [G 1] applied to the map $K \times X \to X$ it follows that there exists a Stein K-manifold $\tilde{X}$ which contains X as a totally real K-submanifold. By the Complexification Theorem, there exists a holomorphic Stein manifold $X^{\mathbb{C}}$ which contains X as a totally real K-submanifold. Since X is totally real, the $K^{\mathbb{C}}$-orbits through the points of X are closed. The quotient X/K is a real semi-analytic subspace of $X\mathbb{C}//K$.

From the Embedding Theorem in [H 1] it follows that:

There exists a linearly equivariant closed real analytic embedding of X into some $\mathbb{R}^N$ if and only if the K-orbit type of X is finite.

At this point it should be noted, that the Einbettungssatz 2 in [H 1] is not corrected stated. It says that a holomorphic Stein $K^{\mathbb{C}}$-manifold can be linearly equivariant embedded if and only if the $K^{\mathbb{C}}$-orbit type is finite. But finiteness of the $K^{\mathbb{C}}$-orbit type is only a sufficient condition. The necessary and sufficient condition is finite K-orbit type. It is easy to see that this condition is necessary [J]. On the other hand, the finiteness of the K-orbit type implies finite Slice type and this implies the existence of a linearly equivariant embedding of a holomorphic Stein $K^{\mathbb{C}}$-manifold (this, in fact, is proved in [H 1]).

(f) Let Y be a holomorphic $K^{\mathbb{C}}$-space and $\phi: Y \to \mathbb{R}$ a K-invariant strictly plurisubharmonic proper function. Let $D(\phi) = \{x \in Y; \phi(x) < 1\}$. Then $K^{\mathbb{C}} \cdot D(\phi)$ an open Stein $K^{\mathbb{C}}$-subspace of Y.

References

[C] Cartan, H.: Œuvres, Vol. I. Berlin Heidelberg New York: Springer 1979

[Ch] Chevalley, C.: Theory of Lie groups. Princeton: Princeton University Press 1946

[D, K] Dadok, J., Kac, V.: Polar representations. J. Algebra **92**, 504–524 (1985)

[G 1] Grauert, H.: On Levi's problem and the imbedding of real analytic manifolds. Ann. Math. **68**, 460–472 (1958)

[G 2] Grauert, H.: Set theoretic complex equivalence relations. Math. Ann. **265**, 137–148 (1983)

[H 1] Heinzner, P.: Linear äquivariante Einbettungen Steinscher Räume. Math. Ann. **280**, 147–160 (1988)

[H 2] Heinzner, P.: Kompakte Transformationsgruppen Steinscher Räume. Math. Ann. **285**, 13–28 (1989)

[H 3] Heinzner, P.: Fixpunktmengen kompakter Gruppen in Teilgebieten Steinscher Mannigfaltigkeiten. J. reine angew. Math. **402**, 128–137 (1989)

[Ho] Hochschild, G.: The structure of Lie groups. San Francisco London Amsterdam: Holden-Day 1965

[J] Jänich, K.: Differenzierbare *G*-Mannigfaltigkeiten. (Lect. Notes Math., Vol. 9) Berlin Heidelberg New York: Springer 1968

[K 1] Kaup, W.: Infinitesimale Transformationsgruppen komplexer Räume. Math. Ann. **160**, 72–92 (1965)

[K 2] Kaup, W.: Reelle Transformationsgruppen und invariante Metriken auf komplexen Räumen. Invent. Math. **3**, 43–70 (1967)

[K, N] Kempf, G., Ness, L.: The length of vectors in representations spaces. In: Algebraic geometry. (Lect. Notes Math., Vol. 732, pp. 233–244) Berlin Heidelberg New York: Springer 1979

[Kr] Kraft, H.: Geometrische Methoden in der Invariantentheorie. Braunschweig Wiesbaden: Vieweg 1985

[K, S, S] Kraft, H., Slodowy, P., Springer, T.A.: Algebraische Transformationsgruppen und Invariantentheorie. DMV Seminar Bd. 13. Basel Boston Berlin: Birkhäuser 1989

[L] Lassalle, M.: Séries de Laurent des fonctions holomorphes dans la complexification d'un espace symétrique compact. Ann. Sci. Éc. Norm. Supér. IV Sér. **11**, 167–210 (1978)

[Lu 1] Luna, D.: Slice etales. Bull. Soc. Math. Fr. Mem. **33**, 81–105 (1973)

[Lu 2] Luna, D.: Fonctions différentiables invariantes sous l'opération d'un groupe réductif. Ann. Inst. Fourier **26**, 33–49 (1976)

[Lu 3] Luna, D.: Sur certaines opérations différentiables des groupes de Lie. Am. J. Math. **97**, 172–181 (1975)

[Lu 4] Luna, D.: Sur les orbites fermées des groupes algébriques réductifs. Invent. Math. **16**, 1–5 (1972)

[Mo 1] Mostow, G.D.: Some new decomposition theorems for semi-simple groups. Am. Math. Soc. **14**, 31–51 (1955)

[Mo 2] Mostow, G.D.: On covariant fiberings of Klein spaces. Am. J. Math. **77**, 247–278 (1955)

[P, S] Procesi, C., Schwarz, G.: Inequalities defining orbit spaces. Invent. Math. **81**, 539–554 (1985)

[RH, W] Reese, H.,F., Wells, R.O.: Zero sets of non-negative strictly plurisubharmonic functions. Math. Ann. **201**, 165–170 (1973)

[R] Roberts, M.: A note on coherent G-sheaves. Math. Ann. **275**, 573–582 (1986)

[Ro] Rothaus, O.S.: Envelops of holomorphy of domains in complex Lie groups. In: Problems on analysis, 309–317. Princeton: University Press 1970

[S] Snow, D.M.: Reductive group action on Stein spaces. Math. Ann. **259**, 79–97 (1982)

[S, W] Streater, R.F., Wightman, A.S.: Die Prinzipien der Quantenfeldtheorie. Bibliographisches Institut Mannheim Zürich: Hochschultaschenbücherverlag 1969

[W] Warner, G.: Harmonic analysis on semisimple Lie groups, Vol. I. Berlin New York: Springer 1972

[We] Weyl, H.: The classical groups. Princeton, NJ. Princeton University Press 1946

[Z, S] Zariski, O., Samuel, P.: Commutative algebra, Vol. I. Princeton, NJ: Van Nostrand 1961