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The action of conformal transformations on a Riemannian manifold

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1 Introduction

The following assertion was accepted for twenty years when, in 1992, R.J. Zimmer and K.R. Gutschera discovered a very important gap in the only known proof of it [A1, A2]:

Theorem A *Let $C(M)$ be the whole conformal group of a Riemannian manifold M with $\dim M = n \geqq 2$. If M is not conformally equivalent with S^n or E^n, then $C(M)$ is inessential, i.e. can be reduced to a group of isometries by a conformal change of metric.*

Let us recall that the case of compact manifolds had been previously solved in 1969–71 (cf. [F1] completed by [F4]). (The Obata's proof [O1, O2] completed by [L] was concerning $C_0(M)$ only). Soon after in 1972–73 D.V. Alekseevskii proposed a synthetic general proof, at first in [A1] for $C_0(M)$, then in [A2] for $C(M)$ itself, which stopped all the research arising from the "Lichnérowicz conjecture".

In fact Theorem A can be divided in two parts of unequal difficulty, namely:

Theorem A_1 *If M is not conformally equivalent with S^n or E^n, then $C(M)$ acts properly on M (hence is compact if M is compact).*

Theorem A_2 *If $C(M)$ acts properly on M it is inessential.*

Theorem A_2 is almost obvious when M is compact. In the non-compact case it follows from Theorem 5 and 6 of [A1]. The proof of these theorems is consistent and based on classical arguments (cf. [G] for details).

Theorem A_1 is more difficult, especially in the non-compact case, as it involves global topological properties of the manifold. Alekseevskii's argument is based on the following strong assertion:

(P) *Let C be a closed group of automorphisms of a G-structure of finite type on M. If all the isotropy subgroups of C are compact, then C acts properly on M.*

A counterexample constructed by R.J. Zimmer has recently proved that (P) is not true, even for M compact, and K.R. Gutschera pointed out a non-obvious gap in Alekseevskii' proof of it (cf. [G]). Thus Theorem [A1] was not actually established in [A1, A2] and the major part of the proof of Theorem A was missing.

By thinking over this problem it seems indeed difficult to prove Theorem A_1 by using local properties of connections only. Thus the solution was to be sought in another direction.

The purpose of the present paper is to propose a rigorous proof of this theorem. Our proof is completely independent from Alekseevskii's one and is based on the theory of "global conformal invariants" initiated in [F2] and developed in [F8]. The direct method used in [F1] for the compact case was indeed not extensible to non-compact manifolds as it is based on direct estimations of modulus of continuity.

The new results contained in [F8] will reduce the general proof of Theorem A_1 to a surprisingly elementary discussion of the behaviour of sequences in $C(M)$. This discussion will be divided in three parts respectively relative to the following cases:

a) M is of class II, b) M is of class I, c) M is compact,

where, for brevity's sake, a non-compact manifold M is said to be of class I [resp. II] if its ideal boundary ∂M satisfies $\mathrm{Cap}\,\partial M = 0$ [resp. $\mathrm{Cap}\,\partial M > 0$] (cf. [F8] Sect. 6).

The arguments will indeed not be the same in those three cases and the necessity of this distinction follows from the fact that a correct proof of Theorem A_1 must involve the conformal geometry at infinity of M.

In the case c) the result is not new, but the present proof is much shorter and probably more pleasant than the one proposed in [F1].

It must be noticed that the case $n = 2$, which was separately treated in [A_1], is included in our proof.

We emphasize that our arguments will not use the structure of group of $C(M)$ and that we always deal with sequences of $C(M)$ instead of subgroups.

In fact, Theorem A_1 is not actually concerning the theory of Lie groups and may be considered as a mere theorem of Analysis.

This will appear more clearly when we extend this theorem to closed sets of uniformally quasiconformal maps [F9] as we have done for the compact case in [F1].

2 Topological preliminaries

In all sections 2 to 6, M will always denote a non-compact manifold with dimension $n \geqq 2$, and we shall deal with compact manifolds in Sect. 7 only. In all cases we will consider $C(M)$ as a subset of $\mathscr{C}(M,M)$ with the compact-open topology, and the assumption that M is C^∞ will only be used to obtain the closure of $C(M)$ in $\mathscr{C}(M,M)$ (theorem 2.8).

Notations. a) $\hat{M} = M \cup \{\infty\}$ is the Alexandrov compactification of M; ω_M is the constant infinite map $M \to \hat{M}$, $x \to \infty$.

b) $\mathscr{C}(M)$ is the linear space of continuous real-valued functions on M.

c) $\mathscr{C}(M,M)$ is the space of continuous maps of M into itself with the compact-open topology (c-topology for brevity). As M is locally compact and satisfies the second axiom of countability, $\mathscr{C}(M,M)$ is metrizable and we are allowed to use sequences in proving the compactness of subsets of $\mathscr{C}(M,M)$.

The c-topology can be extended to $\mathscr{C}(M,\hat{M})$ and thus we are allowed to set:

2.1 *A sequence (f_k) in $\mathscr{C}(M,M)$ is c-converging to the constant infinite map ω_M if, and only if, for any compact sets H, K of M, there exists an integer k_0 such that $f_k(H) \cap K = \emptyset$ for all $k \geqq k_0$.*

We shall use the following criterions of convergence in $\mathscr{C}(M,M)$:

2.2 *A sequence (f_k) in $\mathscr{C}(M,M)$ is c-convergent if, and only if, its restriction to every compact set of M is uniformly convergent.*

2.3 *In order that a sequence (f_k) in $\mathscr{C}(M,M)$ be c-converging to some $f \in \mathscr{C}(M,M)$, it is necessary and sufficient that for any $a \in M$ and any sequence (a_k) converging to a in M the sequence $f_k(a_k)$ tends to $f(a)$.*

At last, for applying the Ascoli's theorem, we will use the following lemma:

2.4 *Let F be a subset of $\mathscr{C}(M,M)$ and H, K be two compact sets of M such that, for any $f \in F, f(H) \subset K$. If F is not equicontinuous on H there exists a sequence (f_k) in F and two convergent sequences (a_k) and (b_k) in H with the same limit a, such that the sequences $(f_k(a_k))$ and $(f_k(b_k))$ converge to different limits.*

Proper action on M

The notion of "proper action" is usually defined for groups of transformations only. For what follows it will be convenient to extend it to all subsets of $\mathscr{C}(M,M)$. Then by using the fact that $\mathscr{C}(M,M)$ is metrizable, we can state (Cf. [B] Sect. 4 no 1):

2.5 *A subset F of $\mathscr{C}(M,M)$ is properly acting on M if and only if, for any sequence (f_k) in F and any sequence (a_k) in M such that $a = \lim(a_k)$ and*

$b = \lim f_k(a_k)$ exist, there exists a subsequence $(g_p) = (f_{k_p})$ of (f_k) which is c-converging to some $f \in F$ satisfying $f(a) = b$.

We have also (cf. [B] Sect. 4, no 5, theorem 1):

2.6 *A closed subset F of $\mathscr{C}(M,M)$ is properly acting on M if, and only if, for any compact sets H,K of M, the set $P(H,K) = \{f \in F / f(H) \cap K \neq \emptyset\}$ is relatively compact.*

The following assertion will perhaps be more vivid:

2.7 *A closed subset F of $\mathscr{C}(M,M)$ is properly acting on M if, and only if, every sequence (f_k) in F which is not c-converging to the infinite constant map ω_M, contains a c-convergent subsequence.*

Proof. a) If this condition is satisfied, let denote H, K two compact sets of M and (f_k) a sequence contained in $P(H,K) = \{f \in F / f(H) \cap K \neq \emptyset\}$. Then (f_k) cannot converge to ω_M and contains a c-convergent subsequence, which proves that $P(H,K)$ is relatively compact.

b) Conversely, let us assume that F acts properly on M and let (f_k) be a sequence in $\mathscr{C}(M,M)$ which does not converge to ω_M. Then for all compact sets H,K of M there exists an infinite sequence (k_p) of integers such that, for all $p, f_{k_p}(H) \cap K \neq \emptyset$. The sequence (f_{k_p}) is contained in $P(H,K)$ and therefore admits a convergent subsequence.

Now for applying these results to the conformal group $C(M)$, we need to know when $C(M)$ is closed in $\mathscr{C}(M,M)$. It seems that this problem has not been taken in account in [A_1].

It is however not obvious that the closure of $C(M)$ in $\mathscr{C}(M,M)$ is the same as the one we can obtain by considering the extension of $C(M)$ to the second order fiber bundle $\mathscr{T}^2(M)$, which belongs to $\mathscr{C}(\mathscr{T}^2(M), \mathscr{T}^2(M))$. Thus we will need the following assertion:

2.8 *Let (f_k) be a sequence of conformal automorphisms of a C^∞-manifold M which is c-converging to some homeomorphism f. Then f is conformal.*

Proof. From a classical result (cf. [Va] 37.3 for instance) f is 1-quasi-conformal, or in other words, it is a conformal homeomorphism; then from [F4] (theorem A) f is C^∞, hence f is a conformal diffeomorphism of M.

We do not know whether there exist other proofs of this theorem; let us only recall that the analogous result for isometries has been proved by S.E. Myers and N.E. Steenrod in [MS] by using the geodesics of the manifold.

Remarks. 1) From the discussion of Sects. 6, 7 it will appear that the limit of a pointwise convergent sequence of conformal automorphisms is either a constant or a homeomorphism, unless M be conformally equivalent with S^n or E^n.

2) The proof of (2, 8) is the only one in this paper which, until now, requires that M is C^∞.

3 Conformal invariants

We simply quote here the definitions and properties that we shall use for proving Theorem A_1. Complete proofs are given in [F8].

In what follows M will still denote a non-compact Riemannian manifold of dimension $n \geqq 2$, which is now only assumed to be of class C^1.

We set at first some definitions and notations.

D_1. $H(M) = \mathscr{C}(M) \cap W_n^1(M)$ is the linear space of continuous real-valued functions u on M admitting an L^n-integrable differential distribution, denoted ∇u by identification with a gradient, satisfying

$$I(u,M) = \int_M |\nabla u|^n d\tau < \infty$$

where $|\nabla u|$ is the norm of ∇u and $d\tau$ the volume element defined by the Riemannian structure of M.

D_2. $H_0(M)$ is the subspace of functions $u \in H(M)$ with compact support in M.

D_3. The capacity of a compact continuum C of M is $\mathrm{Cap}(C) = \mathrm{Inf}_u\, I(u,M)$ where $u \in H_0(M)$ satisfies $u = 1$ on C.

D_4. For all $(x,y) \in M^2$ we set $\mu_M(x,y) = \inf_{C \in \alpha(x,y)} \mathrm{Cap}(C)$ where $\alpha(x,y)$ is the set of compact continua of M containing x, y. If $y = x$, $\mu_M(x,x) = 0$.

D_5. Let C_0, C_1 be two continua (i.e. closed connected sets) of M, compact or not, and $A(C_0, C_1)$ the set of functions $u \in H(M)$ satisfying $u = 0$ on $C_0, u = 1$ on C_1. Then the capacity of the condenser $\Gamma(C_0, C_1)$ defined by C_0, C_1 is

$$\mathrm{Cap}(C_0, C_1) = \inf_{u \in A(C_0,\, C_1)} I(u,M)$$

with $\mathrm{Cap}(C_0, C_1) = +\infty$ if $A(C_0,\ C_1) = \emptyset$.

D_6. For all $(x,y,z) \in M^3$ with $z \neq x$, $z \neq y$ we set:

$$\nu_M(x,y,z) = \inf_{C_0, C_1} \mathrm{Cap}(C_0, C_1)$$

where C_0 is a non-compact continuum containing z, hence joining z to ∞, and C_1 a compact continuum containing x, y.

This function ν_M is extended to any point (x,y,z) of $M^3 \backslash \Delta$, where $\Delta = \{(x,x,x)/x \in M\}$ is the diagonal of M^3, by setting $\nu_M(x,y,y) = \nu_M(x,y,x) = +\infty$ if $y \neq x$ and obviously $\nu_M(x,x,z) = 0$ if $x \neq z$.

Basic properties

3.1 *The functions μ_M and ν_M are only dependent on the conformal structure of M and are therefore invariant under any conformal mapping. More precisely, if f is a conformal mapping of a manifold M onto a manifold N, we have for all x, y, z in M:*

$$\mu_N(f(x), f(y)) = \mu_M(x,y),\quad \nu_N(f(x), f(y), f(z)) = \nu_M(x,y,z)\,.$$

Proof. This follows from the fact that the integral $I(u,M)$ is only depending on the conformal class of the Riemannian metric g of M, and is therefore invariant under any conformal mapping.

3.2 a) *The functions* $\mu_M : M^2 \to \mathbb{R}$ *and* $\nu_M : M^3 \setminus \Delta \to \overline{\mathbb{R}}$ *are continuous and* μ_M *is always finite.*

b) *If there exist two points* a, b *of* M *with* $b \neq a$ *such that* $\mu_M(a,b) = 0$, *then* $\mu_M(C) = 0$ *for any compact continuum* C *of* M, *hence* $\mu_M(x,y) = 0$ *for all* $(x,y) \in M^2$, *which is equivalent with* $\operatorname{Cap} \partial M = 0$ *(cf.* F8*).*

Corollary. *The two following classes of non compact* C^1*-Riemannian manifolds are complementary:*

I *The class of manifolds* M *such that* μ_M *is identically zero.*
II *The class of manifolds* M *on which the relation* $\mu_M(x,y) = 0$ *is equivalent with* $y = x$.

Moreover, if two Riemannian manifolds M, N *are conformally equivalent, they belong to the same class.*

Examples. Let N be a compact Riemannian manifold. Then

a) For any finite set $S = \{a_1, \ldots, a_k\}$ of points of N, the punctured manifold $N \setminus S$ is of class I.

b) For any compact continuum C of N, the open submanifold $N \setminus C$ is of class II.

Other examples are given in [F8].

Special properties

3.3 *If* M *is of class* II, *then* μ_M *is a distance on* M, *and the topology defined by this distance is the same as its topology of manifold.*

3.4 *If* M *is of class* I *the function* ν_M *can be extended to* $(M \times M \times \hat{M}) \setminus \Delta$ *by setting, for all* $(x,y) \in M^2 : \nu_M(x,y,\infty) = 0$. *Then* ν_M *is a continuous map of* $(M \times M \times \hat{M}) \setminus \Delta$ *onto* $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ *with the topology of order. Moreover*

a) *the relation* $\nu_M(x,y,z) = 0$ *is equivalent with* $(y = x$ *or* $z = \infty)$.

b) *If* $y \neq x$ *the relation* $\nu_M(x,y,z) = +\infty$ *is equivalent with* $(z = x$ *or* $z = y)$.

4 Proof of theorem A_1 for manifolds of class II

This case is easy since $C(M)$ is a group of isometries for the metric space (M, μ_M). The announced result will follow from the following lemma which is a slight improvement of the theorem 4.7 in [KN].

4.1 *Let* (X,d) *be a connected locally compact metric space and* (f_k) *a sequence of isometries of* X *onto itself. If there exists a convergent sequence*

(a_k) in M such that $(f_k(a_k))$ has a limit $b \in X$, then there exists a subsequence (f_{k_p}) of (f_k) which is c-converging in $\mathscr{C}(X, X)$ to an isometry f and the sequence (f_k^{-1}) is c-converging to f^{-1}.

Proof. Let $a = \lim(a_k)$. Then the inequalities: $d(f_k(a), b) \leqq d(f_k(a_k), b) + d(f_k(a), f_k(a_k)) = d(f_k(a_k), b) + d(a, a_k)$ show that $f_k(a)$ tends to b, and the hypotheses of the theorem 4.7 of [KN] are fulfilled.

With this slight improvement we can give a rapid proof of the convergence of (f_k^{-1}) if we observe that our hypothesis is in fact symmetrical with respect to (f_k) and (f_k^{-1}), by exchanging a_k with $b_k = f_k(a_k)$. The sequence (f_k^{-1}) is therefore also convergent; its limit g must satisfy $g[f(x)] = \lim f_k^{-1}(f_k(x)) = x$, and similarly $f[g(x)] = x$. Hence f is a homeomorphism and $g = f^{-1}$.

Then by using 2.5 and 2.8 we can state:

4.2 Theorem *The conformal group $C(M)$ of a non-compact Riemannian manifold of class* II *acts properly on M.*

Remark. It is possible to define conformally invariant distances on a class of manifolds which are not necessarily of class II (Cf. [F2], [F8], for the definition of the λ-distance). But the conformal invariant ν_M will allow us to study at the same time all the manifolds which are not of class II.

5 Proof of Theorem A_1 for manifolds of class I

The desired result will follow from the discussion of the behaviour of a sequence (f_k) in $C(M)$, where M is of class I. After having set aside the case of convergence to ω_M (Lemma 5.1) we shall distinguish two main cases leading resp. to propositions (5.2) and (5.3).

5.1 Lemma *If the sequence (f_k) is not c-converging to the constant infinite map ω_M there exists a subsequence $(g_p) = (f_{k_p})$ of (f_k) and a convergent sequence (a_p) in M such that the sequence $(g_p(a_p))$ is convergent.*

Proof. From hypothesis there exist two compact sets H, K in M such that the set $\{k \in \mathbb{N} / f_k(H) \cap K \neq \emptyset\}$ is infinite; hence the existence of a subsequence $g_p = (f_{k_p})$ of (f_p) and of a sequence (a_p) in H such that, for all p, $g_p(a_p) \in K$, and the announced result is obtained by a new extraction of subsequence.

5.2 *Let (f_k) be a sequence of conformal mappings of M onto itself. If there exist two convergent sequences (a_k) and (b_k) in M, with different limits a, b, such that the limits $\alpha = \lim f_k(a_k)$ and $\beta = \lim f_k(b_k)$ exist, then:*
i) *if $\beta \neq \alpha$ there exists a subsequence $(g_p) = (f_{k_p})$ of (f_k) which is c-converging to a conformal map f of M onto itself, and the sequence (g_p^{-1}) is c-converging to f^{-1}.*
ii) *if $\beta = \alpha$ the sequence (f_k) is c-converging to the constant α.*

The proof is divided into five lemmas.

5.2a *For any compact set H of M there exists a compact set K of M containing all the compact sets $f_k(H)$.*

Proof. If it were not true, there would exist a sequence (x_p) of points of H and an infinite sequence (k_p) of integers such that $f_{k_p}(x_p)$ tend to ∞. By extraction of a new sequence we could assume that (x_p) has a limit x; then the sequence $v_M(a_{k_p}, b_{k_p}, x_p)$ would tend to $v_M(a,b,x)$ which is strictly positive while from (3.4) the sequence $v_M(f_{k_p}(a_{k_p}), f_{k_p}(b_{k_p}), f_{k_p}(x_p))$ would tend to $v_M(\alpha, \beta, \infty) = 0$. But this is impossible since from (3.1) these sequences have the same values.

5.2b *The family (f_k) is uniformly equicontinuous on all compact set of M.*

Proof. If not, from (2.4), there would exist a subsequence $(g_p) = (f_{k_p})$ of (f_k) and two convergent sequences $(x_p), (y_p)$ in M, with the same limit x such that the sequences $g_p(x_p)$ and $g_p(y_p)$ converge to different limits ξ, η. Then one at least of the points a, b would be different from x, let $a \neq x$ for precision, and the sequence $v_M(x_p, y_p, a_{k_p})$ would tend to $v_M(x,x,a) = 0$ while $v_M(g_p(x_p), g_p(y_p), g_p(a_{k_p}))$ would tend to $v_M(\xi, \eta, \alpha)$ which is strictly positive, but from (3.1) this is impossible.

5.2c *There exists a subsequence (g_p) of (f_k) which is c-converging to some map $g \in \mathscr{C}(M,M)$.*

This follows from (5.2.a), (5.2.b) and Ascoli's theorem.

5.2d *If $\alpha \neq \beta$ the map g defined in (5.2.c) is a conformal automorphism of M and the sequence (g_p^{-1}) is c-converging to g^{-1}.*

Proof. At first we remark that the sequence (g_p^{-1}) satisfies the same kind of hypothesis as (g_p): if we set $\alpha_p = g_p(a_{k_p})$ and $\beta_p = g_p(b_{k_p})$ we have indeed $\lim \alpha_p = \alpha$, $\lim \beta_p = \beta$, $\lim g_p^{-1}(\alpha_p) = a$, $\lim g_p^{-1}(\beta_p) = b$. Hence there exists a subsequence $(h_r) = (g_{p_r}^{-1})$ of (g_p^{-1}) which has a c-limit h in $\mathscr{C}(M,N)$ and for all $x \in M$ we have $h_r[g_{p_r}(x)] = x$; hence $h(g(x)) = x$, and similarly $g(h(x)) = x$; this proves that g is a homeomorphism, and from (2.8) g is conformal.

At last the sequence (g_p^{-1}) itself tends to g^{-1}.

5.2e *If $\alpha = \beta$ the sequence (f_k) is c-converging to the constant α.*

Proof. Let (x_k) be a convergent sequence in M with limit x, and ξ a cluster point of the sequence $f_k(x_k)$ in the compact $\hat{M}$. Assuming $\xi \neq \alpha$ we should have $v_M(a,b,x) = \lim v_M(a_k, b_k, x_k)$, while $v_M(\alpha, \alpha, \xi) = 0$ would be a cluster value of $v_M(f_k(a_k), f_k(b_k), f_k(x_k))$; but from (3.1) this is impossible since $v_M(a,b,x) > 0$.

Hence for all convergent sequence (x_k) in M, the sequence $(f_k(x_k))$ tends to α, which by using (2.3) proves directly that (f_k) is c-converging to the constant α.

Now we complete (5.2) by the following lemma:

5.3 *Let (f_k) be a sequence in $C(M)$, no subsequence of which is c-convergent (not even to ω_M). Then there exist a subsequence $(g_p) = (f_{k_p})$ of (f_k) and a convergent sequence (a_p) in M with limit a such that*

a the sequence $g_p(a_p)$ has a limit α in M.

b The restriction of (g_p) to $M\backslash\{a\}$ is c-converging to infinity.

c The sequence (g_p^{-1}) is c-converging to the constant a.

Proof.

a The assertion *a*) follows from lemma 5.1, since (f_k) is not *c*-converging to ω_M.

b Assuming that the assertion b) is not true, we could apply the lemma 5.1 to $M\backslash\{a\}$: hence there would exist a sequence (b_p) in $M\backslash\{a\}$ converging to some point $b \neq a$ such that the sequence $g_p(b_p)$ would have a finite cluster value β; then we could use (5.2) to prove the existence of a *c*-convergent subsequence of (g_p), in contradiction with our hypothesis.

c Let $\alpha_p = g_p(a_p)$ and (ξ_p) be a sequence in M converging to some point $\xi \in M$. Then for any convergent sequence (y_p) in M with limit $y \neq a$, we know that $g_p(y_p)$ tends to ∞; hence the sequence $v_M(\alpha_p, \xi_p, g_p(y_p))$ tends to $v_M(\alpha, \xi, \infty) = 0$, and from (3.1) the sequence $v_M(a_p, g_p^{-1}(\xi_p), y_p)$ is also converging to zero. From the continuity of v_M, any cluster point ξ of the sequence $g_p^{-1}(\xi_p)$ satisfies $v_M(a, \xi, y) = 0$, hence $\xi = a$. In conclusion, for all convergent sequence (ξ_p) in M, the sequence $g_p^{-1}(\xi_p)$ tends to a, which proves the *c*-convergence of $(g_p)^{-1}$ to the constant a.

Then by using (2.7) the results of the above discussion allow us to state:

5.4 *Let M be a Riemannian manifold of class I and F a closed part of $C(M)$ which is not properly acting on M. Then there exists a sequence (g_p) in F satisfying one of the following conditions:*

i) *The sequence (g_p) is c-converging to some constant $\alpha \in M$ while (g_p^{-1}) is c-converging to infinity on $M\backslash\{\alpha\}$.*

ii) *There exists a point a in M such that the sequence (g_p) is c-converging to infinity on $M\backslash\{a\}$ while (g_p^{-1}) is c-converging to the constant a on M.*

Remarks. 1) We observe that the conditions i) and ii) are exchanged when we exchange (g_p) and (g_p^{-1}). Hence if $F = F^{-1} = \{f \in C(M)/f^{-1} \in F\}$, and particularly if F is a closed subgroup of $C(M)$, the hypothesis of (5.4) implies the existence of a sequence satisfying i) and of a sequence satisfying ii), both sequences being contained in F.

2) If F is a commutative subgroup of $C(M)$, it is easy to see that the point α in i) and the point a) in ii) are fixed points of F.

3) In the general case we cannot say anything about the limit of $(g_p^{-1}(\alpha))$ in i) neither on that of $(g_p(a))$ in ii).

In the case ii) we can choose a sequence (x_p) converging to a such that $g_p(x_p)$ has an arbitrary given limit $y \in M$: we have but to set $x_p = g_p^{-1}(y)$.

The sequence (g_p) is therefore never *c*-converging to ω_M on M, even if $g_p(a)$ tends to ∞. For example let (f_p) and (g_p) be the sequences of similarities defined in E^n by $f_p(x) = p^2x + p$, $g_p(x) = px + 1/p$; both sequences (f_p^{-1}) and (g_p^{-1}) are c-converging to zero, hence we are in case ii) with $a = 0$, but $f_p(x)$ tends to infinity for all x, while $g_p(x)$ is converging to infinity for $x \neq 0$ only, as $g_p(0)$ tends to zero.

6 End of the proof for non compact manifolds

In all this section M is still denoting a non-compact Riemannian manifold of dimension $n \geqq 2$. By collecting (2.8), (4.2) and (5.4) with remark 1, we obtain immediately:

6.1 *The limit of a c-convergent sequence (f_k) of conformal automorphisms is either a constant map (possibly infinite), or a conformal automorphism. Consequently $C(M)$ is closed in $\mathscr{C}(M,M)$ if, and only if, its adherence does not contain any finite constant map.*

6.2 *If $C(M)$ is not properly acting on M there exists a sequence (g_p) of conformal automorphisms of M which is c-converging on M to some constant $a \in M$, while the restriction to $M \backslash \{a\}$ of the sequence (g_p^{-1}) is c-converging to infinity.*

For brevity such a sequence (g_p) will be called *degenerating.*

We have now to prove that the existence of a degenerating sequence (g_p) in $C(M)$ implies the existence of a conformal map of M onto E^n. We will start with the following lemma.

6.3 *Let us assume that $C(M)$ contains a degenerating sequence (g_p) with limit a, and let A be an open neighborhood of a such that $\bar{A}$ is diffeomorphic with an euclidian closed ball $\bar{B}$. Then there exists a subsequence $(h_k) = (g_{p_k})$ of (g_p) such that $(h_k^{-1}(\bar{A}))$ is an increasing exhausting sequence of compact sets for M.*

Proof. Let (H_k) be an exhausting sequence of compact sets for M. From hypothesis, for all integer k there exists an integer p_k such that for all $p \geqq p_k$, $g_p(H_k) \subset A$, hence $g_p^{-1}(A) \supset H_k$. By a process of recurrence we can choose the sequence (p_k) such that the sequence $(g_{p_k}^{-1}(A))$ is increasing, and $(h_k) = (g_{p_k})$ satisfies the wanted condition.

6.4 Corollary. *If $C(M)$ is not properly acting on M, then M is homeomorphic with $\mathbb{R}^n$*

Proof. From (6.3) M is the union of an increasing sequence $(h_k^{-1}(A))$ of open sets, all homeomorphic with an euclidian ball, hence the result.

Now by "blowing up" the maps (h_k^{-1}) of (6.3) as we did in section 8 of [F1] for the maps ϕ_p, we could directly construct a conformal map of M onto E^n. This process being rather elaborate we reserve it for the extension of theorem A_1 to the quasiconformal case, where it is the only possible, and we will confine ourselves to prove that under the hypothesis of (6.3), M is conformally flat. This fact is almost obvious for $n = 2$; and for $n \geqq 3$ we will use an argument analogous with an Obata's one [O2], founded on the Weyl or Schouten tensor. The adaptation of this argument given by P. Pansu [P] will allow us to give a very short proof.

6.5 *If $C(M)$ is not properly acting on M, then M is conformally flat.*

Proof. Let W be the Weyl tensor of M if $n \geqq 4$ and Dh the absolute differential of the Schouten tensor if $n = 3$, and let us set $\rho = |W|^{n/2}$ in the first case, $\rho = |Dh|^{3/2}$ in the second case. As W and Dh are known to be conformally invariant [L], the integrals $\int_X |W|^{n/2} d\tau$ and $\int_X |Dh|^{3/2} d\tau$, where X is a measurable set of M, are invariant under any $f \in C(M)$. Now, $\varepsilon > 0$ given, we can choose A in (6.3) such that $\int_A \rho d\tau < \varepsilon$, and for all compact set H of M, there exists $k \in \mathbb{N}$ such that $H \subset h_k^{-1}(A)$; hence $\int_H \rho d\tau \leqq \int_{h_k^{-1}(A)} \rho d\tau = \int_A \rho d\tau < \varepsilon$ and finally $\rho = 0$ on all compact set of M, which proves that M is conformally flat.

If $n = 2$ we can choose A such that there exists a conformal map of A onto a ball B, hence $M = \bigcup_k h_k^{-1}(A)$ is the union of an increasing sequence of open sets conformally equivalent with B.

In all cases, from (6.4) and Kuiper's theorem [Ku], M is conformally equivalent with a simply connected domain of E^n, and the existence of a degenerating sequence in $C(M)$ shows that M is conformally equivalent with E^n itself. We can state:

6.6 Theorem. *If the conformal group $C(M)$ of a non-compact Riemannian manifold of dimension $n \geqq 2$ is not properly acting on M, then M is conformally equivalent with E^n.*

7 Proof of Theorem A for compact manifolds

In this section M will denote a compact manifold of dimension $n \geqq 2$, and we shall use the conformal invariant ρ_M defined as follows, with the notations of Sect. 3.

7.1. a *For any distinct points a, b, c, d of M we set $\rho_M(a,b,c,d) = \inf_{C_0, C_1} \mathrm{Cap}(C_0, C_1)$ where C_0 is a continuum joining a to b in M, and C_1 a continuum joining c to d.*

b *Denoting Δ the set of points $(a,b,c,d) \in M^4$ three coordinates of which at least are equal, we extend the definition of ρ_M to $M \backslash \Delta$ by setting*

$$\rho_M(a,b,c,d) = 0 \quad \text{if } a = b \text{ or } c = d$$

$$\rho_M(a,b,c,d) = +\infty \text{ if } \quad \{a,b\} \cap \{c,d\} \neq \emptyset .$$

Then with the topology of order on $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$, we have:

7.2. a *The function $\rho_M : M^4 \backslash \Delta \to \bar{\mathbb{R}}_+$ is continuous and conformally invariant under all conformal map of M.*

b *$\rho_M(a,b,c,d) = 0$ is equivalent with $a = b$ or $c = d$.*
c *$\rho_M(a,b,c,d) = +\infty$ is equivalent with $\{a,b\} \cap \{c,d\} \neq \emptyset$.*

These properties of ρ_M will allow us to discuss the convergence of sequences in $C(M)$ by an argument analogous with the ones of Sects. 5 and 6. At first the following Lemma is a special case of (2.4).

7.3 *If F is a subset of $C(M)$ which is not uniformly equicontinuous on M, there exist a sequence (f_k) in F and two convergent sequences (a_k), (b_k) in M with the same limit a such that the sequences $(f_k(a_k))$ and $(f_k(b_k))$ are resp. converging to distinct limits α, β.*

7.4 *Let (f_k) be a sequence as in (7.3). Then there exists a subsequence $(g_p) = (f_{k_p})$ of (f_k) which is c-converging to some constant γ on $M \backslash \{a\}$ while the sequence (g_p^{-1}) is c-converging to a on $M \backslash \{\gamma\}$.*

Proof. a) Let (c_k) be a convergent sequence in M with $\lim(c_k) = c \neq a$, and $(g_p) = (f_{k_p})$ a subsequence of (f_k) such that $g_p(c_{k_p})$ has a limit γ. Then for any convergent sequence (x_p) in M with $\lim(x_p) = x \neq a$ the sequence $\rho_M(a_{k_p}, b_{k_p}, c_{k_p}, x_p)$ tends to $\rho_M(a,a,c,x) = 0$ and from 7.2 a) the sequence $\rho_M(g_p(a_{k_p}), g_p(b_{k_p}), g_p(c_{k_p}), g_p(x_p))$ tends also to zero; hence any cluster point ξ of the sequence $g_p(x_p)$ satisfies $\rho_M(\alpha,\beta,\gamma,\xi) = 0$ which implies $\xi = \gamma$. The sequence (g_p) is therefore c-converging to the constant γ on $M \backslash \{a\}$.

b) for all sequence (ξ_p) in M converging to some point $\xi \neq \gamma$ we have: $\lim \rho_M(a_{k_p}, b_{k_p}, c_{k_p}, g_p^{-1}(\xi_p)) = \lim \rho_M(g_p(a_{k_p}), g_p(b_{k_p}), g_p(c_{k_p}), \xi_p) = \rho_M(\alpha,\beta,\gamma,\xi)$.

Assuming that the sequence $g_p^{-1}(\xi_p)$ has a cluster point $x \neq a$ we should have $\rho_M(\alpha,\beta,\gamma,\xi) = \rho_M(a,a,c,x) = 0$, which is impossible. The sequence $g_p^{-1}(\xi_p)$ is therefore converging to a, and the sequence (g_p^{-1}) is c-converging to the constant a on $M \backslash \{\gamma\}$.

Remarks. 1) The points a and γ are not necessarily distinct, as we can observe on the Möbius group $C(S^n)$.

2) As in the remark 3 following (5.4) we cannot say anything on the limit of $(g_p(a))$; but even if $g_p(a)$ tends to γ, the convergence of (g_p) is not uniform on M, as $g_p(a_{k_p})$ and $g_p(b_{k_p})$ have different limits.

The following lemma will give a criterion for c-compacity analogous with (5.2).

7.5 *Let (f_k) be a sequence in $C(M)$, and (x_k), (y_k), (z_k) three convergent sequence in M with distinct limits x, y, z such that the sequences $f_k(x_k)$, $f_k(y_k)$, $f_k(z_k)$ are resp. converging to u, v, w.*

a) *If u, v, w are distinct, there exists a subsequence $(g_p) = (f_{k_p})$ of (f_k) which is c-converging to a homeomorphism g of M onto itself, while (g_p^{-1}) is c-converging to g^{-1}.*

b) *If u, v, w are not distinct there exist a point a of M and a subsequence (g_p) of (f_k) which is c-converging to some constant on $M \backslash \{a\}$.*

Proof. a) We prove at first that the $f_k\,(k \in \mathbb{N})$ are uniformly equicontinuous on M: if not, from (7.3) and (7.4) there would exist a point a of M and a subsequence (g_p) of (f_k) c-converging to some constant γ on $M\backslash\{a\}$. Then two at least of the points x, y, z would be distinct from a, let x, y for precision, and we should have $\lim g_p(x_{k_p}) = \gamma = \lim g_p(y_{k_p})$ hence $u = v$, in contradiction with the hypothesis. By exchanging f_k with f_k^{-1} and (x, y, z) with (u, v, w) we observe that the $f_k^{-1}\,(k \in \mathbb{N})$ satisfy the same kind of hypothesis as the f_k, hence the f_k^{-1} are also uniformly equicontinuous on M.

As a consequence there exists a subsequence (g_p) of (f_k) which is c-converging to some continuous map g, while the sequence (g_p^{-1}) is c-converging to some continuous map h, hence $h = g^{-1}$ and g is a homeomorphism.

b) If, for instance $u = v$, the sequence (f_k^{-1}) has the properties stated in (7.3) for (f_k); hence, from (7.4), the announced result by exchanging f_k with f_k^{-1}. At last by using (2.8) we can state:

7.6 *If $C(M)$ is not compact there exist a sequence (g_p) in $C(M)$ and a point a of M such that (g_p) is c-converging to some constant b on $M\backslash\{a\}$ while (g_p^{-1}) is converging to a on $M\backslash\{b\}$.*

For brevity such a sequence (g_p) will be called *degenerating.*

Proof. Let us choose three distinct points x_1, x_2, x_3 of M. Every sequence S in $C(M)$ contains a subsequence (f_k) such that $u_i = \lim f_k(x_i)$ exists for $i = 1, 2, 3$. If the sequence S does not contain any c-convergent subsequence it follows from (7.5) that u_1, u_2, u_3 are not distinct and that $C(M)$ contains a degenerating sequence.

Let us remark here that a, b are not necessarily distinct.

Now if (g_p) is a degenerating sequence as in (7.6) and A an open neighborhood of a, $(g_p(\bar{A}))$ is an exhausting sequence of compact sets for $M\backslash\{b\}$. By choosing A homeomorphic with an euclidian ball we see that $M\backslash\{b\}$ is homeomorphic with E^n, hence M is homeomorphic with S^n. At last the same argument as in Sect. 6 proves that M is conformally equivalent with S^n from Kuiper's theorem [Ku]. We can state:

7.7 Theorem. *If the conformal group $C(M)$ of a compact Riemannian manifold M is not compact, then M is conformally equivalent with S^n.*

This assertion was the Lichnerowicz's conjecture, already proved in [F1, F4]. Another proof of the same kind as the above one has been sketched in [P].

Remark. If M is a compact manifold the isotropy subgroup $C_a(M)$ of a point a can be identified with $C(M\backslash\{a\})$. We can check that the discussion of Sects. 5 and 7 give concordant results for the behaviour of this group.

Conversely the conformal group of a non-compact manifold M may be considered in some way as the isotropy group of ∞ in $C(\hat{M})$, and we observe that (7.6) is equivalent with the following statement:

7.8 *If M is a non-compact manifold which is not conformally equivalent with E^n, then $C(M) \cup \{\omega_M\}$ is a compact subset of $\mathscr{C}(M,\hat{M})$.*

Conclusion. By gathering Theorem 4.2, 6.6 and 7.7 we obtain Theorem A_1. As Theorem A_2 has been proved in [A1], the proof of Theorem A is finally complete.

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