Math. Ann. 304, 225–252 (1996)

Mathematische Annalen

Stable triples, equivariant bundles and dimensional reduction

Steven B. Bradlow[1,*], **Oscar García-Prada**[2]

[1] Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
[2] Departamento de Matemáticas, Universidad Autónoma de Madrid, E-28049 Madrid, Spain

Received: 5 January 1994

1 Introduction

The Hitchin-Kobayashi correspondence between stable bundles and solutions to the Hermitian-Einstein equations allows one to apply analytic methods to the study of stable bundles. One such analytic technique, which has not yet been much exploited, is that of dimensional reduction. This is a useful tool for studying certain special solutions to partial differential equations; in particular it is useful for studying solutions which are invariant under the action of some symmetry group. When applied to the Hermitian-Einstein equations, it thus provides a way of looking at holomorphic bundle structures which are both stable and invariant under some group action on the bundle, i.e. of looking at *equivariant stable bundles*.

Such ideas are developed in [GP3], where they are applied to certain $SU(2)$-equivariant bundles over $X \times \mathbb{P}^1$. Here X is a closed Riemann surface and the $SU(2)$-action is trivial on X and the standard one on $\mathbb{P}^1$. In this case, the equivariant holomorphic bundles over $X \times \mathbb{P}^1$ correspond to holomorphic pairs (i.e. bundles plus prescribed global sections) over X. The dimensional reduction of the Hermitian-Einstein equations gives the vortex equations, and the stable equivariant bundles on $X \times \mathbb{P}^1$ correspond (by dimensional reduction) to τ-stable holomorphic pairs on X, with τ-stability as defined in [B2] and [GP3].

However not all the $SU(2)$-equivariant holomorphic bundles over $X \times \mathbb{P}^1$ correspond to holomorphic pairs on X. In fact those that do form a rather restricted subset of the set of all such equivariant bundles. A very natural relaxation of this restriction leads to a class of equivariant bundles on $X \times \mathbb{P}^1$ which still corresponds to data on the (lower dimensional) space X, but not necessarily to holomorphic pairs. Such bundles on $X \times \mathbb{P}^1$ correspond to a pair of bundles on X, together with a holomorphic homomorphism between them. We call such data a holomorphic triple on X.

* Supported in part by a NSF-NATO Postdoctoral Fellowship and NSF grant DMS 93-03545

In this paper we undertake a detailed investigation of holomorphic triples over the closed Riemann surface X. In particular, we define, in Sect. 3, a notion of stability for such objects. We explore the relationship between the stability of a triple and the stability of the corresponding equivariant bundle over $X \times \mathbb{P}^1$. An important feature of the definition is that, like in the case of holomorphic pairs, it involves a real parameter. This can be traced back to the fact that the definition of stability for a bundle over $X \times \mathbb{P}^1$ depends on the polarization (choice of Kähler metric) on $X \times \mathbb{P}^1$. We discuss the nature of this parameter, and its influence on the properties of the stable triples. We show for example that

In all cases, with one exception, the parameter in the definition of triples stability lies in a bounded interval. The interval is partitioned by a finite set of non-generic values.

Our main result is given in Sect. 4. Loosely speaking, it is that the stable triples over X can be considered the dimensional reduction of the stable equivariant bundles over $X \times \mathbb{P}^1$. In other words,

A holomorphic triple over X is stable if and only if the corresponding $SU(2)$-equivariant extension over $X \times \mathbb{P}^1$ is stable.

In [GP3] dimensional reduction is applied to the Hermitian-Einstein equation on equivariant bundles over $X \times \mathbb{P}^1$. The result is that on bundles corresponding to triples over X, the equivariant solutions correspond to solutions to a pair of *Coupled Vortex Equations* on the two bundles in the triple. By combining this result, our dimensional reduction result for stable bundles, and the Hitchin-Kobayashi correspondence, we can thus show

There is a Hitchin-Kobayashi correspondence between stability of a triple and existence of solutions to the Coupled Vortex Equations.

This is discussed in Sect. 5. In Sect. 6 we discuss the moduli spaces of stable triples. By identifying these as fixed point sets of an $SU(2)$-action on the moduli spaces of stable bundles over $X \times \mathbb{P}^1$, we obtain results such as

For fixed value of the stability parameter, the moduli space of stable triples is a quasi-projective variety. For generic values of the parameter, and provided the ranks and degrees of the two bundles satisfy a certain coprimality condition, the moduli space is projective.

In Sect. 2 we have collected together the basic definitions and background material that we will need.

2 Background and preliminaries

2.1 Basic definitions

Let X be a compact Riemann surface. The product $X \times \mathbb{P}^1$ has an $SU(2)$ action in which $SU(2)$ acts trivially on X and via the identification with the homogeneous space $SU(2)/U(1)$ on $\mathbb{P}^1$.

Definition 2.1 *Let F be a C^∞ complex vector bundleover $X \times \mathbb{P}^1$. Then F is said to be $SU(2)$*-**equivariant** *if there is an action of $SU(2)$ on F covering the action on $X \times \mathbb{P}^1$. Similarly a holomorphic vector bundle F is $SU(2)$*-**equivariant** *if it is $SU(2)$-equivariant as a C^∞ bundle and in addition the action of $SU(2)$ on F is holomorphic.*

In order to avoid the introduction of more notation we shall denote a C^∞ vector bundle and the same bundle endowed with a holomorphic structure by the same symbol. The distinction will be made explicit unless it is obvious from the context. The following structure theorems were proved in [GP3]:

Proposition 2.2 *Every $SU(2)$-equivariant C^∞ vector bundle F over $X \times \mathbb{P}^1$ can be equivariantly decomposed, uniquely up to isomorphism, as*

$$F = \bigoplus_i p^* E_i \otimes q^* H^{\otimes n_i} ,$$

where p and q are the projections from $X \times \mathbb{P}^1$ to the first and second factors, E_i is a C^∞ vector bundle over X, H is the C^∞ line bundle over $\mathbb{P}^1$ with Chern class 1, and $n_i \in Z$ are all different.

In this paper we consider only the case where

$$F = p^* E_1 \oplus p^* E_2 \otimes q^* H^{\otimes 2} . \tag{1}$$

Proposition 2.3 *There is a one-to-one correspondence between $SU(2)$-equivariant holomorphic vector bundles F with underlying $SU(2)$-equivariant C^∞ structure given by (1), and holomorphic extensions of the form*

$$0 \to p^* E_1 \to F \to p^* E_2 \otimes q^* \mathcal{O}(2) \to 0 , \tag{2}$$

where E_1 and E_2 are the bundles over X defining (1) equipped with holomorphic structures. Moreover, every such extension is defined by an element $\Phi \in \mathrm{Hom}(E_2, E_1)$.

Proof. This is Proposition 3.9 in [GP3]. For convenience, we remind the reader that extensions over $X \times \mathbb{P}^1$ of the form (2) are parametrized by

$$H^1(X \times \mathbb{P}^1, p^*(E_1 \otimes E_2^*) \otimes q^* \mathcal{O}(-2)) ,$$

and that by the Künneth formula this is isomorphic to

$$H^0(X, E_1 \otimes E_2^*) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong H^0(X, E_1 \otimes E_2^*) .$$

After fixing an element in $H^1(\mathbb{P}^1, \mathcal{O}(-2))$, the homomorphism Φ can thus be identified with the extension class defining F.

In view of Proposition 2.3, there is a one-to-one correspondence between extensions of the form (2) and *holomorphic triples* (E_1, E_2, Φ) *on* X, where

Definition 2.4 *A* **holomorphic triple** *on X is a triple (E_1, E_2, Φ) consisting of two holomorphic vector bundles E_1 and E_2 on X together with a homomorphism $\Phi \colon E_2 \to E_1$, i.e. an element $\Phi \in H^0(\mathrm{Hom}(E_2, E_1))$.*

By Proposition 2.3, a holomorphic triples over X can clearly be regarded as the "*dimensional reduction*" of an $SU(2)$-equivariant holomorphic vector bundles over $X \times \mathbb{P}^1$. In fact, this correspondence between triples on X and bundles on $X \times \mathbb{P}^1$ can be extended more generally to arbitrary coherent sheaves. Indeed, if S_1 and S_2 are two coherent sheaves on X and $\Psi \in \mathrm{Hom}(S_2, S_1)$, then the triple (S_1, S_2, Ψ) defines a coherent sheaf U over $X \times \mathbb{P}^1$, given as an extension

$$0 \to p^*S_1 \to U \to p^*S_2 \otimes \mathcal{O}(2) \to 0 . \qquad (3)$$

The proof is the same as for bundles, but with $H^1(X \times \mathbb{P}^1, p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2))$ replaced by $\mathrm{Ext}^1_{X\times\mathbb{P}^1}(p^*S_2 \otimes q^*\mathcal{O}(2), p^*S_1)$.

2.2 *Equations for special metrics*

The Hermitian-Einstein equations determine special metrics on holomorphic bundles over $X \times \mathbb{P}^1$. Indeed, if (M, ω) is any compact Kähler manifold, and E is a holomorphic bundle over M, then the Hermitian-Einstein equations for a Hermitian metric h are

$$\sqrt{-1}\Lambda F_h = \lambda I_E . \qquad (4)$$

Here F_h is the curvature of the metric connection determined by h, ΛF_h is the contraction of F_h with the Kähler form ω, $I_E \in \Omega^0(\mathrm{End}\,E)$ is the identity and λ is a constant determined by ω, the rank of E, and its degree. Recall that the degree of a complex bundle over a Kähler manifold is defined by

$$\deg E = \frac{1}{(m-1)!}\int_M c_1(E) \wedge \omega^{m-1} ,$$

where m is the dimension of M and $c_1(E)$ is the first Chern class of E.

For a holomorphic triple (E_1, E_2, Φ) on X, special metrics on E_1 and E_2 are determined by the coupled vortex equations introduced in [GP3], i.e. by the equations

$$\left.\begin{aligned} \sqrt{-1}\Lambda F_{h_1} + \Phi\Phi^* &= 2\pi\tau I_{E_1} \\ \sqrt{-1}\Lambda F_{h_2} - \Phi^*\Phi &= 2\pi\tau' I_{E_2} \end{aligned}\right\} . \qquad (5)$$

In these equations, Φ^* is the adjoint of Φ with respect to the metrics on E_1 and E_2, and τ and τ' are real parameters. If the ranks of the bundles are r_1 and r_2 respectively, and we denote their degrees by d_1 and d_2, then the parameters τ and τ' satisfy the constraint

$$r_1\tau + r_2\tau' = \deg E_1 + \deg E_2 . \qquad (6)$$

On $X \times \mathbb{P}^1$ there is a one-parameter family of $SU(2)$-invariant Kähler metrics, with Kähler forms

$$\omega_\sigma = \frac{\sigma}{2} p^*\omega_X \oplus \omega_{\mathbb{P}^1} .$$

Here ω_X is the Kähler form on X, $\omega_{\mathbb{P}^1}$ is the Fubini-Study Kähler form normalized to volume one, and $\sigma \in \mathbb{R}^+$. Using these Kähler forms, the coupled vortex equations can be interpreted as a dimensional reduction of the Hermitian-Einstein equations, i.e.:

Proposition 2.5 [GP3, Proposition 3.11] *Let $T = (E_1, E_2, \Phi)$ be a holomorphic triple and F be the $SU(2)$-equivariant holomorphic bundle over $X \times \mathbb{P}^1$ associated to T, that is given as an extension*

$$0 \to p^*E_1 \to F \to p^*E_2 \otimes q^*\mathcal{O}(2) \to 0\,. \tag{7}$$

Suppose that τ and τ' are related by (6) *and let*

$$\sigma = \frac{(r_1 + r_2)\tau - (\deg E_1 + \deg E_2)}{r_2}\,. \tag{8}$$

Then E_1 and E_2 admit metrics satisfying the coupled τ-vortex equations if and only if F admits an $SU(2)$-invariant Hermitian-Einstein metric with respect to ω_σ.

Remark. The choice of the Kähler metric on $X \times \mathbb{P}^1$ that we have made differs from the one made in [GP3]. There the parameter σ is multiplying the metric on $\mathbb{P}^1$, i.e. $\omega_\sigma = p^*\omega_X \oplus \sigma q^*\omega_{\mathbb{P}^1}$. This, and the fact that the volume of X was not normalized to one, explains why the relation between τ and σ given there is the inverse of (8).

2.3 *Invariant stability and the Hitchin-Kobayashi correspondence*

The existence of a Hermitian-Einstein metric on a holomorphic vector bundle is governed by the algebraic-geometric condition of *stability*. Recall that a holomorphic vector bundle E over a compact Kähler manifold (M, ω) is said to be *stable* if for every non-trivial coherent subsheaf $E' \subset E$,

$$\mu(E') < \mu(E)\,,$$

where $\mu(E') = \dfrac{\deg E'}{\operatorname{rank} E'}$ is the *slope* of E'.

The precise relation between the Hermitian-Einstein condition and stability is given by the so-called *Hitchin-Kobayashi correspondence*, proved by Donaldson [D1, D2] in the algebraic case and by Uhlenbeck and Yau [U-Y] for an arbitrary compact Kähler manifold (see also [Ko, L, A-B, N-S]).

When considering $SU(2)$-invariant solutions to the Hermitian-Einstein equations, we need to introduce the notion of $SU(2)$-invariant stability. More generally, let (M, ω) be a compact Kähler manifold and let G be a compact Lie group acting on M by isometric biholomorphisms. Let E be a G-equivariant holomorphic vector bundle over M. We say that E is *G-invariantly stable* if $\mu(E') < \mu(E)$ for every G-invariant nontrivial coherent subsheaf $E' \subset E$. The basic relation between G-invariant stability and ordinary stability is given by the following theorem (cf. [GP1, Theorem 4]).

Theorem 2.6 *Let E be a G-invariant holomorphic vector bundle as above. Then E is G-invariantly stable if and only if E is G-indecomposable and is of the form*

$$E = \bigoplus_{i=1}^{n} E_i$$

where each E_i is a stable bundle, and is the image of E_i under some element $G_1 \in G$.

As shown in [GP1], there is a G-invariant version of the Hitchin-Kobayashi correspondence. From this, and Theorem 2.6, we can conclude

Theorem 2.7 *Let $T = (E_1, E_2, \Phi)$ be a holomorphic triple over a compact Riemann surface X equipped with a metric. Let $F \to X \times \mathbb{P}^1$ be the bundle associated to T as above. Let σ and τ be real parameters related by (8). Then E_1 and E_2 admit metrics satisfying the coupled τ-vortex equations if and only if F is a $SU(2)$-invariantly polystable bundle with respect to the Kähler form ω_σ defined above.*

3 Stability for triples

3.1 Definition and basic properties

In this section we define an appropriate notion of stability for a triple, say (E_1, E_2, Φ). Keeping our earlier notation, E_1 and E_2 are holomorphic vector bundles over a Riemann surface X, and Φ: $E_2 \to E_1$ is a holomorphic bundle homomorphism, i.e. $\Phi \in H^0(\mathrm{Hom}(E_2, E_1))$. We denote the ranks of E_1 and E_2 by r_1 and r_2 respectively, and their degrees by d_1 and d_2. Before we can define stability, we need to define an appropriate set of subobjects of a triple.

Definition 3.1 *A triple $T' = (E_1', E_2', \Phi')$ is a* **subtriple** *of $T = (E_1, E_2, \Phi)$ if*

(1) *E_i' is a coherent subsheaf of E_i, for $i = 1, 2$*

(2) *we have the commutative diagram*

$$\begin{array}{ccc} E_2 & \xrightarrow{\Phi} & E_1 \\ \uparrow & & \uparrow \\ E_2' & \xrightarrow{\Phi'} & E_1' \end{array} .$$

The zero triple $T' = 0$, obtained taking $E_1' = E_2' = 0$, and the triple $T' = T$ will be called the **trivial** *subtriples.*

Remark. When studying stability criteria, it will suffice, as usual, to consider saturated subsheaves, that is subsheaves whose quotient sheaves are torsion free. On a Riemann surface these are precisely subbundles.

Definition 3.2 *Let $T' = (E_1', E_2', \Phi)$ be a non zero subtriple of (E_1, E_2, Φ), with* rank $E_1' = r_1'$ *and* rank $E_2' = r_2'$. *For any real τ define*

$$\theta_\tau(T') = (\mu(E_1' \oplus E_2') - \tau) - \frac{r_2'}{r_2}\frac{r_1 + r_2}{r_1' + r_2'}(\mu(E_1 \oplus E_2) - \tau) . \tag{9}$$

The triple $T = (E_1, E_2, \Phi)$ is called **τ-stable** *if*

$$\theta_\tau(T') < 0$$

for all nontrivial subtriples $T' = (E_1', E_2', \Phi)$. The triple is called **τ-semistable** *if $\theta_\tau(T') \leqq 0$ for all subtriples.*

It is sometimes convenient to reformulate this definition as follows.

Definition 3.3 *With σ a real number, define the* **σ-degree** *and* **σ-slope** *of a subtriple $T' = (E_1', E_2', \Phi)$ by*

$$\deg_\sigma(T') = \deg(E_1' \oplus E_2') + r_2'\sigma ,$$

and

$$\mu_\sigma(T') = \frac{\deg_\sigma(T')}{r_1' + r_2'} .$$

The triple $T = (E_1, E_2, \Phi)$ is called **σ-stable** *if for all nontrivial subtriples $T' = (E_1', E_2', \Phi)$ we have*

$$\mu_\sigma(T') < \mu_\sigma(T) .$$

The triple is called **σ-semistable** *if $\mu_\sigma(T') \leqq \mu_\sigma(T)$ for all subtriples.*

A straightforward computation shows the equivalence of these two definitions.

Proposition 3.4 *Fix τ and σ such that $\sigma = \frac{r_1 + r_2}{r_2}(\tau - \mu(T))$, or equivalently $\tau = \mu_\sigma(T)$. Let $T' = (E_1', E_2', \Phi)$ be any subtriple of T. Then $\theta_\tau(T') < 0$ if and only if $\mu_\sigma(T') < \mu_\sigma(T)$. That is, the triple is τ-stable if and only if it is σ-stable. A similar result holds with "$<$" replaced by "$=$".*

Remark. There are two special cases where the notion of stability for a triple is especially simple, namely when $\Phi = 0$, and when E_2 is a line bundle.

Lemma 3.5 *Suppose that $\Phi = 0$. The degenerate holomorphic triple $(E_1, E_2, 0)$ is τ-semistable if and only if $\tau = \mu(E_1)$ and both bundles are semistable. Such triple cannot be τ-stable.*

Proof. Subtriples of $T = (E_1, E_2, 0)$ are all of the form $T' = (E_1', E_2', 0)$, with E_1' and E_2' being any holomorphic subbundles of E_1 and E_2 respectively. Applying the condition $\theta_\tau(T') \leqq 0$ to subtriples of the form $T' = (E_1', 0, 0)$ gives $\mu(E_1') \leqq \tau$, while applying the condition to subtriples of the form $T' = (E_1', E_2, 0)$ gives $\mu(E_1/E_1') \geqq \tau$. These two inequalities imply in particular that $\tau = \mu(E_1)$, and hence E_1 is a semistable bundle. Similarly, by considering the subtriples $(0, E_2', 0)$ and $(E_1, E_2', 0)$, we see that E_2 is also semistable. Notice that the inequalities cannot be made strict without leading to a contradiction.

Corollary 3.6 *The map Φ cannot be identically zero in a τ-stable triple.*

Lemma 3.7 *In the case where $E_2 = L$ is a line bundle, i.e. $r_2 = 1$, the above definition is equivalent to the notion of τ-stability defined in* [GP3]. *It thus corresponds to the $(\tau - \deg L)$-stability for the holomorphic pair $(E_1 \otimes L^*, \Phi)$.*

Proof. In this case there are only two types of subtriple possible, corresponding to $r_2' = 0$ or $r_2' = 1$. In the first case the subtriples are of the form $(E_1', 0, 0)$, where E_1' is an arbitrary holomorphic subbundle of E_1. The condition $\theta_\tau(T') < 0$ then reduces to $\mu(E_1') < \tau$. In the second case, the subtriples are of the form (E_1', E_2, Φ) where E_1' is a holomorphic subbundle such that $\Phi(E_2) \subset E_1'$. For such subtriples the condition $\theta_\tau(T') < 0$ is equivalent to $\mu(E_1/E_1') > \tau$.

Definition 3.2 can thus be considered a natural extension of the τ-stability for pairs defined in [B2]. For the more general triples which we are considering here however, the number of different possibilities for subtriples is too large to reformulate the definition of τ-stability in the style of [GP3] or [B2], i.e. in terms of separate slope conditions on the various families of subtriples. The τ-stability of a triple does however imply the following conditions on subtriples:

Proposition 3.8 *Let (E_1, E_2, Φ) be a τ-stable triple. Let τ' be related to τ by*

$$r_1\tau + r_2\tau' = \deg E_1 + \deg E_2 . \tag{10}$$

Then

(1) *$\mu(E_1') < \tau$ for all holomorphic subbundles $E_1' \subset E_1$,*

(2) *$\mu(E_2') < \tau'$ for all holomorphic subbundles $E_2' \subset E_1$ such that $E_2' \subset \mathrm{Ker}(\Phi)$,*

(3) *$\mu(E_2'') > \tau'$ for all holomorphic quotients of E_2,*

(4) *$\mu(E_1'') > \tau$ for all holomorphic quotients of E_1 such that $\pi \circ \Phi(E_2) = 0$, where $\pi\colon E_1 \to E_1''$ denotes projection onto the quotient.*

Proof. These are immediate consequences of the stability condition applied to the following special subtriples (1) $(E_1', 0, \Phi)$, (2) $(0, E_2', \Phi)$, (3) (E_1, E_2', Φ), with $E_2'' = E_2/E_2'$, (4) (E_1', E_2, Φ), with $E_1'' = E_1/E_1'$.

Notice that (10) can be expressed as $\tau' = \mu(E_1 \oplus E_2) - \dfrac{r_1}{r_1 + r_2}\sigma$. An equivalent formulation of Proposition 3.8 is thus

Proposition 3.9 *Let $T = (E_1, E_2, \Phi)$ be a σ-stable triple. Then*

(1) *$\mu(E_1') < \mu(T) + \dfrac{r_2}{r_1 + r_2}\sigma$ for all holomorphic subbundles $E_1' \subset E_1$,*

(2) *$\mu(E_2') < \mu(T) - \dfrac{r_1}{r_1 + r_2}\sigma$ for all holomorphic subbundles $E_2' \subset E_2$ such that $E_2' \subset \mathrm{Ker}\,\Phi$,*

(3) *$\mu(E_2'') > \mu(T) - \dfrac{r_1}{r_1 + r_2}\sigma$ for all holomorphic quotients, E_2'', of E_2,*

(4) *$\mu(E_1'') > \mu(T) + \dfrac{r_2}{r_1 + r_2}\sigma$ for all holomorphic quotients, E_1'', of E_1 such that $\pi \circ \Phi(E_2) = 0$, where $\pi\colon E_1 \to E_1''$ denotes projection onto the quotient.*

3.2 Stable implies simple

An important consequence of stability for holomorphic bundles is that the only automorphisms of a stable bundle are the constant multiples of the identity, i.e. stable bundles are simple. We now show that an analogous result holds true in the case of holomorphic triples. The key result is the following Proposition.

Proposition 3.10 *Let (E_1, E_2, Φ) be a τ-stable holomorphic triple. Let (u, v) be in $H^0(E_1, E_2, \Phi)$. Either (u, v) is trivial, or both u and v are isomorphisms.*

Proof. Suppose that u and v are both neither trivial nor isomorphisms. Consider the triples $K = (\operatorname{Ker} u, \operatorname{Ker} v, \Phi)$ and $I = (\operatorname{Im} u, \operatorname{Im} v, \Phi)$, where Ker and Im denotes the kernels and images of the maps. Since $u\Phi = \Phi v$, these are both proper subtriples of (E_1, E_2, φ). The τ-stability condition thus gives $\theta_\tau(K) < 0$ and $\theta_\tau(I) < 0$. We also have the exact sequences

$$0 \to \operatorname{Ker} u \to E_1 \to \operatorname{Im} u \to 0\,,$$

and

$$0 \to \operatorname{Ker} v \to E_2 \to \operatorname{Im} v \to 0\,.$$

Then from the exact sequences and the definition of θ_τ, we get $r_1\theta_\tau(K) + r_2\theta_\tau(I) = 0$, which is impossible.

Definition 3.11 *Let*

$$H^0(E_1, E_2, \Phi) = \{(u, v) \in H^0(\operatorname{End} E_1) \oplus H^0(\operatorname{End} E_2) \,|\, u\Phi = \Phi v\}\,. \qquad (11)$$

We say a holomorphic triple (E_1, E_2, Φ) is simple if $H^0(E_1, E_2, \Phi) \simeq \mathbb{C}$, i.e. if the only elements in $H^0(E_1, E_2, \Phi)$ are of the form $\lambda(I_1, I_2)$ where λ is a constant and (I_1, I_2) denote the identity maps on E_1 and E_2.

Corollary 3.12 *If (E_1, E_2, Φ) is τ-stable, then it is simple.*

Proof. Let (u, v) be a nontrivial element in $H^0(E_1, E_2, \Phi)$. By the above Proposition, both u and v are isomorphisms. Fix a point p on the base of the bundles, and let λ be an eigenvalue of $v\colon E_2|_p \to E_2|_p$, i.e. of v acting on the fibre over p. Now define $\hat{u} = u - \lambda I_1$, and $\hat{v} = v - \lambda I_2$. Clearly $(\hat{u}, \hat{v})$ is in $H^0(E_1, E_2, \Phi)$, but since $\hat{u}$ is not an isomorphism, it follows from Proposition 3.10 that both are identically zero, i.e. $(u, v) = \lambda(I_1, I_2)$.

A related, but inequivalent notion to simplicity is that of irreducibility. We make the following definitions.

Definition 3.13 *We say the triple $T = (E_1, E_2, \Phi)$ is* **reducible** *if there are direct sum decompositions $E_1 = \bigoplus_{i=1}^n E_{1i}$, $E_2 = \bigoplus_{i=1}^n E_{2i}$, and $\Phi = \bigoplus_{i=1}^n \Phi_i$, such that $\Phi_i \in \operatorname{Hom}(E_{2i}, E_{1i})$. We adopt the convention that if $E_{2i} = 0$ or $E_{1i} = 0$ for some i, then Φ_i is the zero map. With $T_i = (E_{1i}, E_{2i}, \Phi_i)$, we write $T = \bigoplus_{i=1}^n T_i$. Thus T is reducible if it has a decomposition as a direct sum of subtriples.*

If T is not reducible, we say T is irreducible.

Proposition 3.14 *If a triple $T = (E_1, E_2, \Phi)$ is simple, then it is* **irreducible.**

Proof. Suppose T is reducible, with $T = \bigoplus_{i=1}^n T_i$. Then we can define $(u, v) \in H^0(E_1, E_2, \Phi)$ by $u = \bigoplus_{i=1}^n \lambda_i I_{1i}$, $v = \bigoplus_{i=1}^n \lambda_i I_{2i}$, where for each i, $\lambda_i \in \mathbb{C}$ and $I_{1i}(I_{2i})$ is the identity map on $E_{1i}(E_{2i})$. Clearly T is not simple.

We see, in particular, that stable triples are necessarily irreducible. For reducible triples, we can however define a notion of polystability. This will be useful when we consider the relation between stability and the coupled vortex equations.

Definition 3.15 *Let $T = (E_1, E_2, \Phi)$ be a reducible triple, with $T = \bigoplus_{i=1}^n T_i$. Suppose that in each summand $T_i = (E_{1i}, E_{2i}, \Phi_i)$, the map Φ_i is non-trivial unless $E_{1i} = 0$ or $E_{2i} = 0$. Fix value of τ, and let τ' be related to τ as in* (10). *We say that T is τ-polystable if for each summand T_i :*

(1) *if $\Phi_i \neq 0$, then T_i is τ-stable,*

(2) *if $E_{1i} = 0$, then E_{2i} is a stable bundle of slope τ',*

(3) *if $E_{2i} = 0$, then E_{1i} is a stable bundle of slope τ.*

Remark. It is easy to see, using Lemma 4.3, that both simplicity and irreducibility for a triple on X are equivalent to the corresponding $SU(2)$-invariant notions for the associated extension on $X \times \mathbb{P}^1$.

3.3 Duality for triples

Associated to a triple $T = (E_1, E_2, \Phi)$ there is always a *dual triple* $T^* = (E_2^*, E_1^*, \Phi^*)$, where Φ^* is the transpose of Φ, i.e. the image of Φ via the canonical isomorphism

$$\mathrm{Hom}(E_2, E_1) \cong \mathrm{Hom}(E_1^*, E_2^*) .$$

As one would expect, the stability of T is related to that of T^*. More precisely.

Proposition 3.16 *$T = (E_1, E_2, \Phi)$ is τ-stable if and only if $T^* = (E_2^*, E_1^*, \Phi^*)$ is $(-\tau')$-stable, where τ' is related to τ by* (10). *Equivalently, T is σ-stable if and only if T^* is σ-stable.*

Proof. Let $T' = (E_1', E_2', \Phi')$ be a subtriple of T. This defines a quotient triple $T'' = (E_1'', E_2'', \Phi'')$, where $E_1'' = E_1/E_1'$, $E_2'' = E_2/E_2'$, and Φ'' is the morphism induced by Φ. Then $T''^* = (E_2''^*, E_1''^*, \Phi''^*)$ is the desired subtriple of T^*. Since one has the isomorphism $T \cong T^{**}$ we can conclude that there is a one-to-one correspondence between subtriples of T and subtriples of T^*. A straightforward computation verifies that the condition $\theta_\tau(T') < 0$ is equivalent to the condition $\theta_{-\tau'}(T''^*) < 0$. The equivalence of the σ-stability for T and T^* now follows from Proposition 3.4 and the fact that if $\tau = \mu_\sigma(T)$, then

$$-\tau' = -\mu(E_1 \oplus E_2) + \frac{r_1}{r_1 + r_2}\sigma = \mu_\sigma(T^*) .$$

3.4 Range and special values of the parameters

Proposition 3.17 *Let* (E_1,E_2,Φ) *be a* τ*-stable triple, and let* τ' *be as in* (10). *Then* (1) $\tau > \mu(E_1)$, (2) $\tau' < \mu(E_2)$, *and* (3) $\tau - \tau' > 0$.

Equivalently, if (E_1,E_2,Φ) *is* σ*-stable, then* (1) $\sigma > \mu(E_1) - \mu(E_2)$, *and* (2) $\sigma > 0$.

Proof. The first two statement follows from cases (1) and (3) in Proposition 3.8 with $E_1' = E_1$, and $E_2'' = E_2$ respectively. To prove the third statement, let K be the subbundle of E_2 generated by the kernel of Φ, and let I be the subbundle of E_1 generated by the image of Φ. Since the triple is assumed to be τ-stable, Φ, and therefore I, is non-trivial. By (1) in Proposition 3.8 we thus have $\mu(I) < \tau$. But we also have $0 \to K \to E_2 \to I \to 0$. Thus if $K \neq 0$, then I is a quotient of E_2, and it follows from (3) in Proposition 3.8 that $\mu(I) > \tau'$. If $K = 0$, then $\mu(I) = \mu(E_2) > \tau'$, by part (2). The bounds on σ can be obtained from those on τ by substituting $\tau = \mu_\sigma(T)$, and using the fact that $\sigma = \tau - \tau'$ if τ' is as above.

Part (1) of this proposition gives the lower bound on the allowed range for τ. In almost all cases the rank and degree of E_1 and E_2 also impose an upper bound on τ. In fact

Proposition 3.18 *Let* (E_1,E_2,Φ) *be a triple with* $r_1 \neq r_2$. *If the triple is* τ*-stable then*

$$\tau < \mu(E_1) + \frac{r_2}{|r_1 - r_2|}(\mu(E_1) - \mu(E_2)) . \tag{12}$$

Equivalently, if the triple is σ*-stable, then*

$$\sigma < \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right)(\mu(E_1) - \mu(E_2)) . \tag{13}$$

Proof. Let $K = \operatorname{Ker}\Phi$ and $I = \operatorname{Im}\Phi$. Consider the subtriples $T_1 = (0,K,\Phi)$ and $T_2 = (I,E_2,\Phi)$. Since $r_1 \neq r_2$, Φ cannot be an isomorphism and at least one of these must be a proper subtriple. Let $r_2' = \operatorname{rank} K$, $r_2'' = \operatorname{rank} I$, $d_2' = \deg K$ and $d_2'' = \deg I$. A straightforward computation shows that

$$\theta_\tau(T_1) < 0 \Leftrightarrow d_2' - r_2'(d_1 + d_2) + r_1 r_2' \tau < 0 \tag{14}$$

$$\theta_\tau(T_2) < 0 \Leftrightarrow d_2'' - d_1 + (r_1 - r_2'')\tau < 0 . \tag{15}$$

From these equations, plus the fact that $d_2 = d_2' + d_2''$ and $r_2 = r_2' + r_2''$, we get

$$(r_1 - r_2)\tau < d_1 - d_2 .$$

If $r_1 > r_2$, then we get

$$\tau < \mu(E_1) + \frac{r_2}{r_1 - r_2}(\mu(E_1) - \mu(E_2)) .$$

To obtain the bound in the case $r_1 < r_2$, note that by Proposition 3.16 the τ-stability of (E_1,E_2,Φ) is equivalent to the $(-\tau')$-stability of the dual triple

(E_2^*, E_1^*, Φ^*), where τ' is given, as in (10) by $r_1\tau + r_2\tau' = d_1 + d_2$. Applying the above argument to (E_2^*, E_1^*, Φ^*) leads to

$$\tau < \mu(E_1) + \frac{r_2}{r_2 - r_1}(\mu(E_1) - \mu(E_2)) .$$

Combining the lower and upper bounds on τ (or σ) we can deduce

Corollary 3.19 *If* rank E_1 *and* rank E_2 *are unequal, then a triple* (E_1, E_2, Φ) *cannot be stable unless* $\mu(E_2) < \mu(E_1)$.

Furthermore, by the proof of Proposition 3.18 we get the following corollary.

Corollary 3.20 *Let* (E_1, E_2, Φ) *be* τ*-stable and suppose that* $r_1 = r_2$. *If* Φ *is not an isomorphism, then* $d_1 > d_2$. *In particular, in any* τ*-stable triple* (E_1, E_2, Φ), *the bundle map* Φ *is an isomorphism if and only if* $r_1 = r_2$ *and* $d_1 = d_2$.

Proof. If Ker $\Phi \neq 0$, then both $(0, K, \Phi)$ and (I, E_2, Φ) are proper subtriples. The proof of Proposition 3.18 thus gives $(r_1 - r_2)\tau < d_1 - d_2$. If Ker $\Phi = 0$ but Φ is not an isomorphism, then $E_1/\mathrm{Im}\,\Phi$ is a torsion sheaf and it follows that $d_2 > d_1$. In particular, if Φ is not an isomorphism then $d_1 \neq d_2$. Conversely, if Φ is an isomorphism, then clearly $r_1 = r_2$ and $d_1 = d_2$.

Notice that when Φ is an isomorphism that the range for τ can fail to be bounded. For example

Proposition 3.21 *Suppose that* E_1 *and* E_2 *are both stable bundles of rank* r *and degree* d, *and that* $\Phi\colon E_2 \to E_1$ *is non trivial. Then for any* $\tau > \mu(E_1)$ *the holomorphic triple* (E_1, E_2, Φ) *is* τ*-stable.*

Proof. Let $\mu(T) = \mu(E_1 \oplus E_2)$, and for a subtriple $T' = (E_1', E_2', \Phi)$ set $\mu(T') = \mu(E_1' \oplus E_2')$. Since E_1 and E_2 are stable and of equal slope, we have $\mu(T') < \mu(T)$ for all subtriples. Thus

$$\theta_\tau(T') \leqq (\mu(T) - \tau)\frac{r_1' - r_2'}{r_1' + r_2'} .$$

Since Φ is a nontrivial map between stable bundles of the same rank and degree, it must be a multiple of the identity (cf. [O-S-S]). In particular Φ is injective and hence $r_1' - r_2' \geqq 0$. Thus $\theta_\tau(T') \leqq 0$. In fact, $\theta_\tau(T') < 0$ unless $r_1' = r_2'$. But in that case, we can write

$$\theta_\tau(T') = r_1'(\mu(E_1') - \mu(E_1)) + r_1'(\mu(E_2') - \mu(E_2)) ,$$

which is strictly negative.

In principle τ is a continuously varying real parameter. The stability properties of a given triple do not likewise vary continuously, but can change only at certain rational values of τ. This is the same phenomenon as appears in the case of stable pairs. In both cases it is due to the fact that, except for τ itself, all numerical quantities in the definition of stability are rational numbers with bounded denominators. In the case of holomorphic pairs, this has the additional consequence that for the generic choice of τ there is no distinction between stability and semistability. This is in contrast to the case of pure bundles, where the notions of stability and semistability coincide only when the rank and degree of the bundle are coprime. The next proposition shows that for a holomorphic triple both the value of τ and the greatest common divisor of the rank and degree, are relevant.

Proposition 3.22 *Let $T = (E_1, E_2, \Phi)$ be a τ-semistable triple, and let $T' = (E_1', E_2', \Phi')$ be a subtriple such that $\theta_\tau(T') = 0$. Then either*

$$r_1 r_2' = r_2 r_1' \quad \text{and} \quad \mu(E_1' \oplus E_2') = \mu(E_1 \oplus E_2)\,, \tag{16}$$

or

$$\frac{r_2(r_1' + r_2')\mu(T') - r_2'(r_1 + r_2)\mu(T)}{r_2 r_1' - r_1 r_2'} = \tau\,. \tag{17}$$

In particular, if $r_1 + r_2$ and $d_1 + d_2$ are coprime, and τ is not a rational number with denominator of magnitude less than $r_1 r_2$, then all τ-semistable triples are τ-stable.

Proof. From the definition of θ_τ, we see that $\theta_\tau(T') = 0$ is equivalent to

$$\left(\mu(E_1' \oplus E_2') - \frac{r_2'}{r_2}\frac{r_1 + r_2}{r_1' + r_2'}\mu(E_1 \oplus E_2)\right) = \tau\,\frac{r_1' r_2 - r_1 r_2'}{r_1' r_2 + r_2' r_2}\,.$$

If $r_1' r_2 - r_1 r_2' \neq 0$ we get (17), and if $r_1' r_2 - r_1 r_2' = 0$ then

$$\frac{r_2'}{r_2}\frac{r_1 + r_2}{r_1' + r_2'} = 1$$

and we get (16).

Next we compare the stability conditions for a triple and for the two bundles in the triple.

Proposition 3.23 *Let (E_1, E_2, Φ) be a non-degenerate holomorphic triple. There is an $\varepsilon > 0$, which depends only on the degrees and ranks of E_1 and E_2, and such that for $\mu(E_1) < \tau < \mu(E_1) + \varepsilon$ the following is true:*

(1) If (E_1, E_2, Φ) is a τ-stable triple, then both E_1 and E_2 are semistable bundles.

(2) Conversely, if E_1 and E_2 are stable bundles, then (E_1, E_2, Φ) will be a τ-stable triple for any choice of $\Phi \in H^0(\mathrm{Hom}(E_2, E_1))$.

Proof. For all subbundles $E_1' \subset E_1$ the slope $\mu(E_1')$ is a rational number with denominator less than r_1. Clearly, if we pick ε small enough then the interval $(\mu(E_1), \mu(E_1) + \varepsilon)$ contains no rational numbers with denominator less than r_1. The condition $\mu(E_1') < \tau$ is thus equivalent to the condition $\mu(E_1') \leqq \mu(E_1)$, i.e. to the semistability of E_1.

Furthermore, as noted above, if $\tau < \mu(E_1) + \varepsilon$ then $\tau' > \mu(E_2) - \frac{r_1}{r_2}\varepsilon$. Hence if $\frac{r_1}{r_2}\varepsilon$ is small enough, then the condition $\mu(E_2/E_2') > \tau'$ for all subbundles $E_2' \subset E_2$ becomes equivalent to the condition that $\mu(E_2/E_2') \geqq \mu(E_2)$.

Conversely, suppose $\tau = \mu(E_1) + \delta$ for some $\delta > 0$, and that Φ is any section of $H^0(\mathrm{Hom}(E_2, E_1))$. Then for any subtriple (E_1', E_2', Φ) we get

$$(r_1' + r_2')\theta_\tau(E_1', E_2', \Phi') = r_1'(\mu(E_1') - \mu(E_1)) + r_2'(\mu(E_2') - \mu(E_2)) \\ + (r_2 r_1' - r_1 r_2')\delta ,$$

where $r_1' = \mathrm{rank}\, E_1'$ and $r_2' = \mathrm{rank}\, E_2'$. If E_1 and E_2 are stable, and δ is small enough, then it follows from this that $\theta_\tau(E_1', E_2', \Phi') < 0$ for all subtriples.

4 Main theorem

In this section we shall show how the stability of a holomorphic triple over X relates to the stability of the associated (SU(2)-equivariant) bundle over $X \times \mathbb{P}^1$. As in Sect. 2, let $F \to X \times \mathbb{P}^1$ be the extension associated to the triple (E_1, E_2, Φ), i.e. let F be

$$0 \to p^* E_1 \to F \to p^* E_2 \otimes q^* \mathcal{O}(2) \to 0 , \tag{18}$$

where p and q are the projections from $X \times \mathbb{P}^1$ to X and $\mathbb{P}^1$ respectively, and $\mathcal{O}(2)$ is the line bundle of degree 2 over $\mathbb{P}^1$. To relate the τ-stability of (E_1, E_2, Φ) to the stability of F we need to consider some Kähler polarization on $X \times \mathbb{P}^1$. The parameter τ will be encoded in this polarization. Let us choose a metric on X with Kähler form ω_X, with volume normalized to one. The metric we shall consider on $X \times \mathbb{P}^1$ will be, as in Sect. 2, the product of the metric on X with a coefficient depending on a parameter $\sigma > 0$, and the Fubini-Study metric on $\mathbb{P}^1$ with volume also normalized to one. The Kähler form corresponding to this metric depending on the parameter σ is

$$\omega_\sigma = \frac{\sigma}{2} p^* \omega_X \oplus q^* \omega_{\mathbb{P}^1} . \tag{19}$$

We can now state the main result of this section.

Theorem 4.1 *Let (E_1, E_2, Φ) be a holomorphic triple over a compact Riemann surface X. Let F be the holomorphic bundle over $X \times \mathbb{P}^1$ defined by (E_1, E_2, Φ) as in Proposition 2.5, and let*

$$\sigma(\tau) = \frac{(r_1 + r_2)\tau - (\deg E_1 + \deg E_2)}{r_2} . \tag{20}$$

Suppose that in (E_1, E_2, Φ) the two bundles E_1 and E_2 are not isomorphic. Then (E_1, E_2, Φ) is τ-stable (equivalently σ-stable) if and only if F is stable with respect to ω_σ.

In the case that $E_1 \cong E_2 \cong E$, the triple (E, E, Φ) is τ-stable (equivalently σ-stable) if and only if F decomposes as a direct sum

$$F = p^* E \otimes q^* \mathcal{O}(1) \oplus p^* E \otimes q^* \mathcal{O}(1) ,$$

and $p^* E \otimes q^* \mathcal{O}(1)$ is stable with respect to ω_σ.

Proof. As mentioned in Sect. 3, if E_2 is a line bundle the τ-stability of (E_1, E_2, Φ) is equivalent to the τ-stability of the pair $(E_1 \otimes E_2^*, \Phi)$ in the sense of Bradlow [B2]. In this case Theorem 4.1 has been proved in [GP3, Theorem 4.6]. The main ideas of that proof extend to the general case in a rather straightforward manner.

Recall that the bundle F associated to (E_1, E_2, Φ) comes equipped with a holomorphic action of $SU(2)$. It makes sense therefore to talk about the $SU(2)$-invariant stability of F. As explained in Sect. 2, this is like ordinary stability, but the slope condition has to be satisfied only for $SU(2)$-invariant subsheaves of F. In order to prove the theorem we shall prove first the following slightly weaker result.

Proposition 4.2 *Let (E_1, E_2, Φ) be a holomorphic triple over a compact Riemann surface X. Let F be the holomorphic bundle over $X \times \mathbb{P}^1$ defined by (E_1, E_2, Φ) and let σ and τ be related by* (20)*. Then (E_1, E_2, Φ) is τ-stable (equivalently σ-stable) if and only if F is $SU(2)$-invariantly stable with respect to ω_σ.*

Proof. We shall start by proving that there is a one-to-one correspondence between subtriples $T' = (E_1', E_2', \Phi')$ of T and $SU(2)$-invariant coherent subsheaves $F' \subset F$. More precisely, we have the following.

Lemma 4.3 *Let $F \to X \times \mathbb{P}^1$ be the bundle associated to a triple (E_1, E_2, Φ), i.e. let F be an extension of the form in* (18)*. Then there is a bijective correspondence between $SU(2)$-invariant coherent subsheaves $F' \subset F$, and subtriples (E_1', E_2', Φ') of (E_1, E_2, Φ). Moreover, the subsheaf F' defined by T' is an extension of the form*

$$0 \to p^*E_1' \to F' \to p^*E_2' \otimes q^*\mathcal{O}(2) \to 0\,. \tag{21}$$

Proof. We first show that every $SU(2)$-invariant coherent subsheaf $F' \subset F$ is a subextension of the form in (21), with $E_1' \subset E_1$ and $E_2' \subset E_2$ coherent subsheaves. Let $f\colon F' \to p^*E_2 \otimes q^*\mathcal{O}(2)$ be the composition of the injection $F' \to F$ with the surjective map $F \to p^*E_2 \otimes q^*\mathcal{O}(2)$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & p^*E_1 & \to & F & \to & p^*E_2 \otimes q^*\mathcal{O}(2) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \operatorname{Ker} f & \to & F' & \to & \operatorname{Im} f & \to & 0\,. \end{array}$$

The $SU(2)$-invariance of F' implies that of $\operatorname{Ker} f$ and $\operatorname{Im} f$. It suffices therefore to show that if E is a holomorphic vector bundle over X and if p^*E is the pull-back to $X \times \mathbb{P}^1$, then every $SU(2)$-invariant subsheaf of p^*E is isomorphic to a sheaf of the form p^*E' for E' a subsheaf of E. Indeed, the action of $SU(2)$ on p^*E can be extended to an action of $SL(2,\mathbb{C})$. Let $F' \subset p^*E$ be a $SL(2,\mathbb{C})$-invariant coherent subsheaf. Consider the action of subgroup $\mathbb{C}^* \subset SL(2,\mathbb{C})$ on $X \times \mathbb{C} \subset X \times \mathbb{P}^1$ and let $A = H^0(X,E)$ be the space of global sections. Clearly $H^0(X \times \mathbb{C}, F') \subset A[t]$, that is, $H^0(X \times \mathbb{C}, F') = \bigoplus_{k-0}^N B_k$, where an element of

B_k is of the form st^k for $s \in A$. The action of $\alpha \in \mathbb{C}^*$ is given by

$$\alpha(st^k) = s\alpha^k t^k .$$

By choosing another subgroup $\mathbb{C}^* \subset SL(2,\mathbb{C})$, the $SL(2,\mathbb{C})$-invariance of F' implies that $H^0(X \times \mathbb{C}, F') = B_0$ and hence $F' = p^*E'$ for $E' \subset E$ a coherent subsheaf.

To complete the proof of the Lemma, we need establish a bijective correspondence between subextensions of the form in (21), and subtriples of (E_1, E_2, Φ). Let i and j be the inclusions $E_1' \hookrightarrow E_1$ and $E_2' \hookrightarrow E_2$, and consider the induced diagram

$$\mathrm{Hom}(E_2', E_1') \xrightarrow{i} \mathrm{Hom}(E_2', E_1) \xleftarrow{j} \mathrm{Hom}(E_2, E_1) .$$

To say that (E_1', E_2', Φ') is a subtriple of (E_1, E_2, Φ) is equivalent to saying that $i(\Phi') = j(\Phi)$. Under the isomorphisms

$$\begin{aligned}
\mathrm{Hom}(E_2', E_1') &\cong \mathrm{Ext}^1(p^*E_1', p^*E_2' \otimes q^*\mathcal{O}(2)) \\
\mathrm{Hom}(E_2', E_1) &\cong \mathrm{Ext}^1(p^*E_1, p^*E_2' \otimes q^*\mathcal{O}(2)) \\
\mathrm{Hom}(E_2, E_1) &\cong \mathrm{Ext}^1(p^*E_1, p^*E_2 \otimes q^*\mathcal{O}(2)) ,
\end{aligned}$$

$i(\Phi')$ defines an extension $\tilde{F}^{(i)}$ which makes the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \to & p^*E_1 & \to & \tilde{F}^{(i)} & \to & p^*E_2' \otimes q^*\mathcal{O}(2) & \to & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \to & p^*E_1' & \to & F' & \to & p^*E_2' \otimes q^*\mathcal{O}(2) & \to & 0 .
\end{array} \qquad (22)$$

In particular F' is a subsheaf of $\tilde{F}^{(i)}$. Similarly, $j(\Phi)$ defines an extension $\tilde{F}^{(j)}$ which fits in the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & p^*E_1 & \to & F & \to & p^*E_2 \otimes q^*\mathcal{O}(2) & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & p^*E_1 & \to & \tilde{F}^{(j)} & \to & p^*E_2' \otimes q^*\mathcal{O}(2) & \to & 0 ,
\end{array} \qquad (23)$$

and in particular $\tilde{F}^{(j)}$ is a subsheaf of F. Since $i(\Phi') = j(\Phi)$, $\tilde{F}^{(i)} \cong \tilde{F}^{(j)}$ and we can compose the above two diagrams to obtain the desired result.

In terms of the parameter τ' as defined in (10), the relation between σ and τ, given by (20), can be rewritten as $\sigma = \tau - \tau'$. If (E_1, E_2, Φ) is τ-stable (equivalently, σ-stable) it follows from (3) in Proposition 3.17 that σ defined by (20) is positive. The slope of F' with respect to ω_σ is defined as

$$\mu_\sigma(F') = \frac{\deg_\sigma F'}{\operatorname{rank} F'} .$$

Here $\deg_\sigma F'$ is the degree of F', and is given by

$$\deg_\sigma F = \frac{1}{2} \int_{X \times \mathbb{P}^1} c_1(F) \wedge \omega_\sigma .$$

The proposition follows now from the following lemma.

Lemma 4.4 *Let T' be a subtriple of T and F' the corresponding $SU(2)$-invariant subsheaf of F. Let σ be as in Proposition 4.2. The following are equivalent*

(1) $\mu_\sigma(F') < \mu_\sigma(F)$

(2) $\theta_\tau(T') < 0$

(3) $\mu_\sigma(T') < \mu_\sigma(T)$.

Proof. The equivalence between (2) and (3) corresponds, of course, to the two equivalent definitions of stability for T (cf. Proposition 3.4).

From (18) and (21) we obtain that

$$\mu_\sigma(F) = \frac{\deg E_1 + \deg E_2 + \sigma r_2}{r_1 + r_2},$$

and

$$\mu_\sigma(F') = \frac{\deg E_1' + \deg E_2' + \sigma r_2'}{r_1' + r_2'},$$

where $r_1' = \operatorname{rank} E_1'$ and $r_2' = \operatorname{rank} E_2'$. From Definition 3.2 we immediately obtain the equivalence between (1) and (3).

Remark. As usual, in order for F to be $SU(2)$-invariantly stable it is enough to check condition (1) of Lemma 4.4 only for saturated $SU(2)$-invariant subsheaves, that is $SU(2)$-invariant subsheaves F' such that the quotient F/F' is torsion-free. Such a subsheaf F' is subbundle outside of a set of codimension greater or equal than 2. Hence by $SU(2)$-invariance one concludes that F' must be actually a subbundle of F over the whole $X \times \mathbb{P}^1$. It is easy to see that the saturation of F' implies that of p^*E_1' and $p^*E_2' \otimes q^*\mathcal{O}(2)$ in (21), and hence $E_1' \subset E_1$ and $E_2' \subset E_2$ are in fact subbundles. In other words, the one-to-one correspondence between $SU(2)$-invariant subsheaves of F and subtriples of T established in Lemma 4.3 sends saturated subsheaves into saturated subtriples.

To prove the theorem, we first observe that if F is $SU(2)$-invariantly stable then, by Theorem 2.6, it decomposes as a direct sum

$$F = F_1 \oplus F_2 \oplus \cdots \oplus F_k \tag{24}$$

of stable bundles, where F_i is the transformed by an element of $SU(2)$ of a fixed subbundle F_1 of F.

For the remaining parts of the theorem, the proof splits into two cases, corresponding to whether E_1 and E_2 are isomorphic (as holomorphic bundles) or not. We treat the non-isomorphic case first. Notice that in this case, the map Φ certainly cannot be an isomorphism. Clearly if F is stable it is in particular $SU(2)$-invariantly stable and hence by the previous Proposition, the corresponding triple will be τ-stable. Suppose now that (E_1, E_2, Φ) is τ-stable, and that Φ is not an isomorphism. Our strategy to prove the stability of F will be to prove that F is simple, that is $H^0(\operatorname{End} F) \cong \mathbb{C}$, and hence there must be just one summand in the decomposition of F given by (24).

To compute $H^0(\operatorname{End} F) \cong H^0(F \otimes F^*)$ let us tensor (18) with F^*. We obtain the short exact sequence

$$0 \to p^*E_1 \otimes F^* \to F \otimes F^* \to p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^* \to 0\,,$$

and the corresponding sequence in cohomology

$$0 \to H^0(p^*E_1 \otimes F^*) \to H^0(F \otimes F^*) \to H^0(p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^*) \to\ . \tag{25}$$

We first compute $H^0(p^*E_1 \otimes F^*)$. Dualizing (18), tensoring with p^*E_1, and using that $H^0(p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2)) = 0$, we have the sequence in cohomology

$$0 \to H^0(p^*E_1 \otimes F^*) \to H^0(p^*(E_1 \otimes E_1^*)) \xrightarrow{g} H^1(p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2))\,. \tag{26}$$

By the Künneth formula

$$H^0(p^*(E_1 \otimes E_1^*)) \cong H^0(E_1 \otimes E_1^*)$$

and

$$H^1(p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2)) \cong H^0(E_1 \otimes E_2^*)\,.$$

Using these isomorphisms, g can be interpreted as the map $H^0(E_1 \otimes E_1^*) \to H^0(E_1 \otimes E_2^*)$ defined by Φ, i.e. $g(u) = u\Phi$. Furthermore, from the τ-stability of (E_1,E_2,Φ) one has from Corollary 3.12 that (E_1,E_2,Φ) is simple. Thus $\operatorname{Ker} g \cong 0$ and from the exactness of (26) one obtains

$$H^0(p^*E_1 \otimes F^*) = 0\,. \tag{27}$$

To compute $H^0(p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^*)$, we dualize (18) and tensor it with $p^*E_2 \otimes q^*\mathcal{O}(2)$, to get the sequence

$$\begin{aligned} 0 \to H^0(p^*(E_2 \otimes E_2^*)) &\to H^0(p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^*) \\ &\to H^0(p^*(E_1^* \otimes E_2) \otimes q^*\mathcal{O}(2))\,. \end{aligned} \tag{28}$$

Lemma 4.5 *Let (E_1,E_2,Φ) be τ-stable and suppose that Φ is not an isomorphism, then $H^0(E_1^* \otimes E_2) = 0$.*

Proof. Suppose that there is a non-zero homomorphism $\Psi\colon E_1 \to E_2$. Let $u = \Phi\Psi \in H^0(\operatorname{End} E_1)$ and $v = \Psi\Phi \in H^0(\operatorname{End} E_2)$. Then $u\Phi = \Phi v$ and, since (E_1,E_2,Φ) is simple, we have that $u = \lambda I_{E_1}$ and $v = \lambda I_{E_2}$ for $\lambda \in \mathbb{C}$. If $\lambda \neq 0$, we easily see that Φ is an isomorphism, contradicting the assumption of the Lemma. Thus $\lambda = 0$ and then $\operatorname{Im} \Psi \subseteq \operatorname{Ker} \Phi$ and $\operatorname{Im} \Phi \subseteq \operatorname{Ker} \Psi$.

We can therefore consider the subtriples of (E_1,E_2,Φ) given by $T_1 = (K,E_2,\Phi)$ and $T_2 = (0,I,\Phi)$, where $K = \operatorname{Ker} \Psi$ and $I = \operatorname{Im} \Psi$. Let $r_1' = \operatorname{rank} K$, $r_1'' = \operatorname{rank} I$, $d_1' = \deg K$ and $d_1'' = \deg I$. Applying the τ-stability condition to T_1 and T_2 we get the inequalities

$$r_2 d_1'' - r_1''(d_1 + d_2) + r_1 r_1'' \tau < 0$$

$$d_1' - d_1 + (r_1 - r_1')\tau < 0\,.$$

From this and the fact that $r_1 = r_1' + r_1''$ and $d_1 = d_1' + d_1''$, we obtain that $\tau < \mu(E_1 \oplus E_2)$, or, equivalently, $\sigma(\tau) < 0$. This contradicts the τ-stability of (E_1, E_2, Φ).

From the Künneth formula and Lemma 4.5 we get that

$$H^0(p^*(E_1^* \otimes E_2) \otimes q^*\mathcal{O}(2)) \cong H^0(E_1^* \otimes E_2) \otimes H^0(\mathcal{O}(2)) \cong 0\,,$$

and from (28) we get $H^0(p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^*) \cong H^0(E_2 \otimes E_2^*)$. From this and (27), the first three terms in (25) reduce to

$$0 \longrightarrow H^0(F \otimes F^*) \xrightarrow{i} H^0(E_2 \otimes E_2^*)\,.$$

Since F is $SU(2)$-invariantly stable then it is $SU(2)$-invariantly simple, i.e. the only $SU(2)$-invariant endomorphisms are multiples of the identity. Let $\Psi \in H^0(F \otimes F^*)$ be a non $SU(2)$-invariant endomorphism of F, i.e. $\Psi^g \neq \Psi$ for some $g \in SU(2)$. Since i must be compatible with the action of $SU(2)$, we get $i(\Psi^g) = (i(\Psi))^g$. On the other hand, $(i(\Psi))^g = i(\Psi)$ since the action of $SU(2)$ on $H^0(E_2 \otimes E_2^*)$ is trivial. Hence $i(\Psi) = i(\Psi^g)$ contradicting the injectivity of i. Thus $H^0(F \otimes F^*) \cong \mathbb{C}^*$, which concludes the proof of our theorem for the case where E_1 and E_2 are not isomorphic.

Now suppose that $E_1 \cong E_2$. We first prove

Lemma 4.6 *Suppose $E_1 \cong E_2$. Then for any $\tau > \mu(E_1)$, the triple (E_1, E_2, Φ) is τ-stable if and only if Φ is an isomorphism and E_1 is stable.*

Proof. Suppose that the triple is τ-stable. Then (by Corollary 3.20) Φ is an isomorphism. Now consider the subtriples of the form $T' = (\Phi(E_2'), E_2', \Phi)$. These have $r_1' = r_2'$ and, since Φ is an isomorphism, $\mu(T') = \mu(E_2')$. Hence $\theta_\tau(T') = \mu(E_2') - \mu(E_2)$, and thus the τ-stability of the triple implies the stability of E_2. Conversely, if $E_2 \cong E_1$ and both are stable, then all non trivial Φ are isomorphisms. It now follows as a special case of Proposition 3.21 that the triple (E_1, E_2, Φ) is τ-stable.

Suppose that $E_1 \cong E_2 \cong E$ and that the bundle F associated to (E, E, Φ) is of the form

$$F = p^*E \otimes q^*\mathcal{O}(1) \oplus p^*E \otimes q^*\mathcal{O}(1)\,. \tag{29}$$

If we assume now that $p^*E \otimes q^*\mathcal{O}(1)$ is stable, then E is also stable and hence $H^0(E \otimes E^*) \cong \mathbb{C}$. Thus there is only one non-trivial extension class (corresponding to $\Phi = \lambda I$). We must now examine this (unique) non-trivial extension

$$0 \longrightarrow p^*E \longrightarrow F \longrightarrow p^*E \otimes q^*\mathcal{O}(2) \longrightarrow 0\,.$$

This is of course nothing else but the pull-back to $X \times \mathbb{P}^1$ of the non-trivial extension on $\mathbb{P}^1$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(2) \longrightarrow 0\,,$$

tensored with p^*E. Thus the action of $SU(2)$ permutes the two summands in (29) and from Theorem 2.6 we conclude that F is an $SU(2)$-invariantly stable bundle. The τ-stability of (E_1, E_2, Φ) follows now from Proposition 4.2.

Conversely, suppose that (E_1, E_2, Φ) is τ-stable, then from Lemma 4.6 we obtain that $E_1 \cong E_2 \cong E$ is stable and Φ is hence a non-trivial multiple of the identity. From the above discussion we conclude that

$$F = p^*E \otimes q^*\mathcal{O}(1) \oplus p^*E \otimes q^*\mathcal{O}(1)\,.$$

On the other hand, from Proposition 4.2, we argue as before that F is certainly invariantly stable. Also (25),(26) and (27) show that we have an exact sequence

$$0 \to H^0(F \otimes F^*) \to H^0(p^*E \otimes q^*\mathcal{O}(2) \otimes F^*)\,.$$

Using that $H^0(E \otimes E^*) \cong \mathbb{C}$, the exact sequence (28) becomes,

$$0 \to \mathbb{C} \to H^0(p^*E \otimes q^*\mathcal{O}(2) \otimes F^*) \to \mathbb{C}^3\,.$$

From these two exact sequences, we see that

$$1 \leqq h^0(F \otimes F^*) \leqq h^0(p^*E \otimes q^*\mathcal{O}(2) \otimes F^*) \leqq 4\,. \tag{30}$$

But since F is invariantly stable it is given by the direct sum (24). In that case,

$$H^0(F \otimes F^*) \cong GL(k, \mathbb{C})\,,$$

where k is the number of stable summands in F. It follows that since $h^0(F \otimes F^*) \neq 1$, then $h^0(F \otimes F^*) = k^2 - 1$ for some integer k. The only possibility consistent with the constraint (30) is thus $h^0(F \otimes F^*) = 3$, i.e. $k = 2$. Hence the bundle $p^*E \otimes q^*\mathcal{O}(1)$ in the decomposition of F is stable, which finishes the proof of the theorem.

Notice that the conclusion of Proposition 4.2 extends straightforwardly to cover polystable objects. We thus get the following corollary, which will be useful in the next section.

Theorem 4.7 *Let (E_1, E_2, Φ) be a holomorphic triple and F be the corresponding holomorphic bundle over $X \times \mathbb{P}^1$. Let τ and σ be related as above. Then (E_1, E_2, Φ) is τ-polystable if and only if F is $SU(2)$-invariably polystable with respect to ω_σ.*

5 Relation to vortex equations

In this section we relate the τ-stability of a triple on X directly to the existence of solutions to the coupled τ-vortex equations. Using the idea of dimensional reduction, statements about triples on X can be reformulated in terms of equivariant bundles on $X \times \mathbb{P}^1$. The equivariant version of the Hitchin-Kobayashi correspondence on $X \times \mathbb{P}^1$ then yields the following result.

Theorem 5.1 *Let $T = (E_1, E_2, \Phi)$ be a holomorphic triple. Then the following are equivalent.*

(1) *The bundles support Hermitian metrics h_1, h_2 such that the coupled τ-vortex equations are satisfied, i.e. such that*

$$\sqrt{-1}\Lambda F_{h_1} + \Phi\Phi^* = 2\pi\tau I_{E_1}\,, \tag{31}$$

$$\sqrt{-1}\Lambda F_{h_2} - \Phi^*\Phi = 2\pi\tau' I_{E_2}\,, \tag{32}$$

with

$$r_1\tau + r_2\tau' = \deg E_1 + \deg E_2\ , \tag{33}$$

(2) *The triple is τ-polystable.*

Proof. Keeping our earlier notation, let $F \to X \times \mathbb{P}^1$ be the SU(2)-invariant holomorphic bundle corresponding to the triple (E_1, E_2, Φ). Let ω_σ be the Kähler form on $X \times \mathbb{P}^1$, as defined in (19), and with σ determined by (20). By Proposition 2.5 there is a bijective correspondence between solutions to the coupled τ-vortex equations on (E_1, E_2, Φ), and SU(2)-equivariant metrics on F which satisfy the Hermitian-Einstein equations with respect to ω_σ. By [GP3] (Theorems 4 and 5), the bundle F admits an SU(2)-equivariant Hermitian-Einstein metric with respect to ω_σ if and only if F is invariantly polystable with respect to ω_σ. Finally, by Theorem 4.7, F is $SU(2)$ invariantly polystable with respect to ω_σ if and only if the corresponding triple (E_1, E_2, Φ) is τ-polystable (with τ and σ related by (20)).

Notice that the statement and conclusion of this theorem make no mention of $X \times \mathbb{P}^1$ or the SU(2)-equivalent bundle F. One might thus expect a more direct proof that does not use dimensional reduction. We will not attempt to prove both directions of the biconditional in the theorem in this way, but the one direction is quite easily seen. That is, one can show how the τ-stability condition can be derived directly as a consequence of the coupled vortex equations. This can be done as follows.

Given metrics which satisfy the coupled τ-vortex equations, consider any holomorphic saturated sub-triple, say $T' = (E_1', E_2', \Phi)$, of T. Let $E_1 = E_1' \oplus E_1''$, and $E_2 = E_2' \oplus E_2''$ be smooth orthogonal splittings of E_1 and E_2. With respect to these splittings, we get a block diagonal decomposition of $\sqrt{-1}\Lambda F_{h_1}$, $\sqrt{-1}\Lambda F_{h_2}$ as

$$\sqrt{-1}\Lambda F_{h_i} = \begin{pmatrix} \sqrt{-1}\Lambda F_i' + \Pi_i & * \\ * & \sqrt{-1}\Lambda F_i'' - \Pi_i \end{pmatrix}\ , \tag{34}$$

where Π_i is a positive definite endomorphism coming from the second fundamental form for the inclusion of E_i' in E_i. We also get a decomposition of Φ as

$$\Phi = \begin{pmatrix} \Phi' & \Theta \\ 0 & \Phi'' \end{pmatrix}\ . \tag{35}$$

The coupled vortex equations thus split into equations on the summands of E_1 and E_2. An analysis of these, involving straightforward modifications of the arguments used in [B2], shows that

(1) T splits as a direct sum of triples (E_{1i}, E_{2i}, Φ_i), i.e. $E_1 = \bigoplus E_{1i}$, $E_2 = \bigoplus E_{2i}$, and $\Phi = \bigoplus \Phi_i$,

(2) each summand (E_{1i}, E_{2i}, Φ_i) is either a τ-stable triple, or $\Phi_i = 0$ and both bundles are stable. In the latter case, the slope of E_{1i} is τ and the slope of E_{2i} is τ'.

6 Moduli spaces

6.1 Moduli spaces of stable triples

Recall that two triples $T = (E_1, E_2, \Phi)$ and $T' = (E'_1, E'_2, \Phi')$ are isomorphic if there exist isomorphism $u\colon E_1 \to E'_1$ and $v\colon E_2 \to E'_2$ such that $\Phi' \circ v = u \circ \Phi$. After fixing the topological invariants of our bundles, that is the ranks r_1 and r_2 and the first Chern classes d_1 and d_2, let $\mathfrak{M}$ be the set of equivalence classes of holomorphic triples and $\mathfrak{M}_\tau \subset \mathfrak{M}$ be the subset of equivalence classes of τ-stable triples. Our goal in this section is to show that $\mathfrak{M}_\tau$ has the structure of an algebraic variety, more precisely:

Theorem 6.1 *Let X be a compact Riemann surface of genus g and let us fix ranks r_1 and r_2 and degrees d_1 and d_2. The moduli space of τ-stable triples $\mathfrak{M}_\tau$ is a complex analytic space with a natural Kähler structure outside of the singularities. Its dimension at a smooth point is*

$$1 + r_2 d_1 - r_1 d_2 + (r_1^2 + r_2^2 - r_1 r_2)(g - 1)\,. \tag{36}$$

The moduli space, $\mathfrak{M}_\tau$ is non-empty if and only if τ is inside the interval

$$(\mu(E_1), \mu_{MAX}) \tag{37}$$

where

$$\mu_{MAX} = \mu(E_1) + \frac{r_2}{|r_1 - r_2|}(\mu(E_1) - \mu(E_2)) \tag{38}$$

if $r_1 \neq r_2$, and $\mu_{MAX} = \infty$ if $r_1 = r_2$.

Moreover $\mathfrak{M}_\tau$ is in general a quasi-projective variety. It is in fact projective if $r_1 + r_2$ and $d_1 + d_2$ are coprime and τ is generic.

Proof. There are several approaches one can take to prove this theorem. One can use standard Kuranishi deformation methods as done in [B-D1, B-D2] for the construction of the moduli spaces of stable pairs. Alternatively one can use geometric invariant theory methods to give an algebraic geometric construction of our moduli spaces, generalizing the construction of the moduli space of stable pairs given in [T, Be, HL]. We will leave these two direct methods for a future occasion and instead will exploit the relation between τ-stable triples and equivariant bundles over $X \times \mathbb{P}^1$. This method is used in [GP3] to construct the moduli spaces of triples when E_2 is a line bundle, as well as the moduli spaces of stable pairs. Apart from smoothness considerations, which we shall discuss later, the steps in the proof are the same as those in [GP3].

Let σ be related to τ by (20) and let ω_σ be the Kähler form on $X \times \mathbb{P}^1$ defined by (19). Let $\mathcal{M}_\sigma$ be the moduli space of stable bundles with respect to ω_σ whose underlying smooth bundle is defined by (1). Let us exclude for the moment the case $r_1 = r_2$ and $d_1 = d_2$. Let $F \to X \times \mathbb{P}^1$ be the bundle associated to (E_1, E_2, Φ) as in Proposition 2.3. Theorem 4.1 says that the correspondence $(E_1, E_2, \Phi) \mapsto F$ defines a map

$$\mathfrak{M}_\tau \to \mathcal{M}_\sigma\,.$$

The action of $SU(2)$ on $X \times \mathbb{P}^1$ defined in Sect. 2 induces an action on $\mathcal{M}_\sigma$ and, since the bundle F associated to (E_1, E_2, Φ) is $SU(2)$-equivariant the image of the above map is contained in $\mathcal{M}_\sigma^{SU(2)}$ – the set of fixed points of $\mathcal{M}_\sigma$ under the $SU(2)$ action. As proved in [GP3, Proposition 5.3] the set $\mathcal{M}_\sigma^{SU(2)}$ can be described as a disjoint union of a finite number of sets

$$\mathcal{M}_\sigma^{SU(2)} = \bigcup_{i \in I} \mathcal{M}_\sigma^i .$$

The index I ranges over the set of equivalence classes of different smooth $SU(2)$-equivariant structures on the smooth bundle F defined by (1). Of course the way of writing F in (1) already exhibits a particular $SU(2)$-equivariant structure, but in principle the bundle F might admit different ones. The set $\mathcal{M}_\sigma^i$ corresponds to the set of equivalence classes in $\mathcal{M}_\sigma$ admitting a representative which is $SU(2)$-equivariant for the smooth equivariant structure defined by $i \in I$. An equivariant smooth structure defines an action on the space of smooth automorphisms of the bundle F and, as shown in [GP3, Theorem 5.6] the sets $\mathcal{M}_\sigma^i$ can be described as the set of equivalence classes of $SU(2)$-equivariant holomorphic structures on the underlying smooth $SU(2)$-equivariant bundle defined by i, modulo $SU(2)$-equivariant isomorphisms.

Let i_0 be the C^∞ $SU(2)$-equivariant structure on F defined by (1). As shown in Proposition 2.3 there is a one-to-one correspondence

$$\{\text{holomorphic triples}\} \stackrel{1-1}{\longleftrightarrow} \{i_0\text{-equivariant holomorphic vector bundles}\} . \tag{39}$$

Furthermore, the equivariant homomorphisms between two equivariant holomorphic bundles F and F' corresponding to triples T and T', respectively, are in one-to-one correspondence with the morphisms between T and T'. In fact the correspondence (39) descends to the quotient and thus from Theorem 4.1 we can identify $\mathfrak{M}_\tau$ with $\mathcal{M}_\sigma^{i_0}$. The properties of $\mathfrak{M}_\tau$ follow now from standard facts about the more familiar moduli spaces of stable bundles $\mathcal{M}_\sigma$ [D-K, G, M, Ko], and more particularly of the fixed-point sets $\mathcal{M}_\sigma^i$ (see [GP3, Theorem 5.6] for details). Namely,

Theorem 6.2 *$\mathcal{M}_\sigma^i$ is a complex analytic variety. A point $[F] \in \mathcal{M}_\sigma^i$ is non-singular if it is non-singular as a point of $\mathcal{M}_\sigma$. The tangent space at such a point can be identified with the $SU(2)$-invariant part of $H^1(X \times \mathbb{P}^1, \operatorname{End} F)$. $\mathcal{M}_\sigma^i$ has a natural Kähler structure induced from that of $\mathcal{M}_\sigma$. Moreover if σ is a rational number then $\mathcal{M}_\sigma^i$ is a quasi-projective variety.*

From this theorem and the identification of $\mathfrak{M}_\tau$ with $\mathcal{M}_\sigma^{i_0}$ we deduce that $\mathfrak{M}_\tau$ is a complex analytic variety with a Kähler metric outside the singularities. To compute the dimension of the tangent space at a smooth point $[T]$ it suffices to compute the dimension of the $SU(2)$-invariant part of $H^1(X \times \mathbb{P}^1, \operatorname{End} F)$. This can be done in a similar way to that of [GP3, Theorem 5.13] to obtain that

$$\dim \mathfrak{M}_\tau = 1 + \chi(E_1 \otimes E_2^*) - \chi(\operatorname{End} E_1) - \chi(\operatorname{End} E_2) ,$$

which by Riemann-Roch yields (36).

We consider now the case $r_1 = r_2 = r$ and $d_1 = d_2 = d$. In this case by Corollary 3.20 and Lemma 4.6 we can identify the moduli space $\mathfrak{M}_\tau$ with the moduli space of stable bundles of rank r and degree d on X. The theorem follows now from well-known results about this moduli space [A-B, N-S].

The fact that $\mathfrak{M}_\tau$ is empty outside the interval (37) if $r_1 \neq r_2$ and outside $(\mu(E_1), \infty)$ if $r_1 = r_2$ follows from Proposition 3.18. As explained in Proposition 3.22 the non-generic values divide this intervals in subintervals in such a way that the stability properties of a given triple do not change for two values of τ in the same subinterval. Therefore we can always choose τ (and hence σ) to be rational, which by Theorem 6.2 gives that $\mathfrak{M}_\tau$ is quasi-projective.

To show the compactness of $\mathfrak{M}_\tau$ when $r_1 + r_2$ and $d_1 + d_2$ are coprime and τ is generic (we are also assuming that $r_1 \neq r_2$ or $d_1 \neq d_2$) we consider a sequence of points in $\mathcal{M}_\sigma^{i_0}$. This sequence must converge in $\bar{\mathcal{M}}_\sigma$ – the Uhlenbeck compactification of $\mathcal{M}_\sigma$. Using $SU(2)$-invariance one can see that the limit has to correspond to a polystable element, but by Proposition 3.22 this has to be actually stable, that is the limit must be in $\mathcal{M}_\sigma$ and hence in $\mathcal{M}_\sigma^{i_0}$ since this is closed. The compactness when $r_1 = r_2$ and $d_1 = d_2$ follows from the compactness of the moduli space of stable bundles of rank r and degree d when r and d are coprime.

The compactness of $\mathfrak{M}_\tau$ can also be obtained (as it is done for pairs in [B-D1]) from the fact that it can be identified with the moduli space of solutions to the coupled vortex equations and these are moment map equations as we shall explain later.

It was shown in [GP3, Theorem 5.13] that when E_2 is a line bundle our moduli spaces are smooth for every value of τ. This does not seem to be the case when E_2 is of arbitrary rank. However we can show the following

Proposition 6.3 *Let $T = (E_1, E_2, \Phi)$ be a τ-stable holomorphic triple such that Φ is either injective or surjective, then $[T]$ is a smooth point of $\mathfrak{M}_\tau$.*

Proof. Let

$$0 \to p^*E_1 \to F \to p^*E_2 \otimes q^*\mathcal{O}(2) \to 0\,, \tag{40}$$

be the extension over $X \times \mathbb{P}^1$ corresponding to T. To prove the smoothness of $\mathfrak{M}_\tau$ at the point $[(E_1, E_2, \Phi)]$ it suffices to show that $H^2(X \times \mathbb{P}^1, \operatorname{End} F) = 0$. Tensoring (40) with F^* the last terms in the corresponding long exact sequence are

$$H^2(p^*E_1 \otimes F^*) \to H^2(F \otimes F^*) \to H^2(p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^*) \to 0\,. \tag{41}$$

By Serre duality

$$H^2(p^*E_1 \otimes F^*) \cong H^0(p^*(E_1^* \otimes K) \otimes F)^*$$

$$H^2(p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^*) \cong H^0(p^*(E_2^* \otimes K) \otimes q^*\mathcal{O}(-4) \otimes F)^*\,,$$

where K is the canonical line bundle of X.

It is easy to see that $H^0(p^*(E_2^* \otimes K) \otimes q^*\mathcal{O}(-4) \otimes F) \cong 0$. To analyse $H^0(p^*(E_1^* \otimes K) \otimes F)$ we tensor (40) with $p^*(E_1^* \otimes K) \otimes q^*\mathcal{O}(-2)$, and since $H^0(p^*(E_1 \otimes E_1 \otimes K) \otimes q^*\mathcal{O}(-2)) \cong 0$, we obtain

$$0 \to H^0(p^*(E_1^* \otimes K)q^*\mathcal{O}(-2) \otimes F) \to H^0(p^*(E_1^* \otimes E_2 \otimes K)) \\ \xrightarrow{f} H^1(p^*(E_1 \otimes E_1^* \otimes K \otimes \mathcal{O}(-2)). \tag{42}$$

The map f in the above sequence is essentially the map

$$H^0(E_1^* \otimes E_2 \otimes K) \xrightarrow{f} H^0(E_1 \otimes E_1^* \otimes K)$$
$$\Psi \mapsto \Phi \circ \Psi .$$

Assume now that Φ is injective, if we prove that f is also injective, by the exactness of (42) we would be done. Suppose that $\operatorname{Ker} f \neq 0$. This means that there exists a non-zero map Ψ: $E_1 \to E_2 \otimes K$, and since $\Psi \circ \Phi = 0$, $\operatorname{Im} \Psi$ is a non-trivial subsheaf contained in $\operatorname{Ker} \Phi$ contradicting the injectivity.

To prove smoothness when Φ is surjective, we consider the dual triple $T^* = (E_2^*, E_1^*, \Phi^*)$. Φ^* is now injective and the result follows from the fact that $T = (E_1, E_2, \Phi)$ is a smooth point if and only if $T^* = (E_2^*, E_1^*, \Phi^*)$ is a smooth point.

6.2 Abel-Jacobi maps

As shown in Proposition 3.23 there is a range for the parameter τ such that the τ-stability of a triple (E_1, E_2, Φ) implies the semistability of E_1 and E_2. Let $\mathfrak{M}_0$ be the moduli space of τ-stable triples for τ in such a range. Let $N(r,d)$ be the Seshadri compactification of the moduli space of stable bundles of rank r and degree d over X, that is, the space of S-equivalence classes of semistable bundles.

There are natural "Abel-Jacobi" maps π_1 and π_2

$$\begin{array}{ccc} \mathfrak{M}_0 & \xrightarrow{\pi_2} & N(r_2, d_2) \\ {\scriptstyle \pi_1}\downarrow & & \\ N(r_1, d_1) & & \end{array}$$

defined as

$$\pi_1([(E_1, E_2, \Phi)]) = [E_1] \quad \text{and} \quad \pi_2([(E_1, E_2, \Phi)]) = [E_2] .$$

We know also from Proposition 3.23 that if both E_1 and E_2 are stable then the intersection of the fibres $\pi_1^{-1}([E_1])$ and $\pi_2^{-1}([E_2])$ can be identified with $\mathbb{P}(H^0(E_1 \otimes E_2^*))$. In general, though, this intersection for non-stable points is hard to describe.

If $\mu(E_1 \otimes E_2^*) > 2g - 2$, that is if

$$r_2 d_1 - r_1 d_2 > r_1 r_2 (2g - 2),$$

where g is the genus of X, then $H^1(E_1 \otimes E_2^*) = 0$ for E_1 and E_2 stable and the projection from $\mathfrak{M}_0$ to $N(r_1, d_1) \times N(r_2, d_2)$ is a fibration on the stable part.

Recall that if $(r_1,d_1)=1$ and $(r_2,d_2)=1$ then stability and semistability coincide and there exist universal bundles

$$\mathbb{E}_1 \to X \times N(r_1,d_1) \quad \text{and} \quad \mathbb{E}_2 \to X \times N(r_2,d_2).$$

Let us denote by p_1, p_2 and π the projections from $X \times N(r_1,d_1) \times N(r_2,d_2)$ to $X \times N(r_1,d_1)$, $X \times N(r_2,d_2)$, and $N(r_1,d_1) \times N(r_2,d_2)$ respectively. It is clear that $\mathfrak{M}_0$ can be identified with

$$\mathbb{P}(\pi_*(p_1^*\mathbb{E}_1 \otimes p_2^*\mathbb{E}_2^*)). \tag{43}$$

But in the non-coprime situation we have no universal bundles $\mathbb{E}_1$ and $\mathbb{E}_2$ available and the analogue of (43) has to be constructed as a moduli space in its own right.

As explained in Theorem 6.1 the moduli space of τ-stable triples is non-empty if and only if τ is in the interval $I=(\mu(E_1),\mu_{MAX})$. We saw in Sect. 3.4 that the stability properties of a given triple can change only at certain rational values of τ (the critical values) which divide I in a finite number of sub-intervals. The moduli spaces for values of τ in the same open subinterval are then isomorphic, and they might change only when crossing one of the critical values. We expect that, as in the case of stable pairs [B-D-W, T], the moduli spaces for consecutive intervals must be related by some sort of flip-type birational transformation. This, as well as the construction of a "master" space for triples (cf. [B-D-W]) containing the moduli space of triples for all possible values of τ, will be dealt with in a future paper.

6.3 Vortices

Thanks to our existence theorem the moduli space of stable triples can be interpreted as the moduli space of solutions to the coupled vortex equations. To understand the meaning of this statement one needs to regard the vortex equations as equations for unitary connections instead of equations for metrics. This point of view corresponds to the fact that fixing a holomorphic structure and varying the metric on a vector bundle is equivalent to fixing the metric and varying the holomorphic structure – or the corresponding connection. Recall that the space of unitary connections on a smooth Hermitian vector bundle can be identified with the space of $\bar{\partial}$-operators which in turn corresponds with the space of holomorphic structures on the bundle.

Let E_1 and E_2 be smooth vector bundles over X and h_1 and h_2 be Hermitian metrics on E_1 and E_2 respectively. Let $\mathscr{A}_1$ (resp. $\mathscr{A}_2$) be the space of unitary connections on (E_1,h_1) (resp. (E_2,h_2)). Let $(A_1,A_2,\Phi) \in \mathscr{A}_1 \times \mathscr{A}_2 \times \Omega^0(\mathrm{Hom}(E_2,E_1))$. The vortex equations can be regarded as the equations for (A_1,A_2,Φ)

$$\left.\begin{aligned} &\bar{\partial}_{A_1 * A_2}\Phi = 0 \\ &\sqrt{-1}\Lambda F_{A_1} + \Phi\Phi^* = 2\pi\tau I_{E_1} \\ &\sqrt{-1}\Lambda F_{A_2} - \Phi^*\Phi = 2\pi\tau' I_{E_2} \end{aligned}\right\} . \tag{44}$$

The connections A_1 and A_2 induce holomorphic structures on E_1 and E_2 and the first equation in (44) simply says that Φ must be holomorphic.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the gauge groups of unitary transformations of (E_1,h_1) and (E_2,h_2) respectively. $\mathcal{G}_1 \times \mathcal{G}_2$ acts on $\mathcal{A}_1 \times \mathcal{A}_2 \times \Omega^0(\mathrm{Hom}(E_2,E_1))$ by the rule

$$(g_1,g_2) \cdot (A_1,A_2,\Phi) = (g_1A_1g_1^{-1}, g_2A_2g_2^{-1}, g_1\Phi g_2^{-1}).$$

The action of $\mathcal{G}_1 \times \mathcal{G}_2$ preserves the equations and the moduli space of *coupled τ-vortices* is defined as the space of all solutions to (44) modulo this action.

The moduli space of vortices can be obtained as a symplectic reduction (see [GP3, Sect. 2.2]) in a similar way to the moduli space of Hermitian-Einstein connections: $\mathcal{A}_1 \times \mathcal{A}_2 \times \Omega^0(\mathrm{Hom}(E_2,E_1))$ admits a Kähler structure which is preserved by the action of $\mathcal{G}_1 \times \mathcal{G}_2$. Associated to this action there is a moment map given precisely by

$$(A_1,A_2,\Phi) \mapsto (\Lambda F_{A_1} - \sqrt{-1}\Phi\Phi^* + 2\sqrt{-1}\pi\tau, \Lambda F_{A_2} + \sqrt{-1}\Phi^*\Phi + 2\sqrt{-1}\pi\tau'). \tag{45}$$

Let μ be this moment map restricted to the subvariety

$$\mathcal{N} = \{(A_1,A_2,\Phi) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \Omega^0(\mathrm{Hom}(E_2,E_1)) \mid \bar\partial_{A_1*A_2}\Phi = 0\}.$$

The moduli space of τ-vortices is then nothing else but the symplectic quotient

$$\mu^{-1}(0,0)/\mathcal{G}_1 \times \mathcal{G}_2,$$

and Theorem 5.1 can be reformulated by saying that there is a one-to-one correspondence

$$\mu^{-1}(0,0)/\mathcal{G}_1 \times \mathcal{G}_2 \overset{1-1}{\longleftrightarrow} \mathfrak{M}_\tau.$$

Acknowledgements. The authors would like to thank Alastair King and Jun Li for helpful conversations and the following institutions for their hospitality during the course of this project: The Mathematics Institute of the University of Warwick, England; I.H.E.S., France; and C.I.M.A.T., Mexico. The work of the second author was carried out at the Mathematics Departments of UC Berkeley and University of Paris-Sud at Orsay, with the support of a Fulbright scholarship and a European Union postdoctoral fellowship under the HCM programme respectively. He wishes to thank S. Kobayashi and A. Beauville for their hospitality.

References

[A-B] Atiyah, M.F., Bott, R.: The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A **308** (1982), 523–615

[Be] Bertram, A.: Stable pairs and stable parabolic pairs. J. Alg. Geometry **3** (1994), 703–724

[B1] Bradlow, S.B.: Vortices in holomorphic line bundles over closed Kähler manifolds. Commun. Math. Phys. **135** (1990), 1–17

[B2] Bradlow, S.B.: Special metrics and stability for holomorphic bundles with global sections. J. Diff. Geom. **33** (1991), 169–214

[B-D1] Bradlow, S.B., Daskalopoulos, G.: Moduli of stable pairs for holomorphic bundles over Riemann surfaces. Int. J. Math. **2** (1991), 477–513

[B-D2] Bradlow, S.B., Daskalopoulos, G.: Moduli of stable pairs for holomorphic bundles over Riemann surfaces II. Int. J. Math. **4** (1993), 903–925

[B-D-W] Bradlow, S.B., Daskalopoulos, G.D., Wentworth, R.A.: Birational equivalences of vortex moduli, Topology, to appear

[D1] Donaldson, S.K.: Anti-self-dual Yang-Mills connections on a complex algebraic surface and stable vector bundles. Proc. Lond. Math. Soc. **3** (1985), 1–26

[D2] Donaldson, S.K.: Infinite determinants, stable bundles and curvature. Duke Math. J. **54** (1987), 231–247

[D-K] Donaldson, S.K., Kronheimer, P.B.: The Geometry of Four Manifolds. Oxford Mathematical Monographs, Oxford University Press, 1990

[GP1] García-Prada, O.: Invariant connections and vortices. Commun. Math. Phys. **156** (1993), 527–546

[GP2] García-Prada, O.: A direct existence proof for the vortex equations over a compact Riemann surface. Bull. London Math. Soc. **26** (1994), 88–96

[GP3] García-Prada, O.: Dimensional reduction of stable bundles, vortices and stable pairs. Int. J. Math. **5** (1994), 1–52

[G] Gieseker, D.: On moduli of vector bundles on an algebraic surface. Ann. Math. **106** (1977), 45–60

[H] Hitchin, N.J.: The self-duality equations on a Riemann surface. Proc. Lond. Math. Soc. **55** (1987), 59–126

[HL] Huybrechts, D., Lehn, M.: Stable pairs on curves and surfaces. J. Alg. Geometry **4** (1995), 67–104

[Ko] Kobayashi, S.: Differential Geometry of Complex Vector Bundles. Princeton University Press, New Jersey, 1987

[L] Lübke, M.: Stability of Einstein-Hermitian vector bundles. Manuscripta Mathematica **42** (1983), 245–257

[M] Maruyama, M.: Moduli of stable sheaves I and II. J. Math. Kyoto Univ. **17** (1977), 91–126, and **18** (1978), 557–614

[N-S] Narasimhan, M.S., Seshadri, C.S.: Stable and unitary bundles on a compact Riemann surface. Ann. Math. **82** (1965), 540–564

[O-S-S] Okonek, C., Schneider, M., Spindler, H.: Vector Bundles on Complex Projective Spaces. Progress in Mathematics 3. Birkhäuser, Boston, 1980

[T] Thaddeus, M.: Stable pairs, linear systems and the Verlinde formula. Invent. Math. **117** (1994), 317–353

[U-Y] Uhlenbeck, K.K., Yau, S.T.: On the existence of Hermitian-Yang-Mills connections on stable bundles over compact Kähler manifolds. Comm. Pure and Appl. Math. **39-S** (1986), 257–293