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Complex cobordism ring and conformal field theory over Z

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0 Introduction

In conformal field theory for free fermions, the Fock space plays a central role and provides a link among several branches of mathematics such as representation theory of the Virasoro algebra, theory of KP equation, moduli of algebraic curves and formal groups [BeSc, KNTY]. On the other hand, string theory drew much attention to the theory of elliptic genera, cf. [La].

Our original motivation having been to give a systematic account of elliptic genera in the context of (arithmetico-geometric) conformal field theory [KSU1, 2], we are led to a basic understanding of the connection between the complex cobordism ring MU^* and the boson Fock space $\mathscr{H}_{T,0}$. As an application, we give a new interpretation of genera (multiplicative sequences) in terms of the KP hierarchy.

To formulate the main results, let us introduce some notation. The boson Fock space $\mathscr{H}_{T,0}$ is a polynomial ring over $\mathbf{Q}$ in indeterminates $t_1, t_2, \ldots$, cf. (2.2). We use the notation $t=(t_1, t_2, \ldots)$ for short and also $kt=(kt_1, kt_2, \ldots)$ for $k\in\mathbf{Z}$. There is a natural pairing of $\mathscr{H}_{T,0}$ with itself (3.1), which plays an important role in the theory of the KP hierarchy.

The Chern classes followed by the augmentation define the linear functionals $c_1^{\alpha_1}c_2^{\alpha_2}\ldots$ on MU^*, which are linearly independent. In other words, we have the pairing "Chern number" between MU^* and the polynomial ring Ch^* over $\mathbf{Q}$ in the indeterminates "universal Chern classes" $c_1, c_2, \ldots$.

Then the main results are the following:

Theorem 0.1 $(=(3.3))$. *The ring homomorphisms*

$$MU^*\otimes_{\mathbf{Z}}\mathbf{Q} \xrightarrow{K} \mathscr{H}_{T,0}=\mathbf{Q}[t_1, t_2, \ldots]: \mathbf{P}^n \mapsto p_n((n+1)t)$$

$$Ch^*=\mathbf{Q}[c_1, c_2, \ldots] \xrightarrow{K\dagger} \mathscr{H}^{\dagger}_{T,0}=\mathbf{Q}[t_1, t_2, \ldots]: c_i \mapsto (-1)^i p_i(-t)$$

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are bijective and transforms the pairing "Chern number" between $MU^*\otimes_{\mathbf{Z}}\mathbf{Q}$ *and* Ch^* *into the natural one of* $\mathscr{H}_{T,0}$ *with itself. Here* $p_i(t)$ *denotes the i-th Schur polynomial cf.* (2.2).

There is a formula of elementary nature calculating arbitrary genus, as an application of this theorem, cf. (3.13).

Under these identifications, we regard an (arbitrary) multiplicative sequence

$$T(c)=\sum_{n=0}^{\infty} T_n(c_1,\dots,c_n)$$

as a formal power series in the variables t_i's: $\tau_T(t):=K^\dagger(T(c))$, where, by abuse of notation, $K^\dagger$ denotes the natural extension of $K^\dagger$ to the completions $\widehat{Ch^*}\to\hat{\mathscr{H}}_{T,0}$. This function will be named as the τ-function associated to T. It has the following remarkable property.

Theorem 0.2 (=(3.9), 1)).

$$\tau_T(t)=\exp\left(\sum_{i\geq 1}(-1)^{i+1}b_it_i\right).$$

Here b_i *is the coefficient of* c_i *in* $T_i(c_1,\dots,c_i)$.

We also have an expression of $T(c)$ as a determinant of a matrix of infinite size (3.9), 2). This result, together with Theorem (0.2), implies the following:

Theorem 0.3 (=(5.3)). 1) *The power series* $\tau_T(t)$ *can be identified with a* τ*-function in the theory of KP hierarchy.*

2) *The Lax operator associated with the* τ*-function in* 1) *is* ∂_x.

3) *There is a one-to-one correspondence between the set of multiplicative sequences over* $\mathbf{Q}$ *and the subset* $1+\partial_x^{-1}\mathbf{Q}[\![\partial_x^{-1}]\!]$ *of all the wave operators.*

For the precise formulation and the statement of the above theorem, see Theorem 5.3.

We have the following observation which connects the above identification K with the one by Morava [La2, Appendix] and Bukhshtaber-Shokurov [BuSh]:

Theorem 0.4 (=(4.4)). *The ring homomorphism*

$$\varphi:\mathscr{H}_{T,0}\to\mathscr{H}_{T,0}:t_i\mapsto p_i(-t)$$

sits in the commutative diagram:

$$\begin{array}{ccc} MU^*\otimes\mathbf{Q} & \xrightarrow{\beta} & \mathscr{H}_{T,0} \\ & \searrow & \downarrow \varphi \\ & K & \mathscr{H}_{T,0} \end{array},$$

where β *is given by the coefficients of the Chern-Dold character.*

As a corollary of this Theorem, we can determine the image $K(MU^*)$ in $\mathscr{H}_{T,0}$, cf. Theorem (4.7).

The connection of cobordism ring with conformal field theory was also discussed in [Mo, MoSh]. Generalizations of elliptic genera were also discussed by [Hir2, Kr, T].

The contents of this paper are as follows. First we recall basic terminologies: the multiplicative sequences, complex cobordism ring, etc. in Sect. 1 and the boson

Fock space in Sect. 2. The connection between the boson Fock space and the complex cobordism ring is established in Sect. 3. The comparison with the result of Morava and Bukhshtaber-Shokurov is in Sect. 4. In Sect. 5 is the second main result, namely, the interpretation of multiplicative sequences in terms of τ-functions in soliton theory.

1 Preliminaries from topology: multiplicative sequences, complex cobordism rings, etc.

1.1. Let A be a commutative ring with a unit 1.

A multiplicative sequence is a formal power series in z

$$T(c,z)=T\left(\sum_{n\geqq 0} c_n z^n\right)=1+\sum_{j\geqq 1} T_j(c)z^j \in A[c_1,c_2,\ldots]\,[\![z]\!]$$

(c_i's are indeterminates for $i\geqq 1$ and $c_0=1$), which satisfy the multiplicativity condition:

$$\text{If}\quad \left(\sum_{i\geqq 0} c_i z^i\right)\left(\sum_{j\geqq 0} c'_j z^j\right)=\sum_{k\geqq 0} c''_k z^k,$$

$$\text{then}\quad T(c,z)T(c',z)=T(c'',z).$$

Therefore $T_j(c)$ is a homogeneous polynomial of degree j with respect to $\deg c_i=i$, cf. [Hir1].

The group $\Lambda(B)=1+zB[\![z]\!]$ with multiplication of power series defines a group scheme $B\mapsto \Lambda(B)$. (B is a commutative ring with 1.) Then a multiplicative sequence is nothing but an endomorphism of A-group scheme Λ(for B; A-algebras)

$$\Phi_T\colon \sum_{i\geqq 0} c_i z^i \mapsto T(c,z).$$

The A-group scheme Λ is affine and representable by the ring

$$Ch_A^* := A[c_1,c_2,\ldots],$$

where c_i's are indeterminates as above. Thus to a multiplicative sequence T corresponds a ring homomorphism

$$\Phi_T^*\colon Ch_A^* \to Ch_A^*;\qquad c_i\mapsto T_i(c).$$

In Sect. 3 we will use the abbreviation $Ch^*=Ch_{\mathbf{Q}}^*$.

1.2. The complex cobordism ring MU^* is equal to the value $MU^*(pt)$ at the one point space of a complex-oriented cohomology functor with the same notation MU^*.

The Euler class e_{MU} of complex vector bundles over a manifold X defines the cobordism Chern class $c_i\in MU^{2i}(X)$ and the total Chern polynomial $c_t\in MU^*(X)[t_1,t_2,\ldots]$ in the same way as for any complex-oriented cohomology theory.

(MU^*, e_{MU}) is universal in the sense that for any complex-oriented cohomology theory (h^*, e_h), there is a natural transformation $\gamma\colon MU^*\to h^*$ such that $e_h=\gamma\cdot e_{MU}$.

The total Landweber-Novikov operation

$$s_t\colon MU^*\to MU^*[t_1,t_2,\ldots]$$

can be given by the formula

$$s_t(x) := f_*(c_t(N_f)) \in MU^*(X)[t_1, t_2, \ldots],$$

where x is an element represented by a map $f: Z \to X$ and $N_f = f^*TX - TZ$ is the virtual normal bundle of f. To each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots)$ corresponds a cohomology operation s_α of MU^* as a coefficient of $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \ldots$ in s_t.

The totality of s_α's is known to form a subalgebra S of the algebra of all stable cohomology operations, by Landweber and Novikov. S acts on $MU^* = MU^*(pt)$ by definition.

1.3. Next we recall the relation of multiplicative sequences and genera in algebraic topology.

A genus with values in a commutative ring A is, by definition, a ring homomorphism

$$\varphi: MU^* \to A.$$

When A is a **Q**-algebra, this notion is equivalent to a ring homomorphism

$$\varphi: MU^*_{\mathbf{Q}} := MU^* \otimes_{\mathbf{Z}} \mathbf{Q} \to A,$$

or to the following formal power series ("logarithm" of φ)

$$\ell_\varphi(z) = \sum_{n \geqq 1} \frac{\varphi(\mathbf{P}^{n-1})}{n} z^n \in A[\![z]\!],$$

for $MU^*_{\mathbf{Q}}$ is a polynomial ring on the generators $\mathbf{P}^n$ [= the cobordism class of $\mathbf{P}^n(\mathbf{C})$], with the convention $\mathbf{P}^0 = 1$ (Milnor). The logarithm $\ell(z)$ is related to the characteristic power series $Q(z)$ associated to a multiplicative sequence $T(c, z)$ by the formula:

$$Q(z) = \frac{z}{\ell^{-1}(z)} = T(1 + z).$$

Examples. 1) (Todd genus)

$$\ell(z) = -\log(1 - z), \quad Q(z) = \frac{z}{1 - e^{-z}}, \quad \varphi(\mathbf{P}^n) = 1 \ (\forall n).$$

2) (elliptic genus) Consider the equation (Jacobi quartic):

$$y^2 = R(x) = 1 - 2\delta x^2 + \varepsilon x^4.$$

A genus is called elliptic (after Ochanine) if its logarithm is given by the elliptic integral:

$$\ell(z) = \int_0^z \frac{dt}{\sqrt{R(t)}}.$$

Then we have

$$Q(z) = \frac{z}{sn(z)}, \quad \varphi(\mathbf{P}^{2n}) = P_n(\delta/\sqrt{\varepsilon})\varepsilon^{n/2}, \quad \varphi(\mathbf{P}^{2n-1}) = 0,$$

where $sn(z)$ is Jacobi's sine function and $P_n(z)$ is the n-th Legendre polynomial:

$$P_n(z) = \frac{1}{2^n n!} \left(\frac{d}{dz}\right)^n \{(z^2 - 1)^n\}.$$

2 Preliminaries from CFT: boson Fock space

We briefly recall some notions from [KNTY, KSU1].

2.1. Conformal field theory (CFT) is a kind of 2-dimensional quantum field theory (QFT) with conformal symmetry. It is initiated by Belavin, Polyakov, and Zamolodchikov and a model of CFT involves representations of the Virasoro Lie algebra $\mathscr{L}$. It is a Lie algebra over $\mathbf{Q}$, $\mathbf{Q}[z, z^{-1}]\frac{d}{dz} \oplus \mathbf{Q} \cdot c$ with the following bracket:

$$\left[f\frac{d}{dz}, g\frac{d}{dz}\right]_{\mathscr{L}} = (fg' - f'g)\frac{d}{dz} + \frac{1}{12}\operatorname{Res}_{z=0}(f'''g\,dz)c$$

$$c \in \text{the center of } \mathscr{L}.$$

The elements c and $L_n = -z^{n+1}\frac{d}{dz}$ $(n \in \mathbf{Z})$ form a basis of $\mathscr{L}$, familiar in the literature.

We also use the completed version

$$\hat{\mathscr{L}} := \mathbf{Q}((z))\frac{d}{dz} \oplus \mathbf{Q} \cdot c, \qquad \mathbf{Q}((z)) = \mathbf{Q}[[z]][z^{-1}].$$

Fock representations are defined by a free fermion field [$U(1)$-current], namely by an extended Heisenberg algebra, and become $\mathscr{L}$-modules through the so-called Sugawara construction. We need only their bosonized version, L_n's being expressed in terms of differential operators. We refer the reader to [DJKM], [TK, Appendix] for details of the boson-fermion correspondence and to [KSU1] for its arithmetic version.

2.2. The boson Fock space of central charge 1 is defined to be:

$$\mathscr{H}_T = \mathbf{Q}[t_1, t_2, \ldots][u, u^{-1}] \quad (t_1, t_2, \ldots \text{ are indeterminates})$$

$$\mathscr{H}_{T,p} = \mathbf{Q}[t_1, t_2, \ldots]u^p = \text{charge } p \text{ sector of } \mathscr{H}_T \ (p \in \mathbf{Z}).$$

Note that, in [KNTY, KSU1,2], the completions $\widehat{\mathscr{H}_T}, \widehat{\mathscr{H}_{T,0}}$ with respect to $\deg t_i = i$, $\deg u = 0$ are considered and the completed version $\widehat{\mathscr{H}_T}$ is called the boson Fock space.

The action of $\mathscr{L}$ on $\mathscr{H}_T$ is defined by

$$L_n = \sum_{m=1}^{\infty} mt_m \frac{\partial}{\partial t_{n+m}} + \frac{1}{2}\sum_{m=1}^{n-1} \frac{\partial}{\partial t_m}\frac{\partial}{\partial t_{n-m}} + \frac{\partial}{\partial t_n} u \frac{\partial}{\partial u} \qquad (n \geqq 1)$$

$$L_{-n} = \sum_{m=1}^{\infty} (n+m)t_{n+m}\frac{\partial}{\partial t_m} + \frac{1}{2}\sum_{m=1}^{n-1} m(n-m)t_n t_m + nt_n u\frac{\partial}{\partial u}$$

$$L_0 = \sum_{m=1}^{\infty} mt_m \frac{\partial}{\partial t_m} + \frac{1}{2}\left(u\frac{\partial}{\partial u}\right)^2$$

$$c = 1.$$

We are mainly interested in the charge 0 sector $\mathscr{H}_{T,0}$. It has an obvious **Z**-structure

$$\mathscr{H}_{T,0}(\mathbf{Z}) = \mathbf{Z}[t_1, t_2, \ldots].$$

But, for arithmetical purpose (cf. [KSU1]), more natural one is given by

$$\mathcal{H}_0(\mathbf{Z})=\mathbf{Z}[p_1(t), p_2(t), \ldots],$$

where the elementary Schur polynomials $p_j(t)$ ($j \geqq 1$) are defined by the formula:

$$\sum_{j\geqq 0} p_j(t)z^j=\exp\left(\sum_{n\geqq 1} t_n z^n\right).$$

Here t abbreviates $(t_1, t_2, \ldots)$ and $p_0(t)=1$.

We note that a natural $\mathbf{Z}$-basis of $\mathcal{H}_0(\mathbf{Z})$ is given by the Schur polynomials $\chi_\lambda(t)$, λ being a partition $\lambda=(\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_\ell > 0$, $\lambda_i \in \mathbf{Z}$:

$$\chi_\lambda(t)=\det(p_{\lambda_i+j-i}(t))_{1\leqq i,j\leqq \ell}.$$

Considering p_n's as variables, we also denote it by $\Delta_\lambda(p)$. The polynomial $p_j(t)$ corresponds to $\lambda=(j)$ and $g_j(t):=(-1)^j p_j(-t)$ to $(1, \ldots, 1)$ (j times).

2.3. We briefly review the appearance of the boson Fock space in a geometric setting of abelian CFT. We consider the setting over $\mathbf{C}$.

Let $\mathcal{M}_{g,\ell}$ denote the fine moduli space of projective smooth algebraic curves of genus g with level ℓ structure, ℓ being divisible by 4, as in [KSU1, Sect. 4]. Then the universal family of curves $\pi:\mathcal{C}\to\mathcal{M}_{g,\ell}=\mathcal{M}$ and a theta characteristic $\Omega^{1/2}$, i.e. a line bundle on $\mathcal{C}$ with $\Omega^{1/2\otimes 2}\cong\Omega^1_{\mathcal{C}/\mathcal{M}}$, are at our disposal, and we have the determinant line bundle

$$L=\det R\pi_*(\Omega^{1/2})$$

on $\mathcal{M}_g$.

We have the dressed moduli space $\mathcal{M}_g^{(\infty)}$, introduced by Beilinson and Kontsevich, cf. [BeSc, KNTY]. A point of $\mathcal{M}_g^{(\infty)}$ corresponds to a triple $\mathcal{X}=(C,Q,t)$, where $C\in\mathcal{M}_g$, $Q\in C$ and t is a choice of a formal local parameter $\hat{\mathcal{O}}_{C,Q}\xrightarrow{\sim}\mathbf{C}[[z]]$.

An important property of $\mathcal{M}_g^{(\infty)}$ is that the completed Virasoro algebra $\hat{\mathcal{L}}$ acts infinitesimally on $\mathcal{M}_g^{(\infty)}$ with central charge 0:

$$T_{\mathcal{X}}\mathcal{M}_g^{(\infty)}\xleftarrow{\sim}\mathbf{C}((z))\frac{d}{dz}\Big/H^0(\mathcal{C},\mathcal{O}_C(*Q)).$$

The natural projection $\mathcal{M}_g^{(\infty)}\xrightarrow{\tilde{\pi}}\mathcal{M}_g$ factorizes into $\mathcal{M}_g^{(\infty)}\xrightarrow{\pi'}\mathcal{C}\xrightarrow{\pi}\mathcal{M}_g$. The morphism $\pi':\mathcal{M}_g^{(\infty)}\to\mathcal{C}$ is easily seen to be a principal homogeneous space under the affine group scheme

$$D^{(0)}=\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}[[z]]).$$

We have $\tilde{\pi}^*L$ on $\mathcal{M}_g^{(\infty)}$ which is the determinant line bundle for the family $\mathcal{C}\times_{\mathcal{M}}\mathcal{M}_g^{(\infty)}\to\mathcal{M}_g^{(\infty)}$. Then it turns out that the infinitesimal action of $\hat{\mathcal{L}}$ on $\mathcal{M}_g^{(\infty)}$ lifts to the one on $\tilde{\pi}^*L$ with central charge 1, by the following diagram (cf. [KNTY, KSU1]):

$$\begin{array}{ccccccc} & & & & \hat{\mathcal{F}}_0\backslash\{0\} & \xrightarrow{\sim} & \widehat{\mathcal{H}_{T,0}}\backslash\{0\} \\ & & & & \downarrow & & \downarrow \\ \mathcal{M}_g^{(\infty)} & \xrightarrow{Kr} & \mathrm{Grass}(\mathbf{C}((z))) & \xrightarrow{Pl} & \mathbf{P}(\hat{\mathcal{F}}_0) & \xrightarrow{B} & \mathbf{P}(\widehat{\mathcal{H}_{T,0}}) \end{array}$$

Kr is the morphism given by Krichever's construction and Pl denotes the Plücker embedding, $\hat{\mathcal{F}}_0$ is the completed fermion Fock space (of charge 0), and B is the bosonization isomorphism. $\tilde{\pi}^*L$ is the line bundle associated with the pull-back of the $\mathbf{C}^*$-bundle $\hat{\mathscr{F}}_0\backslash\{0\}\to\mathbf{P}(\hat{\mathscr{F}}_0)$.

Let us recall the structure of the group $D^{(0)}$, cf. [KNTY, Sect. 2]. The presentation

$$D^{(0)}(\mathbf{C})=\{z\mapsto uz+t_1z^2+t_2z^3+\dots\}$$

shows that $D^{(0)}$ has the coordinates u, t_i $(i\geqq 1)$. Therefore we have

$$\Gamma(D^{(0)},\mathcal{O}_{D^{(0)}})=\mathbf{C}[t_1,t_2,\dots][u,u^{-1}],$$

which is nothing but $\mathscr{H}_T\otimes\mathbf{C}$.

A final remark about this setting of abelian CFT is that everything goes well over the ring $\mathbf{Z}$ (or at least over $\mathbf{Z}[\frac{1}{2}]$), cf. [KSU1].

3 Identification of the complex cobordism ring with the boson Fock space

In the rest of this paper, Ch^* denotes the ring $Ch^*_{\mathbf{Q}}$, cf. (1.1).

3.1. Let us recall the natural pairing (,) on $\mathscr{H}_{T,0}$, [SN, KNTY]: for $P(t)$, $Q(t)\in\mathscr{H}_{T,0}$, put

$$(P(t),Q(t)):=P(\tilde{\partial})Q(t)|_{t=0}.$$

Here $\tilde{\partial}=\left(\frac{\partial}{\partial t_1},\frac{1}{2}\frac{\partial}{\partial t_2},\dots,\frac{1}{n}\frac{\partial}{\partial t_n},\dots\right)$.

The basis $\{\chi_\lambda\}$ consisting of the Schur polynomials is orthonormal with respect to this pairing: $(\chi_\lambda,\chi_\mu)=\delta_{\lambda,\mu}$.

Because of this pairing we put $\mathscr{H}^\dagger_{T,0}=\mathbf{Q}\left[\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_2},\dots\right]$ (the dual of $\mathscr{H}_{T,0}$). The pairing gives rise to the identification $\mathscr{H}_{T,0}\cong\mathscr{H}^\dagger_{T,0}$; $t_n\mapsto\frac{1}{n}\frac{\partial}{\partial t_n}$.

3.2. Next we recall the natural pairing ("Chern number") between Ch^* and $MU^*_{\mathbf{Q}}$: for $P(c)=P(c_1,c_2,\dots)\in Ch^*$ and a manifold M,

$$\begin{aligned}(P(c),[M]):&=P(c)[M]\\&=\text{degree}\dim M\text{ part of }P(c_1(M),\ c_2(M),\dots).\end{aligned}$$

As a basic example, we recall that the calculation of Chern numbers of $\mathbf{P}^n$. Working in the Chow ring $CH^*(\mathbf{P}^n)$ of $\mathbf{P}^n$ or in $MU^*(\mathbf{P}^n)$, the total Chern class is $c_z(\mathbf{P}^n)=(1+\xi z)^{n+1}$, where $\xi=\xi_n$ is the class of hyperplane in $\mathbf{P}^n$. Hence

$$c_i(\mathbf{P}^n)=\binom{n+1}{i}\xi^i\qquad(0\leqq i\leqq n)$$

and

$$c_{i_1}^{a_1}\dots c_{i_k}^{a_k}(\mathbf{P}^n)=\prod_{j=1}^{k}\binom{n+1}{i_j}^{a_j}$$

for $\sum_{j=1}^{k}a_ji_j=n$, especially $c_n(\mathbf{P}^n)=n+1$.

We are ready to state the first main result.

Theorem 3.3. *The ring homomorphisms*

$$K: MU^*_{\mathbf{Q}} \to \mathscr{H}_{T,0} : \mathbf{P}^n \mapsto p_n((n+1)t)$$
$$K^\dagger : Ch^* \to \mathscr{H}_{T,0} : c_i \mapsto (-1)^i p_i(-t) = q_i(t)$$

are isomorphisms which preserve the pairings in (3.1). *Here p_n is the n-th elementary Schur polynomial* (2.2) *and $kt=(kt_1, kt_2, \ldots)$ for $k=n+1$ or -1.*

Remark 3.4. 1) The correspondence $K^\dagger$ can be written as

$$K^\dagger\left(1+\sum_{i\geqq 1} c_i z^i\right) = \exp\left(\sum_{i\geqq 1} (-1)^{i+1} t_i z^i\right).$$

2) The above ring isomorphisms can be naturally extended to the completions with respect to $\deg c_i = i$, $\deg t_i = i$, $\deg \mathbf{P}^n = n$. They will be denoted by the same symbols.

3) M. Furuta also gave a proof of Theorem (3.3).

3.5 Proof of Theorem (3.3). The bijectivity of K and $K^\dagger$ are clear.

We compare the values of the pairing at the basis elements. Put $u_n = nt_n$ for convenience. Let us calculate

$$((K^\dagger)^{-1}(u_{j_1}\ldots u_{j_r}), \mathbf{P}^{i_1}\times\ldots\times\mathbf{P}^{i_s})$$

and

$$(u_{j_1}\ldots u_{j_r}, K(\mathbf{P}^{i_1}\times\ldots\times\mathbf{P}^{i_s}))$$

with $\sum_{k=1}^{s} i_k = \sum_{\ell=1}^{r} j_\ell = n$.

For simplicity, we omit $(K^\dagger)^{-1}$ during the calculation of

$$((K^\dagger)^{-1}(u_{j_1}\ldots u_{j_r}), \mathbf{P}^{j_1}\times\ldots\times\mathbf{P}^{i_s}).$$

It is the degree n part of $\prod_{\ell=1}^{r} u_{j_\ell}(\mathbf{P}^{i_1}\times\ldots\times\mathbf{P}^{i_s})$ [in $CH^*(\mathbf{P}^{i_1}\times\ldots\times\mathbf{P}^{i_s})_{\mathbf{Q}}$ or in $MU^*(\mathbf{P}^{i_1}\times\ldots\times\mathbf{P}^{i_s})_{\mathbf{Q}}$].

By the example in (3.2) and Remark (3.4), 1), we have

$$\sum_{i\geqq 1} (-1)^{i+1}\frac{u_i(\mathbf{P}^n)}{i} z^i = \log(1+\xi_n z)^{n+1} = (n+1)\sum_{i\geqq 1}(-1)^{i+1}\frac{\xi_n^i}{i} z^i.$$

Hence we get $u_i(\mathbf{P}^n) = (n+1)\xi_n^i$. In a similar manner, from

$$c_z(\mathbf{P}^{i_1}\times\ldots\times\mathbf{P}^{i_s}) = \prod_{k=1}^{s} c_z(\mathbf{P}^{i_k}),$$

we obtain

$$u_j(\mathbf{P}^{i_1}\times\ldots\times\mathbf{P}^{i_s}) = \sum_{k=1}^{s} (i_k+1)\xi_{i_k}^j.$$

Consequently we conclude

$$\prod_{\ell=1}^{r} u_{j_\ell}(\mathbf{P}^{i_1}\times\ldots\times\mathbf{P}^{i_s}) = \prod_{\ell=1}^{r}\sum_{k=1}^{s}(i_k+1)\xi_{i_k}^{j_\ell}.$$

We look for its degree n part. Since $\xi^\ell = 0$ for $l > n$ and $\sum i_k = \sum j_\ell$, only the term $\xi_{i_1}^{i_1}\ldots\xi_{i_s}^{i_s}$ does not vanish in the expansion of the above product. Such a term exists only when $(j_1,\dots,j_r)$ refines $(i_1,\dots,i_s)$, i.e. there exists a partition $\{1,\dots,r\}=\coprod_{k=1}^{s} I_k$ s.t. $i_k=\sum_{j\in I_k} j$ for any k.

In such a case, the coefficient of $\xi_{i_1}^{i_1}\dots\xi_{i_s}^{i_s}$ is

$$\sum_{\{I_k\}}\prod_{k=1}^{s}(i_k+1)^{\# I_k}, \tag{1}$$

where the summation ranges over all the possible partitions as above. This is just the value of $((K^\dagger)^{-1}(u_{j_1}\dots u_{j_r}),\ \mathbf{P}^{i_1}\times\dots\times\mathbf{P}^{i_s})$.

Next we calculate $(u_{j_1}\dots u_{j_r},\ K(\mathbf{P}^{i_1}\times\dots\times\mathbf{P}^{i_s}))$, namely

$$\prod_{\ell=1}^{r}\frac{\partial}{\partial t_{j_\ell}}\cdot\prod_{k=1}^{s}p_{i_k}((i_k+1)t)\,|_{t=0}\,. \tag{2}$$

[Notice that u_j corresponds to $\frac{\partial}{\partial t_j}$ under the identification $\mathscr{H}_{T,0}\cong\mathscr{H}_{T,0}^\dagger$ in (3.1).] Since $p_j(t)$ is a homogeneous polynomial of degree j, putting $t=0$ means taking the constant term.

Note that

$$\frac{\partial}{\partial t_m}p_n(t)=p_{n-m}(t),$$

which easily follows from the definition

$$\sum_{n=0}^{\infty}p_n(t)z^n=\exp\left(\sum_{n\geqq 1}t_nz^n\right),$$

with $p_n=0$ for $n<0$.

A variant: $\frac{\partial}{\partial t_m}p_n(at)=ap_{n-m}(at)$.

By this fact combined with $\sum i_k=\sum j_\ell$, the term

$$\prod_{k=1}^{s}\left\{\left(\prod_{j\in I_k}\frac{\partial}{\partial t_j}\right)p_{i_k}((i_k+1)t)\right\}=\prod_{k=1}^{s}\{(i_k+1)^{\# I_k}\cdot p_0(t)\}$$

for each refinement $(j_1,\dots,j_r)$ of $(i_1,\dots,i_s)$ contributes to the constant term in the expression (2) (without $|_{t=0}$). Therefore, the value of (2) is just equal to the number (1).

3.6. Let us consider the integral structures on $\mathscr{H}_{T,0}$, $MU^*_{\mathbf{Q}}$ and Ch^* and their interrelation through $K, K^\dagger$. $\mathscr{H}_{T,0}$ has two different **Z**-structures $\mathscr{H}_0(\mathbf{Z})$, $\mathscr{H}_{T,0}(\mathbf{Z})$ cf. (2.2). $MU^*_{\mathbf{Q}}$ has a usual **Z**-structure MU^* and another one:

$$MU^*(\mathbf{Z})=\{z\in MU^*_{\mathbf{Q}};(y,x)\in\mathbf{Z}\ \text{for}\ \forall y\in Ch^*_{\mathbf{Z}}\},$$

meanwhile $Ch^*_{\mathbf{Z}}$ is a natural **Z**-structure in $Ch^*=Ch^*_{\mathbf{Q}}$, cf. (1.1).

Then we have the following:

Proposition 3.7. 1) $K^\dagger(Ch^*_{\mathbf{Z}})=\mathscr{H}_0(\mathbf{Z})$.
2) $K(MU^*(\mathbf{Z}))=\mathscr{H}_0(\mathbf{Z})$.

Proof. 1) follows easily from the fact that

$$\mathbf{Z}[q_1(t),q_2(t),\dots]=\mathbf{Z}[p_1(t),p_2(t),\dots]\,.$$

For 2), recall that the **Z**-dual of $\mathscr{H}_0(\mathbf{Z})$ with respect to the pairing on $\mathscr{H}_{T,0}$ is $\mathscr{H}_0(\mathbf{Z})$ itself. Then use 1) and Theorem (3.3).

3.8. We now state the second result. Let $T(c,z)$ be a multiplicative sequence with coefficients in $\mathbf{Q}$ (1.1), and put

$$T(c):=T(c,z)|_{z=1}\in\widehat{Ch^*}.$$

Here, like $\widehat{\mathscr{H}_{T,0}}$, $\widehat{Ch^*}$ denotes the completion of $\widehat{Ch^*}$ with respect to $\deg c_i = i$. Then we define

$$\tau_T(t):=K^\dagger(T(c))\in\widehat{\mathscr{H}_{T,0}}$$

and call it the *tau function* associated to T.

Then we have the following result.

Theorem 3.9. 1) *We have the following relation:*

$$\tau_T(t)=\exp\left(\sum_{i\geqq 1}(-1)^{i+1}b_i t_i\right),$$

where b_i is the coefficient of c_i in the polynomial $T_i(c_1,\dots,c_i)$.

Moreover, there is a one-to-one correspondence between the set of multiplicative sequences $T(c,z)$ and the set of sequences $(b_1,b_2,\dots)$ through the above relation.

2) *$\tau_T(t)$ or $T(c)$ has an expansion of the following form:*

$$T(c)=1+\sum_{n\geqq 1}\sum_{\lambda}\Delta_\lambda(a)\Delta_\lambda(c).$$

Here λ runs through all the partitions of n, $\Delta_\lambda(c)$ is the Schur polynomial (2.2), *and a_i's are the coefficients of the formal power series $\sum_{i\geqq 0}(-1)^i a_i z^i = 1/Q(z)$, where $Q(z)$ is the characteristic power series associated to T* (1.3).

Remark 3.10. Hirzebruch knew the formula (3.9), 2) long ago, which he mentioned in a letter to Todd in the case of Todd genus. It is this letter which motivated Theorem (3.9) and the interpretation as τ-function in Sect. 5. The rest of this section is devoted to the proof of this Theorem.

3.11 Proof of Theorem (3.9), 1). Note first that $K: Ch^* \cong \mathscr{H}_{T,0}$ induces an isomorphism of group scheme over $\mathbf{Q}$ by taking their Spec:

$$K^*:\mathbf{G}_a^\infty\to\Lambda.$$

Since a multiplicative sequence T corresponds to an endomorphism Φ_T of the group scheme Λ, we denote by $\tilde{\Phi}_T$ the induced endomorphism of $\mathbf{G}_a^\infty$.

Consider $\tilde{\Phi}_T:\mathbf{G}_a^\infty(\mathbf{Q})\to\mathbf{G}_a^\infty(\mathbf{Q})$. Since $\mathbf{G}_a^\infty(\mathbf{Q})=\mathbf{Q}^\infty$ is a $\mathbf{Q}$-vector space, the additivity of $\tilde{\Phi}_T$ implies its $\mathbf{Q}$-linearity. Thus $\tilde{\Phi}_T$ should be of the form $t_i\mapsto\sum_{j=1}^{\infty}\alpha_{ij}t_j\ (\forall i)$, $\alpha_{ij}\in\mathbf{Q}$.

On the other hand, K^* and Φ_T are degree-preserving with respect to the degrees on Ch^* and $\mathscr{H}_{T,0}$ ($\deg t_i=\deg c_i=i$). Therefore $\tilde{\Phi}_T$ is also degree-preserving. Finally these two facts imply that $\tilde{\Phi}_T$ is of the form $t_i\mapsto b_i t_i (\forall i)$ for some $b_i\in\mathbf{Q}$.

It remains to prove that these b_i's are (up to sign) the coefficients of $T_i(c_1,\dots,c_i)$. From the commutative diagram

$$\begin{array}{ccc}\mathbf{G}_a^\infty(\mathbf{Q}) & \xrightarrow{K^*} & \Lambda(\mathbf{Q})\\ {\scriptstyle\tilde{\Phi}_T}\downarrow & & {\scriptstyle\Phi_T}\downarrow\\ \mathbf{G}_a^\infty(\mathbf{Q}) & \xrightarrow{K^*} & \Lambda(\mathbf{Q})\end{array},$$

we obtain the relation:

$$\exp\left(\sum_{i\geqq 1}(-1)^{i+1}b_i t_i z^i\right)=1+T_1 z+T_2 z^2+\ldots.$$

Differentiate both sides with respect to t_n and set $(t_1, t_2, \ldots)=(0,0,\ldots)$. Then we get

$$(-1)^{n+1}b_n=\frac{\partial}{\partial t_n}T_n|_{t=0}.$$

Since T_n is homogeneous of degree n and $c_n=(-1)^n p_n(-1)\equiv(-1)^{n+1}t_n$ $(\operatorname{mod} t_1,\ldots,t_{n-1})$, the right hand-side is equal to $(-1)^{n+1}$ times the coefficient of c_n in T_n. This proves the assertion.

3.12 Proof of Theorem (3.9), 2). We have to relate the multiplicative sequence $T(c)$ with the Schur functions. For this purpose, we use the theory of Schur functions as developed in [Li, Chap. VI].

Introduce the virtual Chern roots $\gamma_1, \gamma_2, \ldots$ by

$$1+\sum_{i=1}^{N}c_i z^i=\prod_{i=1}^{N}(1+\gamma_i z)$$

for arbitrary N. Since the series $T(c)$ is obtained in the limit $N\to\infty$ from $\prod_{i=1}^{N}Q(\gamma_i)$, we want an expansion formula for it into Schur functions. But it is nothing but the formula [Li, p. 103, (V)]:

$$\prod_{i=1}^{N}Q(\gamma_i)=1+\sum_{\lambda}\Delta_\lambda(a)\Delta_\lambda(c)$$

where λ runs through all the partitions with $\lambda_i\leqq N$ for all i. (Note that $F(x_r)$, $\{\lambda\}$, $\{x;\lambda\}$ in [Li] correspond to $Q(\gamma_i)$, $\Delta_\lambda(a)=\{\alpha;\tilde\lambda\}$, $\Delta_\lambda(c)=\{\gamma;\lambda\}$ respectively, $\tilde\lambda$ being the conjugate of λ.)

3.13 An application to the calculation of genera. We have the following remark about the calculation of values of a multiplicative sequence. Let T be a multiplicative sequence and M a compact complex manifold. Then we have

$$T(c)[M]=p_M\left(b_1,-\frac{b_2}{2},\ldots,(-1)^{i+1}\frac{b_i}{i},\ldots\right).$$

Here $K^\dagger(T(c))=\tau_T(t)=\exp\left(\sum_{i\geqq 1}(-1)^i b_i t_i\right)$ and $K([M])=p_M(t)$. By Theorem (3.3), we get

$$T(c)[M]=\exp\left(\sum_{i\geqq 1}(-1)^{i+1}\frac{b_i}{i}\frac{\partial}{\partial t_i}\right)p_M(t)|_{t=0},$$

which is equal to the right hand-side of the above formula.

4 Comparison with homotopy-theoretic results

In this section, we show a connection of Theorem (3.3) with the result of Morava [La2, Appendix] and Bukhshtaber and Shokurov [BuSh], and give, as an application, a description of **Z**-structure of $\mathscr{H}_{T,0}$ corresponding to MU^* under the identification K.

4.1. First we review some fundamental facts on complex cobordism, cf. (1.2) and [A].

The complex cobordism ring MU^* is equipped with the one-dimensional formal group law F_{MU}:

$$F_{MU}(e_{MU}(L_1), e_{MU}(L_2)) = e_{MU}(L_1 \otimes L_2),$$

where e_{MU} is the Euler class for MU^*, (1.2).

Then it is known that MU^* with F_{MU} is isomorphic to Lazard's ring with the universal one-dimensional formal group law.

We are going to define a genus with values in $\mathscr{H}_{T,0}(\mathbf{Z})$ (1.1). Consider the formal power series

$$\theta(z) = \sum_{n \geqq 0} t_n z^{n+1} \qquad (t_0 = 1),$$

and put

$$\theta^{-1}(z) = \sum_{n \geqq 0} a_n(t) z^{n+1}.$$

Then the polynomial $a_n(t)$ is homogeneous of total degree n with respect to $\deg t_i = i$ and is given by the formula [A, II.7.5]:

$$(*) \qquad (n+1)a_n(t) = \text{degree } n \text{ part of } b^{-n-1} \qquad (n \geqq 1)$$

$$b := \sum_{i \geqq 0} t_i.$$

We define the genus $\beta: MU^* \to \mathscr{H}_{T,0}(\mathbf{Z})$ to be associated with the one-dimensional formal group law:

$$F(z_1, z_2) = \theta(\theta^{-1}(z_1) + \theta^{-1}(z_2))$$

by the universality of (MU^*, F_{MU}). Thus $\theta^{-1}(z)$ is the logarithm of F.

Note that β can be identified with the Hirewicz homomorphism $\pi_*(MU) \to H_*(MU)$ for the ring spectrum MU as is shown in [A, II].

Next we recall the relation of $\mathscr{H}_{T,0}(\mathbf{Z})$ with the Landweber-Novikov algebra S (1.2). S has a (graded) Hopf algebra structure with the coproduct:

$$\Delta s_\omega = \sum_{\omega_1 + \omega_2 = \omega} s_{\omega_1} \otimes s_{\omega_2}.$$

Then its dual Hopf algebra S_* can be identified with $\mathscr{H}_{T,0}(\mathbf{Z})$, the dual basis of $\{s_\omega\}$ being the monomials $\{t^\omega\}$. For the coproduct of S_*, cf. [A, I.6.5].

Finally we recall the formula [A, I.8.1] showing how the operation s_α acts on the cobordism class $[\mathbf{P}^n]$:

$$(\#) \qquad s_\alpha([\mathbf{P}^n]) = (s_\alpha, b^{-n-1})[\mathbf{P}^{n-\|\alpha\|}], \qquad (n \geqq 0).$$

where we put $b = \sum_{n \geqq 1} t_i$, $\|\alpha\| = \sum_{i \geqq 1} i\alpha_i$ for $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $(,)$ is the evaluation pairing $S \times S_* \to \mathbf{Z}$.

4.2. The work of Morava [La2, Appendix] and Bukhshtaber and Shokurov [BuSh] gives an interpretation of the Landweber-Novikov algebra S and the cobordism ring MU^* by the automorphism group of the formal line and its coordinate ring. We recall it briefly in the following.

First introduce the group subscheme $D^{(1)}$ of $D^{(0)}$ (2.3). For a commutative ring R, its R-valued points are

$$D^{(1)}(R)=\{z\mapsto \phi_t(z)=z+t_1z^2+t_2z^3+\dots\}\subset D^{(0)}(R)=\mathrm{Aut}_R(R[[z]]).$$

Then the coordinate ring of $D^{(1)}$ is just equal to the boson Fock space:

$$\Gamma(D^{(1)},\mathcal{O}_{D^{(1)}})=\mathscr{H}_{T,0}(\mathbf{Z}).$$

Therefore $\mathscr{H}_{T,0}(\mathbf{Z})$ has a structure of Hopf algebra.

Consider the following operation D_α on $\mathscr{H}_{T,0}(\mathbf{Z})$ for $\alpha=(\alpha_1,\alpha_2,\dots)$, $\alpha_i\in\mathbf{Z}_{\geq 0}$, $|\alpha|=\sum_{i\geqq 1}\alpha_i<\infty$:

$$P(u*v)=\sum_\alpha (D_\alpha P)(u)v^\alpha \quad \text{for} \quad P(t)\in\mathscr{H}_{T,0}(\mathbf{Z}),$$

when we put

$$\phi_{u*v}(z)=\phi_u(\phi_v(z)).$$

D_α can be easily expressed as a linear differential operator; e.g. for $e_n=(0,\dots,0,\overset{n}{1},0,\dots)$,

$$D_{e_n}=\frac{\partial}{\partial t_n}+\sum_{k=+1}^{\infty}(k+1-n)t_{k-n}\frac{\partial}{\partial t_k}.$$

The totality of linear combinations of D_α's is closed under the composition and is a subalgebra $\mathscr{S}$ of $\mathrm{End}_{\mathbf{Z}}(\mathscr{H}_{T,0}(\mathbf{Z}))$.

In order to relate S and $\mathscr{S}$, let us define a map

$$\beta_{BS}:MU^*\to S_*\cong\mathscr{H}_{T,0}(\mathbf{Z})$$

by the formula: $(\beta_{BS}(m),s)=\mu(s(m))$ for $m\in MU^*$, $s\in S$, where μ is the augmentation $\mu:MU^*\to\mathbf{Z}$. β_{BS} is a ring homomorphism and extends to an isomorphism

$$\beta_{BS}:MU^*(\mathbf{Z})\xrightarrow{\sim}S_*.$$

See (3.6) for $MU^*(\mathbf{Z})$.

Let us observe the coincidence of β_{BS} and β in (4.1), since the authors could not find a suitable reference.

Lemma. $\beta_{BS}=\beta$.

Proof. Let us check $\beta_{BS}(\mathbf{P}^n)$ for $n\geqq 1$. By [A, II.9.1] and (*) in (4.1),

$$\beta(\mathbf{P}^n)=(n+1)a_n(t)=\text{degree } n \text{ part of } b^{-n-1}.$$

On the other hand, we have by (#) in (4.1)

$$\beta_{BS}(\mathbf{P}^n)=\sum_{\|\alpha\|=n}\mu(s_\alpha(\mathbf{P}^n))t^\alpha=\sum_{\|\alpha\|=n}(s_\alpha,b^{-n-1})t^\alpha$$
$$=\text{degree } n \text{ part of } b^{-n-1}.$$

We are already to state the main result of [BuSh, La2, Appendix].

Theorem. 1) *The map $S\to\mathscr{S}:a_\alpha\mapsto D_\alpha$ is an isomorphism of Hopf algebras.*

2) *β_{BS} is equivariant with respect to the isomorphism of* 1).

3) $\beta_{BS}(MU^*)=\{P\in\mathscr{H}_{T,0}(\mathbf{Z});\ell^*(P)\in\mathscr{H}_{T,0}(\mathbf{Z})\}$. *Here ℓ^* is the automorphism of $\mathscr{H}_{T,0}$ induced by the element $\ell(z)=1-e^{-z}\in D^{(1)}(\mathbf{Q})$.*

The above characterization of the image $\beta_{BS}(MU^*)$ relies on a theorem of Stong and Hattori, and is found only in [BuSh].

Finally we recall the relation of $\mathcal{S}$ with the Virasoro algebra $\mathcal{L}$.

The Lie algebra of the group $D^{(1)}$ as a **C**-group scheme is equal to the Lie subalgebra $\mathcal{L}(1)$ of $\mathcal{L}$ spanned by $L_n (n \geqq 1)$. In terms of S, the operations s_{e_n} satisfy the same commutation relation (up to sign) as L_n's:

$$[s_{e_m}, s_{e_n}] = (n-m) s_{e_{m+n}}.$$

4.3. Now we define an endomorphism of $\mathcal{H}_{T,0}$, which connects K (3.3) and β above.

Consider the following ring homomorphism:

$$\varphi : \mathcal{H}_{T,0} \to \mathcal{H}_{T,0} : t_i \mapsto p_i(-t).$$

It is immediate that φ is an automorphism. Then we have

Theorem 4.4. *The following diagram is commutative:*

$$\begin{array}{ccc} MU^* \otimes_{\mathbf{Z}} \mathbf{Q} & \xrightarrow{\beta} & \mathcal{H}_{T,0} \\ & \searrow & \downarrow \varphi \\ & K & \mathcal{H}_{T,0}. \end{array}$$

Proof. We compare the values $K(\mathbf{P}^n)$ and $\varphi(\beta(\mathbf{P}^n))$. First we remark that $K(\mathbf{P}^n) = p_n((n+1)t)$ is given by

$$p_n((n+1)t) = \frac{1}{2\pi i} \oint \frac{\exp\left(\sum_{i \geqq 1} (n+1) t_i z^i\right)}{z^{n+1}} dz.$$

Similarly for $\beta(\mathbf{P}^n)$:

$$\beta(\mathbf{P}^n) = (n+1) a_n(t) = \frac{n+1}{2\pi i} \oint \frac{\theta^{-1}(z)}{z^{n+2}} dz.$$

To calculate the effect of φ on $\beta(\mathbf{P}^n)$, consider the following:

$$w = \varphi(\theta(z)) = \sum_{i \geqq 0} p_i(-t) z^{i+1} = z \exp\left(-\sum_{i \geqq 1} t_i z^i\right).$$

Then we have

$$dw = \left(1 - \sum_{i \geqq 1} i t_i z^i\right) \exp\left(-\sum_{i \geqq 1} t_i z^i\right) dz.$$

Since $w = \varphi(\theta(z))$ is the inverse power series to $\varphi(\theta^{-1}(z))$, we get

$$\begin{aligned} \varphi(\beta(\mathbf{P}^n)) &= \frac{n+1}{2\pi i} \oint \theta^{-1}(w) \frac{dw}{w^{n+2}} \\ &= \frac{n+1}{2\pi i} \oint \left(1 - \sum_{i \geqq 1} i t_i z^i\right) \exp\left(\sum_{i \geqq 1} (n+1) t_i z^i\right) \frac{dz}{z^{n+1}}. \end{aligned}$$

Now observe the relation:

$$\begin{aligned} \sum_{i \geqq 1} i t_i z^i &= \frac{1}{n+1} z \frac{d}{dz} \left(\sum_{i \geqq 1} (n+1) t_i z^i\right) \\ &= \frac{1}{n+1} z \frac{d}{dz} \log\left(\sum_{j \geqq 1} p_j((n+1)t) z^j\right) \\ &= \frac{1}{n+1} \sum_{j \geqq 1} j p_j((n+1)t) z^j \exp\left(-\sum_{i \geqq 1} (n+1) t_i z^i\right). \end{aligned}$$

Hence we have

$$\varphi(\beta(\mathbf{P}^n)) = \frac{n+1}{2\pi i}\oint \exp\Big(\sum_{i\geqq 1}(n+1)t_i z^i\Big)\frac{dz}{z^{n+1}}$$
$$-\frac{1}{2\pi i}\oint \sum_{j\geqq 1} jp_j((n+1)t)z^j \frac{dz}{z^{n+1}}$$
$$=(n+1)p_n((n+1)t)-np_n((n+1)t)=p_n((n+1)t).$$

This completes the proof.

Remark 4.5. Unlike the map β, the map φ is not equivariant with respect to the action of $\mathscr{S}$. More precisely, the action of Virasoro generators $L_n (n\geqq 1)$ on $\mathscr{H}_{T,0}$ is given by

$$L_n = \sum_{j\geqq n}\left(j-\frac{1}{2}(n+1)\right)p_{j-n}\frac{\partial}{\partial p_j}$$

using $p_j(t)$'s as variables. Meanwhile, the action of L_n on $\mathscr{H}_{T,0}$ as the coordinate ring of $D^{(1)}$ is given by D_{e_n} (4.2).

4.6. We determine the image of MU^* by the map K (3.3) using the above Theorem (4.4), 3), cf. (3.7).

Let us calculate the effect of ℓ^* on the generators t_i's. By definition of ℓ^*, we have

$$\sum_{i\geqq 0}\ell^*(t_i)z^{i+1} = \ell\Big(\sum_{i\geqq 0} t_i z^{i+1}\Big) = 1-\exp\Big(-\sum_{i\geqq 0} t_i z^{i+1}\Big).$$

Therefore we get

$$\ell^*(t_i) = -p_{i+1}(-1,-t_1,\dots,-t_i) \qquad (i\geqq 1).$$

Then the image $K(MU^*)$ is described in the following

Theorem 4.7.
$$K(MU^*) = \mathbf{Z}[r_1(t), r_2(t),\dots]\cap\mathscr{H}_0(\mathbf{Z}).$$

Here $\mathscr{H}_0(\mathbf{Z})$ *denotes* $\mathbf{Z}[p_1(t),p_2(t),\dots]$ (2.2) *and* $r_i(t)$ *is the following polynomial:*

$$r_i(t) := \sum_{0\leqq j\leqq i}\frac{1}{j+1}p_{i-j}(-(j+1)t).$$

Proof. By Theorem (4.4), 3), we have

$$K(MU^*) = \varphi(\beta(MU^*)),$$

while we know, by Bukhshtaber-Shokurov's Theorem, 3) in (4.2), that

$$\beta(MU^*) = \mathscr{H}_{T,0}(\mathbf{Z})\cap(\ell^*)^{-1}(\mathscr{H}_{T,0}(\mathbf{Z})).$$

Thus we get

$$K(MU^*) = \varphi(\mathscr{H}_{T,0}(\mathbf{Z}))\cap\varphi(\ell^*)^{-1}(\mathscr{H}_{T,0}(\mathbf{Z})).$$

But we know that the polynomials $(-1)^i p_i(-1) = q_i(t)$ for all i generate the ring $\mathscr{H}_0(\mathbf{Z})$. Hence we have $\varphi(\mathscr{H}_{T,0}(\mathbf{Z})) = \mathscr{H}_0(\mathbf{Z})$.

So it remains to calculate $\varphi\cdot(\ell^*)^{-1}(t_i)$ $(i\geqq 1)$. Look at the maps:

$$\begin{array}{ccccc}\mathscr{H}_{T,0} & \overset{\ell^*}{\leftarrow} & \mathscr{H}_{T,0} & \overset{\varphi}{\rightarrow} & \mathscr{H}_{T,0}\\ \ell^*(t_i) & \leftarrow & t_i & \mapsto & \varphi(t_i)=p_i(-t)\end{array}.$$

To avoid confusion in the calculation below, we rename the generators in the left $\mathscr{H}_{T,0}$ as s_i's.

In order to calculate $\varphi \cdot (\ell^*)^{-1}$, we set the generating series for ℓ^* and φ equal:

$$\sum_{i \geqq 0} \ell^*(s_i) z^i = \sum_{i \geqq 0} \varphi(t_i) z^i .$$

But the left hand side is

$$\begin{aligned} - \sum_{i \geqq 0} p_{i+1}(-1, -s_1, \ldots, -s_i) z^i &= -z^{-1} \sum_{i \geqq 0} p_{i+1}(-1, -s_1, \ldots, -s_i) z^{i+1} \\ &= -z^{-1} \left(\exp\left(- \sum_{i \geqq 0} s_i z^{i+1} \right) - 1 \right), \end{aligned}$$

while the right hand side is

$$\sum_{i \geqq 0} p_i(-t) z^i = \exp\left(- \sum_{i \geqq 1} t_i z^i \right).$$

Let us solve the equality in s_i's. Then it will give the expression of $\varphi \cdot (\ell^*)^{-1}(t_i)$ in t_i's.

$$\begin{aligned} \sum_{i \geqq 0} s_i z^{i+1} &= -\log \left(1 - z \exp\left(- \sum_{i \geqq 1} t_i z^i \right) \right) \\ &= \sum_{n \geqq 1} \frac{z^n}{n} \exp\left(-n \sum_{i \geqq 1} t_i z^i \right) \\ &= \sum_{n \geqq 1} \frac{z^n}{n} \sum_{m \geqq 0} p_m(-nt) z^m \\ &= \sum_{i \geqq 0} \left\{ \sum_{0 \leqq j \leqq i} \frac{1}{j+1} p_{i-j}(-(j+1)t) \right\} z^{i+1} . \end{aligned}$$

This gives nothing but the definition of $r_i(t)$ in the statement of the Theorem.

It might be interesting to find the generators of the subring $K(MU^*)$ of $\mathscr{H}_{T,0}$, since it is isomorphic to Lazard's ring which is a polynomial ring.

5 Genera and τ-functions

In this section, we explain how the formal power series $\tau_T(t)$ can be interpreted as a τ-function in the theory of the KP hierarchy.

5.1. Here we point out that the expansion of $\tau_T(t)$ of $T(c)$ (3.9), 2) can be interpreted as the following determinant:

$$\det \left(\begin{pmatrix} \ddots & & & & & & & \\ & c_1 & c_2 & c_3 & & & & \\ 0 & 1 & c_1 & c_2 & c_3 & & & \\ & & 1 & c_1 & c_2 & c_3 & & \\ & & & 1 & c_1 & c_2 & c_3 & \end{pmatrix} \cdots \cdots \begin{pmatrix} \ddots & & 0 \\ 1 & & \\ a_1 & 1 & \\ a_2 & a_1 & 1 \\ a_3 & a_2 & a_1 \\ & a_3 & a_2 \\ & & a_3 \\ & \vdots & \end{pmatrix} \right).$$

det is a product of a $\mathbf{Z}_{<0}\times\mathbf{Z}$-matrix and a $\mathbf{Z}\times\mathbf{Z}_{<0}$-matrix, cf. [SN]. Then the above determinant is the limit $m,n\to\infty$ of its truncated version:

$$\det(D_{m,n}(c)^t D_{m,n}(a)).$$

Here $D_{m,n}(c)$ denotes the following $m\times(m+n)$-matrix:

$$m\begin{pmatrix} \ddots & c_1 & c_2 & c_3 & & & & \\ 0 & 1 & c_1 & c_2 & c_3 & & & \\ & & 1 & c_1 & c_2 & c_3 & & \cdots \\ & & & 1 & c_1 & c_2 & c_3 & \end{pmatrix}.$$
$$\qquad\qquad m \qquad\qquad\qquad n$$

Now apply the following formula: for a $p\times q$-matrix A and a $q\times p$-matrix $B\,(p\leqq q)$,

$$\det(AB)=\sum_I A_I B_I,$$

where A_I (resp. B_I) denotes the p-minor of A (resp. B) consisting of the columns (resp. rows) corresponding to $I\subset\{1,\ldots,q\}$ with $\#I=p$.

In our case, the m-minors appearing in the above expansion is of the form

$$\det\begin{pmatrix} c_{k_0} & c_{k_1} & \cdots & c_{k_{m-1}} \\ \vdots & \vdots & & \vdots \\ c_{k_0-m+1} & & \cdots & c_{k_{m-1}-m+1} \end{pmatrix}$$

for $0\leqq k_0<k_1<\ldots<k_{m-1}<m+n$. But this is equal to $\Delta_\lambda(c)$ for $\lambda=(\lambda_1,\ldots,\lambda_m)$, $\lambda_i=k_{m-i}-(m-i)$ and λ runs through all the partitions of depth $\leqq m$. Similarly for $\Delta_\lambda(a)$.

Therefore the expansion in question is equal to

$$\sum_\lambda \Delta_\lambda(a)\Delta_\lambda(c),$$

where λ runs through all the partitions of depth $\leqq m$ and of width $\leqq n$. Taking the limit $m,n\to\infty$, we find the expression of (3.9), 2).

5.2. Let us briefly recall τ-functions in the theory of the KP hierarchy. The proofs of the facts recalled below can be found in [S, SM, SN, Sh].

A series of non-linear equations such as the Kadomtsev-Petviashvili equation ($-4u_t+u_{xxx}+12uu_x=0$) form the so-called KP-hierarchy, which has the following form ("Sato equation"):

$$\partial_{t_n}W=B_nW-W\partial_x^n \qquad (n=1,2,3,\ldots)$$
$$B_n=(W\partial_x^nW^{-1})_+$$
$$W\in 1+\mathscr{E}_{\mathbf{C}}(-1)\otimes\mathbf{C}[[t_1,t_2,\ldots]].$$

Here $\mathscr{E}_{\mathbf{C}}(-1)$ denotes the totality of microdifferential operators of one variable x of order $\leqq -1$, and $(\,)_+$ means the differential operator part (i.e. the part not involving negative powers of ∂_x).

These are equations for the coefficients of W. We can consider a solution of the form $W=\sum_{j=0}^{\infty} w_j(x,t)\partial_x^{-j}$ ($w_0=1$), called the wave operator (for L). Note that

$$L=W\partial_xW^{-1}\in\partial_x+\mathscr{E}_{\mathbf{C}}(-1)\otimes\mathbf{C}[[t_1,t_2,\ldots]]$$

satisfies the equation of Lax type:

$$\partial_{t_n} L = [B_n, L], \quad B_n = (L^n)_+ \quad (n = 1, 2, 3, \ldots).$$

Note also that L is determined by W up to multiplication by an element of $1 + \partial_x^{-1}\mathbf{C}[[\partial_x^{-1}]]$.

We know by [DJKM] that for a solution W, there is an element $\tau(t) \in \widehat{\mathscr{H}_{T,0}}(\mathbf{C}) = \mathbf{C}[[t_1, t_2, \ldots]]$, called the τ-function associated to L, satisfying:

$$W = \tau(x+t)^{-1}\tau\left(x + t_1 - \partial_x^{-1}, t_2 - \frac{1}{2}\partial_x^{-2}, \ldots, t_n - \frac{1}{n}\partial_x^{-n}, \ldots\right)$$
$$= \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\tau(x+1)}{\tau(x+t)} \partial_x^{-n},$$
$$\tilde{\partial} = \left(\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \ldots, \frac{1}{n}\partial_{t_n}, \ldots\right), \quad x + t = (x + t_1, t_2, \ldots).$$

The condition for $\tau(t) \in \widehat{\mathscr{H}_{T,0}}(\mathbf{C})$ to be a τ-function for some L is equivalent to the so-called Hirota's bilinear equations, or the Plücker relations for the coefficients $\{\xi_\lambda\}$ of the expansion of $\tau(t) = \sum_\lambda \xi_\lambda \chi_\lambda(t)$.

The totality of τ-functions has a structure of $\mathbf{C}^*$-bundle over the Grassmannian $Grass(\mathbf{C}((z)))$ (of charge 0), [S, DJKM]. A τ-function can be represented by a frame ξ (a $\mathbf{Z} \times \mathbf{Z}_{<0}$-matrix) for the corresponding subspace of $\mathbf{C}((z))$, [KNTY, Sects. 1, 4]. It is given as the determinant $\tau(\xi, t) = \det({}^t\xi_0 \cdot \xi(t))$ in the sense explained in (5.1), where ξ_0 is the reference frame $(\delta_{i+1,j})_{i\in\mathbf{Z}, j\in\mathbf{Z}_{<0}}$ and $\xi(t) = \exp\left(\sum_{n\geq 1} t_n \Lambda^n\right) \cdot \xi$ is the time-evolution of ξ, $\Lambda = (\delta_{i+1,j})_{i,j\in\mathbf{Z}}$.

Thus the correspondence $\xi \mapsto \tau(\xi, t)$ induces

$$\{\text{frames } \xi\}/SL_{\mathbf{Z}_{<0}}(\mathbf{C}) \xrightarrow{\sim} \{\tau\text{-functions of the form } \tau(\xi, t)\},$$

which is the $\mathbf{C}^*$-bundle over $Grass(\mathbf{C}((z)))$.

Now we are ready to state our theorem.

Theorem 5.3. 1) *If we substitute* $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, \ldots)$ *with* $\tilde{t}_i = (-1)^{i+1} t_i$ *into* $\tau_T(t)$, *then* $\tau(t) = \tau_T(\tilde{t})$ *is the* τ*-function corresponding to the frame* $\xi = {}^tD(a)$ *where* a *is given in Theorem 3.9, 2).*

2) *The Lax operator associated with* $\tau(t) = \tau_T(\tilde{t})$ *is* ∂_x.

3) *There is a one-to-one correspondence between the set of multiplicative sequences over* $\mathbf{Q}$ *(resp.* $\mathbf{C}$*) and the subset* $1 + \partial_x^{-1}\mathbf{Q}[[\partial_x^{-1}]]$ *(resp.* $1 + \partial_x^{-1}\mathbf{C}[[\partial_x^{-1}]]$*) of all the wave operators.*

Remark 5.4. The above theorem says that the time-evolution of W corresponding to the frame ${}^tD(a)$ does not move the Lax operator L. Let us consider a mapping

$$W \mapsto L = W\partial_x W^{-1}$$

from the set of wave operators to the set of Lax operators. Then the fiber over ∂_x of this mapping is $1 + \partial_x^{-1}\mathbf{C}[[\partial_x^{-1}]]$.

5.5 Proof of Theorem (5.3). First we remark the following identity:

$$
{}^t\xi_0 \cdot \exp\left(\sum_{n\geqq 1} t_n \Lambda^n\right) = {}^t\xi_0 \sum_{n=0}^{\infty} p_n(t)\Lambda^n
= \begin{pmatrix} \ddots & & & & & & \\ & p_1 & p_2 & p_3 & & & \\ 0 & 1 & p_1 & p_2 & p_3 & & \cdots \\ & & 1 & p_1 & p_2 & p_3 & \\ & & & 1 & p_1 & p_2 & p_3 \end{pmatrix} =: D(p).
$$

Therefore we have $\tau(\xi, t) = \det(D(p)\cdot\xi)$. Remember that we have $K^\dagger(c_n) = (-1)^n p_n(-t)$, (3.3). If we replace $t = (\ldots, t_n, \ldots)$ by $\tilde{t} = (\ldots, (-1)^{n+1}t_n, \ldots)$, we have

$$
\begin{aligned}
{}^t\xi_0 \cdot \exp\left(\sum_{n\geqq 1} (-1)^{n+1} t_n \Lambda^n\right) &= {}^t\xi_0 \cdot \exp\left(\sum_{n\geqq 1} (-t_n)(-\Lambda)^n\right) \\
&= {}^t\xi_0 \sum_{n=0}^{\infty} p_n(-1)(-\Lambda)^n \\
&= {}^t\xi_0 \sum_{n=0}^{\infty} (-1)^n p_n(-t)(\Lambda)^n \\
&= \sum_{n=0}^{\infty} K^\dagger(c_n)\Lambda^n = K^\dagger(D(c)).
\end{aligned}
$$

The result in (5.1) says that $\tau_T(t) = \det(D(c)\cdot {}^tD(a))$. Therefore, by the above calculation, we have

$$\tau_T(t) = \det(\xi_0 {}^tD(a)(\tilde{t})).$$

Hence

$$\tau_T(\tilde{t}) = \det(\xi_0 {}^tD(a)(t)).$$

This proves the first assertion.

Let us calculate the wave operator for $\tau_T(t)$. By (3.9), 1),

$$
\begin{aligned}
W &= \exp\left(-\sum_{i\geqq 1} (-1)^{i+1} b_i (-1)^{i+1} t_i\right) \exp(-b_1 x) \\
&\quad \times \exp(b_1 x) \exp\left(\sum_{i\geqq 1} (-1)^{i+1} b_i (-1)^{i+1}\left(t_i - \frac{1}{i}\,\partial_x^{-i}\right)\right) \\
&= \exp\left(\sum_{i\geqq 1} -\frac{b_i}{i}\,\partial_x^{-i}\right) = 1 + \sum_{i=1}^{\infty} p_i\left(-b_1, -\frac{b_2}{2}, \ldots, -\frac{b_i}{i}\right)\partial_x^{-i}.
\end{aligned}
$$

Here $p_i\left(-b_1, -\frac{b_2}{2}, \ldots, -\frac{b_i}{i}\right)$ means the value of p_i at $t = \left(-b_1, -\frac{b_2}{2}, \ldots\right)$. This W is clearly of constant coefficients. Thus we get

$$L = W\partial_x W^{-1} = \partial_x.$$

This proves the second assertion.

For the third assertion, recall that a multiplicative sequence is determined by the corresponding sequence $(b_1, b_2, \ldots)$ in (3.9), 1). It means that τ-functions of the form

$$\exp\left(\sum_{i\geqq 1} (-1)^{i+1} b_i t_i\right)$$

bijectively correspond to multiplicative sequences.

Thus it remains to show that τ-functions of the above form up to scalar multiple are in one-to-one correspondence with wave operators with constant coefficients. By 2), it suffices to prove that

$$W=1+\sum_{n\geqq 1} d_n\partial_x^{-n} \quad (d_n\in\mathbf{Q})$$

comes from a τ-function of the above form.

Let us solve the following equation for $\tau=\tau(t)$:

$$W=1+\sum_{n\geqq 1} d_n\partial_x^{-n}=\sum_{n\geqq 0}\frac{p_n(-\tilde{\partial})\tau}{\tau}\partial_x^{-n}.$$

Hence, we have

$$p_n(-\tilde{\partial})\tau=d_n\tau \quad (n\geqq 1).$$

Note that

$$p_n(-\tilde{\partial})=-\frac{1}{n}\partial_{t_n}+(\text{terms in }\partial_{t_1},\ldots,\partial_{t_{n-1}}).$$

In particular, we have $p_1(-\tilde{\partial})=-\partial_{t_1}$. Then, by induction on n, we conclude that $\tau(t)$ has the form

$$\tau(t)=\exp\left(\sum_{i=0}^{n}(-1)^{i+1}b_it_i\right)\cdot\tilde{\tau}(t_{n+1},t_{n+2},\ldots)$$

for some $b_i\in\mathbf{Q}$. This proves the third assertion and completes the proof of Theorem (5.3).

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