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# On differential equations in normal form

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## 0 Introduction

Since their introduction by Poincaré and Dulac, normal forms have been used to investigate non-elementary stationary points of ordinary differential equations, in particular their stability; see Arnold and Anosov [2], Bruno [3], Guckenheimer and Holmes [10], and Takens [15], among others. Although it is known that – due to possible divergence of the normalizing transformation, cf. Bruno [4, 5] – general differential equations may have a more complicated behaviour near a stationary point than differential equations in normal form do, this class certainly deserves to be considered in its own right. While properties of normal forms have been used in many particular cases and there is much work on Hamiltonian systems in normal form (for instance Cushman and Rod [7], van der Meer [16], and Kummer [12]), there seems to be little work – with the exception of Bruno [4, 5] – on general properties of differential equations in normal form and their reduction to systems of lower dimension. The purpose of this article is to present a general theory of this class of differential equations.

By $\mathbb{K}$ we denote the field of real or complex numbers and by $V$ a finite dimensional vector space over $\mathbb{K}$. Let $N$ be some open neighbourhood of 0 in $V$ and $f: N \to V$ analytic with $f(0)=0$. Thus, the series expansion of $f$ about 0 is of the type $f(x)=Bx+\sum_{j\geq 2} f_j(x)$, with $B: V\to V$ linear and the $f_j: V\to V$ homogeneous of degree $j$. Furthermore, let $B=B_s+B_n$ be the decomposition of $B$ into its semisimple and nilpotent parts. The differential equation

$$\dot{x}=f(x)$$

is in *normal form* about 0 if $[B_s, f]=0$ (where $[\cdot,\cdot]$ denotes the usual Lie bracket); this is equivalent to $[B_s, f_j]=0$ for all $j\geq 2$ in view of $[B_s, B_n]=0$. More precisely, this is the normal form with respect to the semisimple part considered by Bruno [3–5]. It is called "preliminary normal form" by Arnold and Anosov [2]. The further normalization in case $B_n \neq 0$ which was extensively studied in recent years will not be investigated here. (On this topic cf., for instance, Arnold and Anosov [2, Part I, Chap. 3, Sects. 3 and 7], the historical notes in van der Meer [16, pp. 42–45] and Cushman and Sanders [8].)

In the following (Proposition 1.9) it will be shown that there is a decomposition of $V$ as a direct sum of two $B_s$-invariant subspaces $U$ and $W$ (one of which may be zero; $W$ is the largest $B_s$-invariant subspace on which every polynomial first integral of $\dot{x}=B_s x$ is constant), which yields a decomposition of the differential equation: For $x=u+w\in U\oplus W$, $\dot{x}=f(x)$ splits into

$$\dot{u}=f(u),$$
$$\dot{w}=g(u,w);$$

more precisely, $U$ is $f$-invariant and $g(u,w):=f(u+w)-f(u)\in W$ for all $u$ and $w$.

If a solution of $\dot{u}=f(u)$ is known, then the remaining equation on $W$ can be reduced to linear differential equations and is therefore, in a sense, harmless.

If $B_s|_U\neq 0$, then there is a solution-preserving map from $\dot{u}=f(u)$ to some differential equation on an algebraic variety of smaller dimension as will be shown in Sect. 3. This follows essentially from the existence of a nontrivial symmetry group for $\dot{u}=f(u)$. If a solution of the reduced equation is known, then solutions of $\dot{u}=f(u)$ can be found by integration alone. More important, many special properties of $\dot{u}=f(u)$ are preserved by the reduction.

## 1 Preliminary results and a decomposition

Given a fixed basis (and system of coordinates) on $V$, the notion of a polynomial map from $V$ to $V$ (or $\mathbb{K}$) with respect to these coordinates is unambiguous.

By $\mathscr{Pol}(V)$ we denote the Lie algebra of all polynomial maps from $V$ to $V$; this is a $\mathbb{Z}$-graded Lie algebra: $\mathscr{Pol}(V)=\bigoplus_{j\in\mathbb{Z}}\mathscr{P}_j$, with $\mathscr{P}_j:=\{p\in\mathscr{Pol}(V): p$ homogeneous of degree $j+1\}$ for $j\geq -1$ and $\mathscr{P}_j:=\{0\}$ for $j<-1$. By $\mathscr{A}(V)$ we denote the Lie algebra of all power series $\sum_{j\geq 0} f_j$, with $f_j\in\mathscr{P}_{j-1}$, which converge in some neighbourhood of 0; we identify $\mathscr{A}(V)$ with the (germs of) analytic functions about 0. Furthermore, the (associative and naturally graded) algebra of all polynomial maps from $V$ to $\mathbb{K}$ is called $S(V)$ and the algebra of all power series $\varphi=\sum_{j\geq 0}\varphi_j$ [$\varphi_j\in S(V)$, homogeneous of degree $j$] which converge near 0 is called $A(V)$; again, the latter is identified with the algebra of functions analytic in 0.

For $\varphi\in S(V)$ [or $A(V)$] and $f\in\mathscr{Pol}(V)$ [or $\mathscr{A}(V)$], let $L_f(\varphi)(x):=D\varphi(x)\cdot f(x)$. Then $L_f$ is a derivation of $S(V)$ [or $(A(V)$] – this is the usual relation between vector-valued functions and vector fields – and the identity $L_{[f,g]}=L_fL_g-L_gL_f$ holds. $L_f$ is called the Lie derivative of $\varphi$ with respect to $f$.

We call $\varphi$ a *first integral* of $\dot{x}=f(x)$ (or, briefly, of $f$) if $L_f(\varphi)=0$; thus all level sets of $\varphi$ are invariant sets of $\dot{x}=f(x)$. In particular, let $I(B_s)$ be the set of all polynomial first integrals of $B_s$ and $\bar{I}(B_s)$ the set of all analytic first integrals near 0 of $B_s$. These are subalgebras of $S(V)$, $A(V)$, respectively, and furthermore $\varphi=\sum_{j\geq 0}\varphi_j\in\bar{I}(B_s)$ if and only if every $\varphi_j\in I(B_s)$.

Finally, let $\mathscr{C}(B_s)=\{f\in\mathscr{Pol}(V): [B_s,f]=0\}$ and $\bar{\mathscr{C}}(B_s)=\{f\in\mathscr{A}(V): [B_s,f]=0\}$. These are Lie subalgebras of $\mathscr{Pol}(V)$ resp. $\mathscr{A}(V)$, and $f=\sum_{j\geq 0} f_j\in\bar{\mathscr{C}}(B_s)$ if and only if every $f_j\in\mathscr{C}(B_s)$.

Let $g$ be analytic in a neighbourhood $\tilde{N}$ of 0 in a finite dimensional $\mathbb{K}$-vector space $\tilde{V}$. Let $H: \tilde{N} \to N$ analytic such that $H(0)=0$. We call $H$ a *solution-preserving* map from $\dot{x}=g(x)$ to $\dot{x}=f(x)$ if $H$ maps parametrized solutions of $\dot{x}=g(x)$ to parametrized solutions of $\dot{x}=f(x)$. Thus, whenever $z(t)$ is a solution of $\dot{x}=g(x)$ in $\tilde{N}$, then $H(z(t))$ is a solution of $\dot{x}=f(x)$. The following is well-known:

**(1.1) Lemma.** *The analytic map $H$ is solution-preserving from $\dot{x}=g(x)$ to $\dot{x}=f(x)$ if and only if $DH(x)\cdot g(x)=f(H(x))$ for all $x\in\tilde{N}$.*

Suppose that $f(x)=Bx+\sum_{j\geqq 2} f_j(x)$ is the Taylor series of $f$ about 0. If $\tilde{V}=V$ and $H$ is invertible, then $g(x)=\tilde{B}x+\sum_{j\geqq 2} g_j(x)$, thus $g(0)=0$ and $B$ and $\tilde{B}$ are conjugate; we will assume that they are equal in the following.

Note that for every homogeneous polynomial $p: V\to V$ of degree $m>1$ there is a unique multilinear and symmetric $\hat{p}: V^m \to V$ such that $\hat{p}(x,\ldots,x)=p(x)$ for all $x$. We will no longer distinguish $p$ and $\hat{p}$ typographically from now on. In particular, $Dp(x)\cdot q(x)=mp(x,\ldots,x,q(x))$ for all $q\in\mathscr{P}o\ell(V)$.

Now suppose that $H=\sum_{j\geqq 1} h_j$ is invertible near 0 (equivalently, $h_1$ is invertible). Using Lemma 1.1 and comparing homogeneous terms of degree $m\geqq 2$ shows, after a few modifications,

**(1.2) Lemma.** *The local analytic diffeomorphism $H$ is solution-preserving from $\dot{x}=g(x)$ to $\dot{x}=f(x)$ if and only if $[B,h_1]=0$ and*

$$[B,h_m](x)=\sum_{j=2}^{m-1}\sum_{\substack{\ell_1,\ldots,\ell_j>0\\ \ell_1+\ldots+\ell_j=m}} f_j(h_{\ell_1}(x),\ldots,h_{\ell_j}(x)) - \sum_{i=2}^{m-1} ih_i(x,\ldots,x,g_{m+1-i}(x))+f_m(x)-g_m(x)$$

*holds for all $m\geqq 2$.*

Next we shall investigate the structure of $\mathscr{C}(B_s)$ and $I(B_s)$. Let

$\lambda_1,\ldots,\lambda_n$ be the eigenvalues of $B_s$ (counted with multiplicity);

$e_1,\ldots,e_n$ a corresponding eigenbasis and

$x_1,\ldots,x_n$ the coordinates with respect to this basis.

(This may require complexification of a real vector space but the results will also be valid for $\mathbb{K}=\mathbb{R}$, as will be indicated separately when it is not trivial.)

For $m\geqq 1$ let us investigate the action of $\operatorname{ad}B=[B,\cdot]$ on $\mathscr{P}_{m-1}$.

For the "monomial"

$$p(x):=x_1^{m_1}\ldots x_n^{m_n}e_j$$

(with $m_1,\ldots,m_n\geqq 0$, $m_1+\ldots+m_n=m$, and $1\leqq j\leqq n$) one computes $[B_s,p]=(m_1\lambda_1+\ldots+m_n\lambda_n-\lambda_j)p$.

Thus, every monomial is an eigenvector of $\operatorname{ad}B_s$ on $\mathscr{P}_{m-1}$. As these monomials form a basis of $\mathscr{P}_{m-1}$, we have proven the first part of the following proposition, the remaining assertions are standard (see Humphreys [11]):

**(1.3) Proposition.** (a) $\operatorname{ad}B_s|_{\mathscr{P}_{m-1}}$ *is semisimple.*

(b) $\operatorname{ad}B_n|_{\mathscr{P}_{m-1}}$ *is nilpotent.*

(c) *The decomposition of* $\operatorname{ad} B|_{\mathscr{P}_{m-1}}$ *into semisimple and nilpotent part is given by* $\operatorname{ad} B_s|_{\mathscr{P}_{m-1}} + \operatorname{ad} B_n|_{\mathscr{P}_{m-1}}$.

In particular, $p \in \mathscr{C}(B_s)$ if and only if $m_1\lambda_1 + \ldots + m_n\lambda_n - \lambda_j = 0$ (in the notation above), and, by linear algebra, every homogeneous $q \in \mathscr{C}(B_s)$ of degree $m$ is a linear combination of monomials in $\mathscr{C}(B_s)$.

In an analogous manner, one can characterize the action of $L_B$ on $S_j \subset S(V)$ (subspace of homogeneous polynomials of degree $j$):

For $\varphi(x) := x_1^{j_1} \ldots x_n^{j_n} (j_1, \ldots, j_n \geqq 0, \quad j_1 + \ldots + j_n = j)$ one has $L_{B_s}(\varphi) = (j_1\lambda_1 + \ldots + j_n\lambda_n)\varphi$, and thus

**(1.4) Proposition.** (a) $L_{B_s}|_{S_j}$ *is semisimple.*

(b) $L_{B_n}|_{S_j}$ *is nilpotent.*

(c) *The decomposition of* $L_B \backslash_{S_j}$ *into semisimple and nilpotent part is given by* $L_{B_s}|_{S_j} + L_{B_n}|_{S_j}$.

Again, with the notation above, $\varphi \in I(B_s)$ if and only if $m_1\lambda_1 + \ldots + m_n\lambda_n = 0$, and every homogeneous $\psi \in I(B_s)$ is a linear combination of monomials in $I(B_s)$.

For $p, q_1, \ldots, q_r \in \mathscr{P}ol(V)$, $p$ homogeneous of degree $r$, $q_i$ homogeneous of degree $s_i$ suppose $[B, p] = [B, q_i] = 0$ for all $i$, hence

$$s_i q_i(x, \ldots, x, Bx) = Bq_i(x) \qquad (1 \leqq i \leqq r)$$

and

$$rp(x, \ldots, x, Bx) = Bp(x);$$

equivalently

$$\sum_{i=1}^{r} p(y_1, \ldots, By_i, \ldots, y_r) = Bp(y_1, \ldots, y_r) \quad \text{for all} \quad y_1, \ldots, y_r \in V.$$

Define $f(x) := p(q_1(x), \ldots, q_r(x))$. Then $[B, f] = 0$. Indeed,

$$\begin{aligned}[B, f](x) &= \sum_{i=1}^{r} p(q_1(x), \ldots, s_i q_i(x, \ldots, x, Bx), \ldots, q_r(x)) - Bp(q_1(x), \ldots, q_r(x)) \\ &= \sum_{i=1}^{r} p(q_1(x), \ldots, Bq_i(x), \ldots, q_r(x)) - Bp(q_1(x), \ldots, q_r(x)) = 0.\end{aligned}$$

This observation (with $B_s$ instead of $B$) is essential in the proof of

**(1.5) Proposition** (Bruno [4, Theorem 2, p. 155]). *If* $\dot{x} = g(x)$ *and* $\dot{x} = f(x)$ *are both in normal form and H is solution-preserving from* $\dot{x} = g(x)$ *to* $\dot{x} = f(x)$, *then* $[B_s, h_m] = 0$ *for all* $m \geqq 1$.

*Proof.* For $m = 1$, this follows directly from (1.2). Proceed with induction: If $[B_s, h_j] = 0$ for all $j < m$, then what was said above shows that the right-hand side of the equation in (1.2) is contained in $\mathscr{C}(B_s)$. Thus $[B_s, [B, h_m]] = 0$, which implies $[B_s, h_m] = 0$ by the Proposition 1.3, since $\operatorname{ad} B$ stabilizes the eigenspaces of $\operatorname{ad} B_s$ and is invertible on the eigenspaces for nonzero eigenvalues. □

The proof given here is different from Bruno's and less complicated.

The following proposition is known; the proof of the first part (generalizing an idea of E. Noether) is due to Weitzenböck [18]:

**(1.6) Proposition.** (a) $I(B_s)$ *is a finitely generated* $\mathbb{K}$*-algebra.*

(b) $\mathscr{C}(B_s)$ *is a finitely generated* $I(B_s)$*-module.*

*Proof.* It is sufficient to prove both parts for $\mathbb{K}=\mathbb{C}$.

(a) Let $J$ be the ideal of $S(V)$ which is generated by the homogeneous elements of positive degree in $I(B_s)$. As $S(V)$ is noetherian, there are homogeneous $\varphi_1,\ldots,\varphi_r\in I(B_s)$ of positive degree which generate $J$, and we contend that $I(B_s)=\mathbb{C}[\varphi_1,\ldots,\varphi_r]$. Indeed, let $\varphi\in I(B_s)$ be homogeneous of degree $m$. If $m=0$, then $\varphi\in\mathbb{C}[\varphi_1,\ldots,\varphi_r]$ trivially, and by induction we may assume that the assertion is true for all degrees $<m$. There are homogeneous (!) $\sigma_1,\ldots,\sigma_r\in S(V)$ such that $\varphi=\sigma_1\varphi_1+\ldots+\sigma_r\varphi_r$. Using (1.5), $\sigma_i=\sum_\beta\sigma_{i,\beta}$ with $L_B(\sigma_{i,\beta})=\beta\sigma_{i,\beta}$ for all $i$, and $L_B(\varphi)=L_B(\varphi_i)=0$ $(1\leqq i\leqq r)$ shows $\varphi=\sigma_{1,0}\varphi_1+\ldots+\sigma_{r,0}\varphi_r$. Thus $\sigma_{i,0}\in I(B_s)$ and $\sigma_{i,0}\in\mathbb{C}[\varphi_1,\ldots,\varphi_r]$ by induction hypothesis.

(b) Fix $\ell\in\{1,\ldots,n\}$ and consider all the monomials $x_1^{m_1}\ldots x_n^{m_n}e_\ell\in\mathscr{C}(B_s)$; equivalently, all the $n$-tuples $(m_1,\ldots,m_n)$ of nonnegative integers such that (*) $m_1\lambda_1+\ldots+m_n\lambda_n-\lambda_\ell=0$. We show that there are finitely many among these monomials such that every monomial of the type above is a product of a first integral and one of the finitely many monomials. If there are only finitely many $n$-tuples of integers satisfying (*), this is obvious. If not, let $\{(m_{k1},\ldots,m_{kn})\colon k\in\mathbb{N}\}$ be the set of $n$-tuples of integers satisfying (*). There is a $r\in\mathbb{N}$ such that for every $k>r$ there is a $j\leqq r$ satisfying $m_{ji}\leqq m_{ki}$ for $1\leqq i\leqq n$. For otherwise, to every $r\in\mathbb{N}$ there would be a $k>r$ such that $m_{ki}<m_{ji}$ for all $j\leqq r$ and some $i\in\{1,\ldots,n\}$ depending on $j$. Then there are sequences $(r_1,r_2,\ldots)$ and $(k_1,k_2,\ldots)$ such that $r_{n+1}\geqq k_n$ for all $n$, and (with $j=k_n$) there is an $i\in\{1,\ldots,n\}$ satisfying $m_{k_{n+1},i}<m_{k_n,i}$. Taking subsequences, if necessary, we may assume that $i$ is constant for all $n$. But then $(m_{k_1,i},m_{k_2,i},\ldots)$ is a strictly destrictly decreasing sequence of nonnegative integers, a contradiction. Therefore, for $k>r$ we have

$$x_1^{m_{k1}}\ldots x_n^{m_{kn}}e_\ell=(x_1^{(m_{k1}-m_{j1})}\ldots x_n^{(m_{kn}-m_{jn})})x_1^{m_{j1}}\ldots x_n^{m_{jn}}e_\ell,$$

and the first factor on the right-hand side is in $I(B_s)$. □

**(1.7) Corollary.** *$\mathscr{C}(B_s)$ is infinite dimensional if and only if $I(B_s)\neq\mathbb{K}$.*

*Proof.* If $\gamma\in I(B_s)$ is not constant, then $\{\gamma^jB_s\colon j\in\mathbb{N}\}$ is an infinite $\mathbb{K}$-linearly independent system in $\mathscr{C}(B_s)$.

The reverse direction follows from (1.6). □

The proof of (1.6a) shows in particular that there are monomials $\varphi_i(x)=x_1^{m_{i1}}\ldots x_n^{m_{in}}$ $(1\leqq i\leqq r)$ in the eigencoordinates which generate $I(B_s)$. We may assume that $\{(m_{11},\ldots,m_{1n}),\ldots,(m_{s1},\ldots,m_{sn})\}$ is a maximal $\mathbb{Q}$-linearly independent system in $\{(m_{11},\ldots,m_{1n}),\ldots,(m_{ri},\ldots,m_{rn})\}$. Then $\varphi_1,\ldots,\varphi_s$ are algebraically independent [consider the contribution to a monomial of highest degree in $x_1,\ldots,x_n$ in a hypothetical relation $\gamma(\varphi_1,\ldots,\varphi_s)=0$, $\gamma\neq0$], and $\varphi_1,\ldots,\varphi_s$, $\varphi_j$ are algebraically dependent for all $j>s$, as some power of $\varphi_j$ is a product of powers of $\varphi_1,\ldots,\varphi_s$ by linear algebra.

Thus, we have a system of generators $\varphi_1,\ldots,\varphi_r$ of $I(B_s)$ and $\varphi_1,\ldots,\varphi_s$ is a maximal algebraically independent subsystem. We shall use this system of generators later on; for $\mathbb{K}=\mathbb{R}$ it is easy to obtain a real system of generators from it.

After we are done with the preliminaries, we turn to properties of differential equations in normal form. From now on suppose that

$$\dot{x}=f(x)=Bx+\sum_{j\geqq2}f_j(x)$$

is in normal form.

**(1.8) Proposition.** *If $\varphi \in A(V)$ is a first integral of $\dot{x}=f(x)$, then $\varphi \in \bar{I}(B_s)$.*

*Proof.* We may assume that $\varphi$ is nonconstant and $\varphi(0)=0$, thus $\varphi = \sum_{j\geq 0} \varphi_{r+j}$ and $r \geq 1$, $\varphi_r \neq 0$.

Decomposing $L_f(\varphi)=0$ into its homogeneous parts, we obtain $L_B(\varphi_r)=0$ and

$$(*) \qquad L_B(\varphi_{r+j})+L_{f_2}(\varphi_{r+j-1})+ \dots +L_{f_{j+1}}(\varphi_r)=0 \quad \text{for all} \quad j\geq 1 .$$

From (1.5) we find $L_{B_s}(\varphi_r)=0$. Assume by induction that $L_{B_s}(\varphi_{r+k})=0$ for all $k<j$. Then applying $L_{B_s}$ to the equation (*) gives

$$0=L_{B_s}L_B(\varphi_{r+j})+L_{B_s}L_{f_2}(\varphi_{r+j-1})+ \dots +L_{B_s}L_{f_{j+1}}(\varphi_r).$$

From $L_{B_s}L_{f_k}=L_{f_k}L_{B_s}$ for all $k\geq 2$ and the hypothesis we find that every term on the right-hand side, except the first, is zero, and thus $L_{B_s}L_B(\varphi_{r+j})=0$. Since $L_B$ is invertible on the $L_{B_s}$-eigenspaces for nonzero eigenvalues, we find $L_{B_s}(\varphi_{r+j})=0$. □

Now let $W$ be the maximal $B_s$-invariant subspace of $V$ on which every $\varphi \in I(B_s)$ is constant, and $U$ the $B_s$-invariant complementary subspace of $W$ in $V$. Using the eigenbasis, we get $U=\langle\{e_k\colon \text{there is a } \varphi=x_1^{m_1}\dots x_n^{m_n} \in I(B_s) \text{ with } m_k>0\}\rangle$ and $W$ is spanned by the remaining $e_j$. Note that, for every eigenvalue, the whole eigenspace is contained either in $U$ or in $W$, and in case $\mathbb{K}=\mathbb{R}$ the eigenspaces for complex conjugate eigenvalues are simultaneously contained either in $U$ or in $W$. This follows from $x_1^{m_1}\dots x_n^{m_n} \in I(B_s) \Leftrightarrow m_1\lambda_1+\dots+m_n\lambda_n=0$. Thus, we have the decomposition $V=U\oplus W$ (possibly with $U=\{0\}$ or $W=\{0\}$) in any case. For the sake of convenience we will assume $U=\langle e_1,\dots,e_r\rangle$ and $W=\langle e_{r+1},\dots,e_n\rangle$ for some $r\in\{0,\dots,n\}$.

From the definition of $U$ we see that there are positive integers $s_1,\dots,s_r$ such that (*) $s_1\lambda_1+\dots+s_r\lambda_r=0$ and thus $x_1^{s_1}\dots x_r^{s_r}\in I(B_s)$.

The importance of this decomposition for differential equations in normal form is illustrated by

**(1.9) Theorem.** *For every $y\in V$, $u\in U$ and $w\in W$, $f(u)\in U$ and $f(y+w)-f(y)\in W$. Thus, one has a decomposition of $\dot{x}=f(x)$ for $x=u+w\in U\oplus W$:*

$$\dot{u}=f(u),$$
$$\dot{w}=g(u,w),$$

*with $g(u,w)=f(u+w)-f(u)$.*

*Proof.* By construction, $U$ and $W$ are $B$-invariant, and it is sufficient to show the assertion for every homogeneous $p\in\mathscr{C}(B_s)$ of degree $m\geq 2$ and even for the monomials $p(x)=x_1^{m_1}\dots x_n^{m_n}e_j$ with $m_1,\dots,m_n\geq 0$, $m_1+\dots+m_n=m$ and $m_1\lambda_1+\dots+m_n\lambda_n-\lambda_j=0$. Now $x\in U \Leftrightarrow x_{r+1}=\dots=x_n=0$ and to show $p(U)\subset U$ it is sufficient to prove that $m_1\lambda_1+\dots+m_r\lambda_r-\lambda_j=0$ (with $m_1+\dots+m_r=m$) implies $j\leq r$. There is a $k\in\mathbb{N}$ such that $k\cdot s_i-m_i\geq 0$ for $1\leq i\leq r$ [with $s_1,\dots,s_r$ as in (*) above]. Using $\lambda_1 s_1+\dots+\lambda_r s_r=0$ and $m_1\lambda_1+\dots+m_r\lambda_r-\lambda_j=0$, we have $\sum_{i=1}^{r}(k\cdot s_i-m_i)\lambda_i+\lambda_j=0$. Therefore, $x_1^{ks_1-m_1}\dots x_1^{ks_r-m_r}\cdot x_j\in I(B_s)$. Since the exponent of $x_j$ is positive, $e_j\in U$ and $j\leq r$ by definition of $U$.

Furthermore, if $\sum_{i=1}^{n} m_i\lambda_i-\lambda_j=0$ and $m_k>0$ for some $k>r$, then $j>r$ because $j\leqq r$ would imply $\sum_{i=1}^{r}(m_i+s_i-\delta_{ij})\lambda_i+\sum_{i=r+1}^{n} m_i\lambda_i=0$, and, therefore, $e_k\in U$, a contradiction. This implies $Dp(x)w\in W$ for all $x\in V$, $w\in W$, and $p(x+w)-p(x)\in W$ follows. □

This decomposition can (in disguise) also be found in Bruno [4, Theorem 4, p. 163], but Bruno's reasoning deals exclusively with the question how the eigenvalues of $B_s$ are situated in the complex plane, and is more complicated. The characterizations of $U$ and $W$ given above and the proof of (1.9) are new.

If a solution of $\dot{u}=f(u)$ is known, then solutions of $\dot{x}=f(x)$ can be found from $\dot{w}=g(u,w)$. Thus, the investigation of differential equations in normal form is split into two tasks, considering the equations on $U$ and $W$. We will first take a closer look at $W$ in the following section and return to $U$ later.

## 2 The equation on $W$

In this section, $W\neq\{0\}$. Recall that for $U\neq\{0\}$ there are positive integers $s_1,\ldots,s_r$ such that $s_1\lambda_1+\ldots+s_r\lambda_r=0$.

**(2.1) Lemma.** *Let $j>r\geqq 1$ and $m_{r+1},\ldots,m_n\in\mathbb{N}_0$.*

(a) *There are $m_1,\ldots,m_r\in\mathbb{N}_0$ such that $p(x):=x_1^{m_1}\ldots x_n^{m_n}e_j\in\mathscr{C}(B_s)$ if and only if $\sum_{i=r+1}^{n} m_i\lambda_i-\lambda_j\in\mathbb{Z}\lambda_1+\ldots+\mathbb{Z}\lambda_r$.*

(b) *$\sum_{i=r+1}^{n} m_i\lambda_i\in\mathbb{Z}\lambda_1+\ldots+\mathbb{Z}\lambda_r$ if and only if $m_{r+1}=\ldots=m_n=0$.*

*Proof.* (a) There are $\ell_1,\ldots,\ell_r\in\mathbb{Z}$ such that $\sum_{i=r+1}^{n} m_i\lambda_i-\lambda_j=\ell_1\lambda_1+\ldots+\ell_r\lambda_r$. Choose $k\in\mathbb{N}$ such that $m_i:=k\cdot s_i-\ell_i\geqq 0$ for $1\leqq i\leqq r$, then $\sum_{i=1}^{n} m_i\lambda_i-\lambda_j=0$, which shows the "if" part. The other direction is trivial.

(b) In the same manner it follows from $\sum_{i=r+1}^{n} m_i\lambda_i\in\mathbb{Z}\lambda_1+\ldots+\mathbb{Z}\lambda_r$ that there are $m_1,\ldots,m_r\in\mathbb{N}_0$ such that $\sum_{i=1}^{n} m_i\lambda_i=0$. Hence $\varphi(x):=x_1^{m_1}\ldots x_n^{m_n}\in I(B_s)$ and $m_{r+1}=\ldots=m_n=0$ from the definition of $W$. □

If $r=0$, i.e. $U=\{0\}$, Lemma 2.1 trivially holds with $\mathbb{Z}\lambda_1+\ldots+\mathbb{Z}\lambda_r$ replaced by $\{0\}$.

Denote by $\bar{z}$ the image of $z\in(\mathbb{C},+)$ in the factor group $\mathbb{C}/(\mathbb{Z}\lambda_1+\ldots+\mathbb{Z}\lambda_r)$, if $r\geqq 1$, and let $\bar{z}=z$ in case $r=0$.

**(2.2) Lemma.** (a) *For every $j>r$ there are at most finitely many $(m_{r+1},\ldots,m_n)\in\mathbb{N}_0^{n-r}$ such that $\sum_{i=r+1}^{n} m_i\bar{\lambda}_i-\bar{\lambda}_j=\bar{0}$.*

(b) *Let $j,\ell>r$ and furthermore $m_{r+1},\ldots,m_n\geqq 0$ with $m_\ell>0$ and $k_{r+1},\ldots,k_n\geqq 0$ with $k_j>0$ such that $\sum_{i=r+1}^{n} m_i\bar{\lambda}_i-\bar{\lambda}_j=\bar{0}$ and $\sum_{i=r+1} k_i\bar{\lambda}_i-\bar{\lambda}_\ell=\bar{0}$.*

*Then $m_\ell=k_j=1$, $m_i=0$ for $i\neq\ell$, $k_i=0$ for $i\neq j$ and $\bar{\lambda}_\ell=\bar{\lambda}_j$.*

*Proof.* (a) follows from (2.1) and the proof of (1.6b).

(b) Adding both relations yields $\sum_{i=r+1}^{n}(m_i+k_i-\delta_{ij}-\delta_{i\ell})\bar{\lambda}_i=\bar{0}$ and the assertion follows from (2.1b). □

In particular, (2.2) implies that there are relations $\sum_{i=r+1}^{n} m_i\bar{\lambda}_i=\bar{\lambda}_j$ of *maximal length*; i.e., with $\sum_{i=r+1}^{n} m_i$ maximal.

After changing indices, we may assume that precisely $\bar{\lambda}_{s+1}, \ldots, \bar{\lambda}_n$ (for some $s\geqq r$) appear on the right-hand sides of relations of maximal length. Obviously, $W_1:=\langle e_{s+1}, \ldots, e_n\rangle$ is $B$-invariant, and one has a corresponding subspace in case $\mathbb{K}=\mathbb{R}$.

The implications for the differential equation $\dot{x}=f(x)$ in normal form are shown by

**(2.3) Lemma.** *For every $y\in V$ and $w_1\in W_1$ we have $f(y+w_1)-f(y)\in W_1$. Furthermore, $D^2f(y)(w,w_1)=0$ for every $y\in V$, $w\in W$, and $w_1\in W_1$.*

*Proof.* It is sufficient to prove the assertions for $p(x)=x_1^{m_1}\ldots x_n^{m_n}e_j\in\mathscr{C}(B_s)$ and $\sum_{i=1}^{n} m_i\geqq 2$. Thus $\sum_{i=1}^{n} m_i\lambda_i-\lambda_j=0$. For the first assertion it is sufficient to show: If $m_\ell>0$ for some $\ell>s$, then $j>s$; this implies $Dp(y)w_1\in W_1$ for all $y, w_1$.

We have $\sum_{i=r+1}^{n} m_i\bar{\lambda}_i-\bar{\lambda}_j=\bar{0}$ and $j>r$ from (2.1), and there is a relation $\sum_{i=r+1}^{n} k_i\bar{\lambda}_i=\bar{\lambda}_\ell$ of maximal length. Substitution yields

$$\sum_{\substack{i=r+1\\ i\neq\ell}}^{n} m_i\bar{\lambda}_i+\sum_{i=r+1}^{n} m_\ell k_i\bar{\lambda}_i=\bar{\lambda}_j.$$

The length of this relation does not exceed $\sum_{i=r+1}^{n} k_i$, hence $m_\ell=1$, $m_i=0$ for $i\neq\ell$ and $\bar{\lambda}_j=\bar{\lambda}_\ell$, showing $j>s$.

If the second assertion were not true, there would be $\ell_1>r$, $\ell_2>s$ and $m_{\ell_1}, m_{\ell_2}\in\mathbb{N}$ such that $\sum_{i=r+1}^{n} m_i\bar{\lambda}_i=\bar{\lambda}_j$ for some $j$. Substituting $\bar{\lambda}_{\ell_2}$ from a relation $\sum_{i=r+1}^{n} k_i\bar{\lambda}_i=\bar{\lambda}_{\ell_2}$ yields a relation of length greater than $\sum_{i=r+1}^{n} k_j$; a contradiction. □

Let $W_0$ be a $B$-invariant complement of $W_1$ in $W$; actually $W_0=\langle e_{r+1}, \ldots, e_s\rangle$ is uniquely determined. We can refine the decomposition of the differential equation given in (1.9): We have $\dot{w}=g(u,w)$, and for $w=w_0+w_1\in W_0\oplus W_1$ we have $g(u,w)=g(u,w_0)+(g(u,w_0+w_1)-g(u,w_0))$, the latter term being in $W_1$ from Lemma 2.3.

Thus we have

$$\dot{w}_0=h_0(u,w_0),$$

$$\dot{w}_1=h_1(u,w_0,w_1),$$

and the second part of (2.2) shows that

$$h_1(u,w_0,w_1)=C_1(u)w_1+\tilde{h}_1(u,w_0)$$

with $C_1(u)\in\mathrm{Hom}(W_1,W_1)$; furthermore, $\tilde{h}_1(u,w_0)$ contains neither constant nor linear terms in $w_0$, for else $\bar{\lambda}_\ell=\bar{\lambda}_n$ for some $\ell\leqq s$.

By Lemma 2.3, the passage from $V$ to $V/W_1$ yields a solution-preserving map from $\dot{x}=f(x)$ into the (well-defined!) differential equation $(\overline{x+W_1})^{\cdot}=f(x+W_1)$; therefore, we can repeat the argument on $V/W_1$ and use induction to obtain

**(2.4) Theorem.** *There are subspaces $W_1, \ldots, W_k$ of $W$ such that $W=W_1\oplus\ldots\oplus W_k$ and $\dot{w}=g(u,w)$ is given by $\dot{w}_i=C_i(u)w_i+h_i(u,w_{i+1}+\ldots+w_k)$ $(1\leqq i\leqq k)$ for*

$w = w_1 + \ldots + w_k \in W_1 \oplus \ldots \oplus W_k$; *where* $C_i(u) \in \mathrm{Hom}(W_i, W_i)$ *and* $h_i$ *contains neither constant nor linear terms in* $w_{i+1} + \ldots + w_k$.

This theorem was proven – with a different method – by Bruno [4, Theorem 6, p. 172]; the access chosen here seems new and less technically involved. A consequence is that in $W$ there is "only" a sequence of linear equations to solve, starting with $\dot{w}_k = C_k(u)w_k$. In the special case $V = W$, the equation is autonomous. Furthermore, we have

**(2.5) Corollary.** *If* $V = W$ *then* $\dot{x} = f(x)$ *can be solved with elementary functions.*

*Proof.* In this case, $C_j = \mu_j \mathrm{Id} + N_j$, with $N_j$ nilpotent, and the terms in $h_j$ are linear combinations of monomials $x_1^{m_1} \ldots x_n^{m_n} e_\ell$ with $m_1\lambda_1 + \ldots + m_n\lambda_n = \mu_j$. Therefore, the equation in $W_k$ is $\dot{x} = \mu_k x + N_k x$, which can be solved by elementary functions, and substituting for $h_j$ from the known solutions in $W_k, \ldots, W_{j+1}$, we get in $W_j$: $\dot{x} = \mu_j x + N_j x + c(t)e^{\mu_j t}$ with $c(t)$ a polynomial in $t$. It is simple to verify that this equation has general solution $d_j(t)e^{\mu_j t}$ with a polynomial $d_j$. □

It may be worthwhile to illustrate the background of (2.4) and (2.5) and exhibit the basic property of the differential equation on $W$.

Let $\tilde{\mathscr{L}} \subset \mathscr{P}o\ell(W)$ be spanned by those monomials $x_{r+1}^{m_{r+1}} \ldots x_n^{m_n} e_j$ $(j > r)$ such that $m_{r+1}\lambda_{r+1} + \ldots + m_n\lambda_n - \lambda_j \in \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_r$; in view of (2.1) these are precisely the monomials $p$ for which there is a $\varphi \in S(V)$ such that $\varphi \cdot p \in \mathscr{C}(B_s)$. Because $\mathscr{C}(B_s)$ is a subalgebra of $\mathscr{P}o\ell(V)$, $\tilde{\mathscr{L}}$ is a subalgebra of $\mathscr{P}o\ell(W)$. By virtue of (2.2), $\tilde{\mathscr{L}}$ is a finite dimensional Lie algebra of vector fields [which inherits the natural grading of $\mathscr{P}o\ell(W)$], and the reasoning leading to (2.4) shows that the subspaces $Z_0 = \{0\}$, $Z_i = W_1 \oplus \ldots \oplus W_i$ $(1 \leq i \leq k)$ satisfy $[p, z_i] \in Z_{i-1}$ for all $p \in \tilde{\mathscr{L}}$, $z_i \in Z_i$ and $1 \leq i \leq k$, furthermore $[\mathrm{ad}\, w_1(\ldots(\mathrm{ad}\, w_\ell)\ldots)p, z_i] \in Z_{i-\ell-1}$ (or $Z_0$ for $\ell > i-1$) for all $w_1, \ldots, w_\ell \in W$. Since $[p, Z_i] \subset Z_j$ and $[q, Z_i] \subset Z_j$ also imply $[[p,q], Z_i] \subset Z_j$ (as is seen by using the Jacobi identity in $W/Z_i$), we obtain

**(2.6) Theorem.** $\tilde{\mathscr{L}}$ *and* $W$ *generate a finite dimensional transitive (i.e. containing all constant maps) and graded subalgebra of* $\mathscr{P}o\ell(W)$.

These algebras were investigated in detail in [17], in particular, (2.4) is a consequence of the reducibility of these algebras.

## 3 Reduction of the equation on $U$

In the following we will discuss the restriction of the differential equation in normal form $\dot{x} = f(x)$ to the subspace $U$; we may – and will – assume that $U = V$ from now on. Thus we have positive integers $s_1, \ldots, s_n$ such that $\psi(x) := x_1^{s_1} \ldots x_n^{s_n} \in I(B_s)$, and $I(B_s)$ is generated by monomials $\varphi_1, \ldots, \varphi_r$ such that $\varphi_1, \ldots, \varphi_s$ form a maximal algebraically independent subset; cf. the remarks following (1.7). (In case $\mathbb{K} = \mathbb{R}$; $\psi$ and the $\varphi_j$ can obviously be chosen real-valued.)

Let $C_j(x) := x_j e_j$ for $1 \leq j \leq n$.

**(3.1) Proposition.** *For* $m \geq 1$ *and every homogeneous* $p \in \mathscr{C}(B_s)$ *there are* $\gamma_1, \ldots, \gamma_n$ *and* $\psi_1, \ldots, \psi_n \in I(B_s)$ *such that* $p(x) = \sum_{j=1}^{n} \frac{\gamma_j(x)}{\psi_j(x)} C_j(x)$ *and* $\psi_1, \ldots, \psi_n$ *divide* $\psi$, *and* $\frac{\gamma_j}{\psi_j} \cdot x_j$ *is a polynomial for* $1 \leq j \leq n$.

*Proof*. It is sufficient to consider $p(x)=x_1^{m_1}\dots x_n^{m_n}e_j$ with $\sum_{i=1}^n m_i\lambda_i-\lambda_j=0$. Therefore $\left(\text{with } \sum_{i=1}^n s_i\lambda_i=0\right)$ $\sum_{i=1}^n (m_i+s_i-\delta_{ij})\lambda_i=0$. This relation has nonnegative coefficients; therefore $\gamma(x):=\prod_{i=1}^n x_i^{(m_i+s_i-\delta_{ij})}\in I(B_s)$ and $p(x)=\frac{\gamma(x)}{\psi(x)}C_j(x)$, which shows the assertion. □

Note that an analogous statement for $\mathbb{K}=\mathbb{R}$ can be obtained by separating the real and the imaginary parts of $C_j$.

A direct consequence of (3.1) is that the differential equation $\dot{x}=f(x)$ is of the type

$$\dot{x}_i=x_i\left(\lambda_i+\sum_{m\geq 1}\frac{\gamma_{im}(x)}{\psi_{im}(x)}\right)\qquad(1\leq i\leq n)$$

with $\gamma_{im}, \psi_{im}\in I(B_s)$ and every $\psi_{im}$ dividing $\psi$.

Another consequence is

**(3.2) Theorem.** *Let $s$ be the maximal number of algebraically independent elements in $I(B_s)$. Then there is a $(n-s)$-dimensional abelian Lie subgroup $G$ of* $\mathrm{GL}(V)$ *such that every $T\in G$ is solution-preserving from $\dot{x}=f(x)$ into itself.*

*Proof*. The number $s$ is equal to the degree of transcendence of the quotient field of $I(B_s)$, hence uniquely determined and equal to the maximal number of algebraically independent monomials among the $\varphi_1,\dots,\varphi_r$ (so the labelling "$s$" is correct).

With $C_j(x):=x_je_j$ for $1\leq j\leq n$, let $\mathcal{M}:=\left\{D=\sum_{i=1}^n\beta_iC_i\colon \varphi_1,\dots,\varphi_s \text{ are first integrals of } D\right\}$. With the notation following (1.7) we see $\sum_{i=1}^n\beta_iC_i\in\mathcal{M}$ if and only if $\sum_{i=1}^n m_{ji}\beta_i=0$ for $1\leq j\leq s$, hence $\mathcal{M}$ is an abelian Lie algebra of dimension $n-s$. Furthermore, $D\in\mathcal{M} \Leftrightarrow \varphi_1,\dots,\varphi_r$ are first integrals of $D \Leftrightarrow$ every element of $I(B_s)$ is a first integral of $D$. From $[D,C_j]=0$ for all $D\in\mathcal{M}$ and all $j$ and $L_D(\varphi)=0$ for all $D\in\mathcal{M}$ and all $\varphi\in I(B_s)$ we get $[D,p]=0$ for all $D\in\mathcal{M}$ and all homogeneous $p\in\mathscr{C}(B_s)$ from (3.1). Therefore $[D,f]=0$, since $\dot{x}=f(x)$ is in normal form. Let $G$ be the (connected) Lie group with Lie algebra $\mathcal{M}$, then $Tf=fT$ for all $T\in G$, which shows the assertion. □

Note that the proof is constructive and can easily be modified for the case $\mathbb{K}=\mathbb{R}$.

Thus, if $B_s\neq 0$, there is a nontrivial symmetry group of the differential equation $\dot{x}=f(x)$. Of course, this is immediately clear from $[B_s,f]=0$, but (3.2) shows that the dimension may be larger. It is common knowledge that the existence of a nontrivial symmetry group should imply (locally) the reducibility of the differential equation; i.e., the existence of a solution-preserving map into some equation of smaller dimension $s$, cf. Olver [14]. However, it is usually assumed that all the $G$-orbits near the point of interest have the same dimension. This is not the case near the interesting point 0 and indeed, the quotient space of $V$ with respect to the action of $G$ may have quite a strange topological structure, for example, the induced topology on $V/G$ need not be Hausdorff. Therefore, direct employment of the quotient space does not seem appropriate. One of the results given by Bruno [4, Theorem 4, p. 163] shows that there is a rational solution-preserving map from

$\dot{x}=f(x)$ to some $s$-dimensional equation (whose right-hand side is a Laurent series); but this rational map does not seem very useful for qualitative analysis, as it is in general not defined in the interesting point 0.

We will show that there is a quite well-behaved reduction map, using the invariants of $G$ [i.e. $I(B_s)$] instead of $V/G$ itself. Note that these two methods are not identical if the invariants do not separate orbits, as is often the case with normal forms. For the Hamiltonian case the approach is essentially the same as the one described, in Arms et al. [19] and Cushman and Sjamaar [9], as will be shown in (3.13).

Let $\varphi_1, \ldots, \varphi_r$ be the homogeneous generators of $I(B_s)$ introduced before. For every homogeneous $\psi \in I(B_s)$ there is a $\sigma \in S(\mathbb{K}^r)$ such that $\psi = \sigma(\varphi_1, \ldots, \varphi_r)$. Consequently, for every $\psi \in \bar{I}(B_s)$ there is a formal power series

$$\gamma = \sum_{k \geqq 0} \sum_{k_1 + \ldots + k_r = k} \beta_{k_1, \ldots, k_r} y_1^{k_1} \ldots y_r^{k_r}$$

such that $\psi = \gamma(\varphi_1, \ldots, \varphi_r)$.

**(3.3) Lemma.** *For every $\psi \in \bar{I}(B_s)$ there is a $\gamma \in A(\mathbb{K}^r)$ such that $\psi = \gamma(\varphi_1, \ldots, \varphi_r)$.*

*Proof.* According to construction, $\varphi_1, \ldots, \varphi_r$ are normalized monomials in $x_1, \ldots, x_n$. Let $s_i := \deg \varphi_i$ for $1 \leqq i \leqq n$.

We have $\psi(x) = \sum_{j \geqq 0} \sum_{j_1 + \ldots + j_n = j} \alpha_{j_1, \ldots, j_n} x_1^{j_1} \ldots x_n^{j_n}$.

Since $\psi$ converges in a neighbourhood of 0, there are positive $M, \varrho$ such that $\sum_{j_1 + \ldots + j_n = j} |\alpha_{j_1, \ldots, j_n}| \leqq \frac{M}{\varrho^j}$ for all $j \in \mathbb{N}_0$.

If $\alpha_{j_1, \ldots, j_n} \neq 0$, then $j_1 \lambda_1 + \ldots + j_n \lambda_n = 0$, and there is a $r$-tuple $(\ell_1, \ldots, \ell_r)$ of nonnegative integers [depending on $(j_1, \ldots, j_n)$] such that $x_1^{j_1} \ldots x_n^{j_n} = \varphi_1(x)^{\ell_1} \ldots \varphi_r(x)^{\ell_r}$.

Let $\gamma(y) = \sum \alpha_{j_1, \ldots, j_n} y_1^{\ell_1} \ldots y_r^{\ell_r}$, where summation extends over all $(j_1, \ldots, j_n) \in \mathbb{N}_0^n$ with $j_1 \lambda_1 + \ldots + j_n \lambda_n = 0$ and $(\ell_1, \ldots, \ell_r)$ is chosen as above for every $(j_1, \ldots, j_n)$. In particular, $j_1 + \ldots + j_n = \ell_1 s_1 + \ldots + \ell_r s_r \leqq (\ell_1 + \ldots + \ell_r) \cdot S$, where $S := \max\{s_1, \ldots, s_r\}$. Therefore, contributions to the coefficient $\beta_{k_1, \ldots, k_r}$ of a given $y_1^{k_1} \ldots y_r^{k_r}$ can only be made by monomials in $x_1, \ldots, x_n$ of degree $\leqq k \cdot S$, with $k = k_1 + \ldots + k_r$. Hence

$$|\beta_{k_1, \ldots, k_r}| \leqq \sum_{j=0}^{k \cdot S} \sum_{j_1 + \ldots + j_n = j} |\alpha_{j_1, \ldots, j_n}| \leqq \sum_{j=0}^{k \cdot S} \frac{M}{\varrho^j} \leqq \max\left\{1, \frac{1}{\varrho^{k \cdot S}}\right\} \cdot (1 + k \cdot S) M .$$

A further estimate for every $k \in \mathbb{N}_0$ shows

$$\sum_{k_1 + \ldots + k_r = k} |\beta_{k_1, \ldots, k_r}| \leqq r^k \cdot \max\left\{1, \frac{1}{\varrho^{k \cdot S}}\right\} (1 + k \cdot S) M ,$$

and it follows that $\gamma(y_1, \ldots, y_r)$ converges for $|y_i| < \min\left\{\frac{1}{r}, \frac{\varrho^S}{r}\right\}$. □

Note that $\gamma$ is, in general, not uniquely determined by $\psi$, as there may be relations between $\varphi_1, \ldots, \varphi_r$, and therefore divergent series $\gamma$ which "represent" (for instance) $\psi = 0$. This explains why it is necessary to prescribe the construction of $\gamma$ in some way. The requirement that $\varphi_1, \ldots, \varphi_r$ be monomials was made to facilitate the proof of (3.3); we can drop it from now on but we still will require $\varphi_1, \ldots, \varphi_r$ to be homogeneous of positive degree and that $\varphi_1, \ldots, \varphi_s$ form a maximal algebraically independent subset.

Now consider the map $\Phi: V \to \mathbb{K}^r$, $\Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_r(x) \end{pmatrix}$.

From $L_{B_s} L_f = L_f L_{B_s}$ we get that $L_f(\varphi_i) \in \tilde{I}(B_s)$ for $1 \leqq i \leqq r$; according to (3.3) there are $\gamma_1, \ldots, \gamma_r \in A(\mathbb{K}^r)$ such that $L_f(\varphi_i)(x) = D\varphi_i(x) \cdot f(x) = \gamma_i(\varphi_i(x), \ldots, \varphi_r(x))$ for all sufficiently small $x \in V$. In other words:

**(3.4) Proposition.** *For every $f \in \mathscr{C}(B_s)$ there is a $g \in \mathscr{A}(\mathbb{K}^r)$ such that $D\Phi(x) \cdot f(x) = g(\Phi(x))$ for all sufficiently small $x \in V$. Thus $\Phi$ is solution-preserving from $\dot{x} = f(x)$ to $\dot{x} = g(x)$.*

So far, we have not necessarily reduced the dimension of the problem, as one can easily exhibit examples where even $r > n$. But the image of $\Phi$ has smaller dimension, as will be seen in Theorem 3.6.

**(3.5) Lemma.** *Let $\gamma \in A(\mathbb{K}^r)$ and $\varphi := \gamma \circ \Phi \in A(V)$.*
*Then $L_f(\varphi)(x) = L_g(\gamma)(\Phi(x))$ for all sufficiently small $x \in V$.*

*Proof.*

$$\begin{aligned} L_f(\varphi)(x) &= D\varphi(x) \cdot f(x) = D((\gamma \circ \Phi)(x)) \cdot f(x) \\ &= D\gamma(\Phi(x)) \cdot D\Phi(x) \cdot f(x) = D\gamma(\Phi(x)) \cdot g(\Phi(x)). \quad \square \end{aligned}$$

For the moment, let $\mathbb{K} = \mathbb{C}$, $J := \{\gamma \in S(\mathbb{K}^r)\colon \gamma(\varphi_1(x), \ldots, \varphi_r(x)) = 0$ for all $x \in V\}$ and $\tilde{J} := \{\gamma \in A(\mathbb{K}^r)\colon \gamma(\varphi_1(x), \ldots, \varphi_r(x)) = 0$ for all sufficiently small $x \in V\}$. Then $J$ is a prime ideal in $S(\mathbb{K}^r)$, because for $\gamma_1, \gamma_2 \in S(\mathbb{K}^r)$ and $\gamma_1 \cdot \gamma_2 \in J$ one has $\gamma_1(\Phi(x)) \cdot \gamma_2(\Phi(x)) = 0$ for all $x \in V$. Hence, $\gamma_1 \circ \Phi = 0$ or $\gamma_2 \circ \Phi = 0$, showing $\gamma_1 \in J$ or $\gamma_2 \in J$. By the same argument, $\tilde{J}$ is a prime ideal in $A(\mathbb{K}^r)$, $\tilde{J} = J \cdot A(\mathbb{K}^r)$ and $Y := \{y \in \mathbb{K}^r\colon \gamma(y) = 0$ for all $\gamma \in J\}$ is an irreducible algebraic variety. By construction, $\Phi(V) \subset Y$. Since $\dim Y = s$ (the maximum number of independent polynomial functions on $Y$) and the rank of $D\Phi$ is equal to $s$ at most points of $V$ (both facts are due to the algebraic independence of $\varphi_1, \ldots, \varphi_s$), $\Phi(V)$ contains a Zariski-open (and dense) subset of $Y$, which is also open and nonempty with respect to the norm topology on $Y$. An analytic map in a neighbourhood of 0 in $Y$, i.e. an element of $A(\mathbb{K}^r)/\tilde{J}$, is therefore uniquely determined by its behaviour on $\Phi(V)$.

Now let $\mathscr{M} := \{h \in \mathscr{A}(\mathbb{K}^r)\colon$ $Y$ is an invariant set for $\dot{x} = h(x)\}$ and $\mathscr{M}^* := \{h \in \mathscr{A}(\mathbb{K}^r)\colon h(y) = 0$ for all sufficiently small $y \in Y\}$. $\mathscr{M}$ can be characterized as follows: $h \in \mathscr{M} \Leftrightarrow L_h(\gamma) \in \tilde{J}$ for all $\gamma \in \tilde{J}$. For "$\Rightarrow$" note that, with a solution $z(t)$ of $\dot{x} = h(x)$ and $z(0) \in Y$, $z(t) \in Y$ for all $t$; hence $\frac{d}{dt}(\gamma(z(t))) = L_h(\gamma)(z(t)) = 0$ for all $t$, and use Hilbert's Nullstellensatz in $A(\mathbb{K}^r)$, cf. Kunz [13]. To prove the reverse direction, take generators $\varrho_1, \ldots, \varrho_k$ of $\tilde{J}$; then $\varrho_1(z(t)), \ldots, \varrho_k(z(t))$ satisfy a system of linear equations with initial value 0. It is easy to verify that $\mathscr{M}$ is a Lie subalgebra of $\mathscr{A}(\mathbb{K}^r)$ and $\mathscr{M}^*$ is an ideal of $\mathscr{M}$. (Hence $\mathscr{M}/\mathscr{M}^*$ may be considered as the set of analytic vector fields near 0 on $Y$.)

We now show $g \in \mathscr{M}$ [notation as in (3.4)]: Let $\gamma \in \tilde{J}$, hence $\varphi := \gamma \circ \Phi = 0$. From Lemma 3.5 we get $0 = L_f(\varphi)(x) = L_g(\gamma)(\Phi(x))$ for all sufficiently small $x \in V$. Since $\Phi(V)$ contains an open (-dense) subset of $Y$, we see $L_g(\gamma) \in \tilde{J}$ and $g \in \mathscr{M}$.

For $\mathbb{K} = \mathbb{R}$, the argument on the dimension of $\Phi(V)$ remains true, and this implies that all what was said above is also true for $\mathbb{K} = \mathbb{R}$ (cf. Kunz [13, Chap. I, Sect. 3, 7] for a real version of the Nullstellensatz).

We have proven

**(3.6) Theorem.** *$\Phi$ is a solution-preserving map from $\dot{x}=f(x)$ on $V$ to $\dot{y}=g(y)$ on the invariant set $Y\subset\mathbb{K}^r$, and $g+\mathcal{M}^*$ is uniquely determined.*

In the sense of this theorem, we have found a solution-preserving map from $\dot{x}=f(x)$ to a differential equation on the algebraic variety $Y$.

We call $\Phi: V\to Y$ a *reduction map*. Note that the construction of $\Phi$ is determined by properties of $B_s$ alone; hence $\Phi$ associates a $g\in\mathcal{M}/\mathcal{M}^*$ with every $f\in\mathscr{C}(B_s)$. The dimension of $Y$ is equal to $s$, as was to be expected from the dimension of the symmetry group in (3.2), and $\dot{x}=f(x)$ can always be reduced unless $B_s=0$.

It follows from (3.1) that – given a solution of $\dot{y}=g(y)$ on $Y$ – solutions of $\dot{x}=f(x)$ can be found by integration alone.

Let us take a look at some examples:

For $B=\operatorname{diag}(i,-i)$ ($i^2=-1$), the Lie algebra of $G$ is spanned by $B$, $I(B)$ is generated by $\varphi(x)=x_1x_2$, and the reduction map is $\varphi:\mathbb{K}^2\to\mathbb{K}$.

For $B=\operatorname{diag}(i,-i,i\omega,-i\omega)$, where $\omega\notin\mathbb{Q}$, the Lie algebra of $G$ is spanned by $\operatorname{diag}(i,-i,0,0)$ and $\operatorname{diag}(0,0,i,-i)$, thus $G$ is two-dimensional. $I(B)$ is generated by $\varphi_1=x_1x_2$ and $\varphi_2=x_3x_4$, which are algebraically independent.

Thus, one has a reduction map $\Phi:\mathbb{K}^4\to\mathbb{K}^2$.

These examples are well-known; cf. Anosov and Arnold [2], Takens [15], where polar coordinates were employed for the equations in normal form. As can be seen, the important point is that the radial coordinates are first integrals of $B$; the angular coordinates are not of particular importance for the reduction. If $\mathbb{K}=\mathbb{R}$ (i.e. the system comes from a real differential equation after transformation to an eigenbasis of $B$), then the reduced equation can be employed to investigate the stability of the stationary point; cf. [2, 15].

Another example is given by $B=\operatorname{diag}(im,-im,ip,-ip)$ with relatively prime positive integers $m$ and $p$ [the $(m,p)$-resonance]. In this case, $G$ is one-dimensional, $I(B_s)$ is generated by $\varphi_1=x_1x_2$, $\varphi_2=x_3x_4$, $\varphi_3=x_1^px_4^m$, and $\varphi_4=x_2^px_3^m$ (this is a minimal system of generators!), and for $\psi(y):=y_1^py_2^m-y_3y_4$ one has the relation $\psi(\varphi_1,\ldots,\varphi_4)=0$. Therefore, the reduction map $\Phi$ maps $\mathbb{K}^4$ to $Y:=\{y\in\mathbb{K}^4:\ \psi(y)=0\}$, a three-dimensional variety. [If a real system of generators of $I(B_s)$ is desired, take $\varphi_1,\varphi_2$, $\operatorname{Re}\varphi_3$ and $\operatorname{Im}\varphi_3$ and replace $\psi$ by $\tilde{\psi}:=y_1^py_2^m-(y_3^2+y_4^2)$.] Note that this setting is again suitable for the determination of stability criteria.

The next proposition shows that analytic functions near $0\in V$ which are constant on the orbits of $G$ can be considered as analytic functions on $Y$.

**(3.7) Proposition.** (a) *If $\psi\in A(V)$ is constant on the orbits of $G$, then there is a $\gamma\in A(\mathbb{K}^r)$ such that $\psi=\gamma\circ\Phi$.*

(b) *If $Z$ is a $\mathbb{K}$-vector space, $h\in\mathscr{A}(Z)$ and $\Psi:N\to Z$ is analytic on the neighbourhood $N$ of 0, constant on the orbits of $G$, and solution-preserving from $\dot{x}=f(x)$ to $\dot{z}=h(z)$, then there is an analytic $\Gamma:\tilde{N}\to Z$ (with $\tilde{N}$ a neighbourhood of 0 in $\mathbb{K}^r$) which is solution-preserving from $\dot{y}=g(y)$ on $Y$ to $\dot{z}=h(z)$.*

*Proof.* (a) Since $B_s$ lies in the Lie algebra of $G$, we have in particular $\psi(\exp(tB_s)x)=\psi(x)$ for all (sufficiently small) $x$ and $t$; differentiation shows $L_{B_s}(\psi)=0$, hence $\psi\in\bar{I}(B_s)$. The assertion follows from (3.3).

(b) From (a) we obtain the existence of a $\Gamma$ such that $\Psi=\Gamma\circ\Phi$, and $D\Gamma(\Phi(x))\cdot g(\Phi(x))=h(\Gamma(\Phi(x)))$ can be verified directly.

Because $\Phi(V)$ contains an open subset of $Y$, the assertion follows. □

Thus, if $\dot{x}=f(x)$ can be reduced further (which will not be the case in general), then this can be seen from $\dot{y}=g(y)$. One should not expect that the reduced equation can be solved in closed form (unless $I(B_s)$ is generated by one element) but, as will be shown in the following, if $\dot{x}=f(x)$ has additional nice properties, these properties will be transferred to the reduced equation:

**(3.8) Proposition.** *There is a bijective correspondence between the analytic first integrals near* 0 *of* $\dot{y}=g(y)$ *on* $Y$ *and the analytic first integrals near* 0 *of* $\dot{x}=f(x)$, *given by* $\gamma \mapsto \gamma \circ \Phi$.

*Proof.* Of course, "first integral on $Y$" means $L_g(\gamma)\in \tilde{J}$.

Thus, with $\varphi := \gamma \circ \Phi$ and (3.5): $0=L_g(\gamma)(\Phi(x))=L_f(\varphi)(x)$, and $\varphi$ is a first integral of $f$. The map $\gamma \mapsto \gamma \circ \Phi$ is obviously linear, it is injective, as $\gamma \circ \Phi = 0$ implies $\gamma \in \tilde{J}$. The nontrivial part is surjectivity, and this follows from (1.8): If $\varphi$ is a first integral of $f$, then $\varphi \in \tilde{I}(B_s)$, and $\varphi=\gamma\circ\Phi$ for some $\gamma$ by (3.3). □

Of course, in general there will be no nonconstant analytic first integrals of $\dot{x}=f(x)$ in a neighbourhood of 0 but if there are, their existence can be verified from the reduced equation. In the same manner, in general there will not be more automorphisms of $\dot{x}=f(x)$ than those in $G$ (and the one given by $f$ itself) but if there are, they are inherited by the reduced equation:

**(3.9) Proposition.** *Let* $H\in\mathscr{A}(V)$ *be invertible near* 0, $H(0)=0$ *and* $DH(x)\cdot f(x) = f(H(x))$ *for all sufficiently small* $x$. *Then there is a* $H^*\in\mathscr{A}(\mathbb{K}^r)$ *which satisfies* $\Phi\circ H = H^*\circ\Phi$ *and is solution-preserving from* $\dot{y}=g(y)$ *on* $Y$ *to itself.*

*Proof.* Let $H=\sum_{i\geq 1} h_i$. Then $[H,B_s]=0$ from (1.3).

If $\varphi\in\tilde{I}(B_s)$, then $\varphi\circ H\in\tilde{I}(B_s)$, as follows from

$$\begin{aligned} D(\varphi\circ H)(x)\cdot B_s x &= D\varphi(H(x))\cdot DH(x)\cdot B_s x \\ &= D\varphi(H(x))\cdot B_s H(x) = L_{B_s}(\varphi)(H(x)) = 0. \end{aligned}$$

Thus, there is a $H^*\in\mathscr{A}(\mathbb{K}^r)$ such that $\Phi\circ H=H^*\circ\Phi$ by (3.3).

The remaining assertions are easily verified. □

The reduced equation can also be employed to find invariant sets for $\dot{x}=f(x)$:

**(3.10) Proposition.** *If* $Z\subset Y$ *is invariant for* $\dot{y}=g(y)$, *then* $\Phi^{-1}(Z)$ *is invariant for* $\dot{x}=f(x)$ *and is furthermore G-invariant.*

*Proof.* The invariance for $\dot{x}=f(x)$ is obvious, the $G$-invariance follows from the fact that $\Phi$ is constant on the orbits of $G$. □

Some invariant sets for $\dot{y}=g(y)$ are found from the variety $Y$ without any work:

**(3.11) Proposition.** *Let* $Z$ *be the set of all singular points of* $Y$. *Then* $Z$ *is invariant for* $\dot{y}=g(y)$.

*Proof.* Let $\varrho_1,\dots,\varrho_m$ be generators of the prime ideal $\tilde{J}$.

Then there are $\mu_{ij}\in A(\mathbb{K}^r)$ such that

$$(*) \qquad L_g(\varrho_i)=\sum_{j=1}^{m}\mu_{ij}\varrho_j \quad \text{for} \quad 1\leq i\leq m,$$

as follows from the proof of (3.6). The singular points of $Y$ are precisely those for which the derivative $\left(\frac{\delta \varrho_i}{\delta x_j}\right)$ does not have maximal rank $s$. Let $J^*$ be generated by $\varrho_1, \ldots, \varrho_m$ and all the $(s\times s)$-subdeterminants of $\left(\frac{\delta \varrho_i}{\delta x_j}\right)$. We will show $L_g(J^*)\subset J^*$, which implies the invariance of $Z$. In view of (*) and the possibility to change indices it is sufficient to prove $L_g(\sigma)\in J^*$ for $\sigma := \det\left(\frac{\delta \varrho_i}{\delta x_j}\right)_{1\leq i,\, j\leq s}$. For $x\in \mathbb{K}^r$ let $g(x)=\begin{pmatrix} g_1(x)\\ \vdots \\ g_r(x)\end{pmatrix}$. Then we have (where summation is extended over all permutations $\pi$ of $s$ numbers)

$$\begin{aligned}
L_g(\sigma) &= \sum_{k=1}^{r} \frac{\partial}{\partial x_k}\left(\sum_{\pi} \operatorname{sgn}\pi \frac{\partial \varrho_1}{\partial x_{\pi(1)}} \cdots \frac{\partial \varrho_s}{\partial x_{\pi(s)}}\right)\cdot g_k \\
&= \sum_{k}\sum_{\pi} \operatorname{sgn}\pi\left(\left(\frac{\partial}{\partial x_{\pi(1)}}\frac{\partial \varrho_1}{\partial x_k}\right)\frac{\partial \varrho_2}{\partial x_{\pi(2)}}\cdots\frac{\partial \varrho_s}{\partial x_{\pi(s)}} + \ldots\right. \\
&\qquad \left.\ldots + \frac{\partial \varrho_1}{\partial x_{\pi(1)}}\cdots\frac{\partial \varrho_{s-1}}{\partial x_{\pi(s-1)}}\left(\frac{\partial}{\partial x_{\pi(s)}}\frac{\partial \varrho_s}{\partial x_k}\right)\right)\cdot g_k \\
&= \sum_{\pi} \operatorname{sgn}\pi\left(\left(\frac{\partial}{\partial x_{\pi(1)}}(L_g(\varrho_1))\right)\frac{\partial \varrho_2}{\partial x_{\pi(2)}}\cdots\frac{\partial \varrho_s}{\partial x_{\pi(s)}} + \ldots\right. \\
&\qquad \left.\ldots + \frac{\partial \varrho_1}{\partial x_{\pi(1)}}\cdots\frac{\partial \varrho_{s-1}}{\partial x_{\pi(s-1)}}\left(\frac{\partial}{\partial x_{\pi(s)}}L_g(\varrho_s)\right)\right) \\
&\quad - \sum_{\pi} \operatorname{sgn}\pi\left(\sum_{k=1}^{r} \frac{\partial g_k}{\partial x_{\pi(1)}}\frac{\partial \varrho_1}{\partial x_k}\frac{\partial \varrho_2}{\partial x_{\pi(2)}}\cdots\frac{\partial \varrho_s}{\partial x_{\pi(s)}} + \ldots\right. \\
&\qquad \left.\ldots + \frac{\partial \varrho_1}{\partial x_{\pi(1)}}\cdots\frac{\partial \varrho_{s-1}}{\partial x_{\pi(s-1)}}\frac{\partial \varrho_s}{\partial x_k}\frac{\partial g_k}{\partial x_{\pi(s)}}\right).
\end{aligned}$$

By virtue of (*), the first term lies in $J^*$. The second term is equal to

$$-\sum_{k=1}^{r}\sum_{\tau}\operatorname{sgn}\tau\left(\frac{\partial g_k}{\partial x_1}\frac{\partial \varrho_{\tau(1)}}{\partial x_k}\frac{\partial \varrho_{\tau(2)}}{\partial x_2}\cdots\frac{\partial \varrho_{\tau(s)}}{\partial x_s} + \ldots + \frac{\partial g_k}{\partial x_s}\frac{\partial \varrho_{\tau(1)}}{\partial x_1}\cdots\frac{\partial \varrho_{\tau(s-1)}}{\partial x_{s-1}}\frac{\partial \varrho_{\tau(s)}}{\partial x_k}\right),$$

and this is a linear combination of $(s\times s)$-minors of $\left(\frac{\delta \varrho_i}{\delta x_j}\right)$ with coefficients $\frac{\delta g_k}{\delta x_i}$, hence in $J^*$. □

Note that this proof works for every analytic differential equation which has an invariant (irreducible, with no loss of generality) algebraic variety. Thus, in our case, it is an advantage to have singular points in the variety!

As an illustration consider again the $(m,p)$-resonance discussed after (3.6), with $Y$ the zero set of $\psi = y_1^p y_2^m - y_3 y_4$ in $\mathbb{K}^4$.

If $m>1$ and $p>1$, then $Z=\{y\in\mathbb{K}^4\colon y_1=y_3=y_4=0 \text{ or } y_2=y_3=y_4=0\}$. Among the $\Phi$-preimages of $Z$ are $\{x\colon x_3=x_4=0\}$ and $\{x\colon x_1=x_2=0\}$. If $\dot{x}=f(x)$ comes from a real system, then these invariant sets correspond to plane orbits of $\dot{x}=f(x)$.

Next, let us show that certain subalgebras of $\mathscr{C}(B_s)$ are also in a sense preserved by the reduction map. This follows from the result below, the proof of which is elementary:

**(3.12) Proposition.** *Let $f, \tilde{f} \in \mathscr{C}(B_s)$ and $g, \tilde{g} \in \mathscr{M}$ such that $D\Phi(x) \cdot f(x) = g(\Phi(x))$ and $D\Phi(x) \cdot \tilde{f}(x) = \tilde{g}(\Phi(x))$ for all sufficiently small $x$. Then $D\Phi(x) \cdot [f, \tilde{f}](x) = [g, \tilde{g}](\Phi(x))$ holds for all sufficiently small $x$. Thus, $\Phi$ induces a Lie algebra homomorphism from $\mathscr{C}(B_s)$ to $\mathscr{M}/\mathscr{M}^*$.*

In particular, the natural grading in $\mathscr{C}(B_s)$ induces a grading in $\mathscr{M}/\mathscr{M}^*$, and graded subalgebras of $\mathscr{C}(B_s)$ are mapped to graded subalgebras of $\mathscr{M}/\mathscr{M}^*$. In general, it will not be true that proper subalgebras of $\mathscr{C}(B_s)$ are mapped to proper subalgebras of $\mathscr{M}/\mathscr{M}^*$, but this is true in the important special case of Hamiltonian systems. We use the terms Hamiltonian system, Poisson bracket, etc. in the sense of Olver [14, Chap. 6]. Furthermore, it should be noticed that there is no problem to extend the definition of a Poisson bracket (Olver [14, Definition 6.1]) to real or complex algebraic varieties and analytic functions on this variety. The example we have in mind is $Y$ and $A(\mathbb{K}^r)/\tilde{J}$. We call a Poisson bracket on $V$ homogeneous with respect to our special system of coordinates $x_1, \ldots, x_n$, if the structure functions (Olver [14, Chap. 6, p. 383]) are homogeneous polynomials of the same degree in $x_1, \ldots, x_n$. Examples of homogeneous Poisson brackets are the canonical Poisson bracket (Olver [14, Example 6.2]) and the Lie-Poisson bracket (Olver [14, Chap. 6, p. 385]).

**(3.13) Proposition.** *Let $\{\cdot, \cdot\}$ be a homogeneous Poisson bracket on $V$, and let $\dot{x} = f(x)$ be Hamiltonian with respect to this bracket. Then there is a Poisson bracket $\{\cdot, \cdot\}'$ on $Y$ such that the reduced equation $\dot{y} = g(y)$ is Hamiltonian with respect to $\{\cdot, \cdot\}'$.*

*Proof.* Let $\mathscr{L}$ be the subalgebra of $\mathscr{C}(B_s)$ which contains all the Hamiltonian vector fields with respect to $\{\cdot, \cdot\}$. Then $\mathscr{L}$ is a graded subalgebra of $\mathscr{C}(B_s)$, in particular $B_s \in \mathscr{L}$.

(i) For $\gamma_1, \gamma_2 \in A(\mathbb{K}^r)$ there is a $\gamma^* \in A(\mathbb{K}^r)$ such that

$$(*) \qquad \{\gamma_1 \circ \Phi, \gamma_2 \circ \Phi\} = \gamma^* \circ \Phi .$$

To see this, let $\psi_i := \gamma_i \circ \Phi$ for $i = 1, 2$ and define the Hamiltonian vector field $q$ by $L_q(\varrho) := \{\varrho, \psi_2\}$ for all $\varrho \in A(V)$ (cf. Olver [14, Chap. 6]). From $L_{B_s}(\psi_2) = 0$ we get $q \in \mathscr{C}(B_s)$, hence $q \in \mathscr{L}$. Let $h \in \mathscr{M}$ such that $D\Phi(x) \cdot q(x) = h(\Phi(x))$.

According to Lemma 3.5 $(L_h(\gamma_1))(\Phi(x)) = L_q(\psi_1)(x) = \{\psi_1, \psi_2\}(x)$. The assertion follows with $\gamma^* := L_h(\gamma_1)$.

(ii) $\gamma^* + \tilde{J}$ is uniquely determined by $(*)$; therefore, we define

$$\{\gamma_1, \gamma_2\}' \in A(\mathbb{K}^r)/\tilde{J} \quad \text{by} \quad \{\gamma_1, \gamma_2\}' \circ \Phi = \{\gamma_1 \circ \Phi, \gamma_2 \circ \Phi\} .$$

In order to prove the properties of a Poisson bracket for $\{\cdot, \cdot\}'$ on $Y$, we first note that it is sufficient to do this for $\mathbb{K} = \mathbb{C}$, and furthermore, it is sufficient to verify the properties for all points of $\Phi(V) \subset Y$, since $\Phi(V)$ contains an open-dense subset of $Y$.

For instance, the Leibniz rule holds because of

$$\begin{aligned}\{\gamma_1, \gamma_2\gamma_3\}' \circ \Phi &= \{\psi_1, \psi_2\psi_3\} = \{\psi_1, \psi_2\}\psi_3 + \{\psi_1, \psi_3\}\psi_2 \\ &= (\{\gamma_1, \gamma_2\}'\gamma_3) \circ \Phi + (\{\gamma_1, \gamma_3\}'\gamma_2) \circ \Phi ,\end{aligned}$$

and the Jacobi identity follows immediately from

$$\{\{\gamma_1, \gamma_2\}', \gamma_3\}' \circ \Phi = \{\{\gamma_1, \gamma_2\}' \circ \Phi, \gamma_3 \circ \Phi\} = \{\{\gamma_1 \circ \Phi, \gamma_2 \circ \Phi\}, \gamma_3 \circ \Phi\} .$$

(iii) Because $f$ is Hamiltonian, there is a $\varphi \in A(V)$ such that $L_f(\psi) = \{\psi, \varphi\}$ for all $\psi \in A(V)$. Since $\varphi$ is a first integral of $\dot{x} = f(x)$, we have $\varphi \in \bar{I}(B_s)$ by (1.8), and there is a $\gamma \in A(\mathbb{K}^r)$ such that $\varphi = \gamma \circ \Phi$ [cf. (3.3)].

For any $\tilde{\gamma} \in A(\mathbb{K}^r)$ and $\tilde{\varphi} := \tilde{\gamma} \circ \Phi$ we get

$$\{\tilde{\gamma}, \gamma\}' \circ \Phi = \{\tilde{\varphi}, \varphi\} = L_f(\tilde{\varphi}) = L_g(\tilde{\gamma}) \circ \Phi$$

from (3.5). Thus, $g$ is the Hamiltonian vector field on $Y$ corresponding to $\gamma$. □

Note that the proof is constructive. The result also follows from Arms et al. [1] or Cushman and Sjamaar [9], but the proof given here is different. As an application, we will discuss the Hamiltonian $(m, p)$-resonance. This example has been treated before by other authors, cf. Cushman and Rod [7], and Cushman et al. [6] for the semisimple and non-semisimple $(1,1)$-resonance, and van der Meer [16], and Kummer [12] for the general case. The purpose of this example is to illustrate that (and how) the reduction can be carried out naturally within the framework of normal forms alone.

Given the standard Poisson bracket

$$\{\varphi, \psi\} = \frac{\partial \varphi}{\partial x_1}\frac{\partial \psi}{\partial x_2} - \frac{\partial \varphi}{\partial x_2}\frac{\partial \psi}{\partial x_1} + \frac{\partial \varphi}{\partial x_3}\frac{\partial \psi}{\partial x_4} - \frac{\partial \varphi}{\partial x_4}\frac{\partial \psi}{\partial x_3}$$

on $\mathbb{C}^4$, let $f$ be a Hamiltonian vector field in normal form with semisimple linear part $B_s = \mathrm{diag}(im, -im, ip, -ip)$, where $i^2 = -1$ and $m, p$ are relatively prime positive integers.

Let $\psi$ be the Hamiltonian of $f$, thus $\psi = \sum_{j \geqq 2} \psi_j$ with $\psi_j$ homogeneous of degree $j$ and in particular $\psi_2 = imx_1x_2 + ipx_3x_4 + \psi_{2,n}$, where $\psi_{2,n} = 0$ for $(m, p) \neq (1, 1)$ and $\psi_{2,n}$ belonging to the nilpotent part of $B = B_s + B_n$ if $m = p = 1$.

We assume that $\dot{x} = f(x)$ comes from a Hamiltonian system in $\mathbb{R}^4$ after transformation to an eigenbasis. Thus, the real coordinates are $\mathrm{Re}\, x_1$, $\mathrm{Im}\, x_1$, $\mathrm{Re}\, x_3$, and $\mathrm{Im}\, x_3$, furthermore $x_2 = \bar{x}_1$ and $x_4 = \bar{x}_3$.

The algebra $I(B_s)$ is generated by $\varphi_1 = x_1x_2$, $\varphi_2 = x_3x_4$, $\varphi_3 = x_1^p x_4^m$ and $\varphi_4 = x_2^p x_3^m$; and $\varrho(\varphi_1, \ldots, \varphi_4) = 0$, with $\varrho(y) = y_1^p y_2^m - y_3y_4$ is (essentially) the only relation satisfied by $\varphi_1, \ldots, \varphi_4$.

An easy computation shows

$$\{\varphi_1, \varphi_2\} = 0, \quad \{\varphi_1, \varphi_3\} = -p\varphi_3, \quad \{\varphi_1, \varphi_4\} = p\varphi_4, \quad \{\varphi_2, \varphi_3\} = m\varphi_3,$$
$$\{\varphi_2, \varphi_4\} = -m\varphi_4, \quad \{\varphi_3, \varphi_4\} = p^2\varphi_1^{p-1}\varphi_2^m - m^2\varphi_1^p\varphi_2^{m-1}.$$

Therefore, $\{y_1, y_2\}' = 0$, $\{y_1, y_3\}' = -py_3$ and so on. With the "reduced" Hamiltonian $\gamma$ on $Y$ (given by $\gamma \circ \Phi = \psi$) we obtain the reduced equation

$$\dot{y}_1 = -py_3\frac{\partial \gamma}{\partial y_3} + py_4\frac{\partial \gamma}{\partial y_4},$$

$$\dot{y}_2 = my_3\frac{\partial \gamma}{\partial y_3} - my_4\frac{\partial \gamma}{\partial y_4},$$

$$\dot{y}_3 = py_3\frac{\partial \gamma}{\partial y_1} - my_3\frac{\partial \gamma}{\partial y_2} + \sigma(y)\frac{\partial \gamma}{\partial y_4},$$

$$\dot{y}_4 = -py_4\frac{\partial \gamma}{\partial y_1} + my_4\frac{\partial \gamma}{\partial y_2} - \sigma(y)\frac{\partial \gamma}{\partial y_3}$$

with $\sigma(y) = p^2 y_1^{p-1} y_2^m - m^2 y_1^p y_2^{m-1}$.

Of course, $Y$ is invariant for this equation (and only the restriction to $Y$ is of interest). Note that $\gamma$ and $my_1+py_2$ are first integrals of the reduced equation, hence $\psi$ and $m\varphi_1+p\varphi_2$ are first integrals of $\dot{x}=f(x)$. The second first integral (reflecting $[B_s, f]=0$) is a specialty of the normal form.

Since $\dot{x}=f(x)$ comes from a real system, it is desirable to obtain a real reduced system. Furthermore, we can use $my_1+py_2$ as one coordinate:

To this end, let $z_1=my_1+py_2$, $z_2=\alpha y_1+\beta y_2$ (with $\alpha, \beta \in \mathbb{R}$ such that $m\beta-p\alpha=1$; a further condition will be introduced later), $z_3=y_3+y_4$ $(=2\operatorname{Re} y_3)$ and $z_4=-i(y_3-y_4)$ $(=2\operatorname{Im} y_3)$.

With $\tau(z_1,\ldots,z_4):=i\cdot\gamma(y_1(z_1,\ldots,z_4),\ldots,y_4(z_1,\ldots,z_4))$ we get in new coordinates

$$\dot{z}_1=0,$$

$$\dot{z}_2=2z_4\frac{\partial\tau}{\partial z_3}+2z_3\frac{\partial\tau}{\partial z_4},$$

$$\dot{z}_3=-2z_4\frac{\partial\tau}{\partial z_2}-2\tilde{\sigma}(z)\frac{\partial\tau}{\partial z_4},$$

$$\dot{z}_4=-2z_3\frac{\partial\tau}{\partial z_2}+2\tilde{\sigma}(z)\frac{\partial\tau}{\partial z_3}$$

with $\tilde{\sigma}(z)=(\beta z_1-pz_2)^{p-1}(-\alpha z_1+mz_2)^{m-1}(p^2(-\alpha z_1+mz_2)-m^2(\beta z_1-pz_2))$. Here we have first integrals $\tau$ and $z_1$, and only the invariant set

$$4(\beta z_1-pz_2)^m(-\alpha z_1+mz_2)^p-z_3^2-z_4^2=0.$$

is of interest.

A further reduction is achieved by considering an "energy surface" $z_1=c=\text{const}$. In order to preserve the stationary point 0 for the remaining equation in any case, we choose $\alpha$ and $\beta$ such that $-\alpha p^2-\beta m^2=0$, hence $\alpha=-\dfrac{m}{p(p+m)}$, $\beta=\dfrac{p}{m(p+m)}$.

The final version is then the above equation with

$$\tilde{\sigma}(z)=\left(\frac{p}{m(p+m)}z_1-pz_2\right)^{p-1}\left(\frac{m}{p(p+m)}z_1+mz_2\right)^{m-1}\cdot mp(m+p)z_2.$$

Now let $z_1=c>0$ and define $\tau_c$ and $\tilde{\sigma}_c$ by putting $z_1=c$. Then we have the system

$$\dot{z}_2=2z_4\frac{\partial\tau_c}{\partial z_3}+2z_3\frac{\partial\tau_c}{\partial z_4},$$

$$\dot{z}_3=-2z_4\frac{\partial\tau_c}{\partial z_2}-2\tilde{\sigma}_c(z_2,z_3,z_4)\frac{\partial\tau_c}{\partial z_4},$$

$$\dot{z}_4=-2z_3\frac{\partial\tau_c}{\partial z_2}+2\tilde{\sigma}_c(z_2,z_3,z_4)\frac{\partial\tau_c}{\partial z_3}$$

on the invariant surface $F_c$ in $\mathbb{R}^3$ given by

$$4\left(\frac{p}{m(p+m)}c-pz\right)^m\left(\frac{m}{p(p+m)}c+mz\right)^p-z_3^2-z_4^2=0$$

and $-\dfrac{c}{p(p+m)}\leqq z_2\leqq\dfrac{c}{m(p+m)}$ (from $y_1\geqq 0$ and $y_2\geqq 0$).

Furthermore, $\tau_c$ is a first integral.

The geometric information obtained from the reduced equation is as important as the equation itself. $F_c$ is a rotation surface in $\mathbb{R}^3$, and its intersection with the $(z_2, z_3)$-plane is easily discussed.

If $m=p=1$, then $F_c$ is an ellipsoid, while for $m>1$ there is a cusp at $z_2 = \frac{c}{m(p+m)}$ and for $p>1$ there is a cusp at $z_2 = -\frac{c}{p(p+m)}$. These cusps are stationary points of the equation (cf. Proposition 3.11) and their $\Phi$-preimages yield plane solutions $\dot{x}=f(x)$. Moreover, these preimages are invariant for $G=\{\exp tB_s\colon t\in\mathbb{R}\}$ and thereby actually plane periodic solutions unless they contain a stationary point. In an analogous manner, periodic solutions of the reduced equation give rise to invariant tori of $\dot{x}=f(x)$, and the qualitative discussion of $\dot{x}=f(x)$ is indeed reduced to the two-dimensional surface $F_c$ and its intersections with the level sets of $\tau_c$.

The work of Cushman and Rod [7] and Cushman et al. [6] on the (1,1)-resonance illustrate that there is always a reduction of the system depending only on the semisimple part of the linearization. Of course, the behaviour of the system will in general be strongly influenced by a nonzero nilpotent part.

Finally, we want to show that reversibility (among more general properties) is preserved by the reduction map. We need some preliminaries first.

For $T\in\mathrm{GL}(V)$, $p\in\mathscr{A}(V)$, and $\varphi\in A(V)$ let $T^*p := TpT^{-1}$ and $T^*\varphi=\varphi T^{-1}$. The following rules are immediately verified:

$T^*[p,q]=[T^*p, T^*q]$, $T^*(\varphi\gamma)=(T^*\varphi)(T^*\gamma)$ for all $p,q\in\mathscr{A}(V)$ and all $\varphi,\gamma\in A(V)$; furthermore, $(ST)^*=S^*T^*$ and $L_{T^*p}(T^*\varphi)=T^*L_p(\varphi)$.

In the following let $G$ be a finite subgroup of $\mathrm{GL}(V)$ which stabilizes $I(B_s)$ and $\mathscr{C}(B_s)$. An important class of examples is given by finite subgroups of $N(B_s) =\{T\in\mathrm{GL}(V)$: there is a $\chi(T)\in\mathbb{C}^*$ such that $T^*B_s=\chi(T)B_s\}$.

**(3.14) Lemma.** *Let $\{\psi_1,\ldots,\psi_r\}$ be a minimal system of homogeneous generators of $I(B_s)$. Then there are homogeneous generators $\varphi_1,\ldots,\varphi_r$ of $I(B_s)$ such that $\mathbb{K}\varphi_1+\ldots+\mathbb{K}\varphi_r$ is $G$-invariant.*

*Proof.* We may assume that $\psi_1,\ldots,\psi_s$ are the elements in $\{\psi_1,\ldots,\psi_r\}$ of smallest degree $m_0>0$. The homogeneous elements of degree $m_0$ in $I(B_s)$ are then the linear combinations of $\psi_1,\ldots,\psi_s$; in particular $\mathbb{K}\psi_1+\ldots+\mathbb{K}\psi_s$ is $G$-invariant. We proceed by induction:

Assume that $\varphi_1,\ldots,\varphi_t$ are of degree $<m$, $\mathbb{K}\varphi_1+\ldots+\mathbb{K}\varphi_t$ is $G$-invariant, $\{\varphi_1,\ldots,\varphi_t,\psi_{t+1},\ldots,\psi_r\}$ generate $I(B_s)$ and $\deg\psi_{t+1}=\ldots=\deg\psi_v=m$, $\deg\psi_j>m$ for $j>v$. Then the subspace $I_m$ of all homogeneous elements of degree $m$ in $I(B_s)$ is the sum of $A_m:=\mathbb{K}\psi_{t+1}+\ldots+\mathbb{K}\psi_v$ and the subspace $B_m$ spanned by all the products of $\varphi_1,\ldots,\varphi_t$ of degree $m$. Due to minimality this sum is direct, and $B_m$ is $G$-invariant according to hypothesis. Now Maschke's theorem shows the existence of a $G$-invariant complement $\mathbb{K}\varphi_{t+1}+\ldots+\mathbb{K}\varphi_v$. □

From now on, suppose that $\mathbb{K}\varphi_1+\ldots+\mathbb{K}\varphi_r$ is $G$-invariant and consider the reduction map $\Phi=\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_r\end{pmatrix}: V\to\mathbb{K}^r$. Then for every $T\in G$ there is a unique $\tilde{T}\in\mathrm{GL}(\mathbb{K}^r)$ such that $\Phi(T^{-1}x)=\tilde{T}^{-1}\Phi(x)$ for all $x\in V$. It is easily verified that

$T \to \tilde{T}$ is a homomorphism from $G$ to $GL(\mathbb{K}^r)$. Differentiation shows $D\Phi(T^{-1}x) \cdot T^{-1} = \tilde{T}^{-1} D\Phi(x)$.
Now, let $\dot{x} = f(x)$ in normal form and $g \in \mathscr{A}(\mathbb{K}^r)$ such that $D\Phi(x) \cdot f(x) = g(\Phi(x))$.

**(3.15) Lemma.** *$D\Phi(x) \cdot (T^* f(x)) = \tilde{T}^* g(\Phi(x))$ for all $T \in G$.*

*Proof.*

$$\begin{aligned}\tilde{T}^{-1} D\Phi(x) \cdot Tf(T^{-1}x) &= D\Phi(T^{-1}x) \cdot T^{-1} \cdot Tf(T^{-1}x) = D\Phi(T^{-1}x) \cdot f(T^{-1}x) \\ &= g(\Phi(T^{-1}x)) = g(\tilde{T}^{-1}\Phi(x)) . \quad \square\end{aligned}$$

If we consider the natural grading on $\mathscr{C}(B_s)$ and the induced grading on $\mathscr{M}/\mathscr{M}^*$ [cf. the remark after (3.12); thus $g + \mathscr{M}^* \in \mathscr{N}_j$ if and only if $D\Phi(x) \cdot f(x) = g(\Phi(x))$ for some $f \in \mathscr{C}_j := \mathscr{C}(B_s) \cap \mathscr{P}_j$], then, according to (3.15), there is a homomorphism of $G$-modules $\mathscr{C}_j \to \mathscr{N}_j$ for every $j$. Since there are only finitely many irreducible $G$-modules (up to isomorphism) and the image of an irreducible $G$-module is either isomorphic to the given module, or trivial, (3.15) also implies

**(3.16) Proposition.** *Let $\dot{x} = Bx + \sum_{j \geq 2} f_j(x)$ in normal form and $\dot{y} = g(y) = \sum_{j \geq 1} g_j(y)$ the reduced equation on $Y$ (with the grading as above).*

*If $B$ and every $f_j$ lies in the same (up to isomorphism) irreducible $G$-module, then this also holds for every $g_j + \mathscr{M}^*$ (resp. $g_j + \mathscr{M}^* = 0$).*

In particular, for abelian groups (where every irreducible $G$-module is one-dimensional) one gets

**(3.17) Corollary.** *If $T^* f = \chi(T) f$ for all $T \in G$, then $\tilde{T}^* g \equiv \chi(T) g \bmod \mathscr{M}^*$ for the reduced equation.*

One can apply these results to prove that certain $\varphi \in \bar{I}(B_s)$ are also first integrals of $\dot{x} = f(x)$ if the situation of (3.16) or (3.17) is given. We illustrate this by two examples:

First, let $B = \operatorname{diag}(1+i, 1-i, -1+i, -1-i)$. Then $I(B)$ is generated by the algebraically independent elements $\gamma_1(x) = x_1 x_4$ and $\gamma_2(x) = x_2 x_3$.

For $T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ one computes $T^* B = iB$.

If the equation $\dot{x} = f(x) = Bx + \sum_{j \geq 2} f_j(x)$ in normal form satisfies $T^* f = if$, then we get an analogous relation for the reduced equation on $\mathbb{K}^2$. Since $T^* \gamma_1 = \gamma_2$, $T^* \gamma_2 = \gamma_1$, we get for the generators $\varphi_{1,2} := \gamma_1 \pm \gamma_2$ that $T^* \varphi_1 = \varphi_1$, $T^* \varphi_2 = -\varphi_2$. Thus, $\tilde{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $g = g_+ + g_-$ in the reduced equation $\dot{y} = g(y)$, with $\tilde{T}^* g_+ = g_+$, $\tilde{T}^* g_- = -g_-$. On the other hand, $\tilde{T}^* g = ig$ and therefore $g = 0$. Thus, $\gamma_1$ and $\gamma_2$ are first integrals of $\dot{x} = f(x)$. [Note that this result also holds if $\dot{x} = f(x)$ comes from a real system; it doesn't matter that the entries of $T$ with respect to a real basis are complex.]

As a second example, we consider once more the $(m,p)$-resonance with $B = \operatorname{diag}(im, -im, ip, -ip)$ and (for example) $p > 1$.

We choose the system of generators $\varphi_1 = x_1x_2$, $\varphi_2 = x_3x_4$, $\varphi_3 = x_1^p x_4^m + x_2^p x_3^m$, $\varphi_4 = -i(x_1^p x_4^m - x_2^p x_3^m)$ of $I(B_s)$, thus $\varphi_3^2 + \varphi_4^2 = 4\varphi_1^p \varphi_2^m$, and this relation defines $Y \subset \mathbb{K}^4$.

Now let $T = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$ with $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

If $T^*f = -f$ for $f = B + \sum_{j \geq 2} f_j$ in normal form (note that this is compatible with $T^*B = -B$), then $\tilde{T}^*g \equiv -g \bmod \mathcal{M}^*$ for the reduced equation $\dot{y} = g(y)$, and we may assume $\tilde{T}^*g = -g$.

Since $T^*\varphi_i = \varphi_i$ for $1 \leq i \leq 3$, and $T^*\varphi_4 = -\varphi_4$, we have $\tilde{T} = \operatorname{diag}(1, 1, 1, -1)$, and $\tilde{T}^*g = -g$ implies that the reduced equation on $Y$ is of the type

$$\begin{aligned} \dot{y}_1 &= y_4\tilde{\gamma}_1(y_1, y_2, y_3, y_4^2) \\ \dot{y}_2 &= y_4\tilde{\gamma}_2(y_1, y_2, y_3, y_4^2) \\ \dot{y}_3 &= y_4\tilde{\gamma}_3(y_1, y_2, y_3, y_4^2) \\ \dot{y}_4 &= \tilde{\gamma}_4(y_1, y_2, y_3, y_4^2) \end{aligned} \quad \text{or} \quad (*) \quad \begin{aligned} \dot{y}_1 &= y_4\gamma_1(y_1, y_2, y_3) \\ \dot{y}_2 &= y_4\gamma_2(y_1, y_2, y_3) \\ \dot{y}_3 &= y_4\gamma_3(y_1, y_2, y_3) \\ \dot{y}_4 &= \gamma_4(y_1, y_2, y_3), \end{aligned}$$

taking into account the relation $y_4^2 = -y_3^2 + 4y_1^p y_2^m$ on $Y$.

If $\gamma_i(0) \neq 0$ for at least one $i$, $1 \leq i \leq 3$ (and it is easy to see that this is the generic case), then the equation $\dot{y}_i = \gamma_i(y_1, y_2, y_3)$, $1 \leq i \leq 3$, has two independent analytic first integrals $\psi_1, \psi_2$ near 0, because 0 is not a stationary point. Then $\psi_1$ and $\psi_2$ are also first integrals of (*), and from (3.8) we see that $\dot{x} = f(x)$ has – in the generic case – two independent first integrals analytic in 0.

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