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Kähler metrics of constant scalar curvature on bundles over $\mathbf{CP}_{n-1}$

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1 Introduction

Investigating the properties of anti-self-dual metrics, LeBrun [Le] constructed complete zero scalar curvature asymptotically flat metrics on $L^{(k+1)}$ for any integer $k \geqq 0$. Here L is the total space of the universal line bundle $L \to \mathbf{CP}_1$ with Chern class -1. Computing the masses of these metrics, he gave counterexamples to the generalized positive action conjecture of Hawking and Pope [HP].

Starting from a different problem we were led to the consideration of LeBrun's ideas and its possible generalizations. In fact, our motivation was primarily to study the behaviour of constant scalar curvature Kähler surfaces under the blow-up procedure, and in particular, the conditions under which the blow-up of a Kähler surface of constant scalar curvature was another manifold of this type. We soon realized that concrete examples were not abundant and even for the seemingly innocent problem above, models for the desired metric is a neighborhood of the exceptional divisor were in general not available. Following LeBrun's approach in dealing with the problem mentioned before in the zero constant scalar curvature case we proved that, depending on the sign of the constant, one can indeed put Kähler metrics of constant scalar curvature either in $\mathbf{C}^n - 0$ or in some punctured neighborhood of the origin in $\mathbf{C}^n$. To obtain complete Kähler manifolds we need to replace the origin with the complex projective plane $\mathbf{CP}_{n-1}$. Thus, the resulting metrics naturally live on the total space of a disk bundle over $\mathbf{CP}_{n-1}$. A careful analysis in the case when the scalar is nonpositive permits us to find these metrics in the blow-up of the origin in $\mathbf{C}^n$, a line bundle over $\mathbf{CP}_{n-1}$, while in the case when the scalar is positive our metrics live on the blow-up of a proper neighborhood of the origin in $\mathbf{C}^n$, a disk bundle over $\mathbf{CP}_{n-1}$. This paper contains the details of our work.

The examples are constructed using a result concerning radially symmetric solutions of the nonlinear differential equation satisfied by the scalar curvature. We present this in Sect. 2 and discuss the examples separately in Sect. 3. The striking form of a particular solution of the nonlinear scalar curvature equation in the case when $n=2$ and the constant is positive allows us to exhibit a disk bundle $D \xrightarrow{\pi} \mathbf{CP}_1$ whose k power $D^{\otimes k}$ is a Kähler manifold of constant scalar curvature for any $k>1$. We also include this result in Sect. 3.

2 Preliminaries

Consider a Kähler potential ϕ in $\mathbf{C}^n$. The (1, 1)-form

$$\omega = i\partial\bar{\partial}\phi$$

is the Kähler form of a Kähler metric. If we let the expression

$$\omega \wedge \omega \ldots \wedge \omega = \omega^{\wedge n} = i^n V dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n \tag{1}$$

define the function $V(u)$, then the Ricci form ϱ is given by

$$\varrho = i\partial\bar{\partial} \log V,$$

and the scalar curvature σ satisfies the equation

$$\sigma\omega^{\wedge n} = \varrho \wedge \omega^{\wedge (n-1)}. \tag{2}$$

Let u stands for the square of the distance to the origin which in coordinates is written as $u = z^1\bar{z}^1 + z^2\bar{z}^2 + \ldots + z^n\bar{z}^n$. The key to our construction is contained in the following result.

Proposition 1. *Let σ be a real constant and consider* (2) *as an equation in ϕ. Then:*

1. *There are radially symmetric solutions of the form*

$$\phi(u) = a \log u + s(u)$$

on some maximal neighborhood $[0, \alpha)$ of $u=0$. Here α is a positive constant and s is a smooth function of u with nonvanishing first derivative at the origin.

2. *If the initial condition $\dot{s}(0)$ is positive, the function $\dot{s}(u)$ remains positive on the domain of definition $[0, \alpha)$. If $\sigma > 0$ the interval $[0, \alpha)$ is bounded, and for $\sigma \geqq 0$ we have*

$$V = n! \left(\frac{a}{u} + \dot{s}\right)^{n-1} (\dot{s} + u\ddot{s}) > 0, \qquad u \in [0, \alpha),$$

where V is the function defined by (1).

3. *When $\sigma \leqq 0$ and $\dot{s}(0) > 0$, the domain of definition of $s(u)$ equals $[0, \infty)$ and the function V is positive on it. Furthermore, solutions of the form*

$$\phi(u) = a \log u + bu + c,$$

for a, b, c constants, occur only when $n=2$ and $\sigma = 0$.

4. *If $n=2$ and $\sigma > 0$ there are radially symmetric solutions of the form*

$$\phi(u) = \frac{3(k-1)}{2} \log u - \frac{3}{\sigma} \log(1 - u^k),$$

where k is an arbitrary positive integer.

Proof. Suppose the Kähler potential is just a function of u, $\phi = \phi(u)$. In solving (2) the symmetry permits to carry the computations on $z^2 = z^3 = \ldots = z^n = 0$. If we set

$\psi = \log V$ then:

$$\partial\bar{\partial}\phi = (\dot{\phi} + u\ddot{\phi})dz^1 \wedge d\bar{z}^1 + \dot{\phi} \sum_{j=2}^{n} dz^j \wedge d\bar{z}^j,$$

$$\partial\bar{\partial}\psi = (\dot{\psi} + u\ddot{\psi})dz^1 \wedge d\bar{z}^1 + \dot{\psi} \sum_{j=2}^{n} dz^j \wedge d\bar{z}^j,$$

at $z^2 = z^3 = \ldots = z^n = 0$. Therefore:

$$\omega^{\wedge n} = i^n n!\, \dot{\phi}^{n-1}(\dot{\phi} + u\ddot{\phi})dz^1 \wedge d\bar{z}^1 \wedge \ldots dz^n \wedge d\bar{z}^n,$$

$$\omega^{\wedge(n-1)} = i^{n-1}(n-1)! \Big[\dot{\phi}^{n-2}(\dot{\phi} + u\ddot{\phi}) \sum_{j \neq 1} dz^1 \wedge \bar{z}^1 \wedge \ldots \widehat{dz^j \wedge \bar{z}^j} \wedge \ldots \wedge dz^n \wedge d\bar{z}^n$$

$$+ \dot{\phi}^{n-1} dz^2 \wedge d\bar{z}^2 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n\Big],$$

where as usual, the $\widehat{dz^j \wedge \bar{z}^j}$ means that the term $dz^j \wedge \bar{z}^j$ is omitted from the product. Then (2) is just:

$$n\sigma\dot{\phi}(\dot{\phi} + u\ddot{\phi}) = \dot{\phi}(\dot{\psi} + u\ddot{\psi}) + (n-1)\dot{\psi}(\dot{\phi} + u\ddot{\phi}),$$

which is clearly equivalent to

$$\frac{\dot{\phi}}{(u\dot{\phi})^{n-1}} \frac{d}{du}(u^n \dot{\phi}^{n-1}\dot{\psi} - \sigma u^n \dot{\phi}^n) = 0.$$

Consequently,

$$u^n \dot{\phi}^{n-1}\dot{\psi} - \sigma u^n \dot{\phi}^n = A,$$

where A is a constant. Set $\zeta = u\dot{\phi}$. Since $V = n!\,\dot{\phi}^{n-1}(\dot{\phi} + u\ddot{\phi})$ we have

$$V = \frac{n!\,\zeta^{n-1}\dot{\zeta}}{u^{n-1}},$$

and therefore:

$$\dot{\psi} = \frac{d}{du}\log V = \frac{1}{V}\frac{dV}{du} = \frac{\ddot{\zeta}}{\dot{\zeta}} + \frac{(n-1)\dot{\zeta}}{\zeta} - \frac{n-1}{u}.$$

Plugging this into the previous expression and writing the result as an equation for ζ, we obtain:

$$\frac{d}{du}(u\zeta^{n-1}\dot{\zeta}) = \sigma\zeta^n\dot{\zeta} + n\zeta^{n-1}\dot{\zeta} + A\dot{\zeta},$$

from which it follows that:

$$\sigma\zeta^{n+1} + (n+1)\zeta^n + (n+1)A\zeta + B = (n+1)u\zeta^{n-1}\dot{\zeta},$$

where B is another constant of integration. This expression in terms of ϕ becomes:

$$\sigma(u\dot{\phi})^{n+1} + (n+1)(u\dot{\phi})^n + (n+1)Au\dot{\phi} + B = (n+1)u(u\dot{\phi})^{n-1}(\dot{\phi} + u\ddot{\phi}). \qquad (3)$$

We want a solution $\phi = a\log u + s(u)$ with $a > 0$, and where in principle we only required $s(u)$ to be C^2. Later we shall see that such function s will be necessarily smooth. Then:

$$\dot{\phi}=\frac{a}{u}+\dot{s}, \quad \ddot{\phi}=-\frac{a}{u^2}+\ddot{s}.$$

Equation (3) in terms of s is:

$$\sigma(a+u\dot{s})^{n+1}+(n+1)(a+u\dot{s})^n+(n+1)A(a+u\dot{s})+B$$
$$=(n+1)u(a+u\dot{s})^{n-1}(\dot{s}+u\ddot{s}).$$

Although it does not appear so in principle, the constants A and B in the previous expression are already determined by the special form of the solutions we search for. Indeed, the right side vanishes at $u=0$ and we thus obtain the relation:

$$\sigma a^{n+1}+(n+1)a^n+(n+1)Aa+B=0.$$

After some simplifications the equation becomes:

$$(n+1)(a+u\dot{s})^{n-1}(\dot{s}+u\ddot{s})$$
$$=\sigma\sum_{j=1}^{n+1}\binom{n+1}{j}a^{n+1-j}u^{j-1}\dot{s}^j$$
$$+(n+1)\sum_{j=1}^{n}\binom{n}{j}a^{n-j}u^{j-1}\dot{s}^j+(n+1)A\dot{s}.$$

Evaluating this expression at $u=0$ and using the nonvanishing of $\dot{s}(0)$ yields the relation:

$$A=-(n-1)a^{n-1}-\sigma a^n,$$

and therefore:

$$B=n\sigma a^{n+1}+(n+1)(n-2)a^n.$$

Simplifying once again, we finally obtain:

$$(n+1)(a+u\dot{s})^{n-1}\ddot{s}$$
$$=\frac{\sigma}{u^2}((a+u\dot{s})^{n+1}-a^{n+1}-(n+1)a^nu\dot{s})$$
$$+\frac{(n+1)a}{u^2}((a+u\dot{s})^{n-1}-a^{n-1}-(n-1)a^{n-2}u\dot{s}).$$

Setting $w=\dot{s}$ this becomes:

$$\dot{w}=\frac{\sigma((a+uw)^{n+1}-a^{n+1}-(n+1)a^nuw)}{u^2(n+1)(a+uw)^{n-1}}$$
$$+\frac{a((a+uw)^{n-1}-a^{n-1}-(n-1)a^{n-2}uw)}{u^2(a+uw)^{n-1}}$$
$$=f(u,w). \tag{4}$$

The function $f(u,w)$ is smooth in a neighborhood of $u=0$. The theorem of existence and uniqueness of solutions to ordinary differential equations allows us to find a smooth function $w(u)$ satisfying this equation for any initial condition $w(0)=\dot{s}(0)$. There is in fact one such function defined on a maximal domain $[0,\alpha)$. Integrating it we obtain s as desired.

If the initial condition $w(0)=\dot{s}(0)=b$ is positive, the solution to (4) remains positive for any u in $[0,\alpha)$. This is so because if $w(u)$ were zero at one positive value of u, both $w(u)$ and $\dot{w}(u)$ would be zero at that point and, by uniqueness of solutions of differential equations, $w(u)$ would have to be identically zero. If we assume furthermore that $\sigma \geqq 0$, the function $f(u, w)$ is clearly positive for $u \in [0, \alpha)$ and positive values of w. It follows that w is always increasing and since

$$V = n!\,\dot{\phi}^{n-1}(\dot{\phi} + u\ddot{\phi}),$$

we have

$$V = n!\left(\frac{a}{u} + w\right)(w + uf(u, w)) > 0$$

on the domain of definition of the solution. It is clear that $\dot{w} \geqq \sigma w^2/(n+1)$ and thus w blows-up in finite time if $\sigma > 0$. Therefore, $[0, \alpha)$ is bounded if $\sigma > 0$.

When $\sigma = 0$ the solution w to (4) cannot go to $+\infty$ in finite time. In fact, if $n > 2$ and we assume otherwise, near the value of u where this occurs, the solution $w(u)$ would have derivative $\dot{w}$ uniformly close to a/u^2, which is a contradiction. Thus, if $\sigma = 0$ we have $\alpha = \infty$ and $w(u)$ is defined on $[0, \infty)$.

If $n > 2$ the function w cannot be constant. This follows easily by looking at the function $f(u, w)$, derivative of w. When $n = 2$ this function vanishes only if $\sigma = 0$. In that case, integration leads to solutions of the form $\phi = a \log u + bu + c$ as stated.

When σ is strictly negative and $w(0) = b > 0$, the solution $w(u)$ cannot go to $+\infty$ in finite time either. In fact, as $w \to +\infty$ and for nonzero u, the function $f(u, w)$ is asymptotically equals to $\sigma w^2/(n+1) < 0$. Thus, w is a decreasing function when its value is large. We then conclude that $\alpha = \infty$ in this case also, but we need to be more careful in the analysis of V. For that we look at (3) as a first order differential equation in $u\dot{\phi}$. Since $u\dot{\phi} \neq 0$ for $u \neq 0$, the only way V could vanish is if the derivative of $u\dot{\phi}$ vanishes. But the polynomial in $u\dot{\phi}$ in the left of the equation, divided by u, is positive as $u \to 0^+$. Hence, for the derivative of $u\dot{\phi}$ to vanish at some point $u_0 > 0$, its value there must be a positive root, $u_0\dot{\phi}(u_0)$, of that polynomial. By the uniqueness of solutions of differential equations, $u\dot{\phi}$ would have to be constant, which is a contradiction if $\dot{s}(0) = b > 0$. Thus, the derivative of $u\dot{\phi}$ does not have a zero on $[0, \infty)$ and V is positive.

To prove the last part of the proposition we go back to (3), use $n = 2$ and consider solutions where the constant B is set to zero. To integrate this equation it is easier to trace back to the variable $\zeta = u\dot{\phi}$. Then if $\sigma\zeta^2 + 3\zeta + 3A = \sigma(\zeta + \alpha_1)(\zeta + \alpha_2)$, we obtain the solution:

$$\gamma u = \left(\frac{\zeta + \alpha_2}{\zeta + \alpha_1}\right)^{\frac{3}{\sigma(\alpha_1 - \alpha_2)}}.$$

Here γ is a positive constant of integration. Notice that $\alpha_1 + \alpha_2 = 3/\sigma$ and $\alpha_1 \cdot \alpha_2 = 3A/\sigma$. Therefore:

$$\alpha_1 = \frac{3}{2\sigma} + \frac{3}{2|\sigma|}\sqrt{1 - \frac{4A\sigma}{3}}, \quad \alpha_2 = \frac{3}{2\sigma} - \frac{3}{2|\sigma|}\sqrt{1 - \frac{4A\sigma}{3}},$$

$$\gamma \stackrel{\text{def}}{=} \frac{\sigma(\alpha_1 - \alpha_2)}{3} = sg\sigma\sqrt{1 - \frac{4A\sigma}{3}},$$

where $sg\sigma$ is the sign function. Solving for ζ we obtain:

$$\zeta = \frac{\alpha_1 u^\gamma - \alpha_2}{1 - u^\gamma} = u\dot{\phi}.$$

If $\sigma > 0$ we choose A such that $\gamma = k$. Solving for ϕ we obtain:

$$\phi = \frac{\alpha_2 - \alpha_1}{k} \log(1-u^k) - \alpha_2 \log u = -\frac{3}{\sigma} \log(1-u^k) - \alpha_2 \log u,$$

where $\alpha_2 = 3(1-k)/2$. This completes the proof. □

Remark 1. The fact that the Kähler potential $a \log u + bu$ in $\mathbf{C}^n - 0$ produces a metric of constant scalar curvature only when $n=2$, in which case the curvature is zero, was apparently discovered by Burns and stated in a 1986 AMS lecture in Charlotte, NC. This result is contained in the third part of the proposition above.

We should observe that the Taylor series expansion of the function $s(u)$ found in the first part of the proposition above is of the form

$$s(u) = s(0) + bu + \left(\frac{\sigma n}{2} + \frac{(n-1)(n-2)}{2a}\right)\frac{b^2u^2}{2} + \dots.$$

Hence, when $b > 0$, it is clear that a nonvanishing σ does play a role in determining the neighborhood of the origin where the function $s(u)$ will be defined and where $\phi = a \log u + s(u)$ is the Kähler potential of a constant scalar curvature Kähler metric. When $n=2$, $\sigma=1$, $a=1$, one can explicitly integrate (4) and see that the interval $[0, \alpha)$ may in general be bounded. This particular example was very helpful in leading us through our work.

3 Construction of the examples

Let $L \xrightarrow{\pi} \mathbf{CP}_{n-1}$ be the universal line bundle [GH] obtained by blowing-up $\mathbf{C}^n$ at the origin. This is just the subvariety $\{(z^1, \dots, z^n, t^1, \dots, t^n): t^i z^j - z^i t^j = 0, \forall i, j\}$ of $\mathbf{C}^n \times \mathbf{CP}_{n-1}$ viewed as a bundle over $\mathbf{CP}^{n-1}$. Here $z^1, \dots, z^n$ are coordinates in $\mathbf{C}^n$ and $t^1, \dots, t^n$ are homogeneous coordinates in $\mathbf{CP}_{n-1}$. Near the zero section of this bundle and on the complement of $t_1 = 0$ we can use $z^1, t^2/t^1, \dots, t^n/t^1$ as the coordinates of a point which projects to $(z^1, z^1t^2/t^1, \dots, z^1t^n/t^1)$ in $\mathbf{C}^n - 0$. Hence,

$$u = \sum_j z^j \bar{z}^j = z^1 \bar{z}^1 \left(1 + \sum_{j \geqq 2} t^j/t^1\right)$$

and therefore,

$$\partial\bar{\partial} \log u = \partial\bar{\partial} \log\left(1 + \sum_{j \geqq 2} t^j/t^1\right),$$

the Kähler form of the Fubini-Study metric on $\mathbf{CP}_{n-1}$. This shows that $\partial\bar{\partial}\log u$ in $\mathbf{C}^n - 0$ pulls back and extends smoothly to a $(1,1)$-form on D. An entirely similar argument shows that $\partial\bar{\partial}s(u)$ defines a smooth $(1,1)$-form on D if s is a smooth function on $\mathbf{C}^n$. Therefore, if we consider a solution ϕ of (2) of the form $\phi(u) = a \log u + s(u)$ with $\dot{s}(0) = b > 0$ and s defined on the interval $[0, \alpha)$ as given by Proposition 1, the $(1,1)$-form $\partial\bar{\partial}\phi(u)$ extends to a smooth $(1,1)$-form on the blow-up D of the open set $U = \{z \in \mathbf{C}^n : \|z\| < \alpha\}$, and the positivity of the function V proven in Proposition 1 shows that this is the Kähler potential of a metric in D. Thus, in the cases where $\alpha = \infty$, i.e. when $\sigma \leqq 0$, we obtain a Kähler metric of constant scalar curvature σ in L as D and L coincide, while otherwise, i.e. when $\sigma > 0$, we obtain one such metric in the disk bundle D which sits properly inside L.

Let us consider once again the coordinates $v = z^1$, $w^1 = z^2/z^1, \dots, w^{n-1} = z^n/z^1$ where the divisor is given by $v = 0$. A more explicit understanding of the metric on D is obtained if we write the potential as $\phi(u) = a \log u + bu + g(u)$ where $g(u) = s(u)$

$-\dot{s}(0)u = s(u) - bu$, and compute to get:

$$\omega = i\left(bv\bar{v} + \frac{a((1+w\bar{w}) - w^j\bar{w}^j)}{(1+w\bar{w})^2}\right) dw^j d\bar{w}^j + ib(w^j\bar{v}dvd\bar{w}^j + \bar{w}^j v dw^j d\bar{v})$$
$$+ ib(1+w\bar{w})dvd\bar{v} + i\partial\bar{\partial}g(u)$$
$$= \omega_S + i\partial\bar{\partial}g(u),$$

where ω_S is the Kähler form of the Kähler metric in L associated with the potential $a\log u + bu$.

To show completeness of the metric it will suffice to compute the distance from the zero section in the normal directions, and prove that it is an exhausting function, i.e., it approaches $+\infty$ as we get closer to the boundary of the fiber, in the case there is one, or as we get to ∞ when there is no boundary at all. This amounts to estimate the function

$$h(u) = \int_0^u (\dot{s}(\tau) + \tau\ddot{s}(\tau))^{\frac{1}{2}} d\tau,$$

and show it blows-up as $u \nearrow \alpha$. If $\sigma > 0$, $\dot{s}(u)$ blows-up in finite time and that result is clear since $\dot{w} = \ddot{s} > 0$ satisfies (4). When $\sigma = 0$ the function $\dot{s}(u)$ is increasing and the result is also obvious because $\alpha = \infty$.

When $\sigma < 0$ we have to be more careful. In general the function $u\dot{s}(u)$ is increasing and when $\sigma < 0$ it is defined on $[0, \infty)$. Furthermore, it is bounded on intervals of the form $[0, p)$. Since $\dot{s} + u\ddot{s}$ is positive, given the differential (4) that $\ddot{s}$ satisfies, $u\dot{s}(u)$ must be globally bounded. Therefore this function must approach a positive limit at ∞. From this, and using the differential equation once again, we conclude that $u^2\ddot{s}$ must also converge and thus, $\dot{s} + u\ddot{s} \to 0$ as $u \to \infty$. We look at the function $q(u) = u^2(\dot{s} + u\ddot{s})$. From (4) and the converge of the various terms above, we conclude that $q(u)$ cannot approach 0 at ∞, and in fact, it remains positively away from zero as u goes to ∞. Therefore, $\sqrt{q(u)/u^2}$ is asymptotically as C/u for some positive constant C and, consequently, its integral growth as $\log u$ near ∞, showing the desired statement for the growth of the function $h(u)$.

This shows that the metrics we obtain are complete. We summarize this discussion in the following

Theorem 1. *Let σ be a real constant. Then blowing-up a sufficiently small symmetric neighborhood of the origin in $\mathbf{C}^n$ we obtain a disk bundle $D \xrightarrow{\pi} \mathbf{CP}_{n-1}$ whose total space carries a complete Kähler metric of constant scalar curvature σ with radially symmetric Kähler potential. If $\sigma \leqq 0$ the bundle D can be taken to be the universal line bundle $L \xrightarrow{\pi} \mathbf{CP}_{n-1}$ of Chern class -1, while in the case where $\sigma > 0$ the bundle D is properly contained in L.*

Suppose now that σ is a positive constant and $n = 2$. Given an integer $k > 1$, by Proposition 1 (2) has solutions of the form

$$\phi(u) = a\log u - \frac{3}{\sigma}\log(1 - u^k),$$

where $a = 3(k-1)/2 > 0$. Expanding in Taylor series we get:

$$\phi(u) = a\log u + \frac{3}{\sigma}u^k + p(u) = \phi_0(u) + p(u),$$

where we let the expression define both, $\phi_0(u)$ and p. We claim that ϕ_0 is the Kähler potential of a complete metric on $L^{\otimes k}$. That will indicate that $\phi(u)$ itself stands a chance of being the Kähler potential of a metric in the blow-up of $\{z \in \mathbf{C}^2 : \|z\| < 1\}$ at the origin.

Indeed, consider the projection map $\mathbf{C}^2 - 0 \xrightarrow{pr} \mathbf{CP}_1$. If $z = (z_1, z_2) \in \mathbf{C}^2 - 0$, the inverse image under pr of $[z_1, z_2] \in \mathbf{CP}_1$ is the punctured complex line $\{\lambda(z_1, z_2) : \lambda \in \mathbf{C} - 0\}$. Therefore on the fiber $\partial\bar{\partial}u^k$ is just $k^2 u^{k-1} d\lambda d\bar{\lambda}$. Hence changing r to $\hat{r} = \|z\|^{k-1} r^k / k$ while changing θ to $\hat{\theta} = k\theta$ simultaneously, the metric on the fiber given by $\|z\| \, \|\lambda\|^{2(k-1)}(dr^2 + r^2 d\theta^2)$ becomes $(d\hat{r}^2 + \hat{r}^2 d\hat{\theta}^2)$. This is a smooth metric in the quotient of the fiber by the discrete group $\mathbf{Z}_k$ and therefore, since the logarithmic term produces a nonnegative contribution, ϕ_0 is the Kähler potential of a smooth metric on $L^{\otimes k}$, as stated. The completeness is rather clear. It is worth mentioning that the arguments here follow closely those related to the discussion of the Eguchi-Hanson metrics [EH] (see also [Le]).

To obtain our example we now proceed as before and consider the bundle $D \xrightarrow{\pi} \mathbf{CP}_1$ obtained by blowing-up $\{z \in \mathbf{C}^2 : \|z\| < 1\}$ at the origin. Since

$$\begin{aligned} \partial\bar{\partial}\phi(u) &= \partial\bar{\partial} a \log u + \frac{3}{\sigma(1-u^k)} \left(\partial\bar{\partial}u^k + \frac{\partial u^k \bar{\partial} u^k}{1-u^k} \right) \\ &= \partial\bar{\partial} a \log u + \frac{3}{\sigma(1-u^k)^2} \\ &\quad \times (k u^{k-1}(1-u^k)\partial\bar{\partial}u + k u^{k-2}(k-1+u^k)\partial u \bar{\partial} u), \end{aligned}$$

it is fairly clear that this will define a Kähler metric in $D^{\otimes k}$ and the boundary of the fibers of this bundle are at ∞, the distance measured from the zero section in the normal directions.

We summarize this discussion into the following

Theorem 2. *Let σ be a positive real constant and k an arbitrary integer greater than 1. Then blowing-up $\{z \in \mathbf{C}^2 : \|z\| < 1\}$ at the origin we obtain a disk bundle $D \xrightarrow{\pi} \mathbf{CP}_1$ such that $D^{\otimes k}$ is a complete Kähler manifold with Kähler potential*

$$\phi(u) = \frac{3(k-1)}{2} \log u - \frac{3}{\sigma} \log(1-u^k),$$

for $u = \|z\|^2$.

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