Math. Ann. 291, 75–86 (1991)

Mathematische Annalen

Isoperimetric inequalities and identities for k-dimensional cross-sections of convex bodies

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Received January 28, 1991; in revised form March 14, 1991

0 Introduction

Let E be an n-dimensional ellipsoid centered at the origin in $\mathbf{R}^n$ and let $\mathrm{Gr}(k,n)$ denote the Grassmann manifold of k-dimensional subspaces L through the origin in $\mathbf{R}^n$. The following formula was discovered by Furstenberg and Tzkoni [8]:

$$c_{k,n}\{\mathrm{Vol}_n(E)\}^k = \int_{\mathrm{Gr}(k,n)} \{\mathrm{Vol}_k(E\cap L)\}^n dL\,.$$

Here dL is the normalized rotation invariant measure on $\mathrm{Gr}(k,n)$. We will sometimes write dL as $d\mu_k(L)$ to emphasize the dependence on k. The constant $c_{k,n}$ is chosen to make this an equality when E is the ball. Thus

$$c_{k,n} = \left\{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right\}^k \Big/ \left\{\frac{k}{2}\Gamma\left(\frac{k}{2}\right)\right\}^n .$$

Furstenberg and Tzkoni give two rather elegant derivations of their formula, based on the representation theory of the group $SL_n(\mathbf{R})$. They indicate that for $k=1$ this volume formula is just integration in polar coordinates and is valid for any symmetric star shaped body in $\mathbf{R}^n$, while for general k the identity does not appear to reduce to any well-known formula, and they do not know in what generality it holds. Miles [15] gives a very simple derivation of this identity based on a classical formula of Blaschke and Petkantschin. He remarks that his derivation suggests a negative answer to the question of extendibility of this formula to more general regions. Finally, in a much earlier paper, Busemann [5] proves the estimate

$$\mathrm{Vol}_n(M)^{n-1} \geqq c_n \int_{\mathrm{Gr}(n-1,n)} \{\mathrm{Vol}_{n-1}(M\cap L)\}^n dL$$

(valid for a convex body $M \subset \mathbf{R}^n$) and shows that equality holds only when M is an ellipsoid centered at the origin. (Actually, this follows from a more general estimate

* Supported in part by a grant from the National Science foundation

involving several convex bodies, as we shall see below.) The methods found in these papers are sufficient to handle the case of general k, but it does not appear that they have been put together before. After treating some preliminaries in Sects. 1–3, including a projection analog of the cross-section integral, we adapt these methods in Sect. 4 to the case $1<k<n$ and give slightly more detailed proofs for some of the basic steps. We show that if E is a convex body in $\mathbf{R}^n$ then

$$c_{k,n}\{\mathrm{Vol}_n(E)\}^k \geqq \int_{\mathrm{Gr}(k,n)} \{\mathrm{Vol}_k(E\cap L)\}^n dL,$$

with equality precisely when E is an ellipsoid centered at the origin. In Sect. 5 we replace $\mathbf{R}^n$ by $\mathbf{C}^n$ and give an estimate which reduces to an identity for *complex* ellipsoids and characterizes these among all convex bodies in $\mathbf{C}^n$. We should also mention the paper of Guggenheimer [10]. There he gives the following variant of the Furstenberg-Tzkoni formula:

$$c_n \mathrm{Vol}_n(E)^{n-1}\cdot \mathrm{Vol}_{n-1}(\partial E) = \int_{\mathrm{Gr}(n-1,n)} \mathrm{Vol}_{n-1}(E\cap L)^{n+1} dL,$$

valid whenever E is an ellipsoid centered at the origin. This derivation is based on a curvature calculation and is somewhat different from the previous formulae; we do not know isoperimetric inequalities associated with it. Some of the results in this paper were announced in an I.H.E.S. preprint (see Grinberg [9]). We thank E. Lutwak for some valuable discussions and the referee for helpful remarks.

1 Quermassintegrals

The $n+1$ *Quermassintegrals* $W_0(K), W_1(K), \ldots, W_n(K)$ of a convex body K in $\mathbf{R}^n$ are defined by letting $W_0(K)=\pi_n$, the volume of the unit n-ball, and for $0<k<n$,

$$W_{n-k}(K)=\pi_n \int_{\mathrm{Gr}(k,n)} \frac{\mathrm{Vol}_k(K|E)}{\pi_k}\, d\mu_k(E).$$

In the above integral $d\mu_k$ is the rotation invariant probability measure on $\mathrm{Gr}(k,n)$, and $\mathrm{Vol}_k(K|E)$ denotes the k-dimensional volume of the orthogonal projection of K onto $E\in \mathrm{Gr}(k,n)$.

The $n+1$ dual Quermassintegrals of K, $\tilde{W}_0(K), \tilde{W}_1(K), \ldots, \tilde{W}_n(K)$ are defined in exactly the same way, except that the projection $K|E$ is replaced by the intersection $K\cap E$.

The inequality between the volume of K and its Quermassintegrals is

$$\tilde{W}_k(K) \leqq \pi_n^{k/n} V(K)^{(n-k)/n} \leqq W_k(K), \qquad (0<k<n)$$

with equality in the right inequality iff K is a ball, and equality in the left inequality iff an origin centered ball (see Burago-Zalgaller [3]). Note that the right inequality is the $(n-1)$-direct extension of the plane isoperimetric inequality. The case $k=1$ is the classical inequality between volume and surface area while the case $k=n-1$ is the classical (Urysohn) inequality between volume and mean width.

The $n+1$ affine Quermassintegrals of a body K in $\mathbf{R}^n$, $\Phi_0(K), \Phi_1(K), \ldots, \Phi_n(K)$ are defined by letting $\Phi_0(K)=V(K)$, $\Phi_n(K)=\pi_n$ and for $0<k<n$ by:

$$\Phi_{n-k}(K)=\pi_n\left[\int_{\mathrm{Gr}(k,n)} [\mathrm{Vol}_k(K|E)/\pi_k]^{-n} d\mu_k(E)\right]^{-1/n}.$$

The $n+1$ dual affine Quermassintegrals of K, $\tilde{\Phi}_0(K)$, $\tilde{\Phi}_1(K), \ldots, \tilde{\Phi}_n(K)$ are defined by letting $\tilde{\Phi}_0(K) = V(K)$, $\tilde{\Phi}_n(K) = \pi_n$ and for $0 < k < n$ by

$$\tilde{\Phi}_{n-k}(K) = \pi_n \left[\int_{\mathrm{Gr}(k,n)} [\mathrm{Vol}_k(K \cap E)/\pi_k]^n d\mu_k(E) \right]^{1/n}.$$

These integrals were first introduced by Lutwak [12–14]. As we shall prove in the next section, his usage of the adjective *affine* is justified. Clearly, from Jensen's inequality it follows that

$$(*) \qquad \tilde{W}_k(K) \leqq \tilde{\Phi}_k(K) \quad \text{and} \quad \Phi_k(K) \leqq W_k(K),$$

for all k. It had been conjectured by Lutwak that a stronger inequality than (*) holds. Specifically, the inequality between the volume of a body and its affine Quermassintegrals is

$$(**) \qquad \tilde{\Phi}_k(K) \leqq \pi_n^{k/n} V(K)^{(n-k)/n} \leqq \Phi_k(K), \qquad (0 < k < n-1)$$

with equality in the right inequality if and only if K is an ellipsoid and equality in the left inequality if and only if K is an origin centered ellipsoid. The case $k = n-1$ of the right inequality follows from the Blaschke-Santaló inequality, see Lutwak [12]. The case $k = 1$ of the right inequality is the Petty projection inequality [16]. The case $k = 1$ of the left inequality is the Busemann intersection inequality [5]. The fact that for ellipsoids there is equality in the left inequality is known as the Furstenberg-Tzkoni formula, while the fact that there is equality in the right inequality is the dual Furstenberg-Tzkoni formula and was first proved by Lutwak.

The cases $1 < k < n-1$ for both the left and right inequalities in (**) had been open. In this article the left inequality, with equality conditions, will be shown to hold for all k. We hope that the right inequality may be handled in the future using similar ideas.

2 Affine invariance

In accordance with common usage in the convexity literature we define the group of *affine transformations* in $\mathbf{R}^n$ to be the group of linear transformations generated by translations and by unimodular linear automorphisms. We will show in this section that the affine Quermassintegrals $\Phi_k(K)$ and the dual affine Quermassintegrals $\tilde{\Phi}_k(K)$ are affine invariants, justifying the adjectives used.

Theorem 1. *Let K be a measurable body in $\mathbf{R}^n$ and let g be an affine transformation. Then for $0 < k < n$, $\tilde{\Phi}_k(K) = \tilde{\Phi}_k(gK)$,* provided g is linear.

Proof. Let H denote a k-plane in $\mathbf{R}^n$ and let g be an affine transformation. Then the function

$$\sigma_k(g, H) \equiv \frac{\mathrm{Vol}_k(g(K \cap H))}{\mathrm{Vol}_k(K \cap H)}$$

is a *multiplier function* on the symmetric space $\mathrm{Gr}(k, n)$, [8]. The Radon-Nikodym derivative

$$\sigma_{\mathrm{Gr}(k,n)}(g, H) = \frac{dg^{-1}H}{dH}$$

is also a multiplier on $\mathrm{Gr}(k,n)$ and a calculation presented in [8] shows that

$$\sigma_{\mathrm{Gr}(k,n)}(g,H)=\sigma_k(g,H)^{-n}.$$

Now

$$\begin{aligned}\tilde{\Phi}_{n-k}(gK)&=\pi_n\Bigg[\int\limits_{H\in\mathrm{Gr}(k,n)}[\mathrm{Vol}_k(gK\cap H)/\pi_k]^n d\mu_k(H)\Bigg]^{1/n}\\
&=\pi_n\Bigg[\int\limits_{gH\in\mathrm{Gr}(k,n)}[\mathrm{Vol}_k(gK\cap gH)/\pi_k]^n d\mu_k(gH)\Bigg]^{1/n}\\
&=\pi_n\Bigg[\int\limits_{H\in\mathrm{Gr}(k,n)}\sigma_k(g,H)^n[\mathrm{Vol}_k(K\cap H)/\pi_k]^n d\mu_k(gH)\Bigg]^{1/n}\\
&=\pi_n\Bigg[\int\limits_{H\in\mathrm{Gr}(k,n)}[\mathrm{Vol}_k(K\cap H)/\pi_k]^n d\mu_k(H)\Bigg]^{1/n}\\
&=\tilde{\Phi}_{n-k}(K).\quad\square\end{aligned}$$

It is clear that the intersection integrals are not translation invariant. To prove affine invariance for projection integrals (i.e. $\Phi_k(K)=\Phi_k(gK)$) we emulate the intersection case and use multipliers. For this we will need a projection volume identity.

Lemma 1. *Let K be a body in $\mathbf{R}^n$ and let ξ be a k-plane. Then for all $\phi\in SL_n(\mathbf{R})$*

$$\mathrm{Vol}_k(\phi K\,|\,\xi)=\mathrm{Vol}_k(K\,|\,\phi^t\xi)\sigma_k(\phi^t,\xi)$$

Proof. If ϕ is an orthogonal matrix then the identity is trivially true. Assume now that ϕ is *lower triangular* with respect to the coordinate system $\{\xi,\xi^\perp\}$. This means that $\phi^t\xi=\xi$ and we have the matrix representation

$$\phi=\begin{pmatrix}A & 0\\ C & D\end{pmatrix}.$$

Now $\mathrm{Vol}_k(K\,|\,\phi^t\xi)=\mathrm{Vol}_k(K\,|\,\xi)$, while

$$\mathrm{Vol}_k(\phi K\,|\,\xi)=\mathrm{Vol}_k(A[K\,|\,\xi]).$$

By definition of $\sigma_k(\cdot,\cdot)$, the right hand side above is $\sigma_k(A,\xi)\cdot\mathrm{Vol}_k(K\,|\,\xi)$. This proves the original assertion for ϕ lower triangular. In the general case we write ϕ as the product of a lower triangular matrix and an orthogonal matrix and combine the two previous observations. $\square$

Theorem 2. *Let K be a measurable body in $\mathbf{R}^n$ and let g be an affine transformation. Then for $0<k<n$, $\Phi_k(K)=\Phi_k(gK)$.*

Proof. We now parallel the intersection integral case:

$$\begin{aligned}\Phi_{n-k}(gK)&=\pi_n\Bigg[\int\limits_{H\in\mathrm{Gr}(k,n)}[\mathrm{Vol}_k(gK\,|\,H)/\pi_k]^{-n}d\mu_k(H)\Bigg]^{-1/n}\\
&=\pi_n\Bigg[\int\limits_{H\in\mathrm{Gr}(k,n)}[\mathrm{Vol}_k(K\,|\,g^tH)/\pi_k]^{-n}\{\sigma_k(g^t,H)\}^{-n}d\mu_k(H)\Bigg]^{-1/n}\\
&=\pi_n\Bigg[\int\limits_{H\in\mathrm{Gr}(k,n)}[\mathrm{Vol}_k(K\,|\,H)/\pi_k]^{-n}\{\sigma_k(g^t,g^{-t}H)\}^{-n}d\mu_k(g^{-t}H)\Bigg]^{-1/n}\\
&=\pi_n\Bigg[\int\limits_{H\in\mathrm{Gr}(k,n)}[\mathrm{Vol}_k(K\,|\,H)/\pi_k]^{-n}\{\sigma_k(g^t,g^{-t}H)\}^{-n}\\
&\qquad\times\{\sigma_k(g^{-t},H)\}^{-n}d\mu_k(H)\Bigg]^{-1/n}\end{aligned}$$

and using the multiplier property of σ_k

$$= \pi_n \left[\int_{H \in \mathrm{Gr}(k,n)} [\mathrm{Vol}_k(K|H)/\pi_k]^{-n} d\mu_k(H) \right]^{-1/n}$$
$$= \Phi_{n-k}(K). \quad \square$$

It should be noted that the *non*-invariance of $\Phi(K)$ under affine transformations has been conjectured a number of times and attempts at counterexamples have been made. With this simple proof the invariance is settled and the exponent $-n$ is distinguished as the "right" one to use. A routine modification yields a complex version of all this.

3 Ellipsoids and Banach norms

Let V be a finite dimensional real Banach space with norm function $\|\cdot\|$. Below we list some well known facts in functional analysis.

Proposition 1. *The following are equivalent:*
i) *The Banach norm $\|\cdot\|$ is induced by a Hilbert space structure.*
ii) *$\|\cdot\|$ satisfies the parallelogram law*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

iii) *The hypersurface $\{v \in V \mid \|v\| = 1\}$ is an ellipsoid.*

Using this we can get another characterization of ellipsoids.

Theorem 3 (Busemann). *Let E be a convex body in $\mathbf{R}^n$. Let k be an integer with $1 < k < n$. Suppose that for every k-plane L through the origin the intersection $E \cap L$ is an ellipsoid. Then E is an ellipsoid.*

Proof. We use Proposition 1 above. Notice that E is symmetric about the origin ($x \in E \Leftrightarrow -x \in E$). Choose a convex *defining function* $N(x)$ for E so that N is positively homogeneous of degree 1 and

$$E = \{x \in \mathbf{R}^n \mid N(x) \leqq 1\}.$$

Then $N(x)$ defines a Banach norm function in $\mathbf{R}^n$. We verify that this norm satisfies the parallelogram law. Select vectors x and $y \in \mathbf{R}^n$. Then $x, y, x+y, x-y$ all lie in some 2-plane through the origin. Take a k-plane L containing this 2-plane. Since $E \cap L$ is an ellipsoid in L the restriction of N to L satisfies the parallelogram law. In particular,

$$N(x+y)^2 + N(x-y)^2 = 2N(x)^2 + 2N(y)^2.$$

Since this is true for arbitrary $x, y \in \mathbf{R}^n$, $N(\cdot)$ satisfies the parallelogram law on all of $\mathbf{R}^n$ and by Proposition 1, E is an ellipsoid. $\square$

A slight variation of this yields a characterization of ellipsoids by *complex* slices. A real ellipsoid in $\mathbf{C}^n$ is called *Hermitian* if it is defined by a Hermitian quadratic form.

Theorem 4. *Let E be a convex body in $\mathbf{C}^n$. Let k be an integer with $1 < k < n$. Suppose that for every complex k-plane L through the origin the intersection $E \cap L$ is an ellipsoid. Then E is an ellipsoid. If each slice is Hermitian then E is Hermitian.*

Proof. Almost the same as the proof of Theorem 1 above. The punchline is that any two vectors x, y in $\mathbf{C}^n$ are contained in *some* complex k-plane L through the origin. □

Note that to detect real ellipsoids we never have to use complex norms but only the complex vector space structure in the proof. In this case the complex planes form a much "thinner" detector than the set of all real $2k$-planes.

The case $k=n-1$ of the real theorem (3 above) is needed in Busemann's paper [5], but is not explicitly mentioned there. On the other hand, Busemann gives a more general theorem in his book [6]. A further generalization may be found in Burton [4]. The proof given here seems to be known but not widely exposed in the literature and usually rediscovered in applications. For this reason, we have chosen to include it.

4 Real ellipsoids

Let $p_1, \ldots, p_k$ be points in a k-dimensional vector space V. We denote by $T(p_1, \ldots, p_k)$ the k-dimensional Euclidean volume of the parallelotope defined by these points and the origin. Let $M_1, \ldots, M_k$ be convex bodies in V. Here and henceforth all convex bodies are assumed compact and with interior. Departing somewhat from Busemann's notation [5] we denote by τ_k^m the functional

$$\tau_k^m(M_1, \ldots, M_k, V) \equiv \int_{p_1 \in M_1} \cdots \int_{p_k \in M_k} T(p_1, \ldots, p_k)^m dV_{p_1} \ldots dV_{p_k}.$$

Theorem 5. *Let $M_1, \ldots, M_k$ be measurable sets in $\mathbf{R}^n$. Then*

$$\mathrm{Vol}_n(M_1) \ldots \mathrm{Vol}_n(M_k) = c_{k,n} \int_{L \in \mathrm{Gr}(k,n)} \tau_k^{n-k}(M_1 \cap L, \ldots, M_k \cap L, L) dL.$$

Here $\mathrm{Gr}(k,n)$ is the real Grassmann manifold of k-planes through the origin in $\mathbf{R}^n$ and dL is its rotation invariant measure with total mass adjusted to make this identity hold in the case where all the bodies in question are spheres.

Proof. Let Z_k be the "incidence manifold"

$$\{(L, x_1^k, \ldots, x_k^k) \mid L \in \mathrm{Gr}(k,n),\ x_1^k \in L, \ldots, x_k^k \in L\}.$$

Then Z_k is just the k-fold power of the tautological bundle over $\mathrm{Gr}(k,n)$. Let X denote $\mathbf{R}^n$ and let X^k denote the k-fold cartesian product of X with itself. Then there is a map $\phi: Z \to X^k$ defined by

$$\phi(L, x_1^k, \ldots, x_k^k) = (x_1^k, \ldots, x_k^k).$$

Clearly, ϕ is onto. Let dV denote the Euclidean volume form on X^k; this is just the k-fold wedge of the volume form on X. Then a calculation presented in Miles [15] and Santaló [17] and attributed to Blaschke and Petkantschin gives

$$\phi^*(dV) = T(x_1^k, \ldots, x_k^k)^{n-k} d\mu(L) \prod_{i=1}^{k} dx_i^k(L).$$

Here $d\mu(L)$ is the unnormalized measure on $\mathrm{Gr}(k,n)$ (with total mass adjusted to make the formula true) and $dx_i^k(L)$ is the Euclidean measure in the k-plane L (which can be viewed as a fibre-density on the tautological bundle over $\mathrm{Gr}(k,n)$. The theorem follows from this last identity and the change of variable formula. □

The case $k=n-1$ can be found in [5] (the case $k=n-2$ is also implicit there).

Recall now the Steiner symmetrization process (see, for example, Guggenheimer [10]). Let E be a convex body in $\mathbf{R}^n$ and let P be a hyperplane through the origin with unit normal v. Then E is a union of 1-dimensional line segments parallel to v. Take each such segment I and slide it parallel to itself until its midpoint lies in P. The resulting set $\bar{E}$ is again convex and has the same volume as E. In general, if p is a point in E we will denote by $\bar{p}$ its image under symmetrization (implicitly, this is with respect to P, but we'll not always say so explicitly). Likewise, the image of the entire set E under symmetrization will be denoted by $\bar{E}$. Steiner symmetrization can also be performed on compact, non-convex bodies. In that case, one assigns to the set $E\cap v$ an interval I through v with center in P and length equal to the 1-dimensional Hausdorff measure of $E\cap v$. See Federer, [7, Sect. 2.10.30].

Theorem 6. *Let $M_1, \dots, M_k$ be compact (but not necessarily convex) bodies in $V=\mathbf{R}^k$. Then for any hyperplane P the functional τ_k^m does not increase under simultaneous symmetrization in P:*

$$\tau_k^m(M_1, \dots, M_k, V) \geqq \tau_k^m(\bar{M}_1, \dots, \bar{M}_k, V).$$

Proof. We give a coordinate free variation of Busemann's original proof; this will be useful when we consider complex analogs. Let v denote a unit normal for the hyperplane P. Assume first that the M_i are convex. Choose points

$$p_1 \in M_1, \dots, p_k \in M_k.$$

Then there are unique points $q_i \in P$ and real numbers t_i $(i=1, \dots, n)$ so that

$$p_i = q_i + t_i v.$$

By standard properties of determinants (multilinearity and antisymmetry), the volume

$$T(p_1, \dots, p_k) \equiv T(q_i, \dots, q_k, t_1, \dots, t_k)$$

is the absolute value of a linear (real) function in $t \equiv (t_i)$ for fixed $q \equiv (q_i)$. We'll denote this function by $f(t)$. For each i let p_i' denote the reflection of p_i about the center of the interval through p_i in M_i parallel to v. Then if $p \equiv (p_i)$, $p' \equiv (p_i')$ then $p = q + tv$, $p' = q + t'v$ and the collection of center vectors c satisfies

$$c = \frac{p+p'}{2} = q + \left(\frac{t+t'}{2}\right) v.$$

Hence

$$2T(p-c) = 2\left|f\left(\frac{t-t'}{2}\right)\right| \leqq |f(t)| + |f(t')| = T(p) + T(p').$$

Since $p_i - c_i = \overline{p_i}$ is the image of the point p_i under P-symmetrization and since

$$\tau_k^m(M_1, \dots, M_k) \equiv \int_{p_1 \in M_1} \dots \int_{p_k \in M_k} T(p_1, \dots, p_k)^m dV_{p_1} \dots dV_{p_k},$$

and

$$\tau_k^m(M_1, \dots, M_k) \equiv \int_{p_1 \in M_1} \dots \int_{p_k \in M_k} T(p_1', \dots, p_k')^m dV_{p_1} \dots dV_{p_k},$$

we see that τ_k does not increase under symmetrization. In the case that the M_i are non-convex, we appropriate the (almost always) measurable sets $M_i \cap (p_i + \mathbf{R}v)$ by finite unions of intervals and apply the same arguments above. $\square$

The theorem for *convex* bodies M_i is given in Busemann's original paper. To decide the case for equality above, we will use an n-dimensional version of the following result.

Bertrand's Theorem. *Let C be a convex plane curve. If the midpoints of parallel chords of C are always collinear then C is a conic.*

See Bertrand, [1]. An n-dimensional result may be obtained from this theorem by applying the arguments of Sect. 1. Blaschke has also given a theorem in this direction (see Blaschke [2]). Here we will give an alternative argument which was hinted at in Busemann's original paper.

Theorem 7. *Let B be a convex body in $\mathbf{R}^n$. If for each vector $v \neq 0$ the chords of B in the direction v have their midpoints in a fixed hyperplane $Q = Q(v)$ then B is an origin centered ellipsoid.*

Proof. Note that the hypothesis on B is stated entirely in terms of the affine structure of $\mathbf{R}^n$ and so it holds for all affine images of B. By a result of C. Löwner (see Guggenheimer, [10, p. 149]) there is a unique ellipsoid E containing B, centred at the origin and of minimal volume. Let P be the hyperplane through the origin with normal v and consider the Steiner symmetrization of B with respect to P. The symmetrization map $p \to \bar{p}$ is realized by a *global affine* transformation T (this is the punchline of the proof). In fact, any point $x \in \mathbf{R}^k$ can be written uniquely as $x = q + tv$, where $q \in Q$ and t is real (since B has interior, the hypothesis on B implies that v is transverse to Q). Then $\bar{x} = \Pi(q) + tv$ where Π is orthogonal projection from $\mathbf{R}^k$ onto P. This affine map takes the minimal circumscribing ellipsoid E into a new ellipsoid $\bar{E}$. Since T is volume preserving, $\mathrm{Vol}(E) = \mathrm{Vol}(\bar{E})$ and this implies that $\bar{E}$ is the minimal circumscribing ellipsoid for $\bar{B}$: if $\bar{B}$ had a smaller circumscribing ellipsoid F then $T^{-1}(F)$ would be a smaller circumscribing ellipsoid for B. Now $\bar{B}$ inherits the properties of B and so this argument may be repeated. There is a sequence of symmetrizations of B which converges to a sphere with volume $\mathrm{Vol}(B)$. This sequence carries the minimal "surrounding" ellipsoid E with it at every step. In the limit, the image of this ellipsoid must coincide with the limit sphere, and hence it has the same volume. Thus the original bounding ellipsoid must have had the same volume as B and so $B = E$, an origin centered ellipsoid. □

Remark. The Löwner ellipsoid is, of course superfluous to the argument. The argument can be made entirely in the realm of a compact set of affine transformations. However, we have kept the ellipsoid construction for a number of reasons. First, it retains the spirit of Busemann's original paper. Second, the proof of compactness in the purely affine-map argument is tantamount to the introduction of the ellipsoid. Third, the conclusion that B approaches and finally coincides with its Löwner ellipsoid has geometric appeal. And finally, it suggests an argument for the complex analog (see the next section).

Theorem 8. *If equality holds in Theorem 6 and each body M_i is convex then the M_i's form a homothetic family of origin centered ellipsoids.*

Proof. We retain the notation of the proof of Theorem 6. Note that when $p = c$, i.e. when the p_i are centers of intervals, $T(p)$ must vanish identically, hence the p_i must all lie in a hyperplane. By varying just one p_i while keeping all the others fixed we see that all the "centers" must lie in just *one* hyperplane. The previous theorem implies that each M_i is an origin centered ellipsoid.

To show that for every i and j the ellipsoid M_i and M_j are homothetic, choose a principal axis v of M_i and let P be the hyperplane through the origin with normal v. Then the centers of all chords of M_i in the direction v lie in P, so the same must be true of M_j. Thus P is a plane of symmetry for M_j and hence v is a principal axis. To show that the lengths of the axes of M_i and M_j are in the same ratios, choose two such axes u and v. Put $w = u + v$. Then the plane composed of centers of chords of M_i parallel to w depends in a one to one fashion on the ratios of the lengths of the axes u and v in M_i. Since this plane coincides with the corresponding one for M_j, the ratio of lengths must be the same. □

Theorem 9. *Let $M_1, \ldots, M_k$ be convex bodies with interior points in $\mathbf{R}^n$. Then*

$$\mathrm{Vol}_n(M_1) \ldots \mathrm{Vol}_n(M_k) \geqq c_{k,n} \int_{\mathrm{Gr}(k,n)} \mathrm{Vol}_k(M_1 \cap L)^{n/k} \ldots \mathrm{Vol}_k(M_k \cap L)^{n/k} dL$$

with equality only for homothetic ellipsoids centered at the origin.

Proof. We start with the identity

$$\mathrm{Vol}_n(M_1) \ldots \mathrm{Vol}_n(M_k) = c_{k,n} \int_{L \in \mathrm{Gr}(k,n)} \tau_k^{n-k}(M_1 \cap L, \ldots, M_k \cap L, L) dL.$$

We now perform a series of simultaneous Steiner symmetrizations of the bodies $M_i \cap L$ in the k-plane L. This does not increase the right hand side above. The symmetrizations can be chosen so that in the limit we obtain from $M_i \cap L$ a ball B_i in L of volume $\mathrm{Vol}_k(M_i \cap L)$. Let $D_i \equiv D_i(L)$ be the ball in $\mathbf{R}^n$ satisfying $B_i = D_i \cap L$. Then we have shown that

$$\mathrm{Vol}_n(M_1) \ldots \mathrm{Vol}_n(M_k) \geqq c_{k,n} \int_{L \in \mathrm{Gr}(k,n)} \tau_k^{n-k}(D_1 \cap L, \ldots, D_k \cap L, L) dL,$$

and by Theorem 5 this is

$$= c_{k,n} \int_{L \in \mathrm{Gr}(k,n)} \mathrm{Vol}_n(D_1(L)) \ldots \mathrm{Vol}_n(D_k(L)) dL$$

$$= c_{k,n} \int_{L \in \mathrm{Gr}(k,n)} \mathrm{Vol}_k(M_1 \cap L)^{n/k} \ldots \mathrm{Vol}_k(M_k \cap L)^{n/k} dL,$$

since $\mathrm{Vol}_n(D_i) = c'_{k,n}\{\mathrm{Vol}_k(B_i)\}^{n/k}$. This gives the inequality. If equality actually holds then, by Theorem 8, for every k-plane L the slice $M_i \cap L$ is an ellipsoid for every i. By Theorem 3, each M_i is an ellipsoid. Moreover, $M_i \cap L$ and $M_j \cap L$ are homothetic for every i and j and any k-plane L. Therefore, M_i and M_j must have the same axes (consider an L containing an axis of M_i and an axis of M_j). Further, the lengths of the axes must be in the same ratios (just look at two axes at a time, as before). □

Corollary 1. *Let M be a convex body in $\mathbf{R}^n$. Then for any k with $1 < k < n$ we have*

$$\mathrm{Vol}_n(M)^k \geqq c_{k,n} \int_{\mathrm{Gr}(k,n)} \mathrm{Vol}_k(M \cap L)^n dL$$

with equality if and only if M is an ellipsoid centered at the origin.

3 Complex ellipsoids

We now replace $\mathbf{R}^k$ by $\mathbf{C}^k$ and consider the corresponding isoperimetric inequalities. Let $p_1, \ldots, p_k$ denote points in a complex k-dimensional vector space V and let $J(p_1, \ldots, p_k)$ denote the absolute value of the complex determinant of the

$k \times k$ matrix with columns $p_1, \ldots, p_k$. We denote by σ_k^m the functional

$$\sigma_k^m(M_1, \ldots, M_k, V) \equiv \int_{p_1 \in M_1} \cdots \int_{p_k \in M_k} J(p_1, \ldots, p_k)^m dV_{p_1} \ldots dV_{p_k}.$$

Then we have the following variant of Theorem 5.

Theorem 10. *Let $M_1, \ldots, M_k$ be bounded measurable sets in $\mathbf{C}^n$. Then*

$$\mathrm{Vol}_n(M_1) \ldots \mathrm{Vol}_n(M_k) = c_{k,n} \int_{L \in \mathrm{Gr}_{\mathbf{C}}(k,n)} \sigma_k^{2(n-k)}(M_1 \cap L, \ldots, M_k \cap L, L) dL.$$

Here $\mathrm{Gr}_{\mathbf{C}}(k,n)$ is the complex Grassmannian of complex k-planes through the origin in $\mathbf{C}^n$.

The proof amounts to a direct emulation of derivations found in [15, 17] which lead to Theorem 5, but wherever a form appears in the real case, we now put the corresponding holomorphic form of the same degree times its complex conjugate. A representation-theoretic proof in the spirit of Furstenberg-Tzkoni is also possible.

Let E be a real convex body in $\mathbf{C}^n$ (i.e. a convex body in the underlying $\mathbf{R}^{2n}$). We say that E is a *complex ellipsoid* centered at the origin if E is a real ellipsoid centered at the origin and E is circular ($z \in E$, $\theta \in \mathbf{R}$, $\Rightarrow e^{i\theta} z \in E$). Equivalently, E is the unit ball of a complex Hilbert space structure in $\mathbf{C}^n$.

Lemma 2. *Let M be a convex body in $\mathbf{C}^n$. Then there is a unique complex ellipsoid E which contains M, is centered at the origin, and has minimal volume.*

Proof. There exists at least one such minimal ellipsoid, by the same proof as in the real case, using the Blaschke convergence and selection theorems, see [10, p. 149]. We have to show that this complex ellipsoid is unique.

If A and B are symmetric positive semi-definite $m \times m$ matrices then we have the following inequality for their determinants:

$$\left|\det\left(\frac{A+B}{2}\right)\right| \geqq |\det(A)|^{1/2} \cdot |\det(B)|^{1/2},$$

as follows from *Brunn's inequality*. If both A and B are non-singular then equality holds precisely when A and B are scalar multiples of one another (again, see [10]). Now if A and B are *complex* $n \times n$ matrices let $A_{\mathbf{R}}$ and $B_{\mathbf{R}}$ denote the real $2n \times 2n$ matrices which represent the corresponding linear transformations in $\mathbf{R}^{2n}$. Then $|\det(A_{\mathbf{R}})| = |\det(A)|^2$, where the first determinant is in the sense of real $2n \times 2n$ matrices, while the second is in the sense of complex $n \times n$ matrices. The map $A \to A_{\mathbf{R}}$ is clearly additive and linear over the reals, so we have for *Hermitian* positive semi-definite matrices A and B

$$\left|\det\left(\frac{A+B}{2}\right)\right| \geqq |\det(A)|^{1/2} |\det(B)|^{1/2}.$$

Let $\{z \mid zAz^* = 1\}$ and $\{z \mid zBz^* = 1\}$ represent the boundaries of two minimal complex ellipsoids E and F containing M. Then A and B are Hermitian matrices and $\mathrm{Vol}(E) = c_n |\det(A)|^{-1}$, while $\mathrm{Vol}(F) = c_n |\det(B)|^{-1}$. As in the real case, the hypersurface $\left\{z \,\middle|\, z\left(\frac{A+B}{2}\right)z^* = 1\right\}$ also bounds a minimal ellipsoid (it contains $E \cap F$) and has smaller volume (by the above inequality) unless E and F are homothetic. □

Theorem 11. *Let $M_1, \ldots, M_k$ be convex bodies in $V = \mathbf{C}^k$. Then for any real hyperplane P the functional σ_k^m does not decrease under symmetrization in P:*

$$\sigma_k^m(M_1, \ldots, M_k, V) \geqq \sigma_k^m(\bar{M}_1, \ldots, \bar{M}_k, V).$$

Furthermore, equality holds above for arbitrary P after any sequence of symmetrizations if and only if the M_i's are complex ellipsoids centered at the origin and are homothetic to one another.

Proof. The non-increase of σ under symmetrization is proved essentially in the same way as in Theorem 6 above, but note that the corresponding function f is now complex valued. We turn to the case of equality.

This part of proof of the real case hinged upon the fact that for arbitrary "center" points $c_1 \in M_1, \ldots, c_k \in M_k$ we have $T(\bar{c}_1, \ldots, \bar{c}_k) = 0$ since the $\bar{c}_i$'s move into a real hyperplane. Here however we are considering a *complex* determinant $J(\bar{c}_1, \ldots, \bar{c}_k)$ and we cannot conclude that it is zero since our symmetrization process is still a *real* process and moves the c_i's into a *real* hyperplane, but not necessarily into a *complex* hyperplane. Our argument will interplay the real and complex geometry.

Fix a real hyperplane P in $\mathbf{C}^k$ with normal v. Put $w = \sqrt{-1}v$ and let R be the dual real hyperplane to w; note that $U \equiv P \cap R$ is a complex hyperplane. Let $c_i \in M_i$ be "center points" with respect to the direction w which are orthogonal to v. These are precisely center points of the set $M_i \cap P$ with respect to the direction w. After symmetrization in the direction w (all in the ambient P) the c_i become real orthogonal to *both* v and w and hence lie in U, the *complex* hyperplane through the origin with (complex) normal v. This is the crux of the argument. Since $J(\bar{c}_1, \ldots, \bar{c}_k) = 0$, such c_i's must all lie in a single *complex* hyperplane Q and, since M_i has interior, the only possibility is $Q = U$. Now center points in U are not moved at all under symmetrization in the direction w, so we have $\overline{M_i \cap P} = M_i \cap P$. In particular, $\overline{M_i} \supset M_i \cap P$. Let E_i (resp. F_i) be a complex, minimal volume, origin centered ellipsoid containing M_i (resp. $\overline{M_i}$). Note that E_i is actually the unique minimal surrounding ellipsoid for the set $S \equiv \{e^{\sqrt{-1}\theta} \cdot M_i \mid \theta \in \mathbf{R}\}$ (which may or may not coincide with M_i). Also, since for any $x \in \mathbf{R}^n$ there is a $\theta \in \mathbf{R}$ so that $e^{\sqrt{-1}\theta} x \in P$, $S = \{e^{\sqrt{-1}\theta} \cdot (M_i \cap P) \mid \theta \in \mathbf{R}\}$. This implies that E_i is the unique minimal ellipsoid containing $M_i \cap P$. But F_i is an ellipsoid containing $M_i \cap P$, so we must have $\mathrm{Vol}(F_i) \geqq \mathrm{Vol}(E_i)$.

Starting with M_i there is a sequence of symmetrizations tending to a ball of volume $\mathrm{Vol}(M_i)$. The previous paragraph shows that the volumes of the minimal surrounding ellipsoids of the symmetrized M_i cannot decrease. These volumes also tend to $\mathrm{Vol}(M_i)$ so the volume of the original ellipsoid E_i must have been precisely $\mathrm{Vol}(M_i)$ and hence $M_i = E_i$. Finally, the M_i are homothetic by the same argument as in the real case (here "axis" means complex axis, however). □

Note the difference between the real and complex proofs: in one case we have a sequence of equalities corresponding to symmetrizations, while in the other case they are inequalities.

Theorem 12. *Let $M_1, \ldots, M_k$ be convex bodies with interior points in $\mathbf{C}^n$. Then*

$$\mathrm{Vol}_n(M_1) \ldots \mathrm{Vol}_n(M_k) \geqq c_{k,n} \int_{Gr_{\mathbf{C}}(k,n)} \mathrm{Vol}_k(M_1 \cap L)^{n/k} \ldots \mathrm{Vol}_k(M_k \cap L)^{n/k} dL$$

width equality if and only if M is an ellipsoid centered at the origin.

This is proved just like in the real case, but using Theorem 4 instead of Theorem 3.

Corollary 2. *Let M be a convex body in* $\mathbf{C}^n$. *Then for any k with* $1<k<n$ *we have*

$$\mathrm{Vol}_n(M)^k \geqq c_{k,n} \int_{\mathrm{Grc}(k,n)} \mathrm{Vol}_k(M \cap L)^n dL$$

with equality if and only if M is a complex ellipsoid centered at the origin.

It appears that a corresponding analysis is possible in the quaternionic category. In the octonionic category it is not possible in general to define a manifold structure for the Grassmannian, but in some special dimensions it may be possible to detect *octonionic* ellipsoids by volume integrals based on the construction of the Cayley plane.

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