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Singular Perturbations of Variational Problems Arising from a Two-Phase Transition Model

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Abstract. Given that α, β are two Lipschitz continuous functions of Ω to $\mathbb{R}_+$ and that $f(x, u, p)$ is a continuous function of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ to $[0, +\infty[$ such that, for every x, $f(x, \cdot, 0)$ reaches its minimum value 0 at exactly two points $\alpha(x)$ and $\beta(x)$, we prove the convergence of $F^\varepsilon(u) = (1/\varepsilon) \int_\Omega f(x, u, \varepsilon Du)\, dx$ when the perturbation parameter ε goes to zero. A formula is given for the limit functional and a general minimal interface criterium is deduced for a wide class of two-phase transition models. Earlier results of [19], [21], and [22] are extended with new proofs.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let α, β be two Lipschitz continuous functions of Ω to $\mathbb{R}_+$ such that $\alpha(x) \le \beta(x)$, $\forall x \in \Omega$. The aim of this paper is to prove the variational convergence as $\varepsilon \to 0_+$ of functionals of the form

$$F^\varepsilon(u) = \frac{1}{\varepsilon} \int_\Omega f(x, u, \varepsilon Du)\, dx, \tag{1.1}$$

where f is a *continuous* function of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ to $[0, +\infty[$ such that

(H_1) $\forall x \in \Omega$, the function $f(x, \cdot, 0)$ achieves its minimum value 0 at exactly two points, namely $\alpha(x)$, $\beta(x)$, that is,

$$\forall x \in \Omega, \qquad f(x, u, 0) = 0 \quad \Leftrightarrow \quad u \in \{\alpha(x), \beta(x)\}.$$

(H$_2$) $\forall(x, u) \in \Omega \times \mathbb{R}$, $f(x, u, \cdot)$ is a convex function on $\mathbb{R}^N$ which is differentiable at 0 where it achieves a *strict minimum*, that is

$$f(x, u, p) > f(x, u, 0), \quad \forall p \neq 0, \qquad f'_p(x, u, 0) = 0.$$

Here the variable p plays the role of a perturbation creating the dependence of $F^\varepsilon(u)$ on the gradient Du. Scaling with respect to the small parameter $\varepsilon > 0$ is chosen so as to obtain a nonvanishing limit for the sequence (F^ε) as ε tends to 0.

In order to provide relative compactness for the minimizing sequences (u_ε), we also assume the following coercivity condition of $f(x, u, 0)$ with respect to u:

(H$_3$) There exists Φ: $\mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} \Phi(t)/t = +\infty$ and $f(x, u, 0) \geq \Phi(|u|)$, $\forall(x, u) \in \Omega \times \mathbb{R}^N$.

Functionals of type (1.1) appear in the study of the equilibrium of elastic rods under tension [2] (one-dimensional nonlinear elasticity theory) and also in the context of the Van der Waals–Cahn–Hilliard theory of phase transitions [6]. Let us define the latter setting: the model is described by two phases with respective mass densities α and β (which depend only on the local temperature) and by the energy functional $F^\varepsilon(u)$ in which the interfacial energy appears through the dependence on the gradient Du of the density distribution. The problem of determining the stable configurations is to minimize $F^\varepsilon(u)$ among all density distributions of a prescribed total mass m, that is $\int_\Omega u(x)\,dx = m$ $(u \geq 0)$. The mathematical problem is then to study the asymptotic behavior as $\varepsilon \to 0_+$ of solutions u_ε as $\varepsilon \to 0_+$. In the classical theory (see, for example, [7] and [17]), the function f takes the form $f(u, p) = W(u) + |p|^2$ where $W(u)$ is deduced (by subtracting an affine function) from the free Gibbs energy at a given temperature. So, under the assumption that this temperature is constant inside Ω, the local energy W does not depend on x; it is a nonconvex function of $\mathbb{R}$ to $\mathbb{R}_+$ which reaches its minimum value 0 at exactly two scalars α and β.

Following a conjecture by Gurtin [16] and using a Γ-convergence argument [12], [20], Modica [19] proved (see also [24] for similar results) that the sequence of minimizers (u_ε) strongly converges in $L^1(\Omega)$ (modulo replacing (u_ε) by a subsequence) to a function u which takes only the values α and β with interface between $\{u = \alpha\}$ and $\{u = \beta\}$ having minimal area. In this remarkable result, the most striking thing is perhaps that, without any uniform estimate on the gradient of u_ε (we do not even know whether the sequence (Du_ε) is bounded in $L^1(\Omega)$), the limit u is shown to have bounded variations so that the interface has Hausdorff dimension $N - 1$ (this dimension could also be greater than $N - 1$!).

In the context of Modica's results, several questions arise:

(1) Does the minimal interface criterium still remain valid if we replace the perturbation term $|p|^2$ by a convex, possibly nonradial, function of p (note that, in fact, for general functions $f(u, p)$, this perturbation depends on u)?
(2) What happens when functions f, α, and β depend on x?
(3) In view of applications to multidimensional nonlinear elasticity theory, is it possible to obtain similar results in the case of vector functions u?

Question 1 has recently been solved by Owen [21] for the case of a function f of class C^4 which depends radially on p: the same minimal interface criterium was obtained. Very recently, an extension to a nonradial function of class C^3 was given [22]. Question 3 has been studied when $f(u, p) = W(u) + |p|^2$ by Sternberg [24], Baldo [4], and Ambrosio [1] (here p stands for the deformation tensor). Situation 2 is met, for example, in the context of the Van der Waals–Cahn–Hilliard theory when the temperature inside Ω is a given function of x. In particular, the difference $\beta(x) - \alpha(x)$ may vanish if x belongs to some subset where the critical temperature is reached. For new thermodynamical models of phase transitions including variations of temperature, we refer to the recent papers by Casal and Gouin [8], [9]. Let us stress the fact that in these new models the local temperature is considered as an unknown as well as the mass density u. So the assumption above (the temperature is a prescribed function of x) may seem simplistic from a physical point of view although it already leads to many mathematical difficulties!

Our main result (Theorem 3.3) solves problems 1 and 2 under very mild assumptions: f need only be *continuous and satisfy assumptions* (H_1), (H_2), *and* (H_3) given before. Using a Γ(epi)-convergence result (Theorem 3.5), we prove that the minimizers (u_ε) for the problem

$$(\mathscr{P}_\varepsilon) \quad \operatorname{Inf}\left\{\int_\Omega f(x, u, \varepsilon Du)\, dx;\ u \geq 0,\ \int_\Omega u(x)\, dx = m\right\}$$

strongly converge in $L^1(\Omega)$ (modulo replacing (u_ε) by a subsequence) to a function u of the form $u = \alpha 1_A + \beta 1_{\Omega \setminus A}$, where A is a subset of $\Omega_0 = \{x \in \Omega;\ \beta(x) > \alpha(x)\}$, which is a solution of the following geometrical problem:

$$(\mathscr{P}) \quad \operatorname{Inf}\left\{\int_{\partial^* A \cap \Omega_0} h(x, \nu_A(x))\, dH^{N-1};\ A \in X_{\mathrm{loc}}(\Omega_0),\right.$$
$$\left.\int_A (\beta - \alpha)\, dx = m - \int_\Omega \alpha\, dx\right\}.$$

Here $X_{\mathrm{loc}}(\Omega_0)$ denotes the sets with a locally finite perimeter in Ω_0 and ν_A denotes the exterior normal to A which is defined on the reduced boundary $\partial^* A$ of A [14]. The complete description of the integrand $h(x, p)$ is clearly defined; it is continuous and sublinear in p.

Our conclusion is that the interface (which of course appears only on Ω_0) is still $(N - 1)$ Hausdorff dimensional but now the selection criterium is weighted *spatially and directionally* according to the original choice of the function f.

This paper is organized as follows: In Section 2 we give some technical background on sets of finite perimeter (Section 2.1) and on sublinear functions of measures (Section 2.2). Then we recall (Section 2.3) the definition and main properties of the epi(or Γ)-convergence of functionals, notions which are used to go to the limit in the sequence of variational problems $(\mathscr{P}_\varepsilon)$.

In Section 3 we first construct the integrand h, using the conical envelope of $f(x, u, \cdot)$, and give an equivalent variational definition (Lemma 3.2). Then we state our main result (Theorem 3.3) which is deduced from the epiconvergence of (F^ε) (Theorem 3.4). Some examples are given in Section 3.4.

Finally, all the proofs are concentrated in Section 4 and in the Appendix.

Let us mention that our method is new since it is fully variational. In particular, we do not need, as was done in [19]–[22], any decay estimate results for initial value differential equations, so we can remove many restrictive assumptions on the regularity of f, which need only be continuous. The extension to a noncontinuous dependence on x seems difficult to obtain and remains, to our knowledge, an open problem.

I am indebted to L. Modica and G. Buttazzo for their interest when I announced the main results of this paper in May 1987. This final version has been partially written during my stay at the Departments of Mathematics of the University of Pisa and of the University of Ferrara in June 1988.

2. Preliminaries and Notations

Henceforth, H^{N-1} denotes the $(N-1)$-dimensional Hausdorff measure and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$. For every subset A of $\mathbb{R}^N$, we denote by 1_A the characteristic function of A (takes value 1 on A and 0 otherwise). $B(x, r)$ is the open ball with center x and radius r.

2.1. Sets with Bounded Perimeter

For what follows the reader should refer to the original papers of De Giorgi [11] and to the books by Giusti [14] and Massari and Miranda [18]. Let Ω be an open subset of $\mathbb{R}^N$, let $u \in L^1_{\text{loc}}(\Omega)$, and let Du be the gradient of u in the sense of distributions on Ω. If Du is a vector-valued Radon measure, we write $u \in BV_{\text{loc}}(\Omega)$. If, moreover, u belongs to $L^1(\Omega)$ and if the total variation of Du on Ω, that is

$$\int_\Omega |Du| = \text{Sup}\left\{\int_\Omega u \operatorname{div} g\, dx;\, g \in C_0^1(\Omega, \mathbb{R}^N), |g| \leq 1\right\}, \tag{2.1}$$

is finite, then we write $u \in BV(\Omega)$. With the norm $\|u\| = \int_\Omega |u|\, dx + \int_\Omega |Du|$, $BV(\Omega)$ is a strict Banach subspace of $W^{1,1}(\Omega)$ which, in case of a bounded Ω, is compactly embedded in $L^1(\Omega)$. It is easy to check that the functional $u \in BV(\Omega) \to \int_\Omega |Du|$ is lower semicontinuous for the L^1-norm. Let us apply the previous definitions in the case $u = 1_E$, where E is a Lebesgue-measurable subset of Ω: we denote by $X_{\text{loc}}(\Omega)$ (or $X(\Omega)$) the sets E for which 1_E belongs to $BV_{\text{loc}}(\Omega)$ (or $BV(\Omega)$). Following the terminology of [14], an element E of $X_{\text{loc}}(\mathbb{R}^N)$ is called a Caccioppoli set and, for any bounded open set ω, its perimeter in ω, defined by $P_\omega(E) = \int_\omega |D1_E|$, is finite. An example of a Caccioppoli set is given by a subset E of $\mathbb{R}^N$ with Lipschitz continuous boundary ∂E. In this case $P_\omega(E) = H^{N-1}(\omega \cap \partial E)$. To obtain this formula in the general case, we have to replace ∂E by the reduced boundary $\partial^* E$ defined below.

Definition 2.1. Let $E \in X_{\text{loc}}(\mathbb{R}^N)$. A point x belongs to the reduced boundary $\partial^* E$ if:

(i) $\int_{B(x,r)} |D1_E| > 0$ for all $r > 0$.

(ii) The limit $\nu_E(x) = \lim_{r\to 0}(\int_{B(x,r)} D1_E / \int_{B(x,r)} |D1_E|)$ exists and verifies $|\nu_E(x)| = 1$.

$\nu_E(x)$ is the *generalized outer normal to E* and, from Besicovitch theorem of differentiation of measures,

$$D1_E = \nu_E(x)|D1_E|. \tag{2.2}$$

Moreover, $\partial^* E$ is an H^{N-1} σ-finite subset of $\mathbb{R}^N$ which is dense and of full H^{N-1} Hausdorff measure in the essential boundary of E (set of points where the Lebesgue density of E lies in $]0, 1[$). Finally, $|D1_E|$ is nothing else but the trace of H^{N-1} on $\partial^* E$, that is

$$|D1_E|(\cdot) = H^{N-1}(\partial^* E \cap \cdot). \tag{2.3}$$

Remark 2.2. All that has been said before for a Caccioppoli set E also applies when taking for E a set of $X_{\text{loc}}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. Of course, Definition 2.1 has to be restricted to the points x which lie in Ω and, to find that $P_\omega(E)$ is finite, we need ω to be relatively compact in Ω.

2.2. *Sublinear Functionals on Measures*

Let Ω be an open subset of $\mathbb{R}^N$ and let $\varphi(x, p)$ be a Borel function of $\Omega \times \mathbb{R}^N$ to $\mathbb{R}_+$ which is sublinear in p. We consider the convex functional defined on the space $M^b(\Omega, \mathbb{R}^N)$ of bounded $\mathbb{R}^N$-valued measures by

$$\Phi: \lambda \in M^b(\Omega, \mathbb{R}^N) \to \int_\Omega \varphi\left(x, \frac{d\lambda}{d\theta}\right) d\theta, \tag{2.4}$$

where θ is a positive measure such that $\lambda \ll \theta$. Using the sublinearity of $\varphi(x, \cdot)$, it can easily be shown [15] that the integral in (2.4) does not depend on the choice of θ. For that reason, we rewrite (2.4) under the condensed form

$$\Phi(\lambda) = \int_\Omega \varphi(x, \lambda). \tag{2.5}$$

Let us now give some important continuity properties of the functional Φ which have been proved by Reschetniak [23] in the case of a compact metric space Ω. The extension to a σ-compact metrizable Ω is straightforward (see, for example, [5]). Recall that the weak topology on M^b is associated with the duality (M^b, C_0), where C_0 denotes the space of continuous functions on Ω tending to 0 at infinity.

Proposition 2.3.

(i) *If φ is lower semicontinuous on $\Omega \times \mathbb{R}^N$, then Φ is weakly lower semicontinuous on $M^b(\Omega, \mathbb{R}^N)$.*

(ii) *Assume that φ is continuous on $\Omega \times \mathbb{R}^N$. If λ_n converges weakly to λ as $n \to \infty$ and if, moreover, $\int_\Omega |\lambda_n| \to \int_\Omega |\lambda|$, then $\Phi(\lambda_n)$ converges to $\Phi(\lambda)$.*

Let us conclude this section with a variant of the coarea formula which can be found in [10] (in particular, Lemma 2.2]).

Proposition 2.4. *Let $\Psi(x, s, p)$ a Borel function of $\Omega \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}_+$ which is sublinear in p. Let u be a Lipschitz continuous function on Ω and denote, for every $t > 0$, $S_t = \{x \in \Omega;\ u(x) < t\}$. Then, for almost all $t \in \mathbb{R}$, S_t belongs to $X(\Omega)$ and we have*

$$\int_\Omega \Psi(x, u, Du)\, dx = \int_{-\infty}^{+\infty} dt \int_\Omega \Psi(x, t, D1_{S_t}). \tag{2.6}$$

Comment. If $u \in BV(\Omega) \setminus W^{1,1}(\Omega)$, the first member of (2.6) does not make sense. Nevertheless, we still have $S_t \in X(\Omega)$ (for almost all t). Hence the second member is well defined and provides a natural extension of the integral functional to all $BV(\Omega)$ (see [10]). When $\Psi(x, s, p) = |p|$, we get

$$\int_\Omega |Du| = \int_{-\infty}^{+\infty} dt \int_\Omega |D1_{S_t}|. \tag{2.6'}$$

In fact, (2.6′) holds true for every $u \in BV(\Omega)$ (with the left member defined by (2.1)). This is Rishel–Fleming's famous theorem [14, p. 20] (coarea formula).

2.3. *The Concept of* Γ *(or Epi)-Convergence*

We refer to [12] or to [3] for further details. Let (X, τ) a metrizable topological space and let (F^ε) be a sequence of functionals of X to $\overline{\mathbb{R}}$ (where the small parameter $\varepsilon > 0$ belongs to a sequence converging to 0). The *lower and upper τ-epilimits F_i and F_s* of the sequence (F^ε), also denoted $\tau\text{-}\underline{\lim}_\varepsilon F^\varepsilon$ and $\tau\text{-}\overline{\lim}_\varepsilon F^\varepsilon$, are defined by

$$F_i(u) = \tau\text{-}\underline{\lim}_\varepsilon F^\varepsilon(u) = \operatorname{Min}\left\{\liminf_{\varepsilon\to 0} F^\varepsilon(u_\varepsilon);\ u_\varepsilon \xrightarrow{\tau} u\right\}, \tag{2.7}$$

$$F_s(u) = \tau\text{-}\overline{\lim}_\varepsilon F^\varepsilon(u) = \operatorname{Min}\left\{\limsup_{\varepsilon\to 0} F^\varepsilon(u_\varepsilon);\ u_\varepsilon \xrightarrow{\tau} u\right\}. \tag{2.8}$$

Of course, $F_i \le F_s$. Moreover, F_i and F_s are τ-lower semicontinuous (see, for example, [3]).

Definition 2.5. If F_i and F_s coincide as functionals with the same functional F (equivalently $F_s \le F \le F_i$), we say that (F^ε) *τ-epiconverges* to F as $\varepsilon \to 0$.

The fundamental variational property of the epiconvergence is summarized in:

Proposition 2.6. *Assume that (F^ε) τ-epiconverges to F and that there exists a τ-relatively compact minimizing sequence (u_ε) (i.e., such that $F^\varepsilon(u_\varepsilon)$-inf $F^\varepsilon \to 0$ as $\varepsilon \to 0_+$). Then:*

(i) Inf $F^\varepsilon \to$ Inf F *as* $\varepsilon \to 0_+$.
(ii) *Every cluster point u of* $\{u_\varepsilon,\ \varepsilon > 0\}$ *minimizes F.*

3. Setting of the Problem. Main Results

3.1.

Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary, let $\alpha, \beta: \Omega \to \mathbb{R}_+$ be Lipschitz continuous functions such that $0 \le \alpha(x) \le \beta(x)$. Define the function $\gamma: \Omega \to \mathbb{R}_+$ and the open subset Ω_0 by

$$\gamma = \beta - \alpha, \qquad \Omega_0 = \{x \in \Omega; \gamma(x) > 0\}. \tag{3.1}$$

Given $f(x, u, p)$ is a *continuous* function of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}_+$ and m is a real number such that[1]

$$\int_\Omega \alpha(x)\, dx < m < \int_\Omega \beta(x)\, dx, \tag{3.2}$$

let us consider, for every $\varepsilon > 0$, the following variational problem:

$$\operatorname{Inf}\left\{\int_\Omega f(x, u, \varepsilon Du)\, dx; u \in K_m \cap \operatorname{Lip}(\Omega)\right\}, \tag{3.3}$$

where $\operatorname{Lip}(\Omega)$ denotes the set of Lipschitz functions of Ω to $\mathbb{R}$ and K_m is defined as

$$K_m = \left\{u \in L^1(\Omega); u \ge 0 \quad \text{and} \quad \int_\Omega u\, dx = m\right\}. \tag{3.4}$$

Following [19] and [20], we use a rescaling of the integral functional defined in (3.3) by setting

$$F^\varepsilon: u \in L^1(\Omega) \to \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega f(x, u, \varepsilon Du)\, dx & \text{if} \quad u \in \operatorname{Lip}(\Omega), \\ +\infty & \text{if} \quad u \notin \operatorname{Lip}(\Omega). \end{cases} \tag{3.5}$$

Our aim is to go to the limit in (3.3) as ε tends to 0_+ by proving the relative compactness of associated minimizers and the epi(or Γ)-convergence of the sequence (F^ε) when $L^1(\Omega)$ is endowed with the norm topology.

3.2.

To describe the limit problem we have to introduce new notations and definitions. First, we introduce f_c, the conical envelope of f, defined by

$$f_c(x, u, p) = \operatorname*{Inf}_{t>0}\left\{\frac{1}{t} f(x, u, tp)\right\}. \tag{3.6}$$

Let us notice that this integrand f_c provides a lower bound for F^ε which is

[1] In the physical context, m, which denotes the mass of fluid inside Ω, is assumed to be constant. Condition (3.2) means that the two possible phases, described by their mass-densities $\alpha(x)$ and $\beta(x)$, will actually coexist in Ω.

independent of ε. Indeed, for any u in Lip(Ω), we have

$$F^\varepsilon(u) \geq \int_\Omega f_c(x, u, Du)\, dx. \tag{3.7}$$

Then define

$$h(x, p) = \int_{\alpha(x)}^{\beta(x)} f_c(x, u, p)\, du, \tag{3.8}$$

$$G(x, p) = \begin{cases} (1/\gamma(x))h(x, p) & \text{if } \gamma(x) > 0 \quad (\text{that is if } x \in \Omega_0), \\ 0 & \text{if } \gamma(x) = 0, \end{cases} \tag{3.8'}$$

$$\lambda(x, u) = \operatorname{Inf}\{f_c(x, u, p); |p| = 1\}. \tag{3.9}$$

Of course smoothness properties assumed on f (see (H_1) and (H_2)) will affect those of f_c, h, and G. This is defined in Lemma 3.1 below. Moreover, the integrand h defined by (3.8) plays a central role in the description of the limit problem. This fact is a consequence of Lemma 2.2 stated below where the link between h and a one-dimensional variational problem is made: roughly, $h(x, p)$ represents the minimum of energy which is needed to go from $\alpha(x)$ to $\beta(x)$ along the half-line starting from x in the direction p.

Lemma 3.1. *Assume that* $f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}_+$ *is continuous and verifies* (H_1) *and* (H_2) *(see Section 1). Then:*

(i) $f_c(x, u, p)$ *is continuous on* $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, *sublinear in* p, *and satisfies*

$$\forall x \in \bar{\Omega}, \qquad f_c(x, \alpha(x), \cdot) = f_c(x, \beta(x), \cdot) = 0.$$

(ii) $h(x, p)$ *and* $G(x, p)$ *are continuous on* $\bar{\Omega} \times \mathbb{R}^N$, *sublinear in* p; $\lambda(x, u)$ *is continuous of* $\bar{\Omega} \times \mathbb{R}$ *to* $\mathbb{R}_+$ *and satisfies*

$$\forall x \in \bar{\Omega}, \qquad \lambda(x, u) > 0 \qquad \text{if} \quad u \notin \{\alpha(x), \beta(x)\}.$$

Proof. See Section A of the Appendix

Lemma 3.2. *Let* $r > 0$ *and consider the function* I_r *of* $\bar{\Omega} \times \mathbb{R}^N$ *to* $\mathbb{R}$ *defined as*

$$I_r(x, p) = \operatorname{Inf}\left\{\int_0^r f(x, v(t), pv'(t))\, dt;\, v \in \operatorname{Lip}\uparrow(\mathbb{R}),\, v(0) = \alpha(x),\, v(r) = \beta(x)\right\}, \tag{3.10}$$

where Lip↑(ℝ) *denotes the Lipschitz continuous and nondecreasing functions from* ℝ *to* ℝ. *Then, under the assumptions of Lemma* 3.1 *and for every* (x, p) *in* $\bar{\Omega} \times \mathbb{R}^N$, $I_r(x, p)$ *is a nonincreasing function of* r *which tends to* $h(x, p)$ *as* $r \to \infty$.

Remark. In fact, $h(x, p) = I_\infty(x, p)$ where $I_\infty(x, p)$ is defined from (3.10) by putting $r = +\infty$ (in this case, $v(+\infty)$ denotes the limit of $v(t)$ as t tend to $+\infty$). For the proof see Section A.2 of the Appendix.

3.3. Main Results

Theorem 3.3. *Assume that f is continuous from $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}_+$ and verifies* (H_1), (H_2), and (H_3). *Denote, for every ε, r_ε is the infimum of problem* (3.3) *and let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of Lipschitz continuous functions which satisfy the constraint K_m (see* (3.4)) *and is minimizing for* (3.3) *in the following sense:*

$$\exists C > 0 \text{ such that } \forall \varepsilon, \qquad r_\varepsilon \le \int_\Omega f(x, u_\varepsilon, \varepsilon D u_\varepsilon)\, dx \le r_\varepsilon + C\varepsilon. \tag{3.11}$$

Then $\{u_\varepsilon, \varepsilon > 0\}$ is strongly relatively compact in $L^1(\Omega)$ and any cluster point u can be written as $u = \alpha 1_A + \beta 1_{\Omega \setminus A}$, where A is a solution in $X_{\mathrm{loc}}(\Omega_0)$ of the following geometrical problem:

$$\operatorname{Inf}\left\{\int_{\Omega_0 \cap \partial^* A} h(x, \nu_A(x))\, dH^{N-1};\ A \in X_{\mathrm{loc}}(\Omega_0),\ \int_A \gamma(x)\, dx = m'\right\}, \tag{3.12}$$

where[2]

$$m' = m - \int_\Omega \alpha\, dx. \tag{3.13}$$

Moreover, the ratio $r_\varepsilon/\varepsilon$ has a limit as $\varepsilon \to 0_+$ which is equal to the infimum of (3.12).

Comments 3.4. (1) The weak relative compactness in $L^1(\Omega)$ of the minimizing sequence (u_ε) is easily deduced from hypothesis (H_3). Hevertheless, as we will show in the proof (see Section 4), we need the strong convergence in $L^1(\Omega)$ of (u_ε) to make sure that the limit u satisfies a.e. $u(x) \in \{\alpha(x), \beta(x)\}$.

(2) When $\inf_\Omega \gamma > 0$ (for example, if α, β are constant functions), then $\Omega_0 = \Omega$ and there exists some $\lambda_0 > 0$ such that $h(x, p) \ge \lambda_0 |p|$, $\forall (x, p) \in \bar{\Omega} \times \mathbb{R}^N$. It follows that the integral in (3.12) is finite if and only if A has a finite perimeter in Ω (that is $A \in X(\Omega)$). Some examples related to this situation are detailed in Section 3.4.

(3) We do now know in general whether problem (3.3) has solutions. Nevertheless, under some additional hypothesis of strong coercivity of f with respect to p such as

$$\exists C > 0: \qquad f(x, u, p) \ge C|p|^k, \qquad k > 1, \tag{3.14}$$

the existence of solutions is easily obtained, for every $\varepsilon > 0$, in the Sobolev space $W^{1,k}(\Omega)$.

The proof of the main theorem relies on a result of epiconvergence which is stated in Theorem 3.5 below. The detailed proofs are rather technical and are found in Section 4.

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[2] Here $\partial^* A$ denotes the reduced boundary of A and ν_A denotes the outer normal defined on $\partial^* A$ (see Section 2.1).

Theorem 3.5. *Assume that f is continuous from $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}_+$ and verifies* (H_1) *and* (H_2). *Define F^ε as in* (3.5), *and Ω_0 and G by* (3.1) *and* (3.8′). *Then*:

(i) *As $\varepsilon \to 0_+$, (F^ε) epiconverges strongly in $L^1(\Omega)$ to F defined as*

$$F: u \in L^1(\Omega) \to \begin{cases} \int_{\Omega_0} G(x, Du) & \text{if} \quad u \in BV_{\mathrm{loc}}(\Omega_0),\ u(x) \in \{\alpha(x), \beta(x)\}, \text{ a.e.}, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.15}$$

(ii) *Denote I_m as the indicator function of K_m (takes value 0 on K_m, $+\infty$ outside). Then $(F^\varepsilon + I_m)$ epiconverges to $F + I_m$ as $\varepsilon \to 0_+$.*

Remark 3.6. (a) Here we do not need to assume (H_3). On the other hand, when $F(u) < +\infty$, the set $A = \{x \in \Omega_0; u(x) = \alpha(x)\}$ belongs to $X_{\mathrm{loc}}(\Omega_0)$. Then, using the notations of Theorem 3.3 and Section 2.1,

$$F(u) = \int_\Omega h(x, D1_A) = \int_{\Omega_0 \cap \partial^* A} h(x, \nu_A)\, dH^{N-1}. \tag{3.16}$$

(b) As G is continuous, we have $F(u) < +\infty$ if $u \in BV(\Omega_0)$ and $u(x) \in \{\alpha(x), \beta(x)\}$ a.e. It will be shown (see Lemma 4.4) that $F + I_m$ is proper and that there exists at least one $u \in K_m$ such that $F(u) < +\infty$.

(c) It is possible to remove the hypothesis that $f(x, u, \cdot)$ is differentiable at 0 (see H_2). Nevertheless, in this case the functions $f_c(x, \alpha(x), \cdot)$ and $f_c(x, \beta(x), \cdot)$ may fail to vanish (see Lemma 3.1). As a consequence, expression (3.16) for $F(u)$ has to be replaced by

$$F(u) = \int_{\Omega_0 \cap \partial^* A} h(x, \nu_A)\, dH^{N-1} + \int_A f_c(x, \alpha, D\alpha)\, dx + \int_{\Omega \setminus A} f_c(x, \beta, D\beta)\, dx. \tag{3.16′}$$

Note that the last two integrals in (3.16′) vanish when α and β are constant.

3.4. *Examples*

Let $\alpha, \beta \in \mathrm{Lip}(\Omega)$ such that $\inf\{\beta(x) - \alpha(x); x \in \Omega\} > 0$ and consider a continuous function $W(x, u)$ of $\bar{\Omega} \times \mathbb{R}$ to $\mathbb{R}_+$ which satisfies $W(x, u) = 0 \Leftrightarrow u \in \{\alpha(x), \beta(x)\}$. Our aim is to specify the integrand h and the limit functional $F(u)$ (see Theorem 3.5) in the case of particular perturbations of W. Recall (see Comment 3.4) that $F(u) < +\infty$ if and only if $u = \alpha 1_A + \beta 1_{\Omega \setminus A}$ with $A \in X(\Omega)$.

(a) Isotropic Perturbations. Given φ a convex continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that

$$\varphi(0) = 0, \qquad \varphi'(0+) = 0, \qquad \varphi(t) > 0 \quad \text{if } t > 0, \tag{3.17}$$

let us consider the following function f:

$$f(x, u, p) = W(x, u) + \varphi(|p|) \qquad (|p| \text{ denotes the Euclidien norm in } \mathbb{R}^N). \tag{3.18}$$

As a consequence of (3.17), the Fenchel conjugate φ^* of φ (defined on $\mathbb{R}_+$ by $\varphi^*(t^*) = \operatorname{Sup}\{tt^* - \varphi(t); t \in \mathbb{R}_+\}$) is strictly increasing and vanishes at 0. So we can define the generalized inverse of φ^* on $\mathbb{R}_+$ by setting

$$(\varphi^*)^{-1}(t) = \operatorname{Sup}\{t^* \in \mathbb{R}_+; \varphi^*(t^*) \leq t\}. \tag{3.19}$$

Note that $(\varphi^*)^{-1}$ coincide with the usual inverse when φ^* is finite-valued or equivalently if $\varphi(t)/t \to \infty$ as $t \to \infty$ (this avoids the case $\varphi(t) = (1 + t^2)^{1/2} - 1$). After an easy computation, we get

$$f_c(x, u, p) = |p|(\varphi^*)^{-1}(W(x, u)). \tag{3.20}$$

Let $u = \alpha 1_A + \beta 1_{\Omega\setminus A}$ with $A \in X(\Omega)$. We have

$$F(u) = \int_{\partial^* A} k(x)\, dH^{N-1} \qquad \text{where} \quad k(x) = \int_{\alpha(x)}^{\beta(x)} (\varphi^*)^{-1}(W(x, u))\, du. \tag{3.21}$$

So the minimal interface criterion is weighted spatially according to $k(x)$. If, for example, $\varphi(t) = t^2$ (the situation of the Van der Waals–Cahn–Hilliard model), we get $k(x) = 2\int_{\alpha(x)}^{\beta(x)} (W(x, u))^{1/2}\, du$. If now $\varphi(t) = (1 + t^2)^{1/2} - 1$, we get $k(x) = \beta(x) - \alpha(x)$. Here the behavior of $W(x, u)$ when u runs between $\alpha(x)$ and $\beta(x)$ has no effect!

(*b*) *Quadratic Perturbations.* Let $a_{i,j}(x, u)$ be a continuous function from $\bar{\Omega} \times \mathbb{R}$ to the $N \times N$ *positive definite* matrices and consider the function f defined as

$$f(x, u, p) = W(x, u) + a_{i,j}(x, u)p_i p_j. \tag{3.22}$$

We get

$$h(x, p) = 2 \int_{\alpha(x)}^{\beta(x)} (a_{ij}(x, u)p_i p_j \cdot W(x, u))^{1/2}\, du. \tag{3.23}$$

Clearly, for nonscalar matrices $(a_{i,j})$, the function $h(x, p)$ depends both on x and p. So the minimal interface criterion is weighted spatially and also *directionally*.

(*c*) *Perturbations with Linear Growth.* Assume now that α, β are independent of x so that the restriction $f'_p(x, u, 0) = 0$ (see (H2)) can be removed (see Remark 3.6(c)). Let $a_{i,j}(x, u)$ be as above and define

$$f(x, u, p) = ([W(x, u)]^2 + a_{i,j}(x, u)p_i p_j)^{1/2}. \tag{3.24}$$

We get

$$h(x, p) = \int_{\alpha}^{\beta} (a_{i,j}(x, u)p_i p_j)^{1/2}\, du. \tag{3.25}$$

So the choice of W does not affect the criterion.

4. Sketch of the Proofs

Theorem 3.5 is a direct consequence of Propositions 4.1 and 4.2 stated below. Then Theorem 3.3 is easily deduced (see Proposition 2.6) provided the minimizing sequence (u_ε) is strongly relatively compact in $L^1(\Omega)$ (this is proved in Section 4.3).

Proposition 4.1. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a family of functions such that u_ε converges to u in $L^1(\Omega)$ as $\varepsilon \to 0_+$.*

(i) *If* $\liminf_\varepsilon F^\varepsilon(u_\varepsilon) < +\infty$, *then $u(x) \in \{\alpha(x), \beta(x)\}$ a.e. on Ω and $u \in BV_{loc}(\Omega_0)$.*
(ii) *If $u(x) \in \{\alpha(x), \beta(x)\}$ a.e. on Ω and if $A = \{x \in \Omega_0 \backslash u(x) = \alpha(x)\}$ belongs to $X_{loc}(\Omega_0)$, then*

$$\liminf_{\varepsilon} F^\varepsilon(u_\varepsilon) \geq \int_{\Omega_0} h(x, D1_A). \tag{4.1}$$

Proposition 4.2. *Let $u \in L^1(\Omega) \cap BV_{loc}(\Omega_0)$ such that $u(x) \in \{\alpha(x), \beta(x)\}$ a.e. on Ω and let $A = \{x \in \Omega_0 \backslash u(x) = \alpha(x)\}$. Then there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ of Lipschitz continuous functions on Ω such that u_ε converges to u in $L^1(\Omega)$ as $\varepsilon \to 0_+$, $\alpha \leq u_\varepsilon \leq \beta$ for every $\varepsilon > 0$ and*

(i) $\limsup_\varepsilon F^\varepsilon(u_\varepsilon) \leq \int_{\Omega_0} h(x, D1_A)$,
(ii) $\int_\Omega u_\varepsilon(x)\, dx = \int_\Omega u(x)\, dx$, $\forall \varepsilon > 0$.

Comment. From Proposition 4.1 and (3.16), $\underline{\lim}_e F^\varepsilon \geq F$. Since I_m is lower semicontinuous we also have $\underline{\lim}_e(F^\varepsilon + I_m) \geq F + I_m$. On the other hand, from Proposition 4.2, $\overline{\lim}_e(F^\varepsilon + I_m) \leq F + I_m$ for *every* $m > 0$ which obviously implies that $\overline{\lim}_e F^\varepsilon \leq F$.

4.1. Proof of Proposition 4.1

(i) There exists a sequence (ε_n) of positive numbers converging to 0 as $n \to \infty$ and a positive constant C such that $u_{\varepsilon_n} \to u$ a.e. on Ω and

$$F^{\varepsilon_n}(u_{\varepsilon_n}) \leq C. \tag{4.2}$$

Since, from (H_2), $f(x, u, p) \geq f(x, u, 0) \geq 0$, $\forall (x, u, p)$, (4.1) implies

$$0 \leq \int_\Omega f(x, u_{\varepsilon_n}, 0)\, dx \leq C\varepsilon_n. \tag{4.2'}$$

Using Fatou's lemma and the continuity of f, we deduce $f(x, u(x), 0) = 0$ a.e., that is, from (H_1), $u(x) \in \{\alpha(x), \beta(x)\}$ a.e.. Let $A = \{x \in \Omega_0;\ u(x) = \alpha(x)\}$. Since $\alpha(x) = \beta(x)$ when $x \notin \Omega_0$, we can, of course, write $u = \alpha 1_A + \beta 1_{\Omega \backslash A}$.

Our task now is to prove that $A \in X_{loc}(\Omega_0)$. Let ω be an open subset such that $\Omega_0 \supset \bar{\omega}$ and let us associate to ω the function $k(t)$ of $\mathbb{R}$ to $[0, 1]$ defined as

$$k(t) = \operatorname{Inf}\{k_0(t), 1\}, \quad k_0(t) = \operatorname{Inf}\{f_c(x, \alpha(x)t + \beta(x)(1 - t), p);\ x \in \omega, |p| = 1\}. \tag{4.3}$$

Thanks to the continuity of f_c (Lemma 3.1(i)), the infimum in the last expression is attained and, from Lemma 3.1(ii),

$$k(t) = 0 \quad \Leftrightarrow \quad t \in \{0, 1\}. \tag{4.4}$$

Let v_n from Ω_0 to $\mathbb{R}$ be defined by

$$v_n = \frac{1}{\gamma}(\beta - u_{\varepsilon_n}) \qquad (\text{recall } \gamma = \beta - \alpha > 0 \text{ on } \Omega_0) \tag{4.5}$$

and let v_n^* of Ω_0 to $[0, 1]$, the truncated function, be defined by

$$v_n^*(x) = \mathrm{Max}\{\alpha(x), \mathrm{Min}\{v_n(x), \beta(x)\}\}. \tag{4.6}$$

Since $f_c(x, \alpha(x), \cdot) = f_c(x, \beta(x), \cdot) = 0$ (see Lemma 3.1(i)) and noting that

$$u_{\varepsilon_n} = \alpha v_n + \beta(1 - v_n), \qquad Du_{\varepsilon_n} = D\alpha v_n + D\beta(1 - v_n) - \gamma Dv_n,$$

we have, for a.a. $x \in \Omega_0$,

$$f_c(x, u_{\varepsilon_n}, Du_{\varepsilon_n}) \geq f_c(x, \alpha v_n^* + \beta(1 - v_n^*), D\alpha v_n^* + D\beta(1 - v_n^*) - \gamma Dv_n^*). \tag{4.7}$$

Using (4.3) gives

$$f_c(x, u_{\varepsilon_n}, Du_{\varepsilon_n}) \geq k(v_n^*)\gamma |Dv_n^*| - \mathrm{Max}\{|D\alpha|, |D\beta|\}. \tag{4.8}$$

From the definition of f_c (see (3.6)),

$$F^{\varepsilon_n}(u_{\varepsilon_n}) \geq \int_\omega f_c(x, u_{\varepsilon_n}, Du_{\varepsilon_n})\,dx \tag{4.9}$$

which, combined with (4.8) and (4.2), implies that there exists some $C' > 0$ such that

$$\int_\omega k(v_n^*)\gamma |Dv_n^*| \leq C'. \tag{4.10}$$

Now define

$$M(t) = \int_0^1 k(s)\,ds, \qquad w_n = M(v_n^*). \tag{4.11}$$

Clearly, v_n and v_n^* converge to 1_A in $L^1(\omega)$ as $n \to \infty$ (note that $\mathrm{Inf}_\omega \gamma > 0$). Since $k(t) \in [0, 1]$, we have $0 \leq M(t) \leq t$, hence the sequence (w_n) is bounded in $L^1(\omega)$. On the other hand, from (4.10) and (4.11) and denoting γ^* as the greatest lower bound of γ on ω ($\gamma^* > 0$),

$$\int_\omega |Dw_n| \leq \frac{C'}{\gamma^*}, \qquad \forall n, \tag{4.12}$$

which implies that $\{w_n, n \in N\}$ is bounded in $W^{1,1}(\omega)$, hence strongly relatively compact in $L^1(\omega)$ (see Section 2.1). Moreover, since M is continuous, the unique possible limit w, which lies in $BV(\omega)$, is given by

$$w(x) = M(1)1_{A \cap \omega}(x). \tag{4.13}$$

Hence, since $M(1) > 0$ (this from (4.4)), the restriction of $D1_A$ to the open set ω is a measure of bounded variation. This being true for any relatively compact subset of Ω_0, we have proved that $A \in X_{\text{loc}}(\Omega_0)$ (that is $u \in BV_{\text{loc}}(\Omega_0)$).

(ii) We may assume that $\liminf_\varepsilon F^\varepsilon(u_\varepsilon) < +\infty$ (otherwise inequality (4.1) is trivial). There exists a sequence (ε_n) such that $\varepsilon_n \to 0_+$ and

$$\liminf_{\varepsilon} F^\varepsilon(u_\varepsilon) = \lim_{n\to\infty} \int_\Omega f(x, u_{\varepsilon_n}, Du_{\varepsilon_n})\, dx < +\infty. \tag{4.14}$$

Let ω be a relatively compact subset of Ω_0 and define v_n and v_n^* as in (4.5) and (4.6). We have, from (4.7) (recall $f_c \geq 0$),

$$\int_\Omega f_c(x, u_{\varepsilon_n}, Du_{\varepsilon_n})\, dx$$
$$\geq \int_\omega f_c(x, \alpha v_n^* + \beta(1 - v_n^*), D\alpha v_n^* + D\beta(1 - v_n^*) - \gamma Dv_n^*)\, dx.$$

From the subadditivity of f_c,

$$\int_\Omega f_c(x, u_{\varepsilon_n}, Du_{\varepsilon_n})\, dx \geq I_n - J_n, \tag{4.15}$$

where

$$I_n = \int_\omega f_c(x, \alpha v_n^* + \beta(1 - v_n^*), -\gamma Dv_n^*)\, dx, \tag{4.16}$$

$$J_n = \int_\omega f_c(x, \alpha v_n^* + \beta(1 - v_n^*), -D\alpha v_n^* - D\beta(1 - v_n^*))\, dx. \tag{4.16'}$$

Since $v_n^*(x)$ lies in $[0, 1]$ and converges a.e. to 1_A, we can compute the limit of J_n using the dominated convergence theorem and Lemma 3.1(i):

$$\lim_{n\to\infty} J_n = \int_{\omega \cap A} f_c(x, \alpha, -D\alpha)\, dx + \int_{\omega\setminus A} f_c(x, \beta, -D\beta)\, dx = 0. \tag{4.17}$$

Let us now consider I_n. Using formula (2.6) of Proposition 2.4, we can write $I_n = \int_{-\infty}^{+\infty} \varphi_n(t)\, dt$ where

$$\varphi_n(t) = \int_\omega f_c(x, \alpha t + \beta(1 - t), \gamma D1_{S_t^n}); \qquad S_t^n = \{x \in \omega \setminus v_n^*(x) < t\}. \tag{4.18}$$

From Fatou's lemma and noting that $D1_{S_t^n}$ vanishes when $t \notin [0, 1]$,

$$\liminf_{n\to\infty} I_n \geq \int_0^1 \left(\liminf_{n\to\infty} \varphi_n(t)\right) dt. \tag{4.19}$$

Let $m(t)$ be defined by (4.3). We have $\varphi_n(t) \geq k(t) \int_\omega \gamma |D1_{S_t^n}|$. Hence for a.e. $t \in]0, 1[$, the sequence $(1S_t^n)$ is weakly relatively compact in $BV(\omega)$. Moreover, since $\int_\omega |v_n^* - 1_A|\, dx \geq \min(t, 1 - t)|S_t^n \Delta(A \cap \omega)|$ (see, for example, Lemma 1.25 of [14]), the unique possible limit of $1_{S_t^n}$ is a.e. equal to $1_{A\cap\omega}$. Then, thanks to the lower semicontinuity of f_c, we deduce from the Proposition 2.3(i) that a.e.

$$\liminf_{n\to\infty} \varphi_n(t) \geq \int_\omega f_c(x, \alpha t + \beta(1-t), \gamma D1_A). \tag{4.20}$$

Finally, collecting (4.14), (4.15), (4.17), (4.19), and (4.20),

$$\liminf_{\varepsilon} F^\varepsilon(u_\varepsilon) \geq \liminf_{n\to\infty} I_n \geq \int_\omega h(x, D1_A) \tag{4.21}$$

(for the last inequality we use Fubini's theorem and the definition of integrand h (see (3.8)). Assertion (4.21) being true for every open subset ω such that $\Omega_0 \supset \bar{\omega}$, we conclude the proof of (4.1) by taking an increasing sequence (ω_n) such that $\Omega_0 = \bigcup_n \omega_n$. □

4.2. Proof of Proposition 4.2

In a first step we assume that $u = \alpha 1_{\Omega\cap A} + \beta 1_{\Omega\setminus A}$, where A is an open subset of $\mathbb{R}^N$ with smooth boundary ∂A such that $H^{N-1}(\partial A \cap \partial\Omega) = 0$. We conclude thanks to a density argument.

(*a*) *First Step.* Let A be a smooth subset as above and denote by v the outer normal to ∂A. To prove (i) it is enough to show, for any lower-semicontinuous function $\xi(x)$ such that $h(x, v(x)) < \xi(x)$ on ∂A, that it is possible to construct a sequence (u_ε) of Lipschitz continuous functions on Ω converging to u, which satisfy (ii) and:

$$\limsup_{\varepsilon} F^\varepsilon(u_\varepsilon) \leq \int_{\Omega\cap\partial A} \xi(x)\, dH^{N-1} \tag{4.22}$$

(recall that $h(x, \cdot)$ vanishes as $x \notin \Omega_0$, hence the integral in the second member of (i) can be taken on the whole Ω).

From Lemma 3.2 there exists, for every x of $\Omega \cap \partial A$, a real $r > 0$ and a Lipschitz nondecreasing function w on $\mathbb{R}$ such that $w(s) = 0$ if $s < 0$, $w(s) = 1$ if $s > r$, and

$$\int_0^r f(x, \alpha(x) + \gamma(x)w(s), \gamma(x)v(x)w'(s))\, ds < \xi(x) \tag{4.23}$$

(note that when $\gamma(x) = 0$, the integral in (4.23) vanishes; when $\gamma(x) > 0$, (4.23) is deduced from (3.10) by taking $v(s) = w(s)\gamma(x)$). Using the continuity of the first member in (4.23) with respect to x, the inequality still holds in a neighborhood of x. Hence thanks to the compactness of $\partial A \cap \bar{\Omega}$, there exists a finite family $(\Sigma_i)_{i\in I}$ of disjoint open subsets of $\partial A \cap \bar{\Omega}$ and a corresponding family of real numbers (r_i) and of Lipschitz functions (w_i) on Ω such that

$$\begin{cases} H^{N-1}(\partial A \cap \bar{\Omega} \setminus \bigcup \Sigma_i) = 0, \\ w_i(s) = 0 \quad \text{if } s < 0, \qquad w_i(s) = 1 \quad \text{if } s > r_i, \\ \displaystyle\int_0^{r_1} f(x, \alpha(x) + \gamma(x)w_i(s), \gamma(x)v(x)w_i'(s))\, ds < \xi(x), \qquad \forall x \in \Sigma_i. \end{cases} \tag{4.24}$$

In fact, as $f(x, \alpha(x), 0) = f(x, \beta(x), 0) = 0$ (see (H_1)), r_i in (4.24) can be replaced by $r = \operatorname{Max}\{r_i; i \in I\}$.

Let us consider the function $g: \mathbb{R}^N \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} -\operatorname{dist}(x, \partial A) & \text{if} \quad x \in A, \\ +\operatorname{dist}(x, \partial A) & \text{if} \quad x \notin A \end{cases} \tag{4.25}$$

and, for every real t,

$$S_t = \{x \in \mathbb{R}^N; g(x) = t\}, \qquad V_t = \{x \in \mathbb{R}^N; |g(x)| < t\}. \tag{4.26}$$

Using as in [19] an argument of Gilbarg and Trudinger [13], there exists for small t a diffeomorphism Φ of V_t to $\partial A x]-t, +t[$ such that

$$\exists \mu > 0, \qquad \det|D\Phi| \geq \mu. \tag{4.27}$$

Moreover, g is smooth in $\overline{V_t}$ and denoting $\hat{\Phi}$ as the component of Φ on ∂A, we have

$$\forall x \in V_t, \qquad Dg(x) = \nu(\hat{\Phi}(x)). \tag{4.28}$$

The idea of the proof is now to take $u_\varepsilon = \alpha + \gamma w_i(g(x)/\varepsilon)$ on the subset $\{x \in V_t; \hat{\Phi}(x) \in \Sigma_i\}$. Unfortunately we have to interpolate u_ε between Σ_i and Σ_j when $i \neq j$! Thus the construction of u_ε becomes rather technical. Let $\delta > 0$ (after δ will tend to 0) and define $\Sigma_i^\delta = \{x \in \Sigma_i; \operatorname{dist}(x, \partial\Sigma_i) > \delta\}$. Let $\hat{\varphi}_i^\delta \in \mathscr{D}(\partial A, [0, 1])$ such that

$$\sum_i (\hat{\varphi}_i^\delta) = 1 \quad \text{on } \partial A, \qquad \hat{\varphi}_i^\delta = 1 \quad \text{on } \overline{\Sigma_i^\delta}. \tag{4.29}$$

We deduce a smooth partition of the unity on V_t by setting

$$\varphi_i^\delta = \hat{\varphi}_i^\delta \circ \hat{\Phi}. \tag{4.30}$$

Then put, for every $\varepsilon < t/r$,

$$v_\varepsilon^\delta = \begin{cases} \alpha & \text{on} \quad A \setminus V_t, \\ \alpha + \gamma \sum_i \varphi_i^\delta w_i\left(\dfrac{g(x)}{\varepsilon} + v_\varepsilon\right) & \text{on} \quad \Omega \cap V_t, \\ \beta & \text{on} \quad \Omega \setminus (A \cup V_t), \end{cases} \tag{4.31}$$

where v_ε is a real number of $[0, r]$ we choose[3] in order to satisfy the integral constraint $\int_\Omega v_\varepsilon^\delta \, dx = \int_\Omega u \, dx$ (in fact v_ε depends also on δ).

Clearly, v_ε^δ is Lipschitz continuous and verifies $\alpha \leq v_\varepsilon^\delta \leq \beta$. Moreover, there exists some constant $M > 0$ (which depends only on δ) such that

$$\left| Dv_\varepsilon^\delta \right| \leq \frac{M}{\varepsilon}. \tag{4.32}$$

[3] Let us briefly show the existence of v_ε. Denoting by $v_\varepsilon(t, \cdot)$ the function v_ε^δ obtained from (4.31) by taking $v_\varepsilon = t$, we have $v_\varepsilon(0, x) \leq u(x)$ and $v_\varepsilon(r, x) \geq u(x)$ (indeed $w_i(g(x)/\varepsilon) = 0$ if $x \in A$ and $w_i(g(x)/\varepsilon + r) = 1$ if $x \notin A$). Hence the function k defined as $k(t) = \int_\Omega v_\varepsilon(t, x) \, dx$ is continuous (recall w_i is Lipschitz) and verifies $k(0) \leq \int_\Omega u(x) \, dx \leq k(1)$. The existence of v_ε follows.

For a fixed δ, let us denote

$$W_{\varepsilon,i} = \{x \in \Omega;\ -r\varepsilon < g(x) < r\varepsilon,\ \hat{\Phi}(x) \in \Sigma_i^\delta\}, \tag{4.33}$$

$$\Delta_{\varepsilon,i} = \{x \in \Omega;\ -r\varepsilon < g(x) < r\varepsilon,\ \hat{\Phi}(x) \in \Sigma_i \backslash \Sigma_i^\delta\}, \tag{4.33'}$$

$$A_\varepsilon = \{x \in \Omega;\ g(x) < -r\varepsilon\}, \qquad B_\varepsilon = \{x \in \Omega;\ g(x) > r\varepsilon\}. \tag{4.34}$$

We have

$$F^\varepsilon(v_\varepsilon^\delta) = \int_{A_\varepsilon} \varphi_\varepsilon + \int_{B_\varepsilon} \varphi_\varepsilon + \sum_i \int_{W_{\varepsilon,i}} \varphi_\varepsilon + \sum_i \int_{\Delta_{\varepsilon,i}} \varphi_\varepsilon, \tag{4.35}$$

where

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon} f(x, v_\varepsilon^\delta(x), \varepsilon D v_\varepsilon^\delta(x)). \tag{4.36}$$

From the dominated convergence theorem, $\int_{A_\varepsilon} \varphi_\varepsilon$ tends to 0 as $\varepsilon \to 0$. Indeed, $\varphi_\varepsilon 1_{A_\varepsilon} = (1/\varepsilon) f(x, \alpha, \varepsilon D\alpha)$ converges a.e. to $f'_p(x, \alpha(x), 0) \cdot D\alpha(x) 1_A$ which vanishes (see (H_2)) and, from convexity, $0 \le \varphi_\varepsilon 1_{A_\varepsilon} \le f(x, \alpha, D\alpha)$. For the same reason, the integral $\int_{B_\varepsilon} \varphi_\varepsilon$ vanishes as $\varepsilon \to 0$.

Let us now give an upper bound for $\int_{\Delta_{\varepsilon,i}} \varphi_\varepsilon$. From $\alpha \le v_\varepsilon^\delta \le \delta$, the continuity of f, and (4.32), there exists a constant $M' > 0$ (depends only on δ) such that

$$\varphi_\varepsilon(x) \le \frac{M'}{\varepsilon}, \qquad \forall x \in \Omega. \tag{4.37}$$

On the other hand,

$$|\Delta_{\varepsilon,i}| = \int_{-r\varepsilon}^{r\varepsilon} ds \left(\int_{\Sigma_i \backslash \Sigma_i^\delta} |\det(D\Phi)^{-1}|\, dH^{N-1} \right). \tag{4.38}$$

Hence

$$\int_{\Delta_{\varepsilon,i}} \varphi_\varepsilon(x)\, dx \le \frac{2M'r}{\mu} H^{N-1}(\Sigma_i \backslash \Sigma_i^\delta). \tag{4.39}$$

Let us finally compute the limit of $\int_{W_{\varepsilon,i}} \varphi_\varepsilon$. Noting that on $W_{\varepsilon,i}$, $v_\varepsilon^\delta = \alpha + \gamma w_i(g(x)/\varepsilon + v_\varepsilon)$ (this because $\varphi_i^\delta = 1$) and that $|D(g(x)/\varepsilon)| = 1/\varepsilon$ a.e., we have

$$\int_{W_{\varepsilon,i}} \varphi_\varepsilon\, dx = \int_{W_{\varepsilon,i}} \Phi_\varepsilon(x, k_\varepsilon(x)) |Dk_\varepsilon(x)|, \tag{4.40}$$

where

$$k_\varepsilon(x) = \frac{g(x)}{\varepsilon} + v_\varepsilon, \tag{4.40'}$$

$$\Phi_\varepsilon(x, s) = f(x, \alpha(x) + \gamma(x) w_i(s), \varepsilon[D\alpha(x) + w_i(s) D\gamma(x)] + \gamma(x) w_i'(s) Dg(x)).$$

Using formula (2.6) of Proposition 2.4, we have

$$\int_{W_{\varepsilon,i}} \varphi_\varepsilon\, dx = \int_0^r \eta_\varepsilon(s)\, ds, \tag{4.41}$$

where

$$\eta_\varepsilon(s) = \int_{\{x \in W_{\varepsilon,i};\, k_\varepsilon(x) = s\}} \Phi_\varepsilon(x, s)\, dH^{N-1}. \tag{4.41'}$$

For fixed s, the sequence of continuous functions $(\Phi_\varepsilon(\cdot, s))_{\varepsilon>0}$ converges uniformly to the function defined as $\Phi(\cdot, s) = f(\cdot, \alpha + \gamma w_i(s), \gamma w_i'(s)Dg)$. Let $\mu_\varepsilon = H^{N-1}(\Omega \cap S_{t_\varepsilon} \cap \cdot)$ where $t_\varepsilon = \varepsilon(s - v_\varepsilon)$ (S_t defined by (4.26)) and the open set $\Omega_i^\delta = \{x \in V_t \cap \Omega;\, \hat{\Phi}(x) \in \Sigma_i^\delta\}$. We can rewrite (4.41′) as

$$\eta_\varepsilon(s) = \int_{\Omega_i^\delta} \Phi_\varepsilon(x, s)\mu_\varepsilon(dx). \tag{4.41''}$$

From our assumption on subset A, $H^{N-1}(\partial A \cap \partial\Omega) = 0$ hence $H^{N-1}(\Sigma_i^\delta \cap \partial\Omega_i^\delta) = 0$. Owing to Lemma 4 of [19], we deduce that μ_ε converges tightly as $\varepsilon \to 0$ to $\mu = H^{N-1}(\Sigma_i^\delta \cap \Omega \cap \cdot)$. As a consequence, the sequence (η_ε) converges pointwise to the function η defined by

$$\eta(s) = \int_{\Sigma_i^\delta \cap \Omega} f(x, \alpha + \gamma w_i(s), \gamma w_i'(s)Dg)\, dH^{N-1}. \tag{4.42}$$

From the dominated convergence theorem and Fubini's formula, we are led to

$$\lim_{\varepsilon\to 0} \int_{W_{\varepsilon,i}} \varphi_\varepsilon\, dx = \int_0^r \eta(s)\, ds = \int_{\Sigma_i^\delta \cap \Omega} dH^{N-1}\left(\int_0^r f(x, \alpha + \gamma w_i(s), \gamma w_i'(s)Dg)\, ds\right). \tag{4.43}$$

So, thanks to (4.23) and since $Dg(x) = \nu(x)$ on ∂A (see (4.28)),

$$\lim_{\varepsilon\to 0} \int_{W_{\varepsilon,i}} \varphi_\varepsilon\, dx \le \int_{\Sigma_i^\delta \cap \Omega} \xi(x)\, dH^{N-1}. \tag{4.44}$$

Finally, collecting (4.35), (4.39), and (4.44);

$$\limsup_{\varepsilon\to 0} F^\varepsilon(v_\varepsilon^\delta) \le \sum_i \left(\frac{2M'r}{\mu} H^{N-1}(\Sigma_i \setminus \Sigma_i^\delta) + \int_{\Sigma_i^\delta \cap \Omega} \xi(x)\, dH^{N-1}\right).$$

Hence

$$\limsup_{\delta\to 0}\left(\limsup_{\varepsilon\to 0} F^\varepsilon(v_\varepsilon^\delta)\right) \le \int_{\partial A \cap \Omega} \xi(x)\, dx. \tag{4.45}$$

On the other hand, for every δ, the sequence $(v_\varepsilon^\delta)_{\varepsilon>0}$ converges to u, in $L^1(\Omega)$ as $\varepsilon \to 0$ (use the inequality $\alpha \le v_\varepsilon^\delta \le \beta$ together with the dominated convergence theorem).

We conclude (i) using a diagonalization argument [3, p. 33]: there exists a sequence (δ_ε) such that $\delta_\varepsilon \to 0$ and such that $u_\varepsilon = v_\varepsilon^{\delta_\varepsilon}$ satisfies (4.22) and converges to u as $\varepsilon \to 0$. Recall that (ii) is satisfied provided a good choice of the parameter η_ε is used (see (4.31) and footnote 2).

(*b*) *Second Step.* Let $A \in X_{\text{loc}}(\Omega_0)$ and let $u = \alpha 1_A + \beta 1_{\Omega \setminus A}$. Let us denote by Φ_s the $L^1(\Omega)$-upper epilimit (see Section 2) of $(F^\varepsilon + I_m)$ where $m = \int_\Omega u\,dx$. We have to prove

$$\Phi_s(u) \le \int_{\Omega_0} h(x, DI_A). \tag{4.46}$$

Let us use the approximation u_k of u defined by $u_k = \alpha 1_{A_k \cap \Omega} + \beta 1_{\Omega \setminus A_k}$, where (A_k) is the sequence of smooth subsets of $\mathbb{R}^N$ obtained in Lemma 4.3 below. Clearly, from assertion (iii) of Lemma 4.3, $\forall k$,

$$\int_\Omega u_k\,dx = \int_\Omega \alpha\,dx + \int_{A_k} \gamma\,dx = \int_\Omega \alpha\,dx + \int_A \gamma\,dx = \int_\Omega u\,dx.$$

Then, from step (a),

$$\forall k, \qquad \Phi_s(u_k) \le \int_{\Omega_0} h(x, D1_{A_k})$$

and, from assertion (i) of Lemma 4.3, $u_k \to u$ in $L^1(\Omega)$. Finally (4.46) follows using the lower semicontinuity of Φ_s (see Section 2.3) and assertion (ii) of Lemma 4.3:

$$\Phi_s(u) \le \liminf_k \Phi_s(u_k) \le \limsup_k \int_{\Omega_0} h(x, D1_{A_k}) \le \int_{\Omega_0} h(x, D1_A). \qquad \square$$

Lemma 4.3. *Let A be a subset of Ω_0 such that $A \in X_{\text{loc}}(\Omega_0)$. Then there exists a sequence (A_k) of bounded subsets of $\mathbb{R}^N$ with smooth boundaries such that*:

(i) $\lim_{k\to\infty} |(A_k \cap \Omega_0)\Delta A| = 0$,

(ii) $\limsup_{k\to\infty} \int_{\Omega_0} h(x, D1_{A_k}) \le \int_{\Omega_0} h(x, D1_A)$,

(iii) $\int_{A_k \cap \Omega_0} \gamma(x)\,dx = \int_A \gamma\,dx$, $H^{N-1}(\partial\Omega \cap \partial A_k) = 0$ *for large* k.

Remark on Lemma 4.3. Recall $h(x, p) = \gamma(x)G(x, p)$ where G is given by (3.8′). In fact, in this lemma G can be replaced by any continuous function of $\bar{\Omega} \times \mathbb{R}^N$ to $\mathbb{R}_+$ which is sublinear in p. When $G(x, p) = |p|$ and $\gamma(x) =$ constant, we get $\Omega_0 = \Omega$ and, from (ii) and (iii), $\lim_{k\to\infty} \int_\Omega |D1_{A_k}| = \int_\Omega |D1_A|$, $|A_k \cap \Omega| = |A|$ (as a consequence of (i) and of the lower semicontinuity of the total variation, $\liminf_{n\to\infty} \int_\Omega |D1_{A_k}| \ge \int_\Omega |D1_A|$). We then recover Lemma 4 of [19]; note that in [19] it was assumed that both A and $\Omega \setminus A$ have nonempty interiors.

Proof. See Section A.3 of the Appendix.

4.3. *Proof of Theorem 3.3*

Owing to the variational properties of epiconvergence stated in Proposition 2.6 of Section 2.3, we have only to prove that any minimizing sequence in the sense of (3.11) is strongly relatively compact in $L^1(\Omega)$. We need a first estimate on the ratio $r_\varepsilon/\varepsilon$:

Lemma 4.4. *Assume* m *satisfies* (3.2). *Then there exists* $u \in K_m$ *such that* $F(u) < +\infty$. *Consequently,*

$$\limsup_{\varepsilon \to 0_+} \frac{r_\varepsilon}{\varepsilon} < +\infty. \tag{4.47}$$

Let (u_ε) be a minimizing sequence. By (3.11) and $f(x, u, p) \geq f(x, u, 0)$, $\forall p$, we have

$$0 \leq \int_\Omega f(x, u_\varepsilon, 0)\, dx \leq \int_\Omega f(x, u_\varepsilon, Du_\varepsilon)\, dx \leq r_\varepsilon + C\varepsilon. \tag{4.48}$$

Using (4.47), we deduce

$$\int_\Omega f(x, u_\varepsilon, 0)\, dx \to 0 \qquad \text{in } L^1(\Omega), \tag{4.49}$$

$$\exists C' > 0, \quad \exists \varepsilon_0 > 0, \qquad F^\varepsilon(u_\varepsilon) \leq C', \qquad \forall \varepsilon < \varepsilon_0. \tag{4.50}$$

From (4.50) and hypothesis (H_3), by the De La Vallée-Poussin criterion, the set $\{u_\varepsilon, \varepsilon > 0\}$ is equi-integrable on Ω, hence weakly relatively compact in $L^1(\Omega)$. So there exists a sequence (ε_n) converging to 0 and $u \in L^1(\Omega)$ such that

$$u_{\varepsilon_n} \to u, \qquad w - L^1(\Omega), \tag{4.51}$$

$$f(x, u_{\varepsilon_n}(x), 0) \to 0 \qquad \text{a.e. on } \Omega. \tag{4.52}$$

We conclude with the strong convergence in $L^1(\Omega)$ provided u_{ε_n} converges in measure (or a.e.) to u (use Vitali's theorem). To prove this, we first argue on Ω_0, then on $\Omega \setminus \Omega_0$.

(*a*) *Convergence in Measure on* Ω_0. Let ω be an open subset of Ω_0 such that $\Omega_0 \supset \bar{\omega}$. Define the functions $v_n(x)$, $k(t)$, and $M(t)$ by (4.5), (4.3), and (4.11) and proceed exactly as in the proof of Proposition 4.1. From (4.50), we then obtain that the sequence (w_n) defined by $w_n = M(v_n)$ is strongly relatively compact in $L^1(\Omega)$, hence for the convergence in measure. Noticing that $M(t)$ is Lipschitz continuous and verifies (see (4.4)) $0 < k(t) = M'(t) \leq 1$ a.e. on $\mathbb{R}$, it is a strictly increasing function whose inverse M^{-1} exists and is *continuous*. Consequently, as, from (4.5), $u_{\varepsilon_n}(x) = \beta - \gamma M^{-1}(w_n)$, the sequence (u_{ε_n}) is relatively compact for the convergence in measure and, from (4.51), the unique possible cluster point is u. This being true for any open subset ω such that $\Omega_0 \supset \bar{\omega}$, the conclusion actually holds on Ω_0.

(*b*) *Pointwise Convergence on* $\Omega \setminus \Omega_0$. From (4.52) and (H_3), there exists a negligible subset N such that, for every $x \notin N$, $\{u_{\varepsilon_n}(x), n \in N\}$ is bounded in $\mathbb{R}$. Since f is continuous, any cluster point z_x verifies $f(x, z_x, 0) = 0$. When $x \notin \Omega_0$, this implies, from (H_1), that z_x is unique and $z_x = \alpha(x) = \beta(x)$. Hence $u_{\varepsilon_n}(x) \to \alpha(x)$, $\forall x \in \Omega \setminus (\Omega_0 \cup N)$ (recall that $u = \alpha = \beta$ a.e. on $\Omega \setminus \Omega_0$). □

Proof of Lemma 4.4. Let $J = \{t \in \mathbb{R}_+; \exists A \in X(\Omega), \int_A \gamma(x)\, dx = t\}$. By a standard approximation argument, any element B of the Lebesgue σ-algebra $\mathscr{B}(\Omega)$ can be approached by smooth subsets A_k in such a way that $\lim_{k \to \infty} |(A_k \cap \Omega) \Delta B| = 0$.

Hence we can rewrite J as $\{t \in \mathbb{R}_+;\ \exists A \in \mathscr{B}(\Omega) \int_A \gamma(x)\,dx = t\}$. From Liapunov's convexity theorem, we deduce that J is nothing else but the closed interval $[0, \delta]$ where $\delta = \int_\Omega \gamma(x),\ dx = \int_{\Omega_0} \gamma(x)\,dx$. Consequently, by condition (3.2), the real $m' = m - \int_\Omega \alpha(x)\,dx$ does belong to J, that is there exists $A \in X(\Omega)$ such that $m' = \int_A \gamma(x)\,dx$.

Let $u = \alpha 1_A + \beta 1_{\Omega \setminus A}$. Then $\int_\Omega u(x)\,dx = m$ (that is $u \in K_m$) and $F(u) < +\infty$ (see Remark 3.6(b)). On the other hand, since $F^\varepsilon + I_m$ epiconverges to $F + I_m$ (see Theorem 3.5(ii)), there exists a sequence (v_ε) which converges to u and such that $v_\varepsilon \in K_m$ and $F^\varepsilon(v_\varepsilon) \to F(u)$. Thus

$$\limsup_{\varepsilon \to 0_+} \frac{r_\varepsilon}{\varepsilon} \le \limsup_{\varepsilon \to 0_+} F^\varepsilon(v_\varepsilon) < +\infty. \qquad \square$$

5. Appendix

5.1. *Proof of Lemma 3.1*

(i) For any (x, u) in $\bar{\Omega} \times \mathbb{R}$, we have $\operatorname{epi} f_c(x, u, \cdot) = \bigcup_{\lambda > 0} \lambda \operatorname{epi} f(x, u, \cdot)$. Since $f(x, u, \cdot)$ is convex, the second member in the previous equality is a convex cone. It follows that f_c is a sublinear function of p ($f_c(x, u, \cdot)$ is finite-valued as it is upper bounded by $f(x, u, \cdot)$). On the other hand, when $u \in \{\alpha(x), \beta(x)\}$, we have, for every p in $\mathbb{R}^N$,

$$0 \le f_c(x, u, p) \le \lim_{t \to 0_+} \frac{1}{t} f(x, u, tp) = f'_p(x, u, 0) \cdot p.$$

Then, from (H_2), $f_c(x, \alpha(x), \cdot) = f_c(x, \beta(x), \cdot) = 0$.

Let us prove that f_c is continuous. Since f_c is the infimum of a family of continuous functions, it is upper semicontinuous. To show that f_c is lower semicontinuous, let us consider a sequence (x_n, u_n, p_n) and a real $r \ge 0$ such that

$$x_n \to x, \qquad u_n \to u, \qquad p_n \to p, \qquad f_c(x_n, u_n, p_n) \le r,$$

and let us show that $f_c(x, u, p) \le r$. For every $\varepsilon > 0$, by (3.6), there exists some $t_n > 0$ such that

$$\frac{1}{t_n} f(x_n, u_n, t_n p_n) < r + \varepsilon. \tag{5.1}$$

Then consider two cases according to whether or not the sequence (t_n) tends to $+\infty$.

Case 1: t_n tends to $+\infty$. From the convexity assumption, for every $t > 0$, the following inequality eventually holds:

$$\frac{1}{t}[f(x_n, u_n, tp_n) - f(x_n, u_n, 0)]$$
$$\le \frac{1}{t_n}[f(x_n, u_n, t_n p_n) - f(x_n, u_n, 0)] < r + \varepsilon - \frac{1}{t_n} f(x_n, u_n, 0). \tag{5.2}$$

Since $f(x_n, u_n, 0)$ converges to $f(x, u, 0)$ and $f(x_n, u_n, tp_n)$ converges to $f(x, u, tp)$, we deduce, from (5.2),

$$\forall t > 0, \quad \forall \varepsilon > 0, \qquad f_c(x, u, p) \leq \frac{1}{t} f(x, u, tp) \leq r + \varepsilon + \frac{1}{t} f(x, u, 0).$$

The conclusion follows going to the limit as $t \to \infty$ and $\varepsilon \to 0$.

Case 2: t_n does not tend to $+\infty$. There exists $t \in \mathbb{R}_+$ and a subsequence (t_{n_k}), still denoted (t_n), such that $t_{n_k} \to t$. If $t > 0$, the conclusion follows going to the limit in (5.1). If $t = 0$, we use the inequality $(1/t_n) f(x_n, u_n, 0) \leq r + \varepsilon$ which is derived from (5.1) (recall that, from (H_2), $f(x_n, u_n, 0) \leq f(x_n, u_n, t_n p_n)$). It follows that $f(x, u, 0) = \lim_{n \to 0} f(x_n, u_n, 0) = 0$, that is $u \in \{\alpha(x), \beta(x)\}$ (see (H_1)). We conclude, thanks to (i), $f_c(x, u, p) = 0$ so $f_c(x, u, p) \leq r$.

(ii) The continuity of G and h follows obviously from that of f_c. The continuity of λ is deduced from the compactness of $\bar{\Omega} \times \{p \in \mathbb{R}^N;\ |p| = 1\}$. Assume now that $\lambda(x, u) = 0$ for some (x, u). Then there exists some p such that $|p| = 1$ and $f_c(x, u, p) = 0$. We have to prove that $f(x, u, 0) = 0$, which from (H_1) is equivalent to $u \in \{\alpha(x), \beta(x)\}$. Assume that $f(x, u, 0) > 0$. Then, for every $R > 0$,

$$\operatorname*{Inf}_{0 < t < R} \frac{1}{t} f(x, u, tp) \geq \frac{1}{R} f(x, u, 0) > 0.$$

Consequently, $\forall R > 0$,

$$\operatorname*{Inf}_{t \geq R} \frac{1}{t} f(x, u, tp) = f_c(x, u, p) = 0. \tag{5.3}$$

Going to the limit in (5.3) as $R \to +\infty$, we get that the nondecreasing function φ of $\mathbb{R}_+$ to $\mathbb{R}_+$ defined by $\varphi(t) = (1/t)[f(x, u, tp) - f(x, u, 0)]$ tends to 0 as $t \to +\infty$. Hence $\varphi \equiv 0$ which contradicts the fact that $f(x, u, \cdot)$ has a strict minimum at 0 (see (H_2)).

5.2. *Proof of Lemma 3.2*

Let $(x, p) \in \bar{\Omega} \times \mathbb{R}^N$ and let $r' > r$. We have to show that $I_{r'}(x, p) \leq I_r(x, p)$. For any v of Lip$\uparrow(\mathbb{R})$ such that

$$v(s) = \alpha(x) \quad \text{if } s \leq 0, \qquad v(s) = \beta(x) \quad \text{if } s \geq r, \tag{5.4}$$

we have

$$\int_0^{r'} f(x, v(s), pv'(s))\, ds = \int_0^r f(x, v(s), pv'(s))\, ds + (r' - r) f(x, \beta(x), 0).$$

Since, from (H_1), $f(x, \beta(x), 0)$ vanishes on Ω, we deduce

$$I_{r'}(x, p) \leq \int_0^r f(x, v(s), pv'(s))\, ds. \tag{5.5}$$

We get the inequality required by taking the infimum of the second member of (5.5) when v runs over the subset of Lip$\uparrow(\mathbb{R})$ which satisfy (5.4).

Let us now show that the limit $I_\infty(x,p)$ of $I_r(x,p)$ as $r \to +\infty$ verifies $I_\infty(x,p) = h(x,p)$. First we notice that if v is feasible for the variational problem associated with $I_r(x,p)$ (then $v' \geq 0$ a.e.), we have

$$\int_0^r f(x, v(s), pv'(s))\, ds \geq \int_0^r f_c(x, v(s), p)v'(s)\, ds.$$

Using the change of variable $u = v(s)$ in the last integral, we are led to

$$\int_0^r f(x, v(s), pv'(s))\, ds \geq \int_{\alpha(x)}^{\beta(x)} f_c(x, u, p)\, du = h(x, p). \tag{5.6}$$

This being true for every feasible v, we get, $\forall r > 0$, $I_r(x, p) \geq h(x, p)$.

Let us now prove the opposite inequality, that is $I_\infty(x,p) \leq h(x,p)$. Let $\varepsilon > 0$; for any $s \in [\alpha(x), \beta(x)]$ there exists (use (3.6) and the continuity of f) an open interval I_S and a real $t_S > 0$ such that

$$\forall u \in I_s, \qquad f_c(x, u, p) > \frac{1}{t_s} f(x, u, t_s p) - \varepsilon. \tag{5.7}$$

By compactness, there exists a finite subdivision of $[\alpha(x), \beta(x)]$, namely

$$\alpha_0 = \alpha(x) < \alpha_1 < \alpha_2 < \cdots < \alpha_k = \beta(x)$$

and a corresponding family $\{t_i;\, i = 1, k\}$ in $]0, +\infty[$ such that

$$\forall u \in [\alpha_i, \alpha_{i+1}], \qquad f_c(x, u, p) > \frac{1}{t_i} f(x, u, t_i p) - \varepsilon. \tag{5.8}$$

Let $s_0 = 0 < s_1 < s_2 < \cdots < s_k$ be the subdivision of $\mathbb{R}_+$ defined by $s_{i+1} = s_i + (\alpha_{i+1} - \alpha_i)/t_i$ and let us consider the piecewise affine function of $\mathbb{R}$ to $\mathbb{R}_+$ defined by

$$\begin{aligned} &v(s) = \alpha(x) \quad \text{if } s \leq 0, \qquad v(s) = \beta(x) \quad \text{if } s \geq s_k, \\ &v'(s) = t_i \quad \text{if } s \in]s_i, s_{i+1}[. \end{aligned} \tag{5.9}$$

Clearly, v belongs to $\text{Lip}\uparrow(\mathbb{R})$ and, from (5.8),

$$\begin{aligned} &\int_0^{s_k} f(x, v(s), pv'(s))\, ds \\ &\quad = \sum_{i=0}^{i=k-1} \left(\int_{s_i}^{s_{i+1}} \cdots \right) \\ &\quad = \sum_{i=0}^{i=k-1} \left(\int_{\alpha_i}^{\alpha_{i+1}} f(x, v, t_i p)\, dv \right) \qquad (dv = t_i\, ds \text{ on }]s_i, s_{i+1}[) \\ &\quad \leq \sum_{i=0}^{i=k-1} \left(\int_{\alpha_i}^{\alpha_{i+1}} f_c(x, v, p)\, dv \right) + \varepsilon(\beta(x) - \alpha(x)). \end{aligned}$$

Consequently, for every $\varepsilon > 0$,

$$I_\infty(x, p) \leq I_{s_k}(x, p) \leq \int_{\alpha(x)}^{\beta(x)} f_c(x, v, p)\, dv + \varepsilon(\beta(x) - \alpha(x)). \tag{5.10}$$

We conclude by taking the limit in (5.10) as $\varepsilon \to 0_+$. □

5.3. Proof of Lemma 4.3

(a) Let us first assume that both A and $\Omega_0 \backslash A$ have nonempty interior and let $B(x_1, \delta)$ and $B(x_2, \delta)$ be two open balls such that

$$A \supset B(x_1, \delta), \qquad \Omega_0 \backslash A \supset B(x_2, \delta). \tag{5.11}$$

First we approximate A by subsets of finite perimeter in Ω (recall that we have only $A \in X_{\mathrm{loc}}(\Omega_0)$). From formula (2.6) (Section 2.2), we have

$$\int_\Omega G(x, D\gamma(x))\, dx = \int_0^{+\infty} \varphi(t)\, dt < +\infty, \tag{5.12}$$

where φ is defined for almost all t of $]0, +\infty[$ by

$$\varphi(t) = \frac{1}{t} \int_\Omega h(x, D1_{S_t}), \qquad S_t = \{x \in \Omega \backslash \gamma(x) < t\}. \tag{5.13}$$

Since φ is integrable, $\liminf_{t \to 0_+} t\varphi(t) = 0$. Hence we can choose a sequence (t_k) such that

$$t_k \to 0_+, \qquad t_k \varphi(t_k) \to 0, \qquad S_{t_k} \in X(\Omega). \tag{5.14}$$

Let Ω_k be the open set $\{x \in \Omega, \gamma(x) > 1/t_k\}$ and let $A'_k = A \cap \Omega_k$. Clearly, A'_k has a finite perimeter in Ω and $(\partial * A \cap \Omega_k) \cup \partial * \Omega_k \supset \partial * A'_k$. Hence

$$\begin{aligned} \int_\Omega h(x, D1_{A'_k}) &\leq \int_{\Omega_k} h(x, D1_A) + \int_\Omega h(x, D1_{\Omega_k}) \\ &\leq \int_{\Omega_k} h(x, D1_A) + t_k \varphi(t_k) \qquad \text{(use (5.13))}. \end{aligned}$$

From the monotone convergence theorem and (5.14), we deduce

$$\limsup_k \int_\Omega h(x, D1_{A'_k}) \leq \int_{\Omega_0} h(x, D1_A). \tag{5.15}$$

Since A'_k has a finite perimeter in Ω, it can be approximated as was done in [19, Lemma 1] (see also p. 22 of [14]); there exists an open subset $\tilde{A}_k$ of $\mathbb{R}^N$ with a smooth boundary such that

$$|\tilde{A}_k \Delta A'_k| < \frac{1}{k}, \qquad \tilde{A}_k + B\left(0, \frac{1}{k}\right) \supset A'_k, \qquad A'_k + B\left(0, \frac{1}{k}\right) \supset \tilde{A}_k, \tag{5.16}$$

$$H^{N-1}(\partial \tilde{A}_k \cap \partial \Omega) = 0, \tag{5.17}$$

$$\int_\Omega h(x, D1_{\tilde{A}_k}) < \int_\Omega h(x, D1_{A'_k}) + \frac{1}{k} \tag{5.18}$$

(for (5.18) we use the fact that $|D1_{A'_k}|$ is reached tightly on Ω and assertion (ii) of Proposition 2.3; note that integrals can be taken on Ω_0 since $h(x, \cdot)$ vanishes when $x \notin \Omega_0$).

From (5.16), for any $\delta' < \delta$, we have, for large k,

$$\tilde{A}_k \supset B(x_1, \delta'), \qquad \Omega_0 \backslash \tilde{A}_k \supset B(x_2, \delta'). \tag{5.19}$$

Moreover,

$$\lambda_k = \int_{\tilde{A}_k} \gamma(x)\,dx - \int_A \gamma(x)\,dx \qquad \text{tends to 0 as } k \to \infty. \tag{5.20}$$

Now, set as in Lemma 1 of [19],

$$A_k = \begin{cases} \tilde{A}_k \backslash B(x_1, r_k) & \text{if } \lambda_k > 0, \\ \tilde{A}_k & \text{if } \lambda_k = 0, \\ \tilde{A}_k \backslash B(x_2, r'_k) & \text{if } \lambda_k < 0, \end{cases} \tag{5.21}$$

where r_k and r'_k are choosen in order to satisfy

$$\int_{B(x_1, r_k)} \gamma(x)\,dx = \int_{B(x_2, r'_k)} \gamma(x)\,dx = \lambda_k. \tag{5.22}$$

Since the function $r \to \int_{B(x_i, r)} \gamma(x)\,dx$ $(i = 1, 2)$ is continuous and strictly increasing for $0 \le r \le \operatorname{dist}(x_i, \partial\Omega_0)$, the reals r_k and r'_k are unique and tend to 0 as $k \to +\infty$. Then from (5.19), we have, for large k,

$$\int_{A_k \cap \Omega} \gamma(x)\,dx = \int_{\tilde{A}_k} \gamma(x)\,dx - \lambda_k = \int_A \gamma(x)\,dx$$

and

$$H^{N-1}(\partial\Omega \cap \partial A_k) = 0 \qquad \text{(this from (5.17) and (5.19))}.$$

So (iii) is satisfied. Assertion (i) is trivial using (5.16). On the other hand,

$$\int_{\Omega_0} h(x, D1_{A_k}) \le \int_{\Omega_0} h(x, D1_{\tilde{A}_k}) + C'' H^{N-1}(\partial B(x_1, r_k) \cup \partial B(x_2, r'_k)), \tag{5.23}$$

where

$$C'' = \sup\{h(x, p);\, x \in \bar{\Omega}, |p| = 1\}.$$

From (5.15) and (5.18), we then deduce

$$\limsup_{k\to\infty} \int_{\Omega_0} h(x, D1_{A_k}) \le \limsup_{k\to\infty} \int_{\Omega} h(x, D1_{A_k}) \le \int_{\Omega_0} h(x, D1_A).$$

(b) Let us now remove the restriction that both A and $\Omega_0 \backslash A$ have nonempty interior. First we notice that if $|A| = 0$ (or if $|A| = |\Omega_0|$), the result is obvious by taking $A_k = \varnothing$ (or $A_k \supset \bar{\Omega}$, $\forall k$) (in both cases the left and right members in inequality (ii) vanish). So we can assume $0 < |A| < |\Omega_0|$ and there exists two points x_1, x_2 such that, in Ω_0, x_1 is a point of density of A and x_2 is a point of density of $\Omega_0 \backslash A$.

Consider the function $\Phi(\delta_1, \delta_2) = \int_{\Omega_{1,2}} \gamma(x)\,dx - \int_{\Omega_0} \gamma(x)\,dx$, where $\Omega_{1,2} = \Omega_0 \cup B(x_1, \delta_1) \backslash B(x_2, \delta_2)$. Since $\gamma(x) > 0$ on Ω_0, we have, for any $\delta > 0$, $\Phi(\delta, 0) < 0$, $\Phi(0, \delta) > 0$. Hence, as Φ is continuous, there exists some t in $]0, 1[$ (t depends on δ) such that $\Phi(t\delta, (1-t)\delta) = 0$. Define $A_\delta = A \cup B(x_1, (1-t)\delta) \backslash B(x_2, t\delta)$ and $u_\delta = \alpha 1_{A_\delta} + \beta 1_{\Omega \backslash A_\delta}$. Clearly, both A_δ and $\Omega_0 \backslash A_\delta$ have nonempty interiors and $\int_{A_\delta} \gamma(x)\,dx = \int_A \gamma(x)\,dx$. Moreover, $|A_\delta \Delta A|$ tend to 0 as $\delta \to 0_+$ and, using an inequality similar to (5.23), we get

$$\limsup_{\delta \to 0} \int_{\Omega_0} h(x, D1_{A_\delta}) \leq \int_{\Omega_0} h(x, D1_A).$$

Finally, we apply the construction of step (a) to every A_δ and conclude thanks to a diagonalization argument. □

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