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The Obstacle Problem for an Elastoplastic Body*

B. D. Reddy[1] and F. Tomarelli[2]

[1] Department of Applied Mathematics, University of Cape Town, 7700 Rondebosch, Republic of South Africa

[2] Dipartimento di Matematica, Università di Pavia, Strada Nuova 65, 27100 Pavia, Italy

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Abstract. The obstacle problem for elastoplastic bodies is considered within the framework of general existence results for unilateral problems recently presented by Baiocchi *et al.* Two models of plasticity are considered: one is based on a displacement-plastic strain formulation and the second, a specialization of the first, is the standard Hencky model. Existence theorems are given for the Neumann problem for a body constrained to lie on or above the half-space $\{x \in \mathbb{R}^3 \colon x^3 \leq 0\}$. For hardening materials the displacements are sought in the Sobolev space $H^1(\Omega, \mathbb{R}^3)$ while for perfectly plastic materials they are sought in $\mathrm{BD}(\Omega)$, the space of functions of bounded deformation. Conditions for the existence of solutions are given in terms of compatibility and safe load conditions on applied loads.

1. Introduction

There has been considerable progress in the last decade in the qualitative study of boundary- and initial-boundary-value problems in plasticity. The existence theory for the quasi-static rate problem for perfect plasticity was first attempted by Duvaut and Lions [DL], and subsequently improved upon by Johnson [Jo1], who later [Jo2] extended his work to include hardening. These authors have

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formulated the problem in terms of stresses, and in [Jo1] the problem of finding velocities is subsequently considered, though the result is too weak to obtain information about the strains. Suquet [S2], [S3], on the other hand, has approached the evolution problem for perfect plasticity and proved the existence of velocities in the space BD(Ω): this space, introduced independently in [S1] and [MSC], consists of integrable vector-valued functions for which certain combinations of derivatives, corresponding to the strain, are bounded measures. Related work has been carried out by Anzellotti [A1] in the context of the quasi-static rate problem.

It is worth pointing out that when hardening is present, the problems which lead, in perfect plasticity, to the abandonment of the Sobolev spaces in favor of BD(Ω), are not present. The existence theory for the rate problem may then be approached within the conventional framework of a functional which is coercive and weakly lower semicontinuous in a Sobolev space. This has been carried out by Jiang [Ji] and by Reddy *et al.* [RGM].

Another problem which has evoked much interest is the deformation theory or holonomic theory problem of plasticity, in which plastic behavior is approximated by a constitutive relation for total stress in terms of total strain. Here, too, there are no fundamental difficulties if the perfectly plastic problem is approached using a formulation in terms of stress, as in [DL] and [OW], or if hardening is present, as in [RG]. But the displacement formulation for perfect plasticity runs into difficulties which are again circumvented by seeking solutions in the space BD(Ω); for the Hencky model [DL] this problem has been investigated extensively by Temam [T1], [T2] and by Temam and Strang [TS1], these authors adopting an approach which draws on duality arguments. Independently, Anzellotti and Giaquinta [AG1], [AG2] and Anzellotti [A2] have investigated the same problem, exploiting the similarities between this problem and the minimal surface problem posed on the space of functions of bounded variation [G]. Further contributions have been made by Hardt and Kinderlehrer [HK].

Unilateral problems in plasticity have been all but ignored, at least as far as the qualitative theory is concerned. As far as we know the only work in this area has been that of Haslinger and Hlaváček [HH], who have considered the problem of contact between two perfectly plastic bodies, using a stress formulation.

The aim of this contribution is to consider the unilateral problem for elastic–perfectly plastic bodies, within the framework of a general existence theory for unilateral problems developed in [BBGT1] and [BGT]. We are concerned with the deformation theory problem, and use a displacement formulation. To be specific, we consider the Neumann problem for an elastoplastic body which is constrained to lie in the upper half-space.

Two models of plasticity are considered: one is based on a displacement-plastic strain formulation [RMG] and the other, the Hencky model, is a specialization of the first and is a displacement formulation (see, for example, [DL]). In both cases we prove the existence of solutions provided that the data satisfy natural compatibility conditions on the resultant force and resultant moment of external forces, and a safe load condition involving the geometry of the body, its plastic modulus, and the intensity of the forces.

This paper is organized as follows. In Section 2 the unilateral problem is formulated for the elastoplastic material law proposed by Reddy *et al.* Both the hardening and perfectly plastic cases are treated, and the problems based on these two cases are formulated as minimization problems. The existence theorems are stated. Section 3 is devoted to the proofs of the theorems in Section 2. Section 4 treats the Hencky problem, which is viewed as a special case of the problem treated in Sections 2 and 3. In Section 5 we give an indication of the relationship between our methods and the methods of limit analysis [ET], [T2], [TS2].

In a forthcoming paper a Signorini-type problem with constraint acting only on the boundary ($v_3(x)\geq 0$, a.e. $H^{n-1}x\in\partial\Omega\cap\{x_3=0\}$) for an elastoplastic body is solved (see [To3]).

2. Formulation and Statement of Results

Consider a body which in its undeformed state occupies a bounded domain Ω in $\mathbb{R}^3_+=\{x\in\mathbb{R}^3: x_3>0\}$. The boundary $\partial\Omega$ of Ω is assumed to be Lipschitz. On $\partial\Omega$ a surface traction vector field $g(x)$ is prescribed, and there is a vector field $f(x)$ of body forces defined on Ω.

A rigid obstacle occupies the lower half-space $\{x\in\mathbb{R}^3: x_3<0\}$ and the body is constrained to lie on or above the obstacle. The body is assumed to obey a holonomic elastic–plastic constitutive law in which total stress is related to total strain. We make use here of such a law discussed in [RMG]; based on linear kinematic hardening and the von Mises yield criterion. For simplicity, and with little loss in generality, we confine attention to initially unstressed, undeformed bodies (the constitutive law in [RMG] assumes in general that the body has initial stress and deformation fields).

A classical (formal) description of this problem is as follows:

Problem P1. Find the vector displacement field $u(x)$ and the plastic strain tensor field $p(x)$ which satisfy

$$\left.\begin{aligned}
&\{\sigma_{ij}(u,p)\}_{,j}+f_i=0 &&(2.1)\\
&\sigma_{ij}=a_{ijhk}(\varepsilon_{hk}(u)-p_{hk}) &&(2.2)\\
&\sigma_{ij}-\eta p_{ij}\in\partial\Psi(p) &&(2.3)\\
&p_{ji}=p_{ij},\qquad p_{ii}=0 &&(2.4)
\end{aligned}\right\}\quad \text{in } \Omega,$$

$$\begin{aligned}
&t_i=g_i \quad \text{on } \partial\Omega, \qquad i=1,2,\\
&t_3\geq g_3 \quad \text{on } \partial\Omega, \qquad t_3=g_3 \quad \text{on } \partial\Omega\cap\{x_3+u_3>0\},
\end{aligned}\qquad (2.5)$$

$$x_3+u_3(x)\geq 0 \qquad \text{for all} \quad x\in\bar{\Omega}. \qquad (2.6)$$

Here and henceforth conventional indicial notation is used, including the summation convention. A subscript j following a comma denotes partial differentiation with respect to x_j. Cartesian coordinates $\{x_i\}_{i=1}^3$ are used.

In (2.1)-(2.6), σ_{ij} is the symmetric stress tensor, ε_{ij} is the strain tensor with

$$\varepsilon_{ij}(u)=\tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{2.7}$$

and a_{ijhk} is the elasticity tensor; a_{ijhk} has the symmetries

$$a_{ijhk}=a_{jihk}=a_{hkij}. \tag{2.8}$$

The variable η is a given scalar field called the hardening parameter; in the case of a perfectly plastic material we have

$$\eta=0. \tag{2.9}$$

The function Ψ: $M^3\to\mathbb{R}$, where M^3 is the set of real symmetric 3×3 matrices, is called the plastic work function, and is given by

$$\Psi(p)=k|p|=k(p_{ij}p_{ij})^{1/2}, \tag{2.10}$$

where k is a positive scalar field. The subdifferential $\partial\Psi(p)$ of Ψ at p is given by

$$\begin{aligned}\partial\Psi(p)&=\{\tau\in M^3\colon \Psi(q)-\Psi(p)-\tau:(q-p)\geq 0, \forall q\in M^3\}\\ &=\begin{cases}\{kp/|p|\}, & p\neq 0\\ \{\tau\in M^3\colon \tau:q\leq\Psi(q), \forall q\in M^3\}, & p=0,\end{cases}\end{aligned}$$

where $a:b=a_{ij}b_{ij}$ denotes the inner product in M^3.

Equations (2.4) indicate that p is symmetric and traceless, the latter arising from the physical observation that no volume change occurs as a result of plastic deformation. In (2.5), $t_i=\sigma_{ij}\nu_j$ where ν_j is the outward unit normal to $\partial\Omega$.

We now proceed with a variational formulation of Problem P1. For this purpose it is necessary to define certain spaces. We denote by $H^1(\Omega,\mathbb{R}^3)$ the Sobolev space of vector-valued distributions which together with their first derivatives are in $L^2(\Omega)$, and we define the space

$$V=\{w=(v,q)\colon v\in H^1(\Omega,\mathbb{R}^3), q\in L^2(\Omega, M_D^3)\} \tag{2.11}$$

which is a Hilbert space with the norm

$$\|w\|_V=(\|v\|^2_{H^1(\Omega,\mathbb{R}^3)}+\|\varepsilon(v)-q\|^2_{L^2(\Omega,M^3)})^{1/2}. \tag{2.12}$$

Here M_D^3 denotes the set of real, symmetric matrices with zero trace.

The space of M^3-valued measures on Ω with bounded total variation is denoted by $\mathcal{M}$. This space is endowed with the norm

$$\|m\|_{\mathcal{M}}=\sup\left\{\int_\Omega m_{ij}\varphi_{ij}\colon \varphi\in\mathcal{D}(\Omega, M^3), \varphi_{ij}\varphi_{ij}\leq 1\right\}, \tag{2.13}$$

where $\mathcal{D}(\Omega, M^3)$ is the space of M^3-valued test functions on Ω. We note that the action of a bounded measure on a continuous function is shown by an integral without the differential term dx.

The space $\mathrm{BD}(\Omega)$ of functions of bounded deformation is defined by

$$\mathrm{BD}(\Omega)=\{v\in L^1(\Omega,\mathbb{R}^3)\colon \varepsilon(v)\in\mathcal{M}\}, \tag{2.14}$$

and is endowed with the norm

$$\|v\|_{\mathrm{BD}(\Omega)}=\|v\|_{L^1(\Omega,\mathbb{R}^3)}+\|\varepsilon(v)\|_{\mathcal{M}}. \tag{2.15}$$

Finally, we define the product space

$$W=\{w=(v, q)\colon v\in \mathrm{BD}(\Omega),\ q\in \mathcal{M},\ \mathrm{tr}\, q=0,\ \varepsilon(v)-q\in L^2(\Omega, M^3)\} \tag{2.16}$$

with the norm

$$\|w\|_W=\|v\|_{\mathrm{BD}(\Omega)}+\|\varepsilon(v)-q\|_{L^2(\Omega,\mathbb{R}^3)}. \tag{2.17}$$

The set of admissible displacements and plastic strains is the closed convex set

$$K=\{w=(v, q)\colon (x+v(x))\cdot e^3\geq 0 \text{ a.e. in } \Omega\}. \tag{2.18}$$

It is to be understood later that $K\subset V$ for problems with $\eta>0$, and $K\subset W$ for problems with $\eta=0$.

With regard to the smoothness of the various material coefficients we assume that

$$a_{ijhk}\in L^\infty(\Omega) \tag{2.19}$$

and that a_{ijhk} is strongly elliptic: there exists a constant $\alpha>0$ such that

$$a_{ijhk}(x)\zeta_{ij}\zeta_{hk}\geq \alpha|\zeta|^2, \qquad \forall \zeta\in M^3, \quad \text{a.e. in } \Omega. \tag{2.20}$$

We also require that

$$\eta\in L^\infty(\Omega), \qquad \eta(x)\geq 0 \qquad \text{a.e. in } \Omega, \tag{2.21}$$

and that

$$k\in L^\infty(\Omega), \tag{2.22}$$

$$0<k_0\leq k(x)\leq k_1<\infty \qquad \text{a.e. in } \Omega, \tag{2.23}$$

for constants k_0, k_1. We note here that assumption (2.22) needs to be strengthened in the case of perfect plasticity ($\eta=0$) (see the discussion after (2.25)).

We define the functionals

$$F\colon W\to \mathbb{R},\qquad F(w)=\tfrac{1}{2}\int_\Omega a_{ijhk}(\varepsilon_{ij}(v)-q_{ij})(\varepsilon_{hk}(v)-q_{hk})\,dx \tag{2.24}$$

and

$$\Gamma\colon \mathcal{M}\to \mathbb{R},\qquad \Gamma(q)=\int_\Omega k|q|+\tfrac{1}{2}\int_\Omega \eta|q|^2\,dx. \tag{2.25}$$

Remark. If $\eta(x)\geq\eta_0>0$ (the case of hardening), then the set $\{w\colon F(w)+\Gamma(q)<\infty\}$ turns out to be V, so we consider the hardening problem in V; the first integral in (2.25) is then the usual Lebesgue integral. If $\eta=0$, the case of perfect plasticity, the problem has to be considered in W and the first integral in (2.25) represents the total variation of the bounded measure $k|q|$. In order for (2.25) to be meaningful when $\eta=0$, we assume that

$$k\in C^0(\bar{\Omega}) \qquad \text{when} \quad \eta=0. \tag{2.26}$$

The prescribed forces f and g are assumed to be related through the expressions

$$f_i = \ell_i - G_{ij,j}, \qquad g_i = G_{ij}\nu_j, \tag{2.27}$$

where G is a symmetric tensor. If ℓ and G are sufficiently smooth then Green's formula

$$\int_\Omega f\cdot v\,dx + \int_{\partial\Omega} g\cdot v\,ds = \int_\Omega \ell\cdot v\,dx + \int_\Omega G:\varepsilon(v)\,dx \tag{2.28}$$

holds. We wish the right-hand side of (2.28) to be meaningful for $v\in \mathrm{BD}(\Omega)$, so we assume that

$$\ell \in L^3(\Omega, \mathbb{R}^3), \qquad G\in C^0_0(\Omega, M^3) \tag{2.29}$$

(for v in $H^1(\Omega, \mathbb{R}^3)$, say $\eta \geq \eta_0 > 0$, G needs not to vanish at the boundary), and define the linear functional

$$L: \mathrm{BD}(\Omega)\to\mathbb{R}, \qquad Lv = \int_\Omega \ell\cdot v\,dx + \int_\Omega G:\varepsilon(v). \tag{2.30}$$

We observe that (2.29)-(2.30) are the most general assumptions which ensure that L is weak*-continuous on W. It is also important to note that the resultant force and moment do not depend on G: indeed, for smooth enough ℓ and G,

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,ds = \int_\Omega \ell\,dx$$

and

$$\int_\Omega f_\wedge(x-\bar{x})\,dx + \int_{\partial\Omega} g_\wedge(x-\bar{x})\,ds = \int_\Omega \ell_\wedge(x-\bar{x})\,dx \qquad \text{for any} \quad \bar{x}\in\mathbb{R}^3.$$

We are now ready to give a variational formulation of the problem defined by (2.1)-(2.7). Set

$$J(w) = F(w) + \Gamma(q) - Lv; \tag{2.31}$$

we define

Problem P2. Find $z = (u, p)\in K$ such that

$$J(z) \leq J(w), \qquad \forall w\in K.$$

The formal equivalence of Problem P2 to the classical Problem P1 is consistent with the equivalence established for the case $\eta > 0$, without unilateral conditions, in [RG]. We now record the main results of the present paper.

Theorem 1. *Assume that* $\eta(x) \geq \eta_0 > 0$, *and that*

(i) $e^1\cdot\int_\Omega \ell\,dx = e^2\cdot\int_\Omega \ell\,dx = 0$,
(ii) $e^3\cdot\int_\Omega \ell\,dx < 0$, *and*
(iii) *there exists* $x_0\in\Omega$ *such that* $\int_\Omega (x-x_0)_\wedge \ell(x)\,dx = 0$.[1]

Then there is a solution (u, p) *of Problem* P2 *such that* $(u,p)\in H^1(\Omega,\mathbb{R}^3)\times L^2(\Omega, M^3_D)$.

[1] It is enough assuming x_0 in the interior of the convex hull of Ω: $\overset{\circ}{\widehat{\mathrm{co}\,\Omega}}$.

Theorem 2. *Assume that* $\eta = 0$, *that conditions* (i), (ii), *and* (iii) *of Theorem* 1 *hold, and that there exists* $\varepsilon_0 > 0$ *such that*

(iv) $\|G\|_{L^\infty(\Omega, M^3)} + C_\Omega \|\ell\|_{L^3(\Omega, \mathbb{R}^3)} \le (1 - \varepsilon_0) k_0$,

where C_Ω *is the Sobolev–Poincaré constant* (*see* (3.13)) *and* k_0 *is as defined in* (2.23). *Then Problem* P2 *has a solution.*

Remark. Condition (iv) is referred to as a *safe load condition* (see [AG1]). As will become evident in the proof of Theorem 2 (see Lemma 3.1 and its consequences) the safe load condition can be weakened at the expense of strengthening assumptions (i) and (ii), in order to balance the relationship between the applied forces, the geometry of Ω, and the constant k_0. We give two examples, whose proofs are easily obtained by slightly modifying the proof of Theorem 2.

Theorem 2′. *Assume that* $\eta = 0$, *that condition* (iii) *of Theorem* 1 *holds, and that*

(i)′ $e^1 \cdot \ell(x) = e^2 \cdot \ell(x) = 0$ *a.e. in* Ω,
(ii)′ $e^3 \cdot \ell(x) \le 0$ *a.e. in* Ω *and* $e^3 \cdot \int_\Omega \ell \, dx < 0$, *and*
(iv)′ *there exists* $\varepsilon_0 > 0$ *such that* $\|G\|_{L^\infty(\Omega, M^3)} \le (1 - \varepsilon_0) k_0$.

Then there is a solution of Problem P2.

Theorem 2″. *Assume that* $\eta = 0$, *that conditions* (i), (ii), *and* (iii) *of Theorem* 1 *hold, and that*

(iii)′ $\int_\Omega \tilde{\ell}_\wedge (x - x_0) \, dx = 0$, *and*
(iv)″ $\|G\|_{L^\infty(\Omega, M^3)} + C_\Omega \|\tilde{\ell}\|_{L^3} \le (1 - \varepsilon_0) k_0$,

where $\tilde{\ell} = (e^1 \cdot \ell, e^2 \cdot \ell, (e^3 \cdot \ell)^+)$. *Then Problem* P2 *has a solution.*

Remarks. 1. It is mathematically obvious that the functional J is not bounded from below on K, without some smallness assumption on L (or at least on G) and this seems physically reasonable (see Remark 2 below). This is also the case in the problem of finding graphs of prescribed mean curvature [G]:

$$\int_\Omega \sqrt{1 + |\nabla v|^2} + \int_\Omega f v + \int_{\Gamma_N} g v \to \inf,$$

where v is scalar-valued, subject in a suitable sense to a Dirichlet condition on $\partial\Omega \backslash \Gamma_N$. The situation here is somewhat different, though: in the present case the main difficulty arises since we do not assume any Dirichlet condition.

2. The safe load condition is mechanically reasonable since it is a bound on the intensity of that part of the applied force which is not controlled by either the elastic energy or the rigid obstacle. Such a bound is given in terms of the constant k_0 (see (2.23)) and the geometry of Ω coupled with the term ℓ which gives the contribution to the resultant force. Note also that C_Ω is large if the diameter of Ω is large and hence for a prescribed volume it is larger when Ω has outward stings. This fact could be interpreted by saying that plasticity does not support even small loads applied on thin branches of the body, if these loads are not pointing toward the rigid obstacle.

3. Assumptions of the type (i)–(iv), (i)′, (ii)′, and (iv)′, known in the literature as compatibility, or safe load conditions, are explicit conditions which imply the condition

$$J^\infty \geq 0 \qquad \text{on } K,$$

which is in turn necessary in order to have the inferior boundedness of J (see definition (3.1)). More precisely we can say that

Proposition 1. *Conditions* (i) *and* (iii) *of Theorem* 1 *and*

(ii)″ $e^3 \cdot \int_\Omega \ell \, dx \leq 0$, *and*
(iv)‴ $Lv \leq \Gamma(\varepsilon(v)), \quad \forall v\colon v(x) \cdot e^3 \geq 0$ *a.e. in* Ω

are necessary in order to have a minimum.

Proof. Violation of (i) or (ii)″ gives the existence of a translation $\tau \in K^\infty$ subject to $J^\infty(\tau) < 0$.[2] Violation of (iii) implies that either the system of forces is a couple (which cannot be equilibrated by an obstacle) or there is a central axis which does not cross $\widehat{\mathrm{co}\,\overset{0}{\Omega}}$, and, for this last case, Fichera has given a general counterexample even for the elastic case (see [F]). Violation of (iv)‴ implies that there is $\bar{w} \in K^\infty$ such that

$$\bar{w} = (\bar{v}, \varepsilon(\bar{v})), \qquad L\bar{v} > \Gamma(\varepsilon(\bar{v})),$$

with

$$J^\infty(\bar{w}) < F^\infty(\bar{w}) = 0. \qquad \square$$

3. Proofs and Further Remarks

Set $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We recall the following classical definitions of convex analysis (see [R]): assuming that (Y, σ) is a topological vector space, for any $\Lambda\colon Y \to \bar{\mathbb{R}}$ convex, proper, and σ-lower semi-continuous (σ-l.s.c. in short) the recession functional Λ^∞ of Λ is defined by (assuming 0-neighborhoods are absorbing)

$$\Lambda^\infty(y) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} \Lambda(y_0 + \lambda y), \qquad \forall y \in Y, \tag{3.1}$$

for any $y_0 \in \operatorname{dom} \Lambda$; and, for any nonempty convex σ-closed subset T of Y, the *recession cone* T^∞ of T is defined by

$$T^\infty = \bigcap_{\lambda > 0} \frac{1}{\lambda} (T - y_0), \tag{3.2}$$

where y_0 is any element of T. Note that $(\Lambda_1 + \Lambda_2)^\infty = \Lambda_1^\infty + \Lambda_2^\infty$, for all Λ_1, Λ_2 convex and σ-l.s.c. such that $\operatorname{dom}(\Lambda_1 + \Lambda_2) \neq \varnothing$. Moreover, $y \in T^\infty$ iff $y \in Y$ and $\bar{y} + y \in T$ for all $\bar{y} \in T$.

[2] See definitions (3.1) and (3.2).

The following definition, introduced by Baiocchi *et al.* in [BBGT2], holds for any map $\Lambda\colon Y \to \bar{\mathbb{R}}$, not necessarily convex and/or σ-l.s.c.: the *sequential recession functional* Λ_∞ of Λ, with respect to σ, is defined by

$$\Lambda_\infty(y) = \inf\left\{\liminf_n \frac{1}{\lambda_n}\Lambda(\lambda_n y_n)\colon \lambda_n \to +\infty,\, y_n \xrightarrow{\sigma} y\right\}. \tag{3.3}$$

Referring to [BBGT1], [To1], and [To2] for main properties, we just note that

$$\Lambda_\infty = \Lambda^\infty \qquad \text{when } \Lambda \text{ is convex and } \sigma\text{-l.s.c.} \tag{3.4}$$

Theorem 3. *Let Y be a Banach space, let σ be a topology such that the closed unit ball is sequentially σ-compact and let (Y, σ) be a Hausdorff vector space. Let $T \subset Y$ be convex and sequentially σ-closed, and let $\Lambda\colon Y \to \bar{\mathbb{R}}$ be convex and sequentially σ-l.s.c. Moreover, assume the following*:

Compactness. *There exists $y_0 \in T \cap \operatorname{dom} \Lambda$ such that for all bounded sequences $\{y_m\}$ in Y satisfying $y_m \to^\sigma y_0$ and $\Lambda(y_m) \to \Lambda(y_0)$ we have $y_m \to y_0$ (strongly).* (3.5)

Necessary condition. $\Lambda^\infty(y) \geq 0$ *for all* $y \in T^\infty$. (3.6)

Compatibility. $T^\infty \cap \ker \Lambda^\infty$ *is a subspace.* (3.7)

Then Λ achieves a finite minimum over T.

Remarks. 1. The compactness assumption in the form (3.5) has already been introduced in an analogous abstract minimization context (see Theorem 4.1 of [BBGT2]), but this was done in a nonconvex and nonsequential framework. Hence the compactness was coupled there with a compatibility condition more implicit than (3.7).

2. The statement of Theorem 3 can be deduced as a corollary of the sequential versions of Theorem 3.4 of [BBGT1] by taking into account (3.4) and expressing the compatibility condition in a suitable form for convex functionals as the one showed in Theorem (3.12) of [BBGT1]. Nevertheless, for the sake of simplicity, we give here a direct proof of Theorem 3.

Proof of Theorem 3. (I) For any $R > 0$, define x_R as the solution of minimal norm among solutions of the problem

$$\Lambda(x_R) = \min\{\Lambda(x)\colon x \in T,\, \|x\| \leq R\}.$$

(II) It is a well-known consequence of convexity that any $x_{\bar{R}}$ such that $\|x_{\bar{R}}\| < \bar{R}$ is a global minimizer. So assume by contradiction that there is no such $\bar{R}$, and hence that there is a sequence $\{R_n\}$ such that $R_n \to +\infty$ and $\|x_{R_n}\| = R_n$.

(III) $y_n = x_{R_n}/R_n \to^\sigma y$ as $n \to +\infty$ with $y \in T^\infty \cap \operatorname{Ker} \Lambda^\infty$.

(In fact, for all $\bar{y} \in T$, $(1 - R_n^{-1})\bar{y} + R_n^{-1} x_{R_n} \in T$ and is σ-convergent to $\bar{y} + y$ which is then in T. So $y \in T^\infty$. For all $\lambda > 0$ and for n large enough

$$\begin{aligned}\Lambda(y_0 + \lambda y) &\leq \liminf \Lambda((1 - \lambda R_n^{-1}) y_0 + \lambda R_n^{-1} x_{R_n}) \\ &\leq \liminf[(1 - \lambda R_n^{-1})\Lambda(y_0) + \lambda R_n^{-1}\Lambda(x_{R_n})] \leq \Lambda(y_0),\end{aligned}$$

hence $\Lambda^\infty(y) \leq 0$ and (3.6) gives $\Lambda^\infty(y) = 0$).)

(IV) Now we exploit the compactness condition (3.5): set $w_n = (1-R_n^{-1})(y_0-y)+y_n$; then w_n is a bounded sequence, $w_n \to^\sigma y_0$, and the convexity, σ-l.s.c., and the fact that $T^\infty \cap \operatorname{Ker} \Lambda^\infty$ is a subspace together give

$$\Lambda(y_0) \le \liminf_n \Lambda(w_n) \le \limsup_n \Lambda[(1-R_n)^{-1}(y_0-y)+R_n^{-1}(R_n y_n)]$$

$$\le \limsup_n [(1-R_n)^{-1}\Lambda(y_0-y)+R_n^{-1}\Lambda(x_{R_n})] = \Lambda(y_0-y) = \Lambda(y_0).$$

So

$$\lim_n \Lambda(w_n) = \Lambda(y_0) \qquad (w_n \to y_0 \text{ and } y_n \to y \text{ strongly}).$$

(V) Thus $(x_{R_n}-y)$ is a minimum for Λ over $T \cap B_{R_n}$. But

$$\|x_{R_n}-y\| = \|(1-R_n)^{-1}x_{R_n}+(y_n-y)\| \le (1-R_n)^{-1}\|x_{R_n}\|+\|y_n-y\|$$

$$= \|x_{R_n}\|+\|y_n-y\|-1 < \|x_{R_n}\|$$

for n large enough and this is a contradiction. □

Coming back to our mechanical problem, as a space Y we consider $V = H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, M^3)$ when $\eta > 0$, and $W = \mathrm{BD}(\Omega) \times L^2(\Omega, M^3)$ if $\eta = 0$. As a mapping Λ we have the functional J given by (2.31), together with (2.19)-(2.30) and $T = K$ given by (2.18). It remains only to choose a topology σ, in order to apply Theorem 3. We choose, for $Y = V$,

σ as the product of the weak-$H^1(\Omega, \mathbb{R}^3)$ topology on the first component, with the weak-$L^2(\Omega, M^3)$ topology on the second component $(\varepsilon(v)-q)$, (3.8)

and, for $Y = W$,

σ as the product of the weak*-topology of $\mathrm{BD}(\Omega)$ on the first component, with the weak-$L^2(\Omega, M^3)$ topology on the second component $(\varepsilon(v)-q)$. (3.9)

We recall that $\mathrm{BD}(\Omega)$ is the dual of a Banach space (see [TS1]) and satisfies a Sobolev-type embedding (see [T2])

$$\mathrm{BD}(\Omega) \subset L^p(\Omega, \mathbb{R}^3), \qquad 1 \le p \le \tfrac{3}{2},$$

with compact embedding if $p < \frac{3}{2}$.

Thus there is a weak*-topology on W and the closed unit ball in W is sequentially weak*-compact (since V is reflexive, its closed unit ball is sequentially weakly compact too); hence

the closed unit ball of W is sequentially σ-compact;
the closed unit ball of V is sequentially σ-compact. (3.10)

We say that a sequence $\{w_n\}=\{(v_n, q_n)\}$ in V (respectively W) is σ-convergent to $w=(v,q)$, and write $w_n \to^{\sigma} w$, iff

$$\begin{aligned} &v_n \to v \quad \text{weakly in } H^1(\Omega, \mathbb{R}^3),\\ &\varepsilon(v_n)-q_n \to \varepsilon(v)-q \quad \text{weakly in } L^2(\Omega, M^3) \end{aligned} \tag{3.11}$$

(respectively,

$$\begin{aligned} &v_n \to v \quad \text{weak* in BD}(\Omega),\\ &\varepsilon(v_n)-q_n \to \varepsilon(v)-q \quad \text{weakly in } L^2(\Omega, M^3)). \end{aligned}$$

We recall the following result (see [K]). Set

$$\mathcal{R}=\{v\colon \Omega\to\mathbb{R}^3 \text{ s.t. } v(x)=Ax+b,\ A \text{ skew symmetric}\}. \tag{3.12}$$

For any bounded Lipschitz open set ω in $\mathbb{R}^3$, there are a linear map R and a constant C_ω such that $R\colon \mathrm{BD}(\omega)\to\mathcal{R}$ satisfies $Rv=v$, $\forall v\in\mathcal{R}$, and

$$Rv=\alpha(v)_\wedge(x-x_0)+\beta(v), \qquad \forall v,$$

with $\alpha(v), \beta(v)\in\mathbb{R}^3$ and $\beta(v)=\fint_{B_r(x_0)} v(y)\,dy$ where $B_r(x_0)\subset\subset\omega$, $r>0$, and

$$\|v-Rv\|_{L^{3/2}}\le C_\omega\int_\omega |\varepsilon(v)|, \qquad \forall v\in \mathrm{BD}(\omega). \tag{3.13}$$

In the following we refer to C_ω as the Poincaré-Sobolev constant of ω.

Lemma 3.1. *The compatibility conditions* (i), (ii), *and* (iii) *of Theorem* 1 *imply that*

$$\int_\Omega \ell\cdot v\,dx\le C_\Omega\|\ell\|_{L^3(\Omega,\mathbb{R}^3)}\int_\Omega|\varepsilon(v)|$$

for all $v\in\mathrm{BD}(\Omega)$ *such that* $e^3\cdot v(x)\ge 0$ *a.e. in* Ω.

Proof. By using the map R defined in (3.13) we get

$$Rv=\alpha(v)_\wedge(x-x_0)+\beta(v),$$

where

$$\alpha(v), \beta(v)\in\mathbb{R}^3, \qquad x_0 \text{ is defined as in (iii).}$$

Now

$$\begin{aligned}\int_\Omega \ell\cdot v\,dx &= \int_\Omega \ell\cdot(v-Rv)\,dx+\int_\Omega \ell\cdot[\beta(v)+\alpha(v)_\wedge(x-x_0)]\,dx\\ &\le \|\ell\|_{L^3}\|v-Rv\|_{L^{3/2}}+(e^3\cdot\beta(v))\left(e^3\cdot\int_\Omega \ell\,dx\right)\\ &\quad+\alpha(v)\cdot\int_\Omega (x-x_0)_\wedge\ell\,dx\\ &\le C_\Omega\|\ell\|_{L^3}\int_\Omega|\varepsilon(v)|,\end{aligned}$$

where we have used the Poincaré-Sobolev inequality (3.13), and the fact that

$$(\beta(v)\cdot e^3)=(Rv)(x_0)\cdot e^3=\fint_{B_r(x_0)} e^3\cdot v(y)\,dy\ge 0. \qquad \square$$

Since in both cases ($K \subset V$ and $K \subset W$) the σ-topology is stronger than the strong L^1-topology on the first component,

$$K \text{ is convex and sequentially } \sigma\text{-closed.} \tag{3.14}$$

It is well known (see [A2], [AG1], [G], and [T2]) that

F and Γ are sequentially σ-l.s.c., and L is sequentially σ-continuous on bounded subsets of W if $\eta = 0$, and on bounded subsets of V if $\eta > 0$. J is obviously convex. (3.15)

Finally, we define the set RBM of *rigid body motions*:

$$RBM = \{w = (v, q) \in W \colon \varepsilon(v) = q = 0\}. \tag{3.16}$$

Proof of Theorem 1. We check the assumptions of Theorem 3, with the choice $Y = V$, $\eta(x) \ge \eta_0 > 0$. Because of (3.10), (3.14), and (3.15) it suffices to check (3.5), (3.6), and (3.7).

Compactness. Take $y_0 = 0 \in V$. Then $0 \in K \cap \operatorname{dom} J$. Moreover, $J(w_n) \to 0 = J(0)$ and $w_n \to^{\sigma} 0$ imply $Lw_n \to 0$ and $F(w_n) + \Gamma(q_n) \to 0$, hence $w_n \to 0$ strongly.

Necessary condition. We have

$$J^\infty(w) = F^\infty(w) + \Gamma^\infty(q) - Lv, \qquad \forall w = (v, q) \in V, \tag{3.17}$$

and, due to the homogeneity properties and the fact that $\eta > 0$,

$$F^\infty(w) = \begin{cases} +\infty & \text{if } \ \varepsilon(v) \ne q, \\ 0 & \text{if } \ \varepsilon(v) = q, \end{cases} \tag{3.18}$$

and

$$\Gamma^\infty(q) = \begin{cases} +\infty & \text{if } \ q \ne 0, \\ 0 & \text{if } \ q = 0. \end{cases} \tag{3.19}$$

Thus

$$J^\infty(w) = +\infty \qquad \text{iff} \quad w \notin RBM, \tag{3.20}$$

and we have to check that $J^\infty(w) \ge 0$ only for $w \in K^\infty \cap RBM$.

For definition (2.18) of K, we have

$$K^\infty = \{w = (v, q) \in K \colon v \cdot e^3 \ge 0 \text{ a.e. in } \Omega\}.$$

Now, if $w \in K^\infty \cap RBM$, then $w = (v, 0)$ with $v = \alpha_\wedge(x - x_0) + \beta$ for suitable α, $\beta \in \mathbb{R}^3$ and

$$(\alpha_\wedge(x - x_0) + \beta) \cdot e^3 \ge 0 \qquad \text{a.e. in } \Omega.$$

Since $x_0 \in \Omega$, we also have

$$\beta \cdot e^3 \ge 0$$

and, from (i), (ii), and (iii),

$$-Lv = -(\beta \cdot e^3)\left(e^3 \cdot \int_\Omega \ell \, dx\right) \ge 0, \qquad \forall w = (v, q) \in K^\infty \cap RBM. \tag{3.21}$$

Thus (3.20) and (3.21) give $J^\infty(w) \ge 0$, $\forall w \in K^\infty \cap RBM$.

Compatibility. From (3.17)-(3.19),

$$\operatorname{Ker} J^{\infty} = RBM \cap \operatorname{Ker} L, \tag{3.22}$$

where, by an abuse of notation, $w \in \operatorname{Ker} L$ means $v \in \operatorname{Ker} L$.

For any $w = (v, q) \in K^{\infty} \cap \operatorname{Ker} J^{\infty}$ we thus have

$$e^3 \cdot v \geq 0, \qquad q \equiv 0, \qquad v \equiv \alpha_{\wedge}(x - x_0) + \beta \qquad \text{a.e. in } \Omega. \tag{3.23}$$

Moreover, from (i), (iii), and the fact that $v \in \operatorname{Ker} L$,

$$(e^3 \cdot \beta)\left(e^3 \cdot \int_{\Omega} \ell \, dx\right) = 0. \tag{3.24}$$

Condition (ii) and (2.24) give

$$e^3 \cdot \beta = 0, \tag{3.25}$$

and so (3.23) and (3.25) give

$$e^3 \cdot (\alpha_{\wedge}(x - x_0)) = e^3 \cdot (v - \beta) \geq 0 \qquad \text{a.e. in } \Omega. \tag{3.26}$$

Since $x_0 \in \Omega$, for any ζ in Ω there are $\zeta' \in \Omega$ and $\theta \in (0, 1)$ such that $x_0 = \theta\zeta + (1 - \theta)\zeta'$.

Substitution of ζ and ζ' in (3.26) gives

$$e^3 \cdot (\alpha_{\wedge}(x - x_0)) = 0 \qquad \text{a.e. in } \Omega. \tag{3.27}$$

Finally, (3.25) and (3.27) give $-w \in K^{\infty}$, hence

$$-w \in K^{\infty} \cap RBM \cap \operatorname{Ker} L. \tag{3.28}$$

Hence the cone $K^{\infty} \cap \operatorname{Ker} J^{\infty}$ is a subspace too.

We conclude from the abstract Theorem 3 that the problem (1.29) has a solution. □

Proof of Theorem 2. We again use Theorem 3, but now $Y = W$ (see definition (1.11)). Equations (3.10), (3.14), and (3.15) still hold, and the compactness condition (3.5) holds with $y_0 = 0$ as in the proof of Theorem 1. But now, checking (3.6) and (3.7) is different, since Γ grows only linearly.

Since F is quadratic, for all $w = (v, p)$ we have

$$J^{\infty}(w) = F^{\infty}(w) + \Gamma^{\infty}(q) - Lv, \tag{3.29}$$

$$F^{\infty}(w) = \begin{cases} +\infty & \text{if} \quad F(w) \neq 0 \quad (\varepsilon(v) \neq q), \\ 0 & \text{if} \quad F(w) = 0 \quad (\varepsilon(v) = q), \end{cases} \tag{3.30}$$

and

$$\operatorname{Ker} J^{\infty} = \operatorname{Ker} F \cap \operatorname{Ker}(\Gamma^{\infty} - L).^3 \tag{3.31}$$

[3] Here and in the following there is an abuse of notation which does not create any ambiguity: we write $(\Gamma^{\infty} - L)(w) = \Gamma^{\infty}(q) - Lv$ for all $w = (v, q) \in W$; $\operatorname{Ker} \Gamma^{\infty} = \{w = (v, q) : \Gamma^{\infty}(q) = 0\}$ and $\operatorname{Ker} L = \{(v, q) : Lv = 0\}$.

We now show that

$$K^\infty \cap \operatorname{Ker} F \cap \operatorname{Ker}(\Gamma^\infty - L) = K^\infty \cap \operatorname{Ker} F \cap \operatorname{Ker} \Gamma^\infty \cap \operatorname{Ker} L. \tag{3.32}$$

The embedding $K^\infty \cap \operatorname{Ker} F \cap \operatorname{Ker}(\Gamma^\infty - L) \supset K^\infty \cap \operatorname{Ker} F \cap \operatorname{Ker} \Gamma^\infty \cap \operatorname{Ker} L$ is obvious. On the other hand, assume that

$$w = (v, q) \in K^\infty \cap \operatorname{Ker} F \cap \operatorname{Ker}(\Gamma^\infty - L), \tag{3.33}$$

then $\varepsilon(v) = q$ in $\mathcal{M}$ and $e^3 \cdot v \geq 0$ a.e. in Ω. So (2.23) together with Lemma 3.1 give, for all $w \in K^\infty \cap \operatorname{Ker} F$,

$$\begin{aligned} Lv &= \int_\Omega \ell \cdot v\, dx + \int_\Omega G:\varepsilon(v) \leq (C_\Omega \|\ell\|_{L^3(\Omega,\mathbb{R}^3)} + \|G\|_{L^\infty(\Omega,M^3)}) \int_\Omega |\varepsilon(v)| \\ &\leq (1-\varepsilon_0)k_0 \int_\Omega |\varepsilon(v)| \leq (1-\varepsilon_0)\Gamma^\infty(\varepsilon(v)) = (1-\varepsilon_0)\Gamma^\infty(q). \end{aligned} \tag{3.34}$$

Thus

$$\Gamma^\infty(q) - Lv \geq \varepsilon_0 \Gamma^\infty(q) \geq 0, \qquad \forall w \in K^\infty \cap \operatorname{Ker} F. \tag{3.35}$$

Equations (3.33) and (3.35) give $\Gamma^\infty(q) = 0$, say $q = 0$, and

$$w \in \operatorname{Ker} \Gamma^\infty \cap \operatorname{Ker} L, \tag{3.36}$$

so that (3.32) is proved.

Since $\Gamma \geq 0$, (3.29), (3.30), and (3.35) give

$$J^\infty(w) \geq 0, \qquad \forall w \in K^\infty, \tag{3.37}$$

and the necessary condition (3.6) is thus proved.

Equations (3.31) and (3.32) give

$$\begin{aligned} K^\infty \cap \operatorname{Ker} J^\infty &= K^\infty \cap \operatorname{Ker} F \cap \operatorname{Ker} \Gamma^\infty \cap \operatorname{Ker} L \\ &= K^\infty \cap RBM \cap \operatorname{Ker} L. \end{aligned} \tag{3.38}$$

Starting from (3.38) we can repeat the same argument used to prove the compatibility condition in Theorem 1. □

4. The Obstacle Problem for Hencky Plasticity

The unilateral boundary-value problem based on the Hencky model of perfect plasticity may be recovered from (2.1)-(2.7) by setting $\eta = 0$ and by assuming additionally that the material is elastically isotropic: then the elasticity tensor is given by

$$a_{ijhk} = \lambda \delta_{ij}\delta_{hk} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \tag{4.1}$$

where δ_{ij} is the Kronecker delta and λ, μ are Lamé's constants. To conform with previous investigations of the Hencky problem we assume further that the material is homogeneous, so that λ, μ and the scalar k in (2.10) are constants. According to the von Mises yield criterion, the elastic region is the set of stresses for which $|\sigma^D| < k$: thus, using (4.1) we obtain

$$\left.\begin{aligned} \sigma^S &= (3\lambda + 2\mu)\varepsilon^S \\ \sigma^D &= 2\mu\varepsilon^D \end{aligned}\right\} \quad \text{when} \quad |\varepsilon^D| < k/2\mu. \tag{4.2, 4.3}$$

Here and henceforth a superscript D denotes the deviatoric part of a tensor and a superscript S its spherical part:

$$\sigma = \sigma^D + \sigma^S, \tag{4.4}$$

where

$$\sigma_{ij}^D = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}, \tag{4.5}$$

$$\sigma_{ij}^S = \tfrac{1}{3}\sigma_{kk}\delta_{ij}. \tag{4.6}$$

In the region of plastic behavior $|\sigma^D| = k$, p is a scalar multiple of σ^D (recall that $p^S = 0$) and we find that

$$\varepsilon^D - p = \sigma^D/2\mu \tag{4.7}$$

and

$$|p| = |\varepsilon^D| - k/2\mu, \tag{4.8}$$

so that

$$\left.\begin{aligned} \sigma^S &= (3\lambda + 2\mu)\varepsilon^S \\ \sigma^D &= k\varepsilon^D/|\varepsilon^D| \end{aligned}\right\} \quad \text{when } |\varepsilon^D| \geq k/2\mu. \tag{4.9}$$

The formal boundary-value problem corresponding to the Hencky formulation is then given by equations (2.1), (2.5)-(2.7), (4.2)-(4.3), and (4.8)-(4.9).

Upon substitution in (2.24) and (2.25) we find that the first two terms on the right-hand side of (2.31) become

$$E(v) = \int_\Omega \Phi(\varepsilon^D(v)) + \frac{\chi}{2}(\text{div } v)^2, \tag{4.10}$$

where $\chi = \lambda + \frac{2}{3}\mu$ and $\Phi\colon M^3 \to \mathbb{R}$ is defined by

$$\Phi(s) = \begin{cases} k|s| - k^2/4\mu, & |s| \geq k/2\mu, \\ \mu|s|^2, & |s| < k/2\mu. \end{cases} \tag{4.11}$$

With an appropriate formulation of the minimization problem in mind, we set

$$P(\Omega) = \{v \in \mathrm{BD}(\Omega)\colon \text{div } v \in L^2(\Omega)\}; \tag{4.12}$$

this is a Banach space with the norm (see [AG1])

$$\|v\|_{P(\Omega)} = \|v\|_{\mathrm{BD}(\Omega)} + \|\text{div } v\|_{L^2(\Omega)}. \tag{4.13}$$

If $v \in \mathrm{BD}(\Omega)$ with div $v \in L^2(\Omega)$, then $\varepsilon^D(v)$ belongs to $\mathcal{M}$ (see (2.14)), hence the integral $\int_\Omega \Phi(\varepsilon^D(v))$ is defined by (see [AG2])

$$\int_\Omega \Phi(\varepsilon^D(v)) = \int_\Omega \Phi((\varepsilon^D(v))^a)\,dx + k\int_\Omega |(\varepsilon^D(v))^s|, \tag{4.14}$$

where for any measure m of bounded variation we denote by $m = m^a\,dx + m^s$ its Lebesgue decomposition.

The functional L corresponding to external forces is still given by (2.30), but the additional regularity of functions of $P(\Omega)$ permits the assumptions on G to be weakened: we assume that

$$\begin{aligned} &\ell \in L^3(\Omega, M^3), \\ &G^{\mathrm{D}} \in C_0^0(\Omega, M_D^3), \\ &\operatorname{tr} G \in L^2(\Omega). \end{aligned} \tag{4.15}$$

If in addition G belongs to $C^1(\bar{\Omega}, M^3)$, then as before we can express L in terms of body forces and surface tractions:

$$Lv = \int_\Omega f \cdot v\,dx + \int_{\partial\Omega} g \cdot v\,ds$$

with

$$f = \ell - \operatorname{div} G, \qquad g = \tfrac{1}{3}(\operatorname{tr} G)\nu.$$

We now formulate the unilateral problem for the Hencky material.

Problem P3. Find $u \in P(\Omega) \cap K$ such that

$$I(u) = \inf_{v \in P(\Omega) \cap K} I(v), \tag{4.16}$$

where $I = E - L$.

Remark. The existence of a solution to the *unconstrained* version of Problem P3, that is, minimization over all $v \in P(\Omega)$, has been shown (see [AG1], [AG2], [T1], [T2], and [KH]).

Theorem 4. *Assume that* (i), (ii), *and* (iii) *of Theorem* 1 *hold, and that the following* safe load condition *holds*:

$$\|G^{\mathrm{D}}\|_{L^\infty(\Omega, M^3)} + C_\Omega \|\ell\|_{L^3(\Omega, \mathbb{R}^3)} < k, \tag{4.17}$$

where C_Ω is the Sobolev–Poincaré constant defined in (3.13). *Then Problem* P3 *has a solution.*

As with Theorem 2, alternative conditions for the existence of solutions may be laid down here as well.

Theorem 4′. *Assume that conditions* (iii), (i′), *and* (ii′) *of Theorems* 2 *and* 2′ *hold, and that*

$$\|G^{\mathrm{D}}\|_{L^\infty(\Omega, M^3)} < k.$$

Then there is a solution to Problem P3.

Lemma 4.1. *Conditions* (i), (ii), *and* (iii) *of Theorem* 1 *imply that*

$$\int_\Omega \ell \cdot v\,dx \le C_\Omega \|\ell\|_{L^3(\Omega, \mathbb{R}^3)} \int_\Omega |\varepsilon^{\mathrm{D}}(v)|,$$

$$\forall v \in P(\Omega) \quad \text{such that} \quad e^3 \cdot v(x) \geqq 0 \quad \text{a.e. in } \Omega \text{ and} \quad \operatorname{div} v = 0.$$

Proof. Lemma 3.1 tells us that

$$\int_\Omega \ell \cdot v \, dx \le C_\Omega \|\ell\|_{L^3(\Omega,\mathbb{R}^3)} \int_\Omega |\varepsilon(v)|.$$

Moreover, since $|\sigma|^2 = |\sigma^{\mathrm{D}}|^2 + |\sigma^{\mathrm{S}}|^2$ for all $\sigma \in M^3$ and $\operatorname{div} v = 0$, we have

$$\int_\Omega |\varepsilon(v)| = \int_\Omega |\varepsilon^{\mathrm{D}}(v)|$$

and the result follows. □

Proof of Theorem 4. We still use Theorem 3, with the choices $Y = P(\Omega)$, T is the convex set $K = \{v \in P(\Omega)\colon (x + v(x)) \cdot e^3 \ge 0 \text{ a.e. in } \Omega\}$, σ is the weak*-topology on $P(\Omega)$ (that is, $v_n \to^\sigma v$ iff $v_n \to v$ weak* in $\mathrm{BD}(\Omega)$ and $\operatorname{div} v_n \to \operatorname{div} v$ weakly in $L^2(\Omega)$), and $\Lambda = I$, say

$$\Lambda(v) = I(v) \equiv F(v) + \Gamma(\varepsilon^{\mathrm{D}}(v)) - Lv, \tag{4.18}$$

where

$$F(v) = \frac{\chi}{2} \int_\Omega |\operatorname{div} v|^2 \, dx, \tag{4.19}$$

$$\Gamma(\varepsilon^{\mathrm{D}}(v)) = \int_\Omega \Phi(\varepsilon^{\mathrm{D}}(v)). \tag{4.20}$$

Then the closed unit ball of Y is σ-compact, (Y, σ) is a Hausdorff vector space, I is sequentially σ-l.s.c. on bounded subsets of $P(\Omega)$, $K^\infty = \{v \in P(\Omega)\colon v \cdot e^3 \ge 0 \text{ a.e. in } \Omega\}$ which is convex, sequentially σ-closed, $\operatorname{dom} I = P$, and

$$F^\infty(v) = \begin{cases} +\infty & \text{if } \operatorname{div} v \ne 0, \\ 0 & \text{if } \operatorname{div} v = 0, \end{cases} \tag{4.21}$$

$$\Gamma^\infty(\varepsilon^{\mathrm{D}}(v)) = k \int_\Omega |\varepsilon^{\mathrm{D}}(v)|, \tag{4.22}$$

$$\operatorname{Ker} F^\infty = \operatorname{Ker} F = \{v \in P(\Omega)\colon \operatorname{div} v = 0\}, \tag{4.23}$$

$$\operatorname{Ker} \Gamma^\infty = \{v \in P(\Omega)\colon \varepsilon^{\mathrm{D}}(v) = 0\}, \tag{4.24}$$

$$I^\infty(v) = F^\infty(v) + \Gamma^\infty(\varepsilon^{\mathrm{D}}(v)) - Lv. \tag{4.25}$$

Compactness. Take $y_0 = 0 \in K^\infty \cap \operatorname{dom} I = K^\infty$, and a sequence $\{v_n\}_n$ in $P(\Omega)$ with $v_n \to^\sigma 0$, $I(v_n) = I(0)$. Then the Banach-Alaoglu-Bourbaki theorem gives

$$Lv_n \to L0 = 0 \tag{4.26}$$

and since Γ and F are nonnegative,

$$F(v_n) \to 0 \quad \Rightarrow \quad \operatorname{div} v_n \to 0 \quad \text{strongly in } L^2, \tag{4.27}$$

$$\Gamma(v_n) \to 0. \tag{4.28}$$

We claim that (4.28) implies (notice that now Γ is not homogeneous of degree one)

$$\int_\Omega |\varepsilon^{\mathrm{D}}(v_n)| \to 0. \tag{4.29}$$

In fact (4.14) and (4.28) give

$$\int_\Omega \Phi(\varepsilon^{\mathrm{D}}(v_n)^a)\, dx + k \int_\Omega |\varepsilon^{\mathrm{D}}(v_n)^s| \to 0$$

as $n \to \infty$, and, since both terms are nonnegative,

$$\int_\Omega |\varepsilon^{\mathrm{D}}(v_n)^s| \to 0 \tag{4.30}$$

and

$$\int_\Omega \Phi(\varepsilon^{\mathrm{D}}(v_n)^a(x))\, dx \to 0 \tag{4.31}$$

as $n \to \infty$, $x \to \Phi(\varepsilon^{\mathrm{D}}(v_n)^a(x))$ being in $L^1(\Omega)$.

Set $\Omega_n = \{x \in \Omega: \varepsilon^{\mathrm{D}}(v_n)^a(x) < k/2\mu\}$. Then

$$\begin{aligned}\int_\Omega |\varepsilon^{\mathrm{D}}(v_n)^a(x)|\, dx &= \int_{\Omega_n} |\varepsilon^{\mathrm{D}}(v_n)^a(x)|\, dx + \int_{\Omega\setminus\Omega_n} |\varepsilon^{\mathrm{D}}(v)^a(x)|\, dx \\ &\leq |\Omega_n|^{1/2} \|\varepsilon^{\mathrm{D}}(v_n)\|_{L^2(\Omega_n)} \\ &\quad + \frac{1}{k}\int_{\Omega\setminus\Omega_n} \Phi(\varepsilon^{\mathrm{D}}(v_n)^a(x))\, dx + \frac{k}{4\mu}\int_{\Omega\setminus\Omega_n} dx \\ &\leq |\Omega_n|^{1/2}\mu^{-1/2}\left[\int_{\Omega_n} \Phi(\varepsilon^{\mathrm{D}}(v_n)^a)(x)\, dx\right]^{1/2} \\ &\quad + \frac{1}{k}\int_{\Omega\setminus\Omega_n} \Phi(\varepsilon^{\mathrm{D}}(v_n)^a(x))\, dx + \frac{k}{4\mu}|\Omega\setminus\Omega_n|. \end{aligned} \tag{4.32}$$

The first and second terms on the right-hand side of inequality (4.32) tend to zero thanks to (4.31). The third term tends to zero too, since otherwise we can argue by contradiction: by assuming that, up to subsequences $|\Omega\setminus\Omega_n| \geq \delta > 0$, we get

$$\int_\Omega \Phi(\varepsilon^{\mathrm{D}}(v_n)^a(x)) \geq \int_{\Omega\setminus\Omega_n} \Phi(\varepsilon^{\mathrm{D}}(v_n)^a(x))\, dx \geq k^2\delta/4\mu > 0$$

which contradicts (4.31). So

$$\int_\Omega |(\varepsilon^{\mathrm{D}}(v_n)^a(x))|\, dx \to 0 \quad \text{as} \quad n \to \infty, \tag{4.33}$$

and (4.30), (4.33) prove (4.29).

So (4.27), (4.29), and the fact that the σ-convergence entails $v_n \to v$ strongly in $L^1(\Omega, \mathbb{R}^3)$, prove that $v_n \to v$ strongly in $P(\Omega)$.

Necessary condition. Due to (4.21)-(4.25), Lemma 4.1, and the safe load condition (4.17),

$$I^\infty(v) \geq \Gamma^\infty(\varepsilon^{\mathrm{D}}(v)) - Lv \geq \varepsilon_0 \int_\Omega |\varepsilon^{\mathrm{D}}(v)| \geq 0, \qquad \forall v \in K^\infty, \tag{4.34}$$

for some $\varepsilon_0 > 0$.

Compatibility. From (4.25) and (4.34), referring to (3.12), we get

$$K^\infty \cap \operatorname{Ker} I^\infty = K^\infty \cap \operatorname{Ker} F \cap \operatorname{Ker} \Gamma^\infty \cap \operatorname{Ker} L = K^\infty \cap \mathcal{R} \cap \operatorname{Ker} L \tag{4.35}$$

and we can repeat the argument used in the proof of Theorem 1 to show that $K^\infty \cap \operatorname{Ker} I^\infty$ is a subspace. □

5. Necessary Conditions and Limit Analysis

As we have already noted, whenever a functional E has only linear growth at infinity (or possibly only linear in some directions and superlinear in the others, as in our case (Problem P3): $E = F + \Gamma$ with F quadratic and Γ having linear behavior) we have to add additional conditions on the dead loads L, in order that the problem

$$(\mathcal{P}) \quad \inf_{v \in K} I(v) \tag{5.1}$$

admits a finite minimum over the admissible set K, even if K is the whole space.

Usually such conditions are obtained through the method of limit analysis (see [ET], [T2], and [TS2]). The method is as follows: for $\lambda \in \mathbb{R}$, $\lambda \geq 0$, introduce the problem

$$(\mathcal{P}_\lambda) \quad \inf_K (E - \lambda L)$$

and check that $\inf \mathcal{P}_\lambda > -\infty$ for $\lambda = 1$. The set of values of λ satisfying the requirement of finiteness of the infimum is an interval containing at least 0.

Under rather mild conditions, by using the definition of a polar function, dual problems, and extremality conditions of convex analysis, we find that $\inf_K \mathcal{P} > -\infty$ corresponds to

$$\inf_{K^\infty} (A - L) > -\infty, \tag{5.2}$$

where A is the part of the functional E having linear growth. (In our case $A(v) = \int_\Omega \psi_s(\varepsilon^{\mathrm{D}}(v))$ where ψ_s is the support function of the set of admissible stresses $S = \{\eta \in M^3 : |\eta^{\mathrm{D}}| \leq k\}$.

Note also that the (dual) functional

$$\mathscr{E}(\sigma) = \frac{\chi}{18} \int_\Omega (\operatorname{tr} \sigma)^2 + \frac{1}{4\mu} \int_\Omega |\sigma^{\mathrm{D}}|^2$$

has quadratic growth, hence it is coercive and it has a minimum provided the admissible set

$$\mathcal{S}=\{\sigma: |\sigma^{\mathrm{D}}|\leq k,\ \sigma_{ij}=a_{ijhk}\varepsilon_{hk}(v),\ v\in K\}$$

is nonempty.

Since A is positively homogeneous and K^{∞} is a cone the requirement (5.2) is equivalent to

$$\inf_{K^{\infty}}(A-L)\geq 0.$$

But $A=\Gamma^{\infty}$ (when finite) since F^{∞} is either 0 or $+\infty$. So the condition may be rewritten in the equivalent form (again referring to Problem P3)

$$\inf_{v\in K^{\infty}:\ \mathrm{div}\ v=0,\, Lv\neq 0}\frac{\Gamma^{\infty}(\varepsilon^{\mathrm{D}}(v))}{Lv}\geq 1$$

or

$$\inf_{v\in K^{\infty}:\ \mathrm{div}\ v=0,\, Lv=1}\Gamma^{\infty}(\varepsilon^{\mathrm{D}}(v))\geq 1 \tag{5.3}$$

which is the usual limit analysis problem.

The above analytic procedure has a self-explanatory physical meaning in terms of the limit load condition of mechanics and it actually produces the "safe load condition" of engineering. On the other hand, the argument is rather involved and depends on conjugate functions which are often difficult to compute, and lead to implicit conditions.

We underline that, in a different language (in terms of recession functionals), (5.3) simply implies the necessary condition

$$E^{\infty}(v)\geq Lv, \qquad \forall v\in K^{\infty},$$

which is actually a weaker condition (remember that $E^{\infty}=F^{\infty}+\Gamma^{\infty}$ and $F^{\infty}(v)=0$ if $F(v)=0$, $F^{\infty}(v)=+\infty$ if $F(v)\neq 0$).

We stress that $\Lambda^{\infty}\geq 0$ is a necessary condition for finiteness of the infimum of any functional Λ which is convex and l.s.c. on bounded sets, without any other assumption. On the other hand, the assumption $\Lambda^{\infty}\geq 0$ alone is not sufficient to get $\inf\Lambda>-\infty$. But together with the compatibility condition ($\mathrm{Ker}\,\Lambda^{\infty}$ is a subspace), the necessary condition becomes sufficient in order to have $\inf\Lambda>-\infty$.

We notice that, coming back to Problem P3, if

$$\inf_{v\in K^{\infty}:\ \mathrm{div}\ v=0,\, Lv=1}\Gamma^{\infty}(\varepsilon^{\mathrm{D}}(v))>1 \tag{5.4}$$

then the minimizing sequences of $E-L$ have bounded energy E: actually assumption (5.4) is a compactness assumption and is essential in our proof of existence of equilibria in the presence of a rigid constraint. An analogous assumption was used in [AG1] and [T1] for the existence of displacements in the unconstrained case, though they used different techniques.

Finally, we notice that this viewpoint of finding explicitly conditions of solvability in terms of recession functionals extends naturally to nonconvex constraints and functionals (see [BBGT1] and [BBGT2]).

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