Math. Ann. 296, 687–708 (1993)

Mathematische Annalen

Moduli of half conformally flat structures

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Received August 15, 1990; in revised form Juli 21, 1992

Mathematics Subject Classification (1991): 58D27, 58H15, 58D17, 53A30, 32G13, 58G05

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We would like to study in this article the moduli of "conformal structures" on a given 4-manifold. Here the moduli of conformal structures or more precisely the moduli of half conformally flat structures means the set of all half conformally flat structures $[g]$ on a 4-manifold M modulo the action of the gauge group Diff(M), the diffeomorphism group of M.

There is a significant notion in conformal geometry, the conformal flatness.

A Riemannian n-manifold (M, g) is called conformally flat if (M, g) has at every point a locally defined conformal map into a Euclidean space $\mathbb{R}^n$. When $n \geq 4$ this is equivalent to vanishing of the Weyl conformal tensor W.

In four dimensional case one has another notion "half conformally flatness", in other words, vanishing of a half part of W, W^+ or W^-.

Let (M, g) be an oriented Riemannian 4-manifold. Then a 2-form $\alpha \in \Omega^2$ splits with respect to the star operator $*$ into the self-dual part $\alpha^+ = (\alpha + *\alpha)/2$ and the anti-self-dual part $\alpha^- = (\alpha - *\alpha)/2$, $\alpha = \alpha^+ + \alpha^-$.

The Weyl conformal tensor W viewed as an End(TM)-valued 2-form decomposes into $W = W^+ + W^-$ and we say (M, g) is self-dual or anti-self-dual (or simply half conformally flat) if $W^- = 0$ or $W^+ = 0$.

Obviously a conformally flat 4-manifold is self-dual and anti-self-dual.

Examples of conformally flat manifolds which are well known are manifolds of constant curvature and Riemann surfaces. These manifolds are divided into spaces of positive, negative and zero curvature. Similarly the sign of the scalar curvature divides the set of all Riemannian 4-manifolds up to conformal change into three classes (see Sect. 2 for the details and also [5, 46]) so that a half conformally flat structure $[g]$ is called type positive, zero or negative according to the sign of the scalar curvature.

We denote by $\mathscr{C}_M$ the set of smooth conformal structures on a given compact connected oriented 4-manifold M and define an action $\mathscr{W}: \mathscr{C}_M \to \mathbb{R}$; $\mathscr{W}(\gamma) = 1/2 \int_M |W(g)|_g^2\, dv_g = 1/2 \int_M \operatorname{Tr} W \wedge *W$ for $W = W(g)$, the Weyl conformal tensor of a representative g of γ.

The topological identity $\tau(M) = 1/(12\pi^2)\int(|W^+|^2 - |W^-|^2)dv_g$ then indicates the absolute inequality; $\mathscr{W}(\gamma) \geqq 6\pi^2|\tau(M)|$ for the Hirzebruch signature of M, $\tau(M)$, and the equality holds if and only if γ and g is self-dual (necessarily $\tau(M) \geqq 0$) or anti-self-dual ($\tau(M) \leqq 0$).

The moduli $\mathscr{M} = \mathscr{M}_M$ of anti-self-dual conformal structures on M is defined as all equivalence classes of anti-self-dual conformal structures. Her $\gamma, \gamma_1 \in \mathscr{C}_M$ are equivalent if $g_1 = \varphi^* g$ for a diffeomorphism φ of M and for some representatives g and g_1 of γ and γ_1, respectively and we write $\gamma_1 = \varphi^*\gamma$.

Definition 1. The *moduli of anti-self-dual conformal structures* $\mathscr{M}_M$ is defined as the quotient

$$\mathscr{M}_M = \{\gamma = [g] \in \mathscr{C}_M; W(g)^+ = 0\} / \mathrm{Diff}^+(M),$$

modulo the group of orientation preserving diffeomorphisms of M, $\mathrm{Diff}^+(M)$.

To simplify the argument we deal mainly with anti-self-dual case, since reversing the orientation transfers each anti-self-dual conformal structure into a self-dual conformal structure.

Another type of definition of the moduli is

$$\tilde{\mathscr{M}}_M = \{[g] \in \mathscr{C}_M; W(g)^+ = 0\} / \mathrm{Diff}^0(M),$$

where $\mathrm{Diff}^0(M)$ denotes the group of diffeomorphisms isotopic to the identity id_M.

Then $\tilde{\mathscr{M}}_M \to \mathscr{M}_M$ is a fibration whose fibre is the "mapping class group".

The moduli $\tilde{\mathscr{M}}_M$ corresponds to the Teichmüller moduli of Riemann surfaces.

Works for moduli of some special geometric structures, for instance the moduli of Einstein metrics on 4-manifolds, are recently done by several geometers [32, 1, 44] and our investigation of the moduli of half conformally flat structures seems to be an approach along the similar lines. However, there are other moduli spaces which share common feature with our moduli from conformal geometric viewpoint, namely the moduli of Riemann surfaces and the moduli of Yang-Mills instantons [7, 16].

Being guided by established theories of these moduli spaces one can develop the study of our moduli. Like the Yang-Mills instanton case our moduli has a "quantum number", τ corresponding to the instanton number. It admits also an elliptic complex describing the local data.

We have few examples of manifolds for which the moduli is completely known.

For S^4 the moduli consists of a single point, the standard conformally flat structure [36].

The complex projective plane $\mathbb{C}P^2$ has the Fubini-Study metric as an isolated point in $\mathscr{M}$ [27, 45].

The conformally flat case is another example whose moduli is somewhat known. In fact each conformally flat structure has by making use of the developing map a holonomy correspondence $\pi_1(M) \to SO(5,1)$, the conformal group of S^4 with the standard metric, so that the moduli of conformally flat structures is mapped into the representation space $\mathscr{R}(\pi_1(M); SO(5,1))$, the space of conjugacy classes of representations $\pi_1(M) \to SO(5,1)$.

A product 4-manifold $\Sigma_k \times \mathbb{C}P^1$ with metrics of opposite constant curvatures is a nontrivial example of conformally flat 4-manifold. Here Σ_k denotes a genus $k(> 1)$ compact Riemann surface.

By counting the dimensions the moduli of conformally flat structures on $\Sigma_k \times \mathbb{C}P^1$ is naturally embedded in $\mathscr{R}(\pi_1(\Sigma_k); SO(5,1))$, since $\dim \mathscr{R} = 30(k-1)$ is the minus sign of the index (1.1).

As in the Yang-Mills instanton case $\tilde{\mathcal{M}}$ is, in a sense of local moduli, described locally as a conformal group quotient of a real analytic subvariety in a finite dimensional vector space, the first cohomology group $\mathbb{H}^1$ of the elliptic complex: $C^\infty(TM) \to C^\infty(\mathrm{Hom}(\Omega^+, \Omega^-)) \to C^\infty(S_0(\Omega^+))$ (see Sect. 3, (ii) for the precise definition).

This complex has the index

$$\dim \mathbb{H}^0 - \dim \mathbb{H}^1 + \dim \mathbb{H}^2 = 1/2(29\tau(M) + 15\chi(M)) \tag{1.1}$$

from the Atiyah-Singer index theorem ($\chi(M)$ is the Euler characteristic of M).

The 0-th cohomology group $\mathbb{H}^0 = \mathrm{Ker}\, L$ at $\gamma \in \mathscr{C}_M$ is the Lie algebra of the conformal group $C^0(\gamma) = \{\varphi \in \mathrm{Diff}^0(M); \varphi^*\gamma = \gamma\}$.

By applying a slice theorem (Theorem 3.4 in Sect. 3) and the Kuranishi map (Theorem 3.5 in Sect. 3) one has indeed

Theorem 2. *For any $\bar{\gamma} \in \tilde{\mathcal{M}}_M$ there exists a neighborhood $U_{\bar{\gamma}}$, in the sense of local moduli, represented by the group quotient of the zero's of a map $\Phi: \mathbb{H}^1_\varepsilon \to \mathbb{H}^2$;*

$$U_{\bar{\gamma}} = \mathrm{Zero}(\Phi: \mathbb{H}^1_\varepsilon \to \mathbb{H}^2)/C^0(\gamma),$$

where $\mathbb{H}^1_\varepsilon$ is a neighborhood of 0 *in $\mathbb{H}^1$.*

Note that one can define the local moduli of half conformally flat structures by $\{[g] \in \mathscr{C}_M; W(g)^+ = 0\}$ modulo the action of a germ in $\mathrm{Diff}^0(M)$ around id_M. The set $U_{\bar{\gamma}}$ in Theorem 2 means a neighborhood of the local moduli and one says that it gives a neighborhood of $\tilde{\mathcal{M}}_M$ in the sense of local moduli.

The topology of $\mathcal{M}_M$ and $\tilde{\mathcal{M}}_M$ is one naturally induced from the following diagram

$$\begin{array}{c} \mathscr{R}_M / \mathrm{Diff}^+(M) \\ \downarrow \\ \mathcal{M}_M \subset \mathscr{C}_M / \mathrm{Diff}^+(M). \end{array}$$

Here $\mathscr{R}_M$ is the space of all Riemannian metrics on M.

We remark that $\mathcal{M}_M$ and $\tilde{\mathcal{M}}_M$ are Hausdorff [30]. This Hausdorff property is shown by applying the Yamabe problem.

By virtue of the formulation of $\mathscr{C}_M$ given in Sect. 3, the tangent space $T_\gamma \mathscr{C}_M$ is identified with $C^\infty(\mathrm{Hom}(\Omega^+, \Omega^-))$. A positive definite inner product on it is defined as

$$\|A\|^2 = \int_M (-\mathrm{Tr}\, AA^*)(x)\, d\mathbb{V}_g(x), \qquad A \in C^\infty(\mathrm{Hom}(\Omega^+, \Omega^-)) \tag{1.2}$$

in terms of a "canonical" volume form $d\mathbb{V}_g$, where A^* is the adjoint of A.

The notion "canonical" requires $d\mathbb{V}_g$ to satisfy the conformal invariance and the naturality with respect to diffeomorphisms, from which the inner product $\|A\|^2$ is $\mathrm{Diff}^+(M)$-invariant.

By using a basis of $\mathrm{H}^+ = \{$self-dual harmonic 2-forms$\}$, for instance, which is orthonormal with respect to the cup product on $\mathrm{H}^2(M; \mathbb{Z})$ one can exhibit such a canonical volume form (see Sect. 3, v) for the details).

Thus this L^2-inner product is able to descend to the quotient $\mathscr{C}_M/\mathrm{Diff}^0(M)$. By restricting this inner product to the moduli we have

Theorem 3. *If a* 4*-manifold* M *has* $b^2(M) > 0$, *then the moduli* $\tilde{\mathscr{M}}_M$ *of anti-self-dual conformal structures is endowed with a Riemannian metric at each point* $\bar{\gamma}$ *even when it has a quotient singularity.*

We would like to state several consequences and applications of our theorems.

The first one is a local Torelli-type theorem on a "period map".

There is a natural map, the period map, $\mathrm{p}: \mathscr{C}_M \to G^+_{\mathrm{b}^+}(\mathrm{H}^2) = \{$positive b^+-planes in $\mathrm{H}^2(M;\mathbb{R}) \cong \mathbb{R}^{\mathrm{b}^2}\}$, where $\mathrm{H}^2(M;\mathbb{R})$ is equipped with the cup product of type $(\mathrm{b}^+, \mathrm{b}^-)$ [17, Appendix]. At a tangential level this is

$$\mathrm{p}_* : C^\infty(\mathrm{Hom}(\Omega^+, \Omega^-)) \to \mathrm{Hom}(\mathrm{H}^+, \mathrm{H}^-) \tag{1.3}$$

for the spaces $\mathrm{H}^\pm$ of self-dual (anti-self-dual) harmonic 2-forms.

Theorem 4. *For a* K3 *surface the map* p_* *restricted to the tangent space of* $\tilde{\mathscr{M}}_M$ *at any Ricci flat metric becomes an isometry with respect to the* L^2*-metric and the invariant metric on* $\mathrm{Hom}(\mathrm{H}^+, \mathrm{H}^-)$, *so that the component of* $\tilde{\mathscr{M}}_M$ *containing a Ricci flat metric (and hence a type zero anti-self-dual conformal structure) is isometric onto some open domain in the symmetric space* $SO(3,19)/SO(3) \times SO(19)$.

This theorem is already shown in terms of polarized Ricci flat Kähler metrics ([32, 8] and see also for a brief survey [2]). However this theorem will be verified from our formulation of $\tilde{\mathscr{M}}_M$ in Sect. 5 (Proposition 5.2).

As a consequence of this theorem there is no type negative anti-self-dual conformal structure on a K3 surface M, close to any Ricci flat metric.

The moduli $\tilde{\mathscr{M}}_M$ is divided into disjoint three parts

$$\tilde{\mathscr{M}}_M = \tilde{\mathscr{M}}_M^{(+)} \coprod \tilde{\mathscr{M}}_M^{(0)} \coprod \tilde{\mathscr{M}}_M^{(-)}$$

according to the sign of constant scalar curvature.

The presence of each piece implies a geometric restriction on M. In fact, if $\tilde{\mathscr{M}}_M^{(+)}$ is not empty, then M is homeomorphic to $\overline{\mathbb{C}P^2} \sharp \ldots \sharp \overline{\mathbb{C}P^2}$ (b^2-times) provided M is simply connected.

On the other hand, if $\tilde{\mathscr{M}}_M^{(0)} \neq \emptyset$ and $\mathrm{H}^+ \neq 0$, then M must be a Kähler surface with an extremal Kähler metric in the sense of Calabi, namely a Kähler metric of zero scalar curvature [13] (see Sect. 2, and [28] for the classification of candidates of those M's of nonempty $\tilde{\mathscr{M}}_M^{(0)}$).

So we obtain a map from $\tilde{\mathscr{M}}_M^{(0)}$ into the moduli of complex structures on M, $\tilde{\mathscr{T}}_M$.

For a ruled surface, a typical anti-self-dual 4-manifold of which $\tilde{\mathscr{M}}_M^{(0)} \neq \emptyset$ we are able to present $\tilde{\mathscr{M}}_M^{(0)}$ as in the representation space $\mathscr{R}(\pi_1(M); \mathrm{SL}(2,\mathbb{R}) \times PU(2))$, $PU(2) = SU(2)/\mathbb{Z}_2$ whose dimension coincides with the dimension of $\tilde{\mathscr{T}}_M$ (see Theorem 5.1).

The importance of half-conformally flat 4-manifolds is that they are equipped with twistor spaces. It is an interesting question how our moduli relates with the moduli of complex structures on the twistor space, while we only remark on it in Sect. 5.

However, more interesting is an investigation of ends of the moduli of half conformally flat structures. The action $\mathscr{W}(\gamma) = 6\pi^2|\tau(M)|$ holds for any $\bar{\gamma} \in \tilde{\mathscr{M}}_M$ so that a bubbling off phenomenon may occur at points where the Weyl conformal tensor concentrates. A Uhlenbeck's type theorem is expected as in the Yang-Mills instanton case.

The essential difference from the Yang-Mills instanton case is that by bubbling off, a half conformally flat 4-manifold may separate into some half conformally flat 4-orbifolds $M_1, \ldots, M_k$ such that $M = M_1 \sharp \ldots \sharp M_k$ (see also [2, 44]). So possibility of bubbling off is detected by a structure of the quadratic form on $\mathrm{H}^2(M;\mathbb{Z})$ [23]. Here the connected sum is considered as generalized one being attached along S^3/Γ, a finite quotient of the 3-sphere S^3. At any rate the one point blown up of $\mathbb{C}^2$ with anti-self-dual Kähler metric whose conformal compactification is $\overline{\mathbb{C}P^2}$ with the Fubini-Study metric [31, 42] and the Eguchi-Hanson metric on an ALE 4-manifold must play roles as "one-instantons" in the compactification of the moduli.

We discuss in Sect. 2 the scalar curvature type and the connected sum operation. In Sect. 3 we review briefly the fundamental properties of the Weyl conformal tensor and study the moduli of half conformally flat structures to show the main theorems (the real analytic subvariety theorem and the L^2-metric theorem).

We specify our argument in Sect. 4 to the moduli of Ricci flat metrics of unit volume, identified with the moduli of type zero conformal structures when the Hitchin's bound $\chi + 3/2\tau = 0$ is satisfied, and exhibit the detailed proof for the local Torelli-type theorem. Section 5 is devoted to the investigation of the moduli $\tilde{\mathscr{M}}_M^{(0)}$ in terms of complex structures.

We summarize in Appendix several formulae needed in deriving the linearization of the Weyl conformal tensor.

For general references of (half) conformally flat manifolds we refer to [38, 4, 8, 15, 28, 41].

2 Scalar curvature type

(i)

Before discussing the moduli of half conformally flat structures we begin with scalar curvature type.

As is shown as Yamabe problem solved by Aubin and Schoen, a compact connected Riemannian 4-manifold (M, g) admits a constant scalar curvature metric, conformally equivalent to g [5, 46].

A conformal change $g' = f^2 g$, $f \in C^\infty(M) > 0$, has the scalar curvature ϱ' obeying the equation

$$\varrho' f^3 = 6\Delta f + \varrho f \tag{2.1}$$

for the Laplacian $\Delta = \Delta_g$ and the scalar curvature ϱ of g.

From (2.1) one has the following proposition from which the value of constant scalar curvature is unique up to volume-normalized conformal change provided the value is nonpositive.

Proposition 2.1. *Let g and g' be conformally equivalent metrics of same volume. If they have constant scalar curvature $\leqq 0$, then $g' = g$.*

Proof. We assume $\int_M dv_g = 1$. The metric $g' = f^2 g$ is a conformal change. So $\int_M f^4 dv_g = 1$. The proposition is obvious if $\varrho = \varrho' = 0$. So assume $\varrho = \varrho' < 0$. We have $\Delta f = -g^{ij}\partial_i\partial_j f \geqq 0$ at a point $x \in M$ where f has the maximal value. Then

$1 - f^2(x) \geqq 0$ from the equality $(-\varrho)f(1-f^2) = 6\Delta f$ and hence $1 \geqq f$ on M. So $f \equiv 1$ because $\int_M f^4 dv_g = 1$. The case $\varrho' \leqq \varrho < 0$ is similarly proved. qed

Now we divide $\mathscr{C}_M$, the set of conformal structures, into three parts $\mathscr{C}_M^{(+)}$, $\mathscr{C}_M^{(0)}$, $\mathscr{C}_M^{(-)}$ according to the sign of the constant scalar curvature and decompose $\mathscr{M}_M$ as $\mathscr{M}_M = \mathscr{M}_M^{(+)} \coprod \mathscr{M}_M^{(0)} \coprod \mathscr{M}_M^{(-)}$.

To every $\gamma \in \mathscr{C}_M \backslash \mathscr{C}_M^{(+)}$ we choose a representative g of unit volume and assign the value of constant scalar curvature of a conformal change of g within the volume-normalized conformal class. So we get a map, Diff(M)-invariant $\varrho: \mathscr{C}_M \backslash \mathscr{C}_M^{(+)} \to \mathbb{R}$ which descends to a "smooth" function on $(\mathscr{C}_M \backslash \mathscr{C}_M^{(+)})/\,\mathrm{Diff}^0(M)$ in certain Sobolev norm.

(ii) Nonnegative type

The following are known with respect to half conformally flat 4-manifolds of nonnegative type.

Theorem 2.2 [14, 9, 15, 28]. *Let (M, g) be a connected 4-manifold endowed with a complex Kähler structure.* (i) *If (M, g) is compact and self-dual, then (M, g) is a complex space form,* i.e., *$\mathbb{C}P^2$ with a Fubini-Study metric, $\mathbb{C}^2/\Lambda$ with a flat standard metric, D^2/Γ with a standard Kähler metric, or a compact quotient of $D^1 \times \mathbb{C}P^1$ with opposite curvature metrics (here D^1, D^2 are the unit balls).* (ii) *(M, g) is anti-self-dual if and only if the scalar curvature $\varrho = 0$.*

Theorem 2.3 [41]. *Let (M, g) be a compact connected oriented anti-self-dual 4-manifold of type positive or zero. If M admits a harmonic self-dual 2-form $\theta \neq 0$* i.e., $\mathrm{b}^+(M) > 0$, *then (M, g) carries a complex structure for which g is a Kähler metric of type zero and the normalized form $|\theta|^{-1}\theta$ is the Kähler form.*

It follows from Theorem 2.3 that (i) if $\mathscr{M}_M^{(+)} \neq \emptyset$, then $\mathrm{b}^+(M) = 0$, namely the intersection form of $\mathrm{H}_2(M; \mathbb{Z})$ is negative definite or zero so that for such M of $\pi_1 = 1$, M is homeomorphic to the connected sum of $\mathrm{b}^2(M)$ copies of $\overline{\mathbb{C}P^2}$, $\mathbb{C}P^2$ with reversed orientation, due to Donaldson's theorem [16] and (ii) if $\mathscr{M}_M^{(0)} \neq \emptyset$ and $\mathrm{b}^+(M) > 0$, then $\mathscr{M}_M^{(+)} = \emptyset$ and M carries a complex structure with a Kähler metric of zero scalar curvature.

It is concluded moreover from Theorems 2.2, 2.3 that (i) type positive self-dual compact Kähler surface is only $\mathbb{C}P^2$ with a Fubini-Study metric, (ii) type negative self-dual compact Kähler surface is only a complex space form of negative constant holomorphic curvature, (iii) a Kähler metric is anti-self-dual if and only if it is type zero and (iv) compact conformally flat Kähler surfaces are only a Kähler flat torus T^4 and a compact quotient $(D^1 \times \mathbb{C}P^1)/\Gamma$.

The last 4-manifold is in the terminology of algebraic geometry a complex ruled surface M_k, a holomorphic $\mathbb{C}P^1$ bundle over a Riemann surface Σ_k of genus $k(> 1)$.

We remark against this 4-dimensional special feature that every conformally flat Kähler manifold of complex dimension $\geqq 3$ is flat [48].

A Hopf surface, diffeomorphic to $S^1 \times S^3$, is an example of compact conformally flat 4-manifold [11, 40]. Its scalar curvature type is positive.

(iii) Connected sum

A fundamental operation in conformal geometry is taking a connected sum. The class of conformally flat manifolds is closed under the connected sum operation [37]. The subclass, a class of type positive conformally flat manifolds is also closed under this operation [47].

For half conformally flat case the connected sum operation must be specifically important since the "quantum number" τ behaves additively, $\tau(M \sharp N) = \tau(M) + \tau(N)$ and it is reasonably expected that the operation $\sharp$ works on half conformally flat 4-manifolds with "one instanton" $\mathbb{C}P^2$ with a Fubini-Study metric. Actually a connected sum of n copies of $\mathbb{C}P^2$ for arbitrary n is endowed with a self-dual conformal structure [45, 21, 18, 43, 39].

(iv) Negative type case

Type positive manifolds are well investigated because of Lichnerowicz-Hitchin $\hat{A}$-vanishing theorem for spin structure.

However, type negative 4-manifolds seem so far to be less known.

Theorem 2.4. *Let $M = N_1 \sharp N_2$ be a connected sum of compact connected oriented conformally flat 4-manifolds. If N_i, $i = 1, 2$, is a flat torus or a ruled surface M_k, $k > 1$, with a conformally flat structure, then M admits a conformally flat structure and moreover any conformally flat structure on M must be of type negative.*

Proof. From Kulkarni's theorem [37] M admits a conformally flat structure. Let $[g]$ be any conformally flat structure on M. Assume its type is nonnegative. Since $\mathrm{b}^+(M) = \mathrm{b}^+(N_1) + \mathrm{b}^+(N_2) > 0$, (M, g) must be Kähler from Theorem 2.3 so that M is $T^4(\mathrm{b}^2 = 6, \chi = 0)$ or $M_k(\mathrm{b}^2 = 2, \chi = 4(1 - k))$. On the other hand $\mathrm{b}^2(M) = \mathrm{b}^2(N_1) + \mathrm{b}^2(N_2), \chi(M) = \chi(N_1) + \chi(N_2) - 2$. So the topological type of M differes from T^4 and M_k.

Remark. The class of type negative conformally flat 4-manifolds is closed under the connected sum operation, as pointed out by Lafontaine [40].

3 Moduli of anti-self-dual conformal structures

(i)

Let M be a compact connected oriented 4-manifold.

For a smooth metric g on M we denote by $[g]$ the conformal structure represented by g. The volume form of g is $dv_g = \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$.

We note first that any conformal structure γ has the unique representative metric whose volume form coincides with dv_g. We call this metric the volume-normalized representative of γ with respect to the fixed metric g.

Since any g on M is a positive definite symmetric tensor on TM at each point, we regard conformal structures as smooth sections of a fibre bundle $V \to M$ whose fibre at $x \in M$ is $S^+(T_x^*M)/\mathbb{R}^+$. Here $S^+(T_x^*M)$ is the cone of positive definite symmetric bilinear forms on T_xM and $\mathbb{R}^+$ operates by scalar multiplication so that we identify $\mathscr{C}_M \cong C^\infty(M; V)$.

This is the standard description of conformal structures which is valid for arbitrary dimension.

We have another formulation of $\mathscr{C}_M$ from the four dimensionality.

The star operator $*: \Omega^2 \to \Omega^2$ which depends on a conformal structure and the orientation of M give the splitting $\Omega^2_x = \Omega^+_x \oplus \Omega^-_x$, $x \in M$, into $\pm$ eigenspaces Ω^+_x, Ω^-_x satisfying $\Omega^+_x \wedge \Omega^-_x = 0$ in such a way that the wedge product $\cdot \wedge \cdot : \Omega^2 \to \Omega^4 = \mathbb{R}dv$ is positive definite on Ω^+ and negative definite on Ω^-, respectively.

Conversely a choice of an appropriate 3-dimensional subspace U of Ω^2_x on which $\cdot \wedge \cdot$ is positive determines uniquely a conformal structure γ at $x \in M$ so that U and the subspace $U^\perp$ annihilated by U give the splitting $\Omega^2_x = \Omega^+_x \oplus \Omega^-_x$, $\Omega^+_x = U$, $\Omega^-_x = U^\perp$.

So if we have a 4-manifold M, then fixing a conformal structure means equivalently a choice of an appropriate rank 3 subbundle Ω' of $\Omega^2 \to M$ (see [21, 17, Appendix] for this formulation). Thus, once we fix a conformal structure γ with splitting $\Omega^2 = \Omega^+_\gamma \otimes \Omega^-_\gamma$, we can identify $\mathscr{C}_M$ with an open set in $C^\infty(\mathrm{Hom}(\Omega^+_\gamma, \Omega^-_\gamma))$ as

$$\mathscr{C}_M \cong \{A \in C^\infty(\mathrm{Hom}(\Omega^+_\gamma, \Omega^-_\gamma)); \eta \wedge \eta + A\eta \wedge A\eta > 0, \eta \in \Omega^+\}.$$

Remark. These two identifications are very natural because we have an $SO(4)$-isomorphism between the spaces $\Omega^+ \otimes \Omega^-$ and $S_0(T^*M) = \{$traceless symmetric 2-tensors$\}$:

$$\begin{aligned} \Omega^+ \otimes \Omega^- (\cong \mathrm{Hom}(\Omega^+, \Omega^-)) &\to S_0(T^*M) \\ (\eta^+, \eta^-) &\mapsto h = (h_{ij}), \end{aligned} \tag{3.1}$$

$h_{ij} = g^{kl}\eta^+_{ik}\eta^-_{lj}$ [10, Lemma 4.6] and $h \in S_0(T^*M)$ induces a homomorphism $A = A_h : \Omega^+ \to \Omega^-$, $A\eta^+ = (A\eta^+_{ij})$;

$$A\eta^+_{ij} = h^k_i \eta^+_{kj} + h^k_j \eta^+_{ik}, \quad \eta^+ \in \Omega^+ \tag{3.2}$$

giving the inverse.

We adopt the Einstein summation convention throughout this article unless any confusion occurs.

(ii) Elliptic complex

Our next investigation is to derive the linearization of W^+, the self-dual part of the Weyl conformal tensor W.

The tensor W is composed of the Riemannian curvature tensor R, the Ricci tensor Ric and the scalar curvature ϱ.

R is regarded as a self-adjoint operator: $\Omega^2 \to \Omega^2$;

$$R(e_i \wedge e_j) = 1/2 R_{ijkl} e_k \wedge e_l$$

for an orthonormal basis $\{e_i\}$ of 1-forms in such a way that

$$R = \begin{pmatrix} R^{++} & R^{+-} \\ R^{-+} & R^{--} \end{pmatrix}$$

with respect to the splitting $\Omega^2 = \Omega^+ \oplus \Omega^-$. Each of R^{++}, R^{--} has $W^\pm \in C^\infty(S_0(\Omega^\pm))$ as the traceless component and they are written actually as

$$R^{++} = W^+ + 1/12\varrho\,\mathrm{id}, \qquad R^{--} = W^- + 1/12\varrho\,\mathrm{id},$$

where $S_0(\Omega^+)$ denotes the traceless symmtric product of Ω^+ [4, 24].

We denote by $D = D_g : C^\infty(S_0(T^*M)) \to C^\infty(S_0(\Omega^+))$ the directional derivative of W^+ at $\gamma = [g]$, $D_\gamma(h) = (\delta W^+)(h)$ for $h \in T_\gamma \mathscr{C}_M$. The tangent space $T_\gamma \mathscr{C}_M$ is here identified, through the identification of $\mathscr{C}_M$, with the space of traceless symmetric 2-tensors $C^\infty(S_0(T^*M))$, since as we note in (i) we can choose for any $\gamma \in \mathscr{C}_M$ the volume-normalized representative metric with respect to g within γ .

Proposition 3.1. *Let $\gamma = [g]$ be an anti-self-dual conformal structure. Then the directional derivative D is a second order differential operator represented as*

$$D(h) = (\delta W_g(h))^+ , \qquad h \in C^\infty(S_0(T^*M)) , \tag{3.3}$$

that is, $D(h)$ is the self-dual part of the directional derivative of the full Weyl conformal tensor W.

Proof. The proof involves only calculation. The self-dual part W^+ is $W^+ = (P_{+g}, P_{+g})W$ where $(P_{+g}, P_{+g}) : \Omega^2 \otimes \Omega^2 \to \Omega^+ \otimes \Omega^+$ is the product of the projection $P_{+g} = 1/2(\mathrm{id} + *_g) : \Omega^2 \to \Omega^+$. Then

$$(\delta W_g^+)(h) = (P_{+g}, P_{+g})(\delta W_g(h)) + (\delta(P_+, P_+)_g(h))(W) .$$

Since

$$(\delta(P_+, P_+)_g(h))(W) = ((\delta(P_+)_g(h), P_{+g})(W) + (P_{+g}, (\delta P_+)_g(h))(W)$$

and $W = (W_{ijkl}) \in \Omega^2 \otimes \Omega^2$ is anti-self-dual with respect to both (i,j) and (k,l), the term $(\delta(P_+, P_+)_g(h))(W)$ vanishes. So $(\delta W_g^+)(h) = (\delta W_g(h))^+$. qed

The action of diffeomorphisms of M on $\mathscr{C}_M$ yields the Lie derivative operation on the tangent space $T_\gamma \mathscr{C}_M$ by choosing a representative g within γ. Every diffeomorphism φ induces a conformal structure $[\varphi^* g]$ by pulling back the metric g. This conformal structure has the unique representative $f\varphi^* g$ for some $f = f_\varphi \in C^\infty(M) > 0$ in such a way that its volume form coincides with dv_g. The Lie derivative operation $L(X)$, $X \in C^\infty(TM)$ is then defined as $d/dt(f_{\varphi_t}(\varphi_t)^* g)|_{t=0}$, $\varphi_t = \operatorname{Exp} tX$ and is written as

$$\begin{aligned} L = L_g : C^\infty(TM) \to C^\infty(S_0(T^*M)); \qquad X \mapsto L(X) , \\ L(X)_{ij} = \nabla_i X_j + \nabla_j X_i - 1/2(\nabla_i X^i) g_{ij} . \end{aligned} \tag{3.4}$$

We derive then a complex at each anti-self-dual conformal structure $\gamma = [g]$;

$$C^\infty(TM) \xrightarrow{L_g} C^\infty(S_0(T^*M)) \xrightarrow{D_g} C^\infty(S_0(\Omega^+)) \tag{3.5}$$

Proposition 3.2 [21]. *This complex is elliptic.*

This complex has the index $1/2(29\tau(M) + 15\chi(M))$. See also [20].

(iii) Slice theorem

To get a real analytic variety structure theorem for the moduli $\tilde{\mathscr{M}}_M$ we discuss a slice theorem and then a Kuranishi map theorem, even though these theorems are quite common for the Yang-Mills instanton case [22].

We now study the argument given by Ebin in case of the space of Riemannian metrics.

For a fixed smooth Riemannian metric g on M we define spaces

$$C^\infty(\mathrm{End}_s(TM)) = \{h \in C^\infty(\mathrm{End}(TM)); g(h(X), Y) = g(X, h(Y))\}$$

and

$$C^\infty(\mathrm{End}_s^+(TM)) = \{h \in C^\infty(\mathrm{End}_s(TM)); g(h(X), Y) \text{ is positive}\}.$$

These spaces are identified with the space $\mathscr{R}_M$ of Riemannian metrics on M and the tangent space $T_g\mathscr{R}_M$ at g, respectively; $\mathscr{R}_M \cong C^\infty(\mathrm{End}_s^+(TM))$, $T_g\mathscr{R}_M = C^\infty(S(T^*M)) \cong C^\infty(\mathrm{End}_s(TM))$. We notice that there is a bijection between these spaces since we have a bundle isomorphism $\mathrm{End}_s(TM) \to \mathrm{End}_s^+(TM)$, $h \mapsto \exp h$.

An infinitesimal deformation of g in direction of a vector field X, the Lie derivative $L_X g$, defines a linear operator $\mathscr{L}: C^\infty(TM) \to C^\infty(\mathrm{End}_s(TM))$; $X \mapsto (L_X g)^i_j$ where $(L_X g)^i_j = g^{ik}(L_X g)_{kj}$.

The kernel of the L^2-adjoint $\mathscr{L}^*$ of $\mathscr{L}$ in $C^\infty(\mathrm{End}_s(TM))$ gives a slice in $\mathscr{R}_M$.

We need here the Sobolev space completion of $\mathscr{R}_M$ with respect to certain L^2_k-norm.

Choose a sufficiently small ball $\mathscr{V}$ in $\mathrm{Ker}\,\mathscr{L}^*$.

Ebin defined two $\mathrm{Diff}^+(M)$-invariant inner products on the space $\mathscr{R}_M$, an L^2-inner product and a "strong" inner product which defines the same topology with the L^2_k-norm and obtained

Theorem 3.3 [19]. *For any Riemannian metric g in $\mathscr{R}_M$ there is a slice $\mathscr{T}$ in $\mathscr{R}_M$, $g \in \mathscr{T}$ given by $\mathscr{T} = \exp(\mathscr{V})$ which satisfies that* (i) *any $\psi \in I^0_g$ fixes $\mathscr{T}$ invariantly (here $I^0_g = \{\phi \in \mathrm{Diff}^0(M); \phi^* g = g\}$ is the isometry group of g),* (ii) *if $\phi \in \mathrm{Diff}^0(M)$, $\phi^*(\mathscr{T}) \cap \mathscr{T} \neq \emptyset$, then $\phi \in I^0_g$,* (iii) *there is a local section $\chi: \mathrm{Diff}^0(M)/I^0_g \to \mathrm{Diff}^0(M)$ defined on a neighborhood U of the origin such that the map $F: (u, h) \mapsto (\chi(u))^* h: U \times \mathscr{T} \to \mathscr{R}_M$ is a homeomorphism onto a neighborhood of g, diffeomorphic off fixed points of I^0_g.*

We now consider the case of conformal structures.

Suppose that $\gamma \in \mathscr{C}_M$ is a conformal structure on M not conformally equivalent to the standard sphere metric. Then the conformal group of γ is a compact group and γ has a representative g for which $I^0_g = C^0(\gamma)$.

Since each γ_1 has the unique volume-normalized representative g_1 with respect to g (namely, $dv_{g_1} = dv_g$), we have a lift of $\mathscr{C}_M$ to $\mathscr{R}_M$ by assigning g_1 to each γ_1 and then we can identify $\mathscr{C}_M$ with $\mathscr{R}_{M,dv_g}$, the space of volume-normalized Riemannian metric with respect to g and hence with $L^2_{k+1}(\mathrm{End}^+_{s,1}(TM)) = \{h \in L^2_{k+1}(\mathrm{End}^+_s(TM)); \det h = 1\}$.

The tangent space $T_\gamma\mathscr{C}_M$ at γ is also identified with the space of traceless symmetric endomorphisms, $L^2_{k+1}(\mathrm{End}_{s,0}(TM))$.

Define a map

$$\Psi : L^2_{k+1}(\mathrm{End}_{s,0}(TM)) \times (\mathrm{Diff}^0(M)/I^0_g)_o \to L^2_{k+1}(\mathrm{End}_{s,0}(TM))$$

by $\Psi(h, \bar{\varphi}) = h_1$ in such a way that the metric $\varphi^*(g \exp h)$ given by $(h, \bar{\varphi})$ is conformally equivalent to a metric $g \exp h_1$, namely its volume normalized conformal change $f\varphi^*(g \exp h)$, $f = f_{\varphi,h}$, coincides with $g \exp h_1$. Here $(\mathrm{Diff}^0(M)/I^0_g)_o$ denotes a germ at the origin o in the coset space $\mathrm{Diff}^0(M)/I^0_g$ and each $\bar{\varphi}$ in it has a lift in $\mathrm{Diff}^0(M)$ from Ebin's argument.

Consider the following composed map

$$L^*_g \circ \Psi : L^2_{k+1}(\mathrm{End}_{s,0}(TM)) \times (\mathrm{Diff}^0(M)/I^0_g)_o \to L^2_k(TM).$$

The equation $L^*_g(\Psi(h, \bar{\varphi})) = 0$ gives a diffeomorphism gauge fixing condition.

The differential of $L^*_g \circ \Psi$ at $(h, \bar{\varphi}) = (0, o)$ is

$$(L^*_g \circ \Psi)_* (h, X) = L^*_g(h + L_g X)$$

for $X \in T_o(\mathrm{Diff}^0(M)/I^o_g)$. The tangent space $T_o(\mathrm{Diff}^0(M)/I^o_g)$ is the orthogonal complement of $\mathrm{Ker}\, L_g = \{g\text{-Killing vector fields on } M\}$.

So, the partial differential in the second factor is a self-adjoint elliptic operator. Since L_g has trivial kernel over $(\mathrm{Ker}\, L_g)^\perp$, $L^*_g \circ L_g$ is invertible by standard elliptic theory. From the implicit function theorem in Sobolev space one has a neighborhood O_g in $L^2_{k+1}(\mathrm{End}_{s,0}(TM))$ and a map $\bar{\varphi} : O_g \to (\mathrm{Diff}^0(M)/I^o_g)_o\,(\bar{\varphi}(0) = o)$ in the following way: for any $h \in O_g$ there exists a diffeomorphism $\varphi = \varphi(h) \in \mathrm{Diff}^0(M)$ ($\varphi(h)$ is a lift of $\bar{\varphi}(h)$) close to id_M in such a way that the volume normalized metric $f_{\varphi,h}\varphi^*(g \exp h)$ is represented by $g \exp h_1$ for a unique $h_1 \in \mathrm{Ker}\, L^*_g \subset L^2_{k+1}(\mathrm{End}_{s,0}(TM))$.

So, the map

$$\Phi : O_g \to \mathrm{Ker}\, L^*_g \times \mathrm{Diff}^0(M)/I^0_g$$

$\Phi(h) = (h_1, \bar{\varphi}(h))$ gives a local diffeomorphism.

L^*_g coincides with $\mathscr{L}^*$ restricted to the sapce of traceless endomorphisms $L^2_{k+1}(\mathrm{End}_{s,0}(TM))$ so that $\mathscr{S} = \{g \exp h, h \in \mathscr{V}\}$ in $\mathscr{R}_{M,dv_g}$ gives a slice having a local effectiveness.

Theorem 3.4. *For any $\gamma \in \mathscr{C}_M$ which is not the conformal structure represented by the standard sphere, there exists a slice $\mathscr{S}$ in $\mathscr{R}_M$ at a representative g of γ such that* (i) *every $\psi \in C^0(\gamma)$ fixes $\mathscr{S}$ invariantly and* (ii) *there exists a local section $\chi : \mathrm{Diff}^0(M)/C^0(\gamma) \to \mathrm{Diff}^0(M)$ ($\varphi = \chi(\bar{\varphi})$) defined on a neighborhood U of the origin such that the map $F : \mathscr{S} \times U \to \mathscr{R}_{M,dv_g}; (g_1, \bar{\varphi}) \mapsto f_{\bar{\varphi},g_1}\varphi^*(g_1), f_{\bar{\varphi},g_1} \in C^\infty(M), > 0$ is a homeomorphism onto a neighborhood of g, diffeomorphic off fixed points of $C^0(\gamma)$.*

From this theorem the quotient space $\mathscr{S}/C^0(\gamma)$ gives a neighborhood of the local moduli.

Remark. The global effectiveness of our slice, namely the property (ii) of Ebin's theorem is not guaranteed, because $\mathscr{R}_{M,dv_g}$ is not $\mathrm{Diff}^+(M)$-invariant. However, by using the arguments of Yamabe problem we can assert the global effectiveness. In fact, each γ in $\mathscr{C}_M \backslash \mathscr{C}^{(+)}_M$ has a unique volume-normalized Yamabe metric so that one can identify $\mathscr{C}_M \backslash \mathscr{C}_M^{(+)}$ with the subspace in $\mathscr{R}_M$ consisting of Riemannian metrics of unit volume whose scalar curvature is nonpositive constant. The latter space is $\mathrm{Diff}^+(M)$-invariant and one may apply directly Ebin's theorem.

(iv) Kuranishi map

Let γ be an anti-self-dual conformal structure on a 4-manifold M and g a representative of γ.

Consider the anti-self-dual equations in the local slice $\mathscr{S}$

$$W^+(g_1) = 0, \qquad L^*(h) = 0. \tag{3.6}$$

Here g_1 is in $\mathscr{S}$ and $h \in C^\infty(S_0(T^*M))$ is determined by g_1 defined by $g_1((\exp h)(X), Y) = g(X, Y)$.

The star operator is $*_{g_1} = ((\exp h)^*)^{-1} \circ *_g \circ (\exp h)^*$). Then the first equation is replaced by $((\exp h)^* (W(g_1)))^{+_g} = 0$. So we can rewrite (3.6) as

$$((\exp h)^* (W(g_1)))^{+_g} = 0, \qquad L^*(h) = 0. \tag{3.7}$$

Define a map

$$w^+ : S \to L^p_k(S_0(\Omega^+)), \qquad h \mapsto ((\exp h)^* (W(g_1)))^{+_g}. \tag{3.8}$$

We expand $w^+(h)$ as

$$w^+(h) = W^+(g) + D_g(h) + R(h)$$

with a remainder term $R(h) = R_g(h)$. Since γ is anti-self-dual, we have

$$w^+(h) = D_g(h) + R(h). \tag{3.9}$$

As a routine business for solving the equation $w^+(h) = 0$ we introduce a map $\Theta = \Theta_g$, the Kuranishi map, from a small ball in $L^2_{k+1}(S_0(T^*M))$ into $L^2_{k+1}(S_0(T^*M))$;

$$\Theta : h \mapsto h + D_g^* G(R(h)) \tag{3.10}$$

for D_g^*, the adjoint of D_g with respect to the L^2-inner product and $G = G_g$, the Green operator of $D_g D_g^*$ on $L^2_k(S_0(\Omega^+))$.

As was discussed in the deformations of complex structures [35] we can show the following, since the map Θ is locally invertible and $C^0(\gamma)$-equivariant.

Theorem 3.5. (i) *There exists for small $\varepsilon > 0$ a $C^0(\gamma)$-equivariant map Φ from an ε-ball $\mathbb{H}^1_{\gamma,\varepsilon}$ of $\mathbb{H}^1_\gamma \cong \operatorname{Ker} L^* \cap \operatorname{Ker} D_g$ to $\mathbb{H}^2_\gamma \cong \operatorname{Ker} D_g D_g^* : h \mapsto \pi R(\Theta^{-1}(h))$ such that anti-self-dual conformal structures in the slice $\mathscr{S}_\gamma$ are described as* $\mathrm{Zero}(\Phi) = \{h \in \mathbb{H}^1_{\gamma,\varepsilon}; \Phi(h) = 0\}$ *and* (ii) *for each gauge equivalence class $\bar{\gamma}$ the quotient* $\mathrm{Zero}(\Phi)/C^0(\gamma)$ *by the conformal group $C^0(\gamma)$ yields a neighborhood of the local moduli. Here π is the projection of $C^\infty(S_0(\Omega^+))$ onto $\mathbb{H}^2_\gamma$.*

(v) L^2-metric

As a first step towards for defining a Riemannian metric on the moduli $\tilde{\mathscr{M}}_M$ we define a $\mathrm{Diff}^+(M)$-gauge invariant L^2-metric on $\mathscr{C}_M$.

Throughout this section we keep the identification

$$\mathscr{C}_M \subset C^\infty(\mathrm{Hom}(\Omega^+, \Omega^-)).$$

For $A \in C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-))$ define the adjoint $A^*:\Omega^-_\gamma \to \Omega^+_\gamma$ with respect to the volume form dv_h (g is a representative of γ), in other words

$$\eta^+ \wedge A^*\eta^- = (A\eta^+)\wedge\eta^-\,, \qquad \eta^\pm \in \Omega^\pm\,. \tag{3.11}$$

Then the trace $-\operatorname{Tr} AA^*$ is a scalar function on M, positive definite and depends only on γ.

In fact, choose at each point orthonormal bases $\{\eta_i^+\}$, $\{\eta_j^-\}$ of $\Omega^\pm_\gamma$, i.e., $\pm\eta_i^\pm \wedge \eta_j^\pm = \delta_{ij}dv_g$, $i = 1,2,3$ and set $A\eta_i^+ = A_i^j\eta_j^-$. Then $A^*\eta_i^- = (A^*)_i^j\eta_j^+$ has $(A^*)_i^j = -A_j^i$ and hence $-\operatorname{Tr} AA^* = A_j^iA_j^i$ is positive definite.

From this definition the trace is independent of the choice of g.

A diffeomorphism φ acts on $C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-))$ as

$$A \in C^\infty(\mathrm{Hom}(\Omega^+_\gamma,\Omega^-_\gamma)) \mapsto A^\varphi \in C^\infty(\mathrm{Hom}(\Omega^+_{\gamma_1},\Omega^-_{\gamma_1}))\,, \qquad \gamma_1 = \varphi^*\gamma$$

by the following diagram

$$\begin{array}{ccc} \Omega^+_{g,\varphi(x)} & \xrightarrow{\varphi_x^*} & \Omega^+_{\varphi^*g,x} \\ {\scriptstyle A_{\varphi(x)}}\big\downarrow & & \big\downarrow{\scriptstyle (A^\varphi)x} \\ \Omega^-_{g,\varphi(x)} & \xrightarrow{\varphi_x^*} & \Omega^-_{\varphi^*g,x} \end{array}$$

where $x \in M$ and g is a representative of γ. So $(A^\varphi)_x = \varphi_x^* \circ A_{\varphi(x)} \circ (\varphi_x^*)^{-1}$ and $(A^\varphi)_x^* = \varphi_x^* \circ (A^*)_{\varphi(x)} \circ (\varphi_x^*)^{-1}$. Then the pointwise inner product satisfies

$$(-\operatorname{Tr} A^\varphi(A^\varphi)^*)(x) = (-\operatorname{Tr} AA^*)(\varphi(x))\,. \tag{3.12}$$

To define an L^2-inner product on $\mathscr{C}_M$, invariant under the $\mathrm{Diff}^+(M)$-action we need from (3.12) a "canonical" volume form $g \mapsto d\mathbb{V}_g$ satisfying the conformal invariance, $d\mathbb{V}_{fg}(x) = d\mathbb{V}_g(x)$, $f \in C^\infty(M), > 0$, and the naturality, $d\mathbb{V}_{\varphi^*g}(x) = (\varphi^*d\mathbb{V}_g)(x)$.

Assume the existence of the canonical volume form. We then obtain an L^2-inner product on $C^\infty(\mathrm{Hom}(\Omega^+_\gamma,\Omega^-_\gamma))$ as

$$\|A\|^2 = \int_M (-\operatorname{Tr} AA^*)(x)\,d\mathbb{V}_g(x)\,, \qquad A \in C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-))\,, \tag{3.13}$$

integrated in terms of the canonical volume form.

So the remaining problem is to verify the existence of such a volume form.

To investigate it we notice that the quadratic form induced from the cup product:

$$\mathrm{H}^2(M;\mathbb{Z}) \times \mathrm{H}^2(M;\mathbb{Z}) \to \mathrm{H}^4(M;\mathbb{Z}) \cong \mathbb{Z}$$

gives a nondegenerate symmetric form on $\mathrm{H}^2(M;\mathbb{R})$ of type $(\mathrm{b}^+,\mathrm{b}^-)$, identified with the wedge product on the de Rham cohomologies:

$$\mathrm{H}^2(M;\mathbb{R}) \times \mathrm{H}^2(M;\mathbb{R}) \to \mathrm{H}^4(M;\mathbb{R}) = \mathbb{R}[dv]; ([\theta],[\omega]) \mapsto [\theta\wedge\omega]$$

(dv is a volume form of unit volume).

For any metric g $\mathrm{H}^\pm_g$, the space of (anti-)self-dual harmonic 2-forms, are $\mathrm{b}^\pm$-dimensional subspaces of $\mathrm{H}^2(M;\mathbb{R})$, respectively.

To simplify the argument we assume $b^+ > 0$ (when $b^+ = 0$, $b^- > 0$ is assumed so that $b^2 > 0$ is primarily assumed.)

We choose an orthonormal basis $\{\psi^+_{g,i}\}$, $1 \leqq i \leqq b^+$, of H^+_g. The orthonormality is measured by the cup product; $[\psi^+_i] \wedge [\psi^+_j] = \delta_{ij}[dv]$ for $\psi^+_i = \psi^+_{g,i}$.

Define

$$d\mathbb{V}_g = \sum_{i=1}^{b^+} \|\psi^+_i\|^2_g(x)\, dv_g(x)\,, \qquad x \in M\,, \tag{3.14}$$

where $\|\cdot\|_g$ is the norm measured by g.

This does not depend on choices of orthonormal basis. This is conformally invariant since for each i $\|\psi_i\|^2_g dv_g = \psi_i \wedge *\psi_i = \psi_i \wedge \psi_i$.

The canonical volume form (3.14) depends smoothly on the metric g, since $b^+ = \dim H^+_g$ is a topological invariant (see for examle [35, Theorem 4.5, p. 178]).

The naturality of $d\mathbb{V}_g(x)$ is indicated as follows. Any $\varphi \in \mathrm{Diff}^+(M)$ induces a quadratic form isometry $\varphi^* : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z})$ so that $\{\varphi^*\psi^+_i\}$ gives rise to an orthonormal basis of $H^+_{\varphi^* g}$ and hence

$$d\mathbb{V}_{\varphi^* g}(x) = \sum_i \|\varphi^*\psi^+_i\|^2_{\varphi^* g}(x)\, dv_{\varphi^* g}(x) = \sum_i \|\psi^+_i\|^2_g(\varphi(x))\, (\varphi^* dv_g)\, (x)\,.$$

Thus one has

Theorem 3.6. *The inner product* (3.13) *is positive definite and* Diff$^+(M)$*-invariant.*

$d\mathbb{V}_g$ is the Riemannian volume form dv_g multiplied by a nonnegative weight function which has in general a zero locus. The positivity of (3.13) is shown in the following way. Suppose that $\|A\|^2 = 0$ for $A \in C^\infty(\mathrm{Hom}(\Omega^+, \Omega^-))$. Then A must be zero at least at points where $d\mathbb{V}$ is positive and from the result of [3] these points are open dense in M. Therefore A vanishes at every point.

Theorems 2 and 3 in Sect. 1 follow from Theorems 3.4, 3.5 and 3.6.

Remark. In some special case $d\mathbb{V}_g$ coincides with dv_g up to a constant scalar factor. Indeed this is the case when each of ψ^+_j has constant norm.

We remark also that through the identification (3.1) $-\operatorname{Tr} AA^*$ is just $4 \operatorname{Tr} hh$, for $A = A_h$ from (3.2).

4 K3 surfaces

Recall the following formula for a compact connected oriented Riemannian 4-manifold (M, g)

$$\chi(M) + 3/2\tau(M) = \frac{1}{4\pi^2}\int |W^+|^2 + \frac{1}{48\pi^2}\int \{\varrho^2 - 3|\mathrm{Ric}|^2\} \tag{4.1}$$

(see [24, p. 72]). So as an easy observation from (4.1).

Proposition 4.1. *Let M be as before a compact connected oriented 4-manifold. If M satisfies $2\chi(M) + 3\tau(M) = 0$ (this is the case for a complex torus, a quotient of a complex torus, a* K3 *surface, an Enriques surface and the quotient of an Enriques surface by an antiholomorphic involution* [26]*). Then any anti-self-dual Riemannian metric g is of zero scalar curvature if and only if g is Ricci flat.*

The moduli $\tilde{\mathscr{M}}_M^{(0)}$ of type zero anti-self-dual conformal structures on M of $2\chi + 3\tau = 0$ is then identified with the moduli of Ricci flat metrics of unit volume.

Now let M be a K3 surface, a simply connected compact complex surface with the trivial canonical bundle K_M.

The topological invariants are $\chi = 24$, $b^2 = 22$, $(b^+, b^-) = (3, 19)$ so $\tau = -16$ and $2\chi + 3\tau = 0$.

The moduli $\tilde{\mathscr{M}}_M^{(0)}$ is well investigated in terms of the period map. Actually the quadratic form q_M on $H^2(M;\mathbb{Z})$ has type $(3,19)$ and the Grassmannian $G_3^+ = SO(3,19)/SO(3) \times SO(19)$ of oriented positive definite 3-planes in $H^2(M;\mathbb{R})$ gives the Ricci flat Kähler metrics on M provided we ignore the action of $\mathrm{Aut}(H^2(M;\mathbb{Z}); q_M)$; $p: \mathscr{E} \to G_3^+$. Here $\mathscr{E}$ denotes the moduli of Ricci flat metrics of unit volume.

Then $\mathscr{E}$ admits a structure of 57 dimensional symmetric space with an invariant metric. This means that the space $\mathrm{Hom}(H^+, H^-) = H^- \otimes (H^+)^*$ gives the tangent space $T_g\mathscr{E}$ and the invariant metric is $-\operatorname{tr} XX^t$, $X \in \mathrm{Hom}(H^+, H^-)$ from the standard argument of symmetric spaces.

On the other hand the index of the complex (3.5) is -52 and $\dim \mathbb{H}^0 = 0$ and moreover from Corollary A.5 in Appendix $\dim \mathbb{H}^2 = 5$. The virtual dimension of our moduli at each $\bar{\gamma}$ represented by a Ricci flat metric g is then at most 57.

The following proposition asserts as exhibited in Theorem 4, Sect. 1 that $\tilde{\mathscr{M}}_M^{(0)}$ has actually 57 dimension and the connected component of $\tilde{\mathscr{M}}_M$ containing $\tilde{\mathscr{M}}_M^{(0)}$ is itself $\tilde{\mathscr{M}}_M^{(0)}$ and is isometric to the image $p(\mathscr{E})$ in G_3^+. As an easy observation there is no type negative anti-self-dual conformal structure nearby $\tilde{\mathscr{M}}_M^{(0)}$.

Proposition 4.2. *Let g be a Ricci flat metric on a* K3 *surface M. Let $\psi_a^+ \in H^+$, $a = 1,2,3$ and $\psi_b^- \in H^-$, $b = 1, \ldots, 19$ be harmonic 2-forms being orthonormal bases of H^+, H^-, respectively. Then $\psi_b^- \otimes \psi_a^+ \in H^- \otimes H^+$, $1 \leqq a \leqq 3$, $1 \leqq b \leqq 19$ form through the identification $H^+ \cong (H^+)^*$ an orthonormal basis of the tangent space $T_{\bar{\gamma}}\tilde{\mathscr{M}}_M$, $\gamma = [g]$ with respect to the L^2-metric.*

Proof. First we remark that the metric g is Kähler from Theorem 2.3 and each ψ_a^+ is covariantly constant so that $d\mathbb{V}_g = 3dv_g$ and then the L^2-inner product (3.13) is just the ordinary inner product $\|h\|^2 = \int_M \operatorname{Tr} hh dv_g$ of $C^\infty(S_0(T^*M))$ through the identification (3.1).

Let $h \in C^\infty(S_0(T^*M))$ be given via the map (3.1) by $\psi_b^- \otimes \psi_a^+$. Then $h = (h_{ij})$ is $h_{ij} = g^{kl}\psi_{ik}^-\psi_{lj}^+ = \psi_{ik}^-(\psi^+)_j^k$.

We verify $h \in \operatorname{Ker} L^* \cap \operatorname{Ker} D$ at g. Since $d^*\psi^- = 0$ and $\nabla\psi^+ = 0$, $L^*(h)$ is from (3.4) $L^*(h) = -2g^{li}\nabla^j h_{jl} = -2g^{li}(\nabla^j\psi_{jk}^-)\psi_l^{+k} = 0$, so that $h \in \operatorname{Ker} L^*$.

To show $Dh = 0$ we make use of the anti-self-duality of ψ^- and apply (3.3) and (A.1), Appendix. Apply ψ_l^{+k} to $\nabla_i\psi_{jk}^- + \nabla_j\psi_{ki}^- + \nabla_k\psi_{ij}^- = 0$. Then we have

$$\nabla_i h_{jl} - \nabla_j h_{il} + \nabla_s\psi_{ij}^- \cdot \psi_l^{+s} = 0 \tag{4.2}$$

and hence

$$\nabla_k\nabla_i h_{jl} - \nabla_k\nabla_j h_{il} + (\nabla_k\nabla_s\psi_{ij}^-)\psi_l^{+s} = 0\,, \tag{4.3}$$

or interchange k and l

$$\nabla_l\nabla_i h_{jk} - \nabla_l\nabla_j h_{ik} + (\nabla_l\nabla_s\psi_{ij}^-)\psi_k^{+s} = 0\,. \tag{4.4}$$

So the tensor $U \in C^\infty(\Omega^2 \otimes \Omega^2)$ defined in (A.2) is

$$2U_{ijkl} = (\nabla_k \nabla_s \psi^-_{ij})\psi^{+s}_l - (\nabla_l \nabla_s \psi^-_{ij})\psi^{+s}_k . \tag{4.5}$$

$D(h)$ is the $S_0(\Omega^+)$-component of U since g is Ricci flat.

Without loss of generality we can assume $\psi^+_1 = \omega$, the Kähler form and ψ^+_2, ψ^+_3 are the real and imaginary parts of a covariantly constant holomorphic 2-form, respectively.

We use the complex coordinate indices.

For $\psi^+ = \omega$, $\psi^{+i}_j = \sqrt{-1}\,\delta^i_j$, $\psi^{+\bar{i}}_{\bar{j}} = -\sqrt{-1}\,\delta^i_j$, $i, j = 1, 2$ and others are zero. Then $U_{ijkl} = 0$ for $k, l \in \{1, 2\}$ and $i, j \in \{1, 2, \bar{1}, \bar{2}\}$ since $[\nabla_k, \nabla_l] = 0$, and also $U_{ijkl} = 0$ for $i, j \in \{1, 2\}$ and $k, l \in \{1, 2, \bar{1}, \bar{2}\}$ since ψ^- is a (1, 1)-form.

Similarly $U_{ijkl} = 0$ for all indices running over $\bar{1}, \bar{2}$. Therefore the components of U in $\Omega^+ \otimes \Omega^+$ remain to be shown to be zero are only the $\omega \otimes \omega$-component. But it is $g^{\bar{j}i} g^{\bar{l}k} U_{i\bar{j}k\bar{l}} = g^{\bar{j}i} g^{\bar{l}k} (\nabla_k \nabla_{\bar{l}} + \nabla_{\bar{l}} \nabla_k)\psi_{i\bar{j}}$ which vanishes from the fact that ψ^- is a primitive form.

The similar argument works for other ψ^+_2, ψ^+_3 so that $\psi^-_b \otimes \psi^+_a \in \operatorname{Ker} L^* \cap \operatorname{Ker} D$ for any a, b.

That $\psi^-_b \otimes \varphi^+_a$, $1 \leqq a \leqq 3$, $1 \leqq b \leqq 19$ enjoy an L^2-orthonormal basis of $T_\gamma \tilde{\mathscr{M}}_M$ follows from the definition of the L^2-inner product (3.13) and the remark mentioned at the beginning of the proof. qed

5 Half conformal flatness and complex structures

(i) Moduli on ruled surface

The Kodaira-Spencer complex for complex structure deformations for a compact complex surface M has the index $1/6(7c_1^2(M) - 5c_2(M)) = 1/6(21\tau(M) + 9\chi(M))$ [34].

This index is for $M = M_k$, a ruled surface, $6(1-k)$, so that $\mathrm{H}^1(M_k, \mathbb{T}M)$ has the virtual complex dimension $6(k-1)$. This dimension will coincide from Theorem 5.1 with the "complex dimension" of $\tilde{\mathscr{M}}^{(0)}_M$.

This phenomenon is fortunately not accidental.

Let M be a compact surface of $\mathrm{p}_g = 0$ (or equivalently $\mathrm{b}^+(M) = 1$). Then from Theorem 2.3 every type zero anti-self-dual structure $\bar{\gamma} \in \tilde{\mathscr{M}}^{(0)}$ yields the unique complex structure J_γ (up to diffeomorphisms) such that one has a map

$$\mathrm{j}\colon \tilde{\mathscr{M}}^{(0)}_M \to \bar{\mathscr{T}}_M = \{\text{complex structures on } M\}/\operatorname{Diff}^0(M); \qquad \bar{\gamma} \mapsto [J_\gamma] .$$

Relative to a fixed complex structure there are two possibilities of conformal structure deformations. One is a deformation fixing a complex structure and varying a metric and another is a deformation varying a complex structure.

We postpone investigating in the forthcoming paper how the moduli of complex structures affects our moduli.

Suppose that M is now a ruled surface.

Since any ruled surface $M = M_k$ has $\tau = 0$, every anti-self-dual structure is conformally flat. $\tilde{\mathscr{M}}^{(+)}_M = \emptyset$ because $\mathrm{b}^+ = 1$. The moduli of "conformally flat" structures on M_k, $\tilde{\mathscr{M}}_M = \tilde{\mathscr{M}}_M^{(0)} \coprod \tilde{\mathscr{M}}_M^{(-)}$, is considered to lie inside the representation space $\mathscr{R}(\pi_1(M); SO(5,1))$, as explained in Sect. 1.

Now we are interested in $\tilde{\mathscr{M}}_M^{(0)}$, the moduli of type zero conformally flat structures on M_k.

Let $\bar{\gamma} \in \tilde{\mathscr{M}}_M^{(0)}$. Then one has from Theorem 2.3 a representative g of γ, a Kähler metric of zero scalar curvature. From Theorem 2.2 (M_k, g) is then covered by the Kähler product $D^1 \times \mathbb{C}P^1$; $(M_k, g) = D^1 \times \mathbb{C}P^1/\Gamma$ for a discrete subgroup Γ of $\mathrm{Aut}(D^1 \times \mathbb{C}P^1) = \mathrm{SL}(2,\mathbb{R}) \times PU(2)$ acting freely and properly discontinuously. Since every $a \in PU(2)$ has a fixed point on $\mathbb{C}P^1$, Γ is the graph of a homomorphism $\phi: \Gamma_1 \subset \mathrm{SL}(2,\mathbb{R}) \to PU(2) = \mathrm{Aut}(\mathbb{C}P^1)$, where Γ_1 is a subgroup isomorphic to $\pi_1(\Sigma_k)$ acting on D^1 freely and properly discontinuously.

It follows then that every type zero conformally flat structure $\bar{\gamma} \in \tilde{\mathscr{M}}_M^{(0)}$ one-to-one corresponds to an appropriate conjugacy class of representation $\pi_1(\Sigma_k) \to \mathrm{SL}(2,\mathbb{R}) \times PU(2)$. More precisely, $\tilde{\mathscr{M}}_M^{(0)}$ is exactly the set of all conjugacy classes $[\phi]$ containing $\phi: \pi_1(\Sigma_k) \to \mathrm{SL}(2,\mathbb{R}) \times PU(2)$ satisfying that ϕ is the composite of $\phi_1: \pi_1(\Sigma_k) \to \mathrm{SL}(2,\mathbb{R})$ and $\phi_2: \mathrm{Im}(\phi_1) \subset \mathrm{SL}(2,\mathbb{R}) \to PU(2)$ and ϕ_1 acts on the disk D^1 freely and properly discontinuously.

Since the homomorphism ϕ_2 induces a $PU(2)$ flat connection on a complex vector bundle over the Riemann surface $\Sigma_k = D^1/\mathrm{Im}\,\phi_1$; $D^1 \times_{\phi_2} \mathbb{C}^2 \to \Sigma_k$, the following fibration structure theorem is available.

Theorem 5.1. *The moduli $\tilde{\mathscr{M}}_M^{(0)}$ on a ruled surface $M = M_k$, $k > 1$ has a structure of fibration $\tilde{\mathscr{M}}_M^{(0)} \to \tilde{\mathscr{M}}_{\Sigma_k}$, the Teichmüller moduli of Riemann surfaces, whose fibre over a Riemann surface represented by $[\phi_1]$, $\phi_1: \pi_1(\Sigma_k) \to \mathrm{SL}(2,\mathbb{R})$ is the moduli of $PU(2)$ flat connections on the complex smooth vector bundle induced by $\phi_2: \mathrm{Im}\,\phi_1 \to PU(2)$.*

From this theorem it is expected that the fibration yields a Riemannian submersion with respect to the L^2-metric and the Weil-Petersson metric on $\tilde{\mathscr{M}}_{\Sigma_k}$ such that the L^2-metric restricted to each fibre is the metric introduced in [29].

Since $\mathrm{SL}(2,\mathbb{R}) \times PU(2)$ is immersed in $SO(5,1)$ as a proper subgroup, $\mathscr{R}(\pi_1(\Sigma_k); \mathrm{SL}(2,\mathbb{R}) \times PU(2))$ and hence $\tilde{\mathscr{M}}_M^{(0)}$ is immersed in $\mathscr{R}(\pi_1(M_k); SO(5,1))$. Therefore

Corollary 5.2. *Any ruled surface admits type negative anti-self-dual structures around any type zero anti-self-dual structure. Namely, if $\tilde{\mathscr{M}}_M^{(0)} \neq \emptyset$, then $\tilde{\mathscr{M}}_M^{(-)}$ is also not empty.*

Remark. There is a ruled surface admitting no type zero anti-self-dual conformal structure [12].

(ii) Remark for twistor spaces

By the twistor correspondence any anti-self-dual conformal structure γ on a 4-manifold induces a complex structure J_γ on the unit sphere bundle $\mathrm{U}(\Omega^+)$ over M, called the twistor space $Z = Z_M = (\mathrm{U}(\Omega^+), J_\gamma)$ [27].

This correspondence induces a canonical map from $\mathscr{M}_M$ to the moduli $\mathscr{T}_Z$ of complex structures on Z. This map is an embedding since there is a twistorial characterization of complex 3-manifold [8, Theorem 13, 69].

Correspondingly to this we have a homomorphism between the complex (3.5) and the Kodaira-Spencer complex of Z (see (3.3) in [21]); (5.10)

$$\begin{array}{ccccccc}
C^\infty(\mathrm{T}M) & \xrightarrow{L} & C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-)) & \xrightarrow{D} & C^\infty(S_0(\Omega^+)) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
C^\infty(\mathbb{T}Z) & \xrightarrow{\bar\partial} & C^\infty(\Omega^{0,1}\otimes\mathbb{T}Z) & \xrightarrow{\bar\partial} & C^\infty(\Omega^{0,2}\otimes\mathbb{T}Z) & \xrightarrow{\bar\partial} & C^\infty(\Omega^{0,3}\otimes\mathbb{T}Z)
\end{array}$$

which induces the injective homomorphism of the first cohomology groups, the "tangent spaces" of $\tilde{\mathscr{M}}_M$ and $\tilde{\mathscr{T}}_Z$.

A conformally flat structure corresponds to a holonomy homomorphism $\pi_1(M) \to SO(5,1)$. As was pointed out in [4, p. 439] the natural homomorphism $SO(5,1) \to SO(6,\mathbb{C}) \to PSL(4,\mathbb{C})$ then defines on the twistor space $Z = \mathrm{U}(\Omega^+)$ a projectively flat complex structure [25, 33].

The twistor space of a conformally flat 4-manifold is in fact represented locally as a neighborhood in $\mathbb{C}P^3$ containing a complex line.

Our investigation of the moduli of conformally flat 4-manifolds yields examples of family of projectively flat complex 3-manifolds.

A projective flat cmpact complex 3-manifold Z_M satisfies for Chern numbers $16c_3(Z) = 8/3\ c_1c_2(Z) = c_1^3(Z) = 32\chi(M)$ ([33, p. 135] and [27]).

Appendix

In this appendix we will show

Proposition A.1. *Let g be an anti-self-dual conformal structure. Then the linear map $C^\infty(S_0(T^*M)) \to C^\infty(S_0(\Omega^+))$; $h \mapsto (\delta W_g(h))^+$ is written as*

$$(\delta W_g(h))^+ = U(h)^+ + V(h)^+, \tag{A.1}$$

where U^+, V^+ are the $S_0(\Omega^+)$-components of $U, V \in C^\infty(\Omega^2\otimes\Omega^2)$ defined by

$$U_{ijkl} = 1/2(\nabla_k\nabla_j h_{il} - \nabla_l\nabla_j h_{ik} - \nabla_k\nabla_i h_{jl} + \nabla_l\nabla_i h_{jk}), \tag{A.2}$$

$$V_{ijkl} = -1/4(R_{kj}h_{il} - R_{lj}h_{ik} - R_{ki}h_{jl} + R_{li}h_{jk}), \tag{A.3}$$

for $h = (h_{ij}) \in C^\infty(S_0(T^*M))$.

The proof needs a straightforward calculation. For two metrics g and $\tilde g$ we calculate the difference of the Christoffel symbols as

$$\widetilde{\{{}_j{}^i{}_k\}} - \{{}_j{}^i{}_k\} = 1/2\tilde g^{il}(g_{ls}\nabla_j h^s_k + g_{js}\nabla_k h^s_l - g_{js}\nabla_l h^s_k) \tag{A.4}$$

for $h = (h^j_k) \in C^\infty(\mathrm{End}(TM))$ satisfying $g(hX,Y) = \tilde g(X,Y)$.

From this one has

$$\begin{aligned}\delta\{{}_j{}^i{}_k\}(h) &= 1/2(\nabla_j h^i_k + \nabla_k h^i_j - \nabla^i h_{jk}),\\ h_{ij} &= d/dt\, g_{ij}(0), \qquad h^i_j = g^{ik}h_{kj}.\end{aligned} \tag{A.5}$$

Applying the chain rule, one gets

$$\delta R_g(h)^i_{jkl} = \nabla_k(\delta\{{}_j{}^i{}_l\}(h)) - \nabla_l(\delta\{{}_j{}^i{}_k\}(h)),$$

and then from (A.5)

$$\begin{aligned}\delta R_g(h)^i_{jkl} &= 1/2(\nabla_k\nabla_l h^i_j - \nabla_l\nabla_k h^i_j)\\ &\quad + 1/2(\nabla_k\nabla_j h^i_l - \nabla_l\nabla_j h^i_k)\\ &\quad - 1/2(\nabla_k\nabla^i h_{jl} - \nabla_l\nabla^i h_{jk})\,. \end{aligned} \tag{A.6}$$

Hence

$$\begin{aligned}(\delta R_g(h))_{ijkl} &= 1/2(\nabla_k\nabla_j h_{il} - \nabla_l\nabla_j h_{ik} - \nabla_k\nabla_i h_{jl} + \nabla_l\nabla_i h_{jk})\\ &\quad + 1/2(h^t_i R_{tjkl} + h^t_j R_{itkl})\,. \end{aligned} \tag{A.7}$$

The Weyl conformal tensor W has three parts

$$W = R + R' + R''\,,$$

$$R'_{ijkl} = -1/2(g_{ik}R_{jl} - g_{il}R_{jk} + R_{ik}g_{jl} - R_{il}g_{jk})\,, \tag{A.8}$$

$$R''_{ijkl} = 1/6\varrho(g_{ik}g_{jl} - g_{il}g_{jk})\,. \tag{A.9}$$

By calculating $\delta R'$ and $\delta R''$ we derive the following formula valid for any metric and any $h \in C^\infty(S^2(T^*M))$.

Formula A.2.

$$\begin{aligned}\delta W_g(h)_{ijkl} &= (h^t_i W_{tjkl} + h^t_j W_{itkl}) + U_{ijkl}\\ &\quad - 1/2(h^t_i R_{tjkl} + h^t_j R_{itkl})\\ &\quad - 1/2(g_{ik}\delta R^t_l g_{jt} - g_{il}\delta R^t_k g_{jt} + g_{it}\delta R^t_k g_{jl} - g_{it}\delta R^t_l g_{jk})\\ &\quad + 1/6(\delta\varrho)(g_{ik}g_{jl} - g_{il}g_{jk})\,. \end{aligned} \tag{A.10}$$

Now assume that g is anti-self-dual and h is traceless. Then the $S_0(\Omega^+)$-component $(\delta W_g(h))^+$ is

$$(\delta W_g(h))^+ = U^+ + V^+\,,$$

where V^+ is the $S_0(\Omega^+)$-component of the third term V of (A.10), since the first term and the last two terms of (A.10) vanish when we take the $S_0(\Omega^+)$-component. Here we characterize the traceless symmetric product $S_0(\Omega^+)$ as

Lemma A.3. *The traceless symmetric product $S_0(\Omega^+_x)$ of Ω^+_x at a point x is the space of Ricci flat curvature like tensor defined at x sastisfying the first Bianchi identity, namely*

$$S_0(\Omega^+_x) = \{Z = (Z_{ijkl});\, g^{ik}Z_{ijkl} = 0, Z_{ijkl} + Z_{iklj} + Z_{iljk} = 0\}\,.$$

We substitute $R = W - R' - R''$ into V as

$$\begin{aligned}V_{ijkl} &= -1/2(h^t_i W_{tjkl} + h^t_j W_{itkl})\\ &\quad + 1/2(h^t_i R'_{tjkl} + h^t_j R'_{itkl})\\ &\quad + 1/2(h^t_i R''_{tjkl} + h^t_j R''_{itkl})\end{aligned}$$

and take its $S_0(\Omega^+)$-component. Then

$$V^+_{ijkl} = -1/4(R_{kj}h_{il} - R_{lj}h_{ik} - R_{ki}h_{jl} + R_{li}h_{jk})^+$$

from which Proposition A.1 follows.

Remark. If g is anti-self-dual and Einstein, then $V^+ = 0$, namely $(\delta W_g(h))^+ = U^+$.

We would like to obtain a formula for the adjoint D^* of D, $D^* : C^\infty(S_0(\Omega^+)) \to C^\infty(S_0(T^*M))$.

Proposition A.4. *For an anti-self-dual conformal structure* $\gamma = [g]$ D^* *has the form*

$$(D^*Z)_{ij} = \nabla^k\nabla^l Z_{iklj} + \nabla^l\nabla^k Z_{iklj} + R^{kl}Z_{iklj}. \tag{A.11}$$

Proof. D^* is defined as

$$\int_M (h, D^*Z)dv_g = \int_M (Dh, Z)dv_g.$$

From Proposition A.1 (Dh, Z) is $(Dh, Z) = (U^+, Z) + (V^+, Z)$. Here

$$(U^+, Z) = (\nabla_k\nabla_j h_{il} - \nabla_k\nabla_i h_{jl})Z^{ijkl}$$

and

$$(V^+, Z) = h_{ij}R_{kl}Z^{iklj}.$$

Then the formula (A.11) is derived from the integration

$$\int (U^+, Z)dv_g = \int h_{ij}(\nabla_k\nabla_l Z^{iklj} + \nabla_l\nabla_k Z^{iklj})dv_g.$$

Remark. This formula is appeared already in [6] as the first variational equation $D^*W = 0$ of the functional $\mathscr{W} : \mathscr{C}_M \to \mathbb{R}$ (see also [15, Lemma 1]).

As a consequence of Proposition A.4.

Corollary A.5. *Let M be a complex 2-torus or a* K3 *surface and g be a Ricci flat (i.e., type zero) anti-self-dual metric on M. Then the second cohomology group of the complex* (3.5) *is* $\mathbb{H}^2_g \cong \mathbb{R}^5$. *In fact* $\sum a_{ij}\psi_i^+ \otimes \psi_j^+$, $a_{ij} \in \mathbb{R}$, $a_{ij} = a_{ji}$, $\sum a_{ii} = 0$, span $\mathbb{H}^2_g$.

Acknowledgement. The author would like to express his gratitude to T. Mabuchi and C. LeBrun for useful conversations and criticisms and also to Max-Planck-Institut für Mathematik for supporting his stay in Bonn.

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