Math. Ann. 296, 595–605 (1993)

Mathematische Annalen

Effective base point freeness

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Received November 10, 1992; in revised form February 16, 1993

Mathematics Subject Classification (1991): 14C20, 14J10

Let X be a smooth projective variety and let L be an ample divisor on X. By definition some multiple of L is very ample. It is frequently useful to know which multiples are very ample. Already for curves one can not give a universal bound; the answer depends on the genus. One way to overcome this problem is to ask about very ampleness of $K_X + mL$. In this form a nice and uniform answer emerges:

If $\dim X = 1$ then $K_X + 3L$ is very ample (easy);

if $\dim X = 2$ then $K_X + 4L$ is very ample [Reider].

In general [Fuj] conjectured that $K_X + (\dim X + 2)L$ is very ample. A major breakthrough was achieved by [Dem] who proved that $2K_X + 12n^nL$ is very ample where $n = \dim X$. His methods involve heavy analysis and are rather intricate.

[Reid] pointed out that the algebraic methods developed around the cone theorem (cf. [CKM, #9]) can be used to provide some effective estimates, especially when $K_X \equiv cL$ for some constant $c \in \mathbb{Q}$. Similar ideas were utilised in dimension three in a series of papers [Ben 1, 2; Matsuki; Ogu].

My attention was turned to this problem by Lazarsfeld who observed (jointly with Ein) that several of the results of [Reider] on surfaces can be obtained using the ideas of the base point free theorem. Their approach was successfully extended to threefolds in [EinLaz] where they show that if $\dim X = 3$ then $|K_X + 5L|$ is base point free. Their results are in fact more precise, see especially [ibid, Theorem 1*].

Unfortunately, I was unable to come close to the conjectured bounds. Instead, I would like to present an algebraic approach to a Demailly-type result and to point out some consequences.

The algebraic method works for X singular and also for certain non ample line bundles. I formulate the most general case; readers interested in smooth varieties should just always set $\Delta = 0$. (See the end of the introduction for some definitions.)

The characteristic is assumed to be zero throughout.

1.1 Theorem (Effective base point freeness). *Let (X, Δ) be proper and* klt *of dimension n. Let L be a nef Cartier divisor on X. Assume that $aL - (K_X + \Delta)$ is nef and big for some $a \geqq 0$.*

Then

$$|2(n+2)!(a+n)L| \text{ is base point free .}$$

An analog of the conjecture [Fuj] says that $|(a+n+1)L|$ is base point free. Thus the main point of (1.1) is the existence of a universal bound rather than the actual value of the coefficient.

If L is ample and X is smooth then $K_X + (\dim X + 2)L$ is also ample and $K_X + (\dim X + 2)L - K_X$ is nef and big. By (1.1) a large multiple of $K_X + (\dim X + 2)L$ is very ample. Thus (1.1) does not imply the result of Demailly mentioned above. However in all applications that I am aware of, they can be used interchangeably.

To go from freeness to very ampleness is rather easy. The following general lemma is essentially due to [Wil, 1.1]:

1.2 Lemma (Very ampleness lemma). *Let (X, Δ) be* klt. *Let H be Cartier, ample and $|H|$ base point free. Let N be a Cartier divisor such that $N - (K_X + \Delta)$ is nef. Then $(\dim X + 3)H + N$ is very ample.*

Proof. Pick $x_1, x_2 \in X$ and let S be a general member of $|H|$ containing both points. S may be singular, but the relevant vanishings descend from X to S and we can apply induction. □

Using some results of [Cat] the above estimates yield explicit bounds for the number of families of certain varieties. (In fact these applications follow already from [Dem].) Earlier finiteness results were due mostly to [Matsusaka 1, 2], however they did not give any explicit bound. The present bounds are certainly very far from being sharp. Therefore in writing the bounds I tried to make the expressions simple, instead of optimising the estimates.

1.3 Theorem. *The number of different irreducible families of n-dimensional smooth Fano varieties is at most*

$$(n+1)^{n^{(4n+3)^4}} .$$

1.4 Theorem. *The number of different irreducible families of n-dimensional smooth polarized varieties (X, L) with $L^n = d$ and $K_X \cdot L^{n-1} = \xi$ is at most*

$$(n^n(\xi + (n+2)d))^{(n+1)^{(4n+3)^3}(\xi+(n+2)d)^{(4n+3)^2}} .$$

Proof. If L is ample, then using (1.1) and (1.2) we produce a divisor which is very ample. Thus we obtain an embedding $X \to \mathbb{P}$ where we can control the degree of the image of X. By generic projection we can always assume that X is embedded into $\mathbb{P}^{2n+1}$. We can now utilize the following:

1.5 Theorem [Cat, 2.24]. *Let* $\mathrm{Hilb}^{\mathrm{smooth}}_{k,d}(\mathbb{P}^m)$ *be the Hilbert scheme parametrizing smooth and irreducible subvarieties of dimension k and degree d. Then the number of irreducible components of* $\mathrm{Hilb}^{\mathrm{smooth}}_{k,d}(\mathbb{P}^m)$ *is bounded by*

$$(dm+d)^{d^{2m+1}(m+1)^{2m}} .$$

In the Fano case we use $L = -K_X$ as our ample divisor. A bound for the selfintersection of $-K_X$ is provided in [KoMiMo]. The rest is just substituting into the above formulas.

In the general polarized case we use the divisor $D = K_X + (n+2)L$ which is ample (cf. [Fuj]). □

1.6 *Remark.* Using the results of [Mil, Thom] and (1.3–4) it is easy to write down explicit bounds for the sum of the Betti numbers of n-dimensional smooth Fano varieties or of smooth polarized varieties with fixed (d, ξ).

1.7 *Notation.*
(1.7.1) The abbreviation "lc" stands for log canonical, and "klt" for Kawamata log terminal [Ko et al., 2.13]. Note that klt is called log terminal in [KaMaMa].)
(1.7.2) A divisor D on a scheme X is called *nef* if $D \cdot C \geqq 0$ for every proper curve $C \subset X$.
(1.7.3) A divisor D on a proper scheme X is called *big* if $|mD|$ gives a birational map for $m \gg 1$. Thus ample implies nef and big.
(1.7.4) Let r be a real number. $\llcorner r \lrcorner$ (or $[r]$) is the largest integer $\leqq r$ and $\{r\} = r - \llcorner r \lrcorner$. $\llcorner r \lrcorner$ is called the integral part of r and $\{r\}$ the fractional part of r. If $D = \sum d_i D_i$ is a linear combination of divisors such that all the D_i are distinct and irreducible then define

$$\llcorner D \lrcorner = \sum \llcorner d_i \lrcorner D_i, \quad \text{and} \quad \{D\} = \sum \{d_i\} D_i .$$

(1.7.5) Bs $|D|$ denotes the base locus of the linear system $|D|$.

2. Effective base point freeness

Aside from the explicit coefficient, (1.1) is just the base point free theorem of [Kaw] and [Sho] (cf. [CKM, #9]). The proof of this result starts with some linear system $|mD|$ and at each step it decreases the base locus Bs$|mD|$ by increasing m. The usual method attacks the "largest multiplicity" point of the base locus (suitably measured), and therefore the number of necessary steps is unclear.

Here I develop a variant which attacks the largest dimensional part of Bs$|mD|$, thus we need at most $\dim X$ steps to ensure freeness. (See [CKM, #10] or [KaMaMa, 3-1] for the method which I follow closely.)

2.1 *Modified base point freeness method.*
(2.1.1) We are given a klt pair (X, Δ), a Cartier divisor N, a nef and big $\mathbb{Q}$-divisor M and an effective and nef $\mathbb{Q}$-divisor B. Assume that

$$N \equiv K_X + \Delta + B + M .$$

Our aim is to relate the singularities of B to sections of N.

Let $X \setminus W$ be the largest open set such that $(X, \Delta + B)$ is log canonical. Assume that $W \neq \emptyset$ and let Z be an irreducible component of W.

(2.1.2) Take a log resolution $f: Y \to X$ (i.e. Y is smooth and all relevant divisors are smooth and cross normally). Let

$$K_Y \equiv f^*(K_X + \Delta) + \sum e_i E_i, \quad (e_i > -1 \text{ by assumption}) ;$$

$$f^*B \equiv \sum b_i E_i ;$$

$$f^*M \equiv A + \sum p_i E_i \quad \text{where } A \text{ is an ample } \mathbb{Q}\text{-divisor and } 0 \leqq p_i \ll 1 .$$

For any real number c,

$$K_Y \equiv f^*(K_X + \Delta + cB) + \sum (e_i - cb_i)E_i .$$

We want to choose the largest value c such that $K_X + \Delta + cB$ is lc at the generic point of Z. For technical reasons we change the coefficients a little and set

$$c = \min \left\{ \frac{e_i + 1 - p_i}{b_i} | Z \subset f(E_i);\ b_i > 0 \right\} .$$

By changing the p_i slightly we may assume that the minimum is achieved for exactly one index. Let us denote the corresponding divisor by E_0.

(2.1.3) *Claim.* (2.1.3.1) $0 < c < 1$;
(2.1.3.2) $f(E_0) = Z$;
(2.1.3.3) *If* $cb_i - e_i + p_i < 0$ *then* E_i *is* f*-exceptional;*
(2.1.3.4) *If* $cb_i - e_i + p_i \geqq 1$ *and* $i \neq 0$ *then* $Z \not\subset f(E_i)$.

Proof. By assumption $(X, \Delta + B)$ is not lc at Z, thus $c < 1$. Therefore $(X, cB + \Delta)$ is klt outside W, thus $cb_i - e_i + p_i \geqq 1$ implies that $f(E_i) \subset W$. Since Z is an irreducible component of W, this shows (2.1.3.2) and (2.1.3.4).

If $cb_i - e_i + p_i < 0$ then $e_i > 0$ hence E_i is f-exceptional. □

(2.1.4) We can write

(2.1.4.1) $\quad f^*N \equiv K_Y + A + (1-c)f^*B + \sum (cb_i - e_i + p_i)E_i,$ and

(2.1.4.2) $\quad \sum \llcorner cb_i - e_i + p_i \lrcorner E_i = E_0 + H'' - H' ,$

where E_0, H', H'' are effective and without common irreducible components. By (2.1.3.3) and (2.1.3.4)

(2.1.4.3) $\quad H'$ is f-exceptional and $Z \not\subset f(H'')$.

(2.1.5) Set $N' = f^*N + H' - H''$ and consider the exact sequence

(2.1.5.1) $\quad 0 \to \mathcal{O}_Y(N' - E_0) \to \mathcal{O}_Y(N') \to \mathcal{O}_{E_0}(N') \to 0 .$

By construction

(2.1.5.2) $\quad N' - E_0 \equiv K_Y + A + (1-c)f^*B + \sum \{cb_i - e_i + p_i\} E_i ,$

thus $h^i(N' - E_0) = 0$ for $i \geqq 1$. In particular,

(2.1.5.3) $\quad H^0(Y, \mathcal{O}_Y(N')) \to H^0(E_0, \mathcal{O}_{E_0}(N'))$ is surjective .

Similarly,

(2.1.5.4) $\quad N'|E_0 \equiv K_{E_0} + (A + (1-c)f^*B)|E_0 + \sum \{cb_i - e_i + p_i\} E_i|E_0 ,$

thus $h^i(N'|E_0) = 0$ for $i \geqq 1$. Therefore

(2.1.5.5) $\quad h^0(E_0, \mathcal{O}_{E_0}(N')) = \chi(\mathcal{O}_{E_0}(N')) .$

In most applications M will be a variable divisor of the form $M_j = M_0 + jL$ where M_0 is nef and big and L is nef. If L is an actual line bundle then we get that

$$h^0(E_0, \mathcal{O}_{E_0}(N'_0 + jL)) = \chi(\mathcal{O}_{E_0}(N'_0 + jL))$$

is a polynomial in j for $j \geqq 0$. Thus it is nonzero for some value of j unless we are very unlucky. This is the point where one usually utilises the nonvanishing theorem. Unfortunately, in our case it does not apply because of the presence of H''.

(2.1.6) Assume for the moment that we established somehow that $h^0(E_0, \mathcal{O}_{E_0}(N')) \neq 0$. By (2.1.4) we can lift sections to $H^0(Y, \mathcal{O}(f^*N + H' - H''))$. Since $E_0 \not\subset \operatorname{Supp} H''$, we get a section $s \in H^0(Y, \mathcal{O}(f^*N + H'))$ which is not identically zero along E_0.

$H^0(Y, \mathcal{O}_Y(f^*N + H')) = H^0(X, \mathcal{O}_X(N))$ since H' is f-exceptional. Thus s descends to a section of $\mathcal{O}_X(N)$ which does not vanish along $Z = f(E_0)$. □

The following is the crucial technical result needed for (1.1):

2.2. Lemma. *Let $g: X \to S$ be a proper and surjective morphism with connected fibers. Assume that X is projective, S is normal and (X, Δ) is* klt *for some $\mathbb{Q}$-divisor Δ. Let D_S^0 be an ample Cartier divisor on S and let $D_S = mD_S^0$ for some $m > 0$. Let $D^0 = g^*D_S^0$ and $D = g^*D_S$. Assume that $aD^0 - (K_X + \Delta)$ is nef and big for some $a \geqq 0$. Assume that $|D_S| \neq \emptyset$ and let $Z_S \subset \operatorname{Bs}|D_S|$ be an irreducible component. Let $k = \operatorname{codim}(Z_S, S)$.*

Then, with at most $\dim Z_S$ *exceptions, $Z_S \not\subset \operatorname{Bs}|kD_S + (j + a + 1)D_S^0|$ for $j \geqq 0$.*

Proof. Pick general $B_i \in |D|$ and let

$$B = \frac{1}{2m} B_0 + B_1 + \ldots + B_k .$$

(2.2.1) *Claim. Notation as above.*
(2.2.1.1) $B \equiv \frac{1}{2}D^0 + kD$;
(2.2.1.2) $(X, \Delta + B)$ *is* lc *outside* $\operatorname{Bs}|D|$;
(2.2.1.3) $(X, \Delta + B)$ *is not* lc *at the generic points of* $g^{-1}(Z_S)$.

Proof. The first part is clear from the construction. If (X, F) is lc and H is a general member of a base point free linear system then $(X, F + H)$ is also lc. The general choice of the B_i implies the second claim.

In order to see the third part assume first that X is smooth. Let $W \subset g^{-1}(Z_S)$ be an irreducible component. Blowing up W we obtain an exceptional divisor E' whose discrepancy with respect to $(K + \Delta + B)$ is < -1. In the singular case we can use [Ko et al., 18.22]. □

We will apply the method of (2.1) with

$$N_j = kD + (j + a + 1)D^0$$

$$M_0 = aD^0 - (K_X + \Delta) + \frac{1}{2}D^0; \quad \text{and}$$

$$M_j = M_0 + jD^0 .$$

Instead of choosing Z directly, we concentrate on Z_S and set

$$c = \min\left\{\frac{e_i + 1 - p_i}{b_i} \,|\, Z_S \subset gf(E_i);\ b_i > 0\right\}.$$

By changing the p_i slightly we may assume that the minimum is achieved for exactly one index. Let us denote the corresponding divisor by E_0. As in (2.1.3.2) and (2.1.4.3) we conclude that $gf(E_0) = Z_S$ and $Z_S \not\subset gf(H'')$ where H', H'' are defined as in (2.1.4.2).

By (2.1.5) the crucial point is to show that

$$H^0(E_0, N'_j) = \chi(E_0, N'_j) = \chi(E_0, (gf)^*(kD_S + (j + a + 1)D_S^0) + H' - H''|E_0)$$

is not identically zero in j.

Let $G \subset E_0$ be a general fiber of $E_0 \to gf(E_0) = Z_S \subset S$. Then $G \cap H'' = \emptyset$, thus

$$N'_0|G = f^*g^*(kD_S + (a + 1)D_S^0) + H' - H''|G = H'|G .$$

Hence $(gf)_*(N'_0)$ is not the zero sheaf, and

$$H^0(E_0, \mathcal{O}(N'_j|E_0)) = H^0(Z_S, (gf)_*\mathcal{O}(N'_0) \otimes \mathcal{O}(D_S^0)^{\otimes j}) \neq 0 \quad \text{for } j \gg 1 .$$

Therefore, $h^0(E_0, \mathcal{O}_{E_0}(N'_j))$ is a nonzero polynomial of degree $\dim Z_S$ in j for $j \geqq 0$. Thus it can vanish for at most $\dim Z_S$ different values of j.

By (2.1) this implies that

$$f(E_0) \not\subset \mathrm{Bs}\,|kD + (j + a + 1)D^0| = g^{-1}\mathrm{Bs}\,|kD_S + (j + a + 1)D_S^0| .$$

Therefore $Z_S = gf(E_0) \not\subset \mathrm{Bs}\,|kD_S + (j + a + 1)D_S^0|$. This is what we wanted. □

(2.2.2) *Remark.* At first sight the $\dim Z_S$ exceptions in (2.2) are a small problem. However in general we would like to apply (2.2) to all the irreducible components of $\mathrm{Bs}\,|D_S|$ simultaneously. Thus if $\mathrm{Bs}\,|D_S|$ has lots of irreducible components, the exceptions may pile up and we may not be able to find a coefficient that works for every component.

It is quite likely that the conclusion of (2.2) can be replaced by

$$Z_S \not\subset \mathrm{Bs}\,|kD_S + (j + a + 1)D^0| \text{ for } j \geqq \dim Z_S .$$

This is true if $\dim Z_S \leqq 2$ [Reid].

The following result shows how to circumvent this problem at the expense of increasing the coefficients more.

2.3 Corollary. *Notation as in* (2.2). *Assume in addition that* $m \geqq a + \dim S$ *and set* $k = \mathrm{codim}(\mathrm{Bs}\,|D_S|, S)$. *Then*

$$\dim \mathrm{Bs}\,|(2k + 2)D_S| < \dim \mathrm{Bs}\,|D_S| .$$

Proof. Clearly $\mathrm{Bs}\,|(2k + 2)D_S| \subset \mathrm{Bs}\,|D_S|$.

Let Z_S be a maximal dimensional irreducible component of $\mathrm{Bs}\,|D_S|$. Then there is a value $0 \leqq j < \dim S$ such that Z_S is not in the base loci of

$$|kD_S + (j + a + 1)D^0| \quad \text{and} \quad |kD_S + (2m - j - a - 1)D^0| .$$

Thus Z_S is not in the base locus of

$$|kD_S + (j + a + 1)D^0 + kD_S + (2m - j - a - 1)D^0| = |(2k + 2)D_S| . \quad \square$$

2.4 *Proof of (1.1).* By the usual base point freeness we know that there exists a morphism $g: X \to S$ such that $L = g^*D_S^0$ for some Cartier divisor D_S^0.

By vanishing, $h^0(X, jL) = \chi(X, jL)$ for $j \geqq a$, thus $h^0(X, jL) \neq 0$ for $j \geqq a$ with at most $\dim S$ exceptions. As in (2.3) this implies that $h^0(X, 2(a+n)L) \neq 0$.

(2.3) can be used repeatedly to lower the dimension of $\mathrm{Bs}\,|mL|$. This way we obtain that

$$|2^{n+1}(n+1)!(a+n)L|$$

is base point free. This is slightly weaker than (1.1) but for the applications (1.4-5) this does not matter. The coefficient will be improved in (3.6). □

3. Effective nonvanishing

The aim of this section is to derive a rather weak nonvanishing result in the spirit of (2.5.1) which can be used to improve the bound obtained in (2.4). Other consequences will be discussed in [Ko1, 2].

3.1 Definition. Let X be a variety. A subvariety $\mathbf{Z} \subset T \times X$ is called a *covering family of* X if T is irreducible, the second projection $p_X: \mathbf{Z} \to X$ is dominant and the first projection $p_T: \mathbf{Z} \to T$ is proper and flat with irreducible and reduced fibers.

We will frequently denote a covering family by $\{Z_t\}$ where $\{Z_t : t \in T\}$ supposed to run through all fibers of $p_T: \mathbf{Z} \to T$. While this is somewhat ambiguous, in the present situation this will not cause any problems.

By the countability of Hilbert scheme we know that there are countably many divisors $D_i \subset X$ such that if $Z \subset X$ is an irreducible and reduced subvariety such that $Z \not\subset \bigcup D_i$ then Z occurs as a fiber in a covering family.

A point $x \in A \setminus \bigcup D_i$ will be referred to as a very general point of X.

3.2. Theorem. *Let $g: X \to S$ be a surjective morphism, X smooth and projective. Let $U \subset S$ be a dense open set. Let L be a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S, N a Cartier divisor on X, M and Δ $\mathbb{Q}$-divisors on X. Assume that:*

(3.2.1) $\mathrm{Supp}\,\Delta$ *is a normal crossing divisor and* $\llcorner \Delta \lrcorner = \emptyset$;
(3.2.2) *If $\{Z_t\}$ is a covering family then* $L^{\dim Z_t} \cdot Z_t \geqq 1$;
(3.2.3) $N|g^{-1}(U)$ *is linearly equivalent to an effective divisor;*
(3.2.4) M *is nef and either big on the general fiber of g or numerically trivial on* X;
(3.2.5) $N \equiv K_X + \Delta + M + sg^*L$ *for some* $s > \binom{\dim S + 1}{2}$.

Then $h^0(X, N) \neq 0$. More generally, if X_g is the generic fiber of g then $H^0(X, N) \to H^0(X_g, N|X_g)$ is surjective.

We will need the following:

3.3 Theorem [Ko 3, 2.1, 2.2; EsnVie, 1.12, 3.1]. *Let $g: X \to S$ be a surjective morphism, X smooth and projective. Let L be a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S, N a Cartier divisor on X and Δ an effective $\mathbb{Q}$-divisor on X such that* $\llcorner \Delta \lrcorner = \emptyset$ *and* $\mathrm{Supp}\,\Delta$ *has normal crossings only. Assume that $N \equiv K_X + \Delta + g^*L$. Then*

(3.3.1) $H^j(S, R^i g_* \mathcal{O}_X(N)) = 0$ *for* $j > 0$, $i \geqq 0$;
(3.3.2) $H^j(X, \mathcal{O}_X(N)) \to H^j(X, \mathcal{O}_X(N + D))$ *is injective if $g: D \to S$ is not dominant.*

3.4 *Proof of (3.2).* We use induction on $\dim S$. If $\dim S = 0$ then $U = S$ hence we are done by (3.2.3).

By shrinking U we may assume that $g:(g^{-1}(U), \Delta) \to U$ is log smooth (i.e. g is smooth on $g^{-1}(U)$, on $\Delta_i \cap g^{-1}(U)$ for every irreducible $\Delta_i \subset \Delta$, on $\Delta_i \cap \Delta_j \cap g^{-1}(U)$ for every irreducible $\Delta_i, \Delta_j \subset \Delta$, etc.)

If $\dim S \geqq 1$ then pick a very general point $x \in U$. Let B be a $\mathbb{Q}$-divisor on S such that $mB \in |mL|$ for some $m \gg 1$ and $\operatorname{mult}_x B \geqq 1 - \varepsilon_B$. (From now on ε with a subscript stands for a very small positive number.)

Choose a log resolution $f_S: Y_S \to S$ and write

$$\varepsilon_L f_S^* L = A + \sum{}' p_i F_i, \quad A \text{ ample}, 0 \leqq p_i \ll 1 . \tag{3.4.1}$$

(Here the $\sum'$ is supposed to remind one that the index set of this sum is not the same as the index set of subsequent sums without $'$.) We may assume that the first step in constructing the resolution was to blow up x. The corresponding divisor will be denoted by F.

Now consider $Y_S \times_S X \to X$. This is a log resolution of $(X, \Delta + g^* B)$ over $g^{-1}(U)$. By further blow-ups outside $g^{-1}(U)$ we can make it into a resolution

$$\begin{array}{ccccc} f: Y & \longrightarrow & Y_S \times_S X & \longrightarrow & X \\ {\scriptstyle q}\downarrow & & \downarrow & & \\ Y_S & =\!= & Y_S . & & \end{array} \tag{3.4.2}$$

Let $E_i = q^\circ(F_i) \subset Y$ be the unique irreducible component of $q^{-1}(F_i)$ which dominates $F_i \cap f_S^{-1}(U)$. For notational simplicity we will denote many other divisors on Y by E_j, we drop the $'$ from the sum notation to indicate this. There will be three kinds of divisors denoted by E_i:

(i) $q^\circ(F_i)$; these will be called U-divisors;

(ii) the proper transform of Δ_i where $x \in g(\Delta_i)$; these will be called Δ-divisors;

(iii) all the other E_j will have the property that $gf(E_j) \subset S \setminus U$. Such divisors will be called negligible.

With this convention in mind let

$$\begin{aligned} K_Y &\equiv f^*(K_X + \Delta) + \sum e_i E_i , \\ f^* g^* B &\equiv \sum b_i E_i ; \\ \varepsilon_L f^* g^* L &\equiv q^* A + \sum p_i E_i . \end{aligned} \tag{3.4.3}$$

Note that the coefficient p_i for a U-divisor $E_i = q^\circ(F_i)$ is the same as the p_i in the formula (3.4.1). We can write

$$f^* N \equiv K_Y + (s - c - \varepsilon_L) f^* g^* L + f^* M + q^* A + \sum (c b_i - e_i + p_i) E_i . \tag{3.4.4}$$

Set

$$c = \min \left\{ \frac{e_i + 1 - p_i}{b_i} \,|\, x \in gf(E_i);\, b_i > 0 \right\} . \tag{3.4.5}$$

If E_j is negligible then $x \notin gf(E_j)$ and if E_j is a Δ-divisor then $gf(E_j) = S$ thus $b_j = 0$. Therefore the value of c is determined by the coefficients of the U-divisors in (3.4.3).

By changing the p_i slightly we may assume that the minimum is achieved for exactly one index. Let us denote the corresponding divisor by E_0. By looking at the divisor $q^\circ(F)$ we conclude that $c \leqq \dim S + \varepsilon_S$. Let

$$\sum \llcorner c b_i - e_i + p_i \lrcorner E_i = E_0 + H'' - H' , \tag{3.4.6}$$

where E_0, H', H'' are effective and without common irreducible components. If E_j is a Δ-divisor then $b_j = 0$ and $0 > e_j > -1$, thus $\llcorner cb_j - e_j + p_j \lrcorner = 0$. As in (2.1.4.3) we get that

$$H' \text{ is } f\text{-exceptional and } \quad x \notin gf(H''). \tag{3.4.7}$$

Set $N' = f^*N + H' - H''$ and consider the exact sequence

$$0 \to \mathcal{O}_Y(N' - E_0) \to \mathcal{O}_Y(N') \to \mathcal{O}_{E_0}(N') \to 0. \tag{3.4.8}$$

By construction

$$\begin{aligned} N'|E_0 \equiv K_{E_0} &+ ((s - c - \varepsilon_L)f^*g^*L + f^*M + q^*A)|E_0 \\ &+ \sum \{cb_i - e_i + p_i\} E_i|E_0. \end{aligned} \tag{3.4.9}$$

Set

$$\begin{aligned} X_0 &= E_0, S_0 = q(E_0) \quad \text{and} \quad g_0 = q|E_0, \\ L_0 &= (1 - \varepsilon_0)\binom{\dim S}{2}^{-1} ((s - c - \varepsilon_L)f_S^*L + A)|E_0 \\ N_0 &= N'|E_0, \\ M_0 &= f^*M|E_0, \\ \Delta_0 &= \sum \{cb_i - e_i + p_i\} E_i|E_0. \end{aligned} \tag{3.4.10}$$

We claim that all the conditions of (3.2) are satisfied by X_0, S_0, etc.

(3.2.1) is clear.

$$(1 - \varepsilon_0)(s - c - \varepsilon_L) \geqq (1 - \varepsilon_0)(s - \dim S - \varepsilon_L - \varepsilon_S) > \binom{\dim S}{2}$$

thus $L_0 - L$ is nef hence (3.2.2) also holds. (3.2.4-5) hold by definition. Finally if $G_0 \subset X_0$ is the generic fiber of g_0, then $N'|G = f^*N + H' - H''|G = f^*N|G + H'|G$ is effective.

Thus by induction

$$h^0(E_0, \mathcal{O}_{E_0}(N')) > 0. \tag{3.4.11}$$

Furthermore

$$\begin{aligned} N' - E_0 \equiv K_Y &+ (s - c - \varepsilon_L)f^*g^*L + f^*M + q^*A \\ &+ \sum \{cb_i - e_i + p_i\} E_i. \end{aligned} \tag{3.4.12}$$

If M is big on the general fiber of g then $f^*M + q^*A$ is nef and big, hence $h^i(N' - E_0) = 0$ for $i \geqq 1$. In particular,

$$H^0(Y, \mathcal{O}_Y(N')) \to H^0(E_0, \mathcal{O}_{E_0}(N')) \tag{3.4.13}$$

is surjective and we are done.

If M is numerically trivial then

$$H^1(Y, \mathcal{O}(N' - E_0)) \to H^1(Y, \mathcal{O}(N'))$$

is injective by (3.3) since

(3.4.14) $$(s - c - \varepsilon_L) f^* g^* L + q^* A = q^*((s - c - \varepsilon_L) f_S^* L + A)$$

is the pull back of a nef and big divisor from Y_S and $E_0 \subset \operatorname{Supp} q^*(f_S^* B)$). Thus again the morphism (3.4.13) is surjective. □

3.5 Corollary. *Notation as in* (2.2). *Assume in addition that* $m \geqq a + 1 + \binom{\dim S + 1}{2}$ *and set* $k = \operatorname{codim}(\operatorname{Bs}|D_S|, S)$. *Then*

$$\dim \operatorname{Bs}|(k+1)D_S| < \dim \operatorname{Bs}|D_S| .$$

Proof. We need to apply (3.2) instead of (2.3) in the method (2.1). □

3.6 *Proof of (1.1).* The argument is essentially the same as in (2.4). First we apply (2.3) until we reach m large enough and then apply (3.5) instead of (2.3). □

Acknowledgement. I would like to thank R. Lazarsfeld for showing me early versions of [EinLaz] and F. Catanese for correcting my original incorrect version of (1.5). Partial financial support was provided by the NSF under grant numbers DMS-8707320 and DMS-8946082 and by an A.P. Sloan Research Fellowship.

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