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Blichfeldt's density bound revisited

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1 Introduction

Let C be a compact set with nonempty interior, a *body*, for short, in the d-dimensional Euclidean space E^d. A *packing* with C is a collection of congruent copies of C with mutually disjoint interiors. The (upper) *density* of a packing $\mathscr{P}$ is defined by an appropriate limit [3, 12] and is, roughly speaking, the portion of the volume of the space which is occupied by the members of $\mathscr{P}$. the *packing density* $\delta(C)$ of C is the supremum of the upper densities of all packings with C. One of the principal problems in the theory of packing is to determine the packing density of specific convex bodies, or, as this goal is out of reach of available methods, to give upper bounds for $\delta(C)$.

One of the first results in this direction is due to Blichfeldt [2] who derived the upper bound

$$\delta(B^d) \leqq \frac{d+2}{2}\, 2^{-d/2}$$

for the packing density of the unit ball B^d in E^d. Blichfeldt's result has been subsequently improved by several authors [10, 11, 13, 8, 7], the best bound presently known being

$$\delta(B^d) \leqq 2^{-d \cdot 0.599 + o(d)}\,,$$

established by Kabatjanskiĭ and Levenštein [7].

In this paper we point out that even though Blichfeldt's bound for the packing density of B^d has been superseded by now, Blichfeldt's method of proving it can be successfully applied to more general cases. In the next section we start by recalling the main idea of Blichfeldt's method.

2 Blichfeldt gauges

Let C be a body in E^d and suppose that $\{\varphi_i\}_{i=1}^{\infty}$ is a collection of isometries such that $\{\varphi_i C\}_{i=1}^{\infty}$ is a packing. We associate with C a material body $\widehat{C}$ described by a real valued Lebesgue integrable function $f(x)$ giving the density of mass at the point $x \in E^d$. The total mass of the body $\widehat{C}$ defined by the density function $f(x)$ is thus given by the integral

$$I(f) = \int_{E^d} f(x)dx\,.$$

We replace each member $\varphi_i C$ of the packing by the corresponding material body $\varphi_i \widehat{C}$ whose density function is $f(\varphi_i^{-1}x)$, where φ_i^{-1} denotes the inverse of the isometry φ_i. We can define the density Δ of the arrangement of the material bodies $\{\varphi_i \widehat{C}\}_{i=1}^{\infty}$ in E^d in almost the same manner as the density of any arrangement of bodies is defined (see, e.g., [3, pp. 55–56]). The only modification needed is the use of the mass $I(f)$ in place of the volume $V(C)$. It is easily seen that the density Δ is related to the (volume) density δ of the packing $\{\varphi_i C\}_{i=1}^{\infty}$ and the volume of C through the equation

$$\Delta = \delta\,\frac{I(f)}{V(C)}\,.$$

Blichfeldt's idea was to introduce a density function for which the density of mass yielded by the arrangement of bodies $\{\varphi_i \widehat{C}\}_{i=1}^{\infty}$ is bounded pointwise. Since the contribution of $\varphi_i \widehat{C}$ to the density of mass at the point x is $f(\varphi_i^{-1}x)$, the total density of mass at x is $\sum_{i=1}^{\infty} f(\varphi_i^{-1}x)$. We say that $f(x)$ is a *Blichfeldt gauge* for the convex body $C \in E^d$, if for any collection $\{\varphi_i\}_{i=1}^{\infty}$ of isometries of E^d such that $\{\varphi_i C\}_{i=1}^{\infty}$ is a packing, we have

$$\sum_{i=1}^{\infty} f(\varphi_i^{-1}x) \leqq 1$$

for all $x \in E^d$. If $f(x)$ is a Blichfeldt gauge, then, of course, $1 \geqq \Delta$. Hence we get the following

Theorem of Blichfeldt. *If $f(x)$ is a Blichfeldt gauge for C, then*

$$\delta(C) \leqq \frac{V(C)}{I(f)}\,.$$

Blichfeldt proved this result only for the case when C is a ball but the proof in the general case is straightforward (see [6, p. 353] for a proof for packing with translates of C).

If C possesses some symmetry, then it is natural to consider gauges f which are invariant under the group of symmetries of C. In fact, it is easily seen that considering gauges more general than those which share the group of symmetry with C cannot improve the bound in Blichfeldt's theorem. In particular, we assume that in the case of the unit ball the value of $f(x)$ depends only on the distance $|x|$ of x from the origin. Blichfeldt obtained his bound by using the gauge

$$f_0(x) = \begin{cases} 1 - \frac{1}{2}|x|^{1/2} & \text{for } |x| \leqq \sqrt{2}\,, \\ 0\,, & \text{for } |x| > \sqrt{2}\,, \end{cases}$$

and pointed out that with the modified gauge

$$f(x) = \begin{cases} f_0(x) & \text{for } |x| \geqq 1\,, \\ 1 - f_0(2 - x) & \text{for } |x| \leqq 1 \end{cases}$$

one can slightly improve on that bound, obtaining the following:

$$\delta(B^d) \leqq \left[\frac{2}{d+2}(\sqrt{2})^d(1+b_d)\right]^{-1},$$

where

$$b_d = \frac{1}{(\sqrt{2})^d(d+1)} - (\sqrt{2}-1)^{d+1}\left(1 + \frac{\sqrt{2}}{d+1}\right).$$

A further improvement was achieved by Rankin [10] by means of a very intricate gauge, depending on d. However, the fact that each of Blichfeldt's gauges f_0 and f is independent from d makes them more useful for our purpose, even though Rankin's gauge yields a better bound for ball packings.

For a body C and a positive real number λ let λC denote the set consisting of the points of the form λx, $x \in C$. In particular, λB^d is the ball of radius λ centered at the origin. We observe that if $f(x)$ is a Blichfeldt gauge for C, then $f(x/\lambda)$ is a Blichfeldt gauge for λC. Let $r(C)$ denote the insphere radius of C. For a positive number $\varrho \leqq r(C)$ let $C_{-\varrho}$ denote the inner parallel body of C at distance ϱ, that is $C_{-\varrho}$ consists of the points x for which $x + \varrho B^d \subset C$. For a body C and a point x we denote by $d(x, C)$ the distance from x to a nearest point of C. Let $f(x)$ be a Blichfeldt gauge for the unit ball B^d such that the value of $f(x)$ depends only on $|x|$. Thus $f(x) = h(|x|)$, where $h(\alpha)$ is a measurable function defined for non-negative reals. For a fixed value of $\varrho \leqq r(C)$ we define the function $g(x)$ as

$$g(x) = h\left(\frac{d(x, C_{-\varrho})}{\varrho}\right).$$

Suppose that $\{\varphi_i\}_{i=1}^{\infty}$ is a collection of isometries such that $\{\varphi_i C\}_{i=1}^{\infty}$ is a packing. Let x be an arbitrary point in space. Let x_i be a point of $\varphi_i C_{-\varrho} = (\varphi_i C)_{-\varrho}$ at distance $d(x, \varphi_i C_{-\varrho})$ from x, and denote by τ_i the translation through the vector $\varphi_i^{-1} x_i$. Further let $\psi_i = \varphi_i \tau_i$ the composition of τ_i and φ_i. We observe that the image of the origin under the isometry ψ_i is x_i. Therefore $\psi_i(\varrho B^d)$ is the ball of radius ϱ centered at x_i. As $x_i \in \varphi_i C_{-\varrho}$, it follows by the definition of the inner parallel body that $\psi_i(\varrho B^d) \subset \varphi_i C$. Since the sets $\{\varphi_i C\}_{i=1}^{\infty}$ constitute a packing, so do the balls $\{\psi_i(\varrho B^d)\}_{i=1}^{\infty}$. Since, furthermore, $f(x/\varrho)$ is a Blichfeldt gauge for ϱB^d, we have

$$\sum_{i=1}^{\infty} f\left(\frac{\psi_i^{-1} x}{\varrho}\right) \leqq 1\,.$$

As we observed above, we have $\psi_i^{-1} x_i = 0$. Therefore

$$|\psi_i^{-1} x| = |\psi_i^{-1} x - \psi_i^{-1} x_i| = |x - x_i| = d(x, \varphi_i C_{-\varrho}) = d(\varphi_i^{-1} x, C_{-\varrho})\,.$$

Thus

$$f\left(\frac{\psi_i^{-1} x}{\varrho}\right) = h\left(\left|\frac{\psi_i^{-1} x}{\varrho}\right|\right) = h\left(\frac{d(\varphi_i^{-1} x, C_{-\varrho})}{\varrho}\right) = g(\varphi_i^{-1} x)\,,$$

and consequently

$$\sum_{i=1}^{\infty} g(\varphi_i^{-1}x) \leqq 1 .$$

Thus we obtained the following

Theorem. *If $h(\alpha)$, $\alpha \geqq 0$, is a real valued function such that $h(|x|)$ is a Blichfeldt gauge for the unit ball, and C is a convex body with insphere radius $r(C)$, then for any $\varrho \leqq r(C)$*

$$g(x) = h\left(\frac{d(x, C_{-\varrho})}{\varrho}\right)$$

is a Blichfeldt gauge for C.

This theorem yields reasonably good upper bounds for the packing density of convex bodies which are either the outer parallel body of a convex body of dimension lower than d or are obtained as the Cartesian product of a ball and a body of lower dimension.

3 Applications

Utilizing the idea of the previous section, we will find some upper bounds for the packing density of two types of bodies in E^d, namely for an outer parallel body of a segment (a sausage-like body) and for a cylinder whose base is the unit ball B^{d-1}. In the computations that follow, we will use the Blichfeldt unit-ball gauge $f(x)$ mentioned above. Under this gauge, we denote

$$A_d = \frac{I(f)}{V(B^d)}$$

and we compute that

$$A_d = \frac{2}{d+2}(\sqrt{2})^d(1 + b_d) .$$

In particular, for $d = 2$, 3, and 4, we get

$$A_2 = (29 - 16\sqrt{2})/6 = 1.062097\ldots ,$$
$$A_3 = (25 - 16\sqrt{2})/2 = 1.186291\ldots ,$$

and

$$A_4 = (609 - 416\sqrt{2})/15 = 1.379143\ldots .$$

A. Packing sausages. Let S_h^d be the outer parallel body of radius 1 of a segment of length h in E^d. Applying the theorem for $K = S_h^d$, $\varrho = 1$, the inradius of S_h^d, and the Blichfeldt gauge $f(x)$ mentioned above we readily get

$$I(g) = hA_{d-1}V(B^{d-1}) + A_d V(B^d) ,$$

while

$$V(S_h^d) = hV(B^{d-1}) + V(B^d) .$$

Thus

$$\delta(S_h^d) \leq \frac{hV(B^{d-1}) + V(B^d)}{hA_{d-1}V(B^{d-1}) + A_d V(B^d)} .$$

In particular, setting $d = 3$ and $d = 4$ we get

$$\delta(S_h^3) \leqq \frac{hV(B^2) + V(B^3)}{hA_2V(B^2) + A_3V(B^3)} = \frac{3h+4}{3hA_2 + 4A_3}$$

and

$$\delta(S_h^4) \leqq \frac{hV(B^3) + V(B^4)}{hA_3V(B^3) + A_4V(B^4)} = \frac{8h + 4\pi}{8hA_3 + 3\pi A_4} .$$

Setting $h = 1$ we get:

$$\delta(S_1^3) \leqq \frac{7}{3A_2 + 4A_3} = 0.88256\ldots$$

and

$$\delta(S_1^4) \leqq \frac{8 + 3\pi}{8A_3 + 3\pi A_4} = 0.77483\ldots .$$

For comparison, a certain lattice packing of E^3 with S_1^3 is of density

$$\frac{7\pi}{6\sqrt{3} + 12\sqrt{2}} = 0.8036\ldots$$

and we conjecture it to be the densest. We conjecture the same for the lattice packing of E^4 with S_1^4 whose density is

$$\frac{3\pi + 8}{24\sqrt{2} + 48}\pi = 0.6680\ldots .$$

B. Packing cylinders. Let C_h^d be the cylinder of radius 1 and height h in E^d, more precisely

$$C_h^d = \left\{ (x_1, x_2, \ldots, x_d) \colon \sum_{i=1}^{d-1} x_i^2 \leqq 1,\ 0 \leqq x_d \leqq h \right\}.$$

We assume $h \geqq 2$ to insure that the inradius of C_h^d is equal to 1. An upper bound for $\delta(C_h^d)$ is obtained by an application of the theorem, using the modified Blichfeldt gauge $f(x)$ as described in Sect. 2 and setting $\varrho = 1$. Observe that in this case $V(C_h^d) = hV(B^{d-1})$ and $I(g) = (h-2)A_{d-1}V(B^{d-1}) + A_dV(B^d)$, which yields

$$\delta(C_h^d) \leqq \frac{hV(B^{d-1})}{(h-2)A_{d-1}V(B^{d-1}) + A_dV(B^d)} .$$

Setting $d = 3$ we get

$$\delta(C_h^3) \leqq \frac{h}{(h-2)A_2 + \frac{4}{3}A_3} ,$$

which gives a meaningful (smaller than 1) upper bound only when h is sufficiently large, namely when

$$h > \frac{2A_2 - \frac{4}{3}A_3}{A_2 - 1} = 8.73\ldots .$$

For $d = 4$ we get

$$\delta(C_h^4) \leqq \frac{h}{(h-2)A_3 + \frac{3}{8}\pi A_4} ,$$

which gives a meaningful upper bound when

$$h > \frac{2A_3 - \frac{3}{8}\pi A_4}{A_3 - 1} = 4.01\ldots .$$

Of course, one might ask if it is true that $\delta(C_h^d) = \delta(B^{d-1})$ for all d and all h, but the answer is not known for any $d \geqq 3$, not even for a single value of h.

Let C_∞^d denote the cylinder of radius 1 in E^d infinite in both directions. C_∞^d is not compact, but its packing density $\delta(C_\infty^d)$ can be defined by an appropriate limit in a similar way as the packing density of compact sets. It is easily seen that $\delta(C_\infty^d) = \lim_{h\to\infty} \delta(S_h^d) = \lim_{h\to\infty} \delta(C_h^d)$. As $h \to \infty$, the upper bound presented here for each of $\delta(S_h^d)$ and $\delta(C_h^d)$ approaches $1/A_{d-1}$, which is the Blichfeldt upper bound for $\delta(B^{d-1})$. Hence

$$\delta(C_\infty^d) \leqq 1/A_{d-1} .$$

The conjecture that

$$\delta(C_\infty^d) = \delta(B^{d-1})$$

(for all d) has been confirmed for $d = 3$ by Bezdek and Kuperberg [1].

4 Remarks

In high dimensions. Let $M(d, \varphi)$ be the maximum number of points on the $(d-1)$-dimensional spherical space S^{d-1}, the boundary of B^d, all of whose mutual angular (geodesic) distances are greater than or equal to φ. Equivalently, $M(d, \varphi)$ is the maximum number of points in E^d such that the angle spanned by any two points at the origin is at least φ.

For dimensions greater than about 50, better bounds for the packing density of balls can be obtained using the observation of Levenštein [9, p. 108], which states that for $1 < \lambda < 2$, the function

$$f_d(x) = \begin{cases} (M(d, \arccos(1 - 2/\lambda^2)))^{-1} & \text{for } |x| < \lambda, \\ 0 & \text{for } |x| \geqq \lambda \end{cases}$$

is a Blichfeldt gauge for B^d.

In order to see this we have to show that for any packing of unit balls in E^d there are at most $M(d, \arccos(1 - 2/\lambda^2))$ centers of balls at a distance of λ or less $(\lambda < 2)$ from a given point $x \in E^d$. Indeed, the distance between any two centers is at least 2, and an easy computation shows that the angle at x spanned by a segment of length greater than or equal to 2 whose ends are within λ from x is at least $\arccos(1 - 2/\lambda^2)$.

Using the Blichfeldt gauge $f_d(x)$ we obtain

$$\delta(B^d) \leqq \lambda^{-d} M(d, \arccos(1 - 2/\lambda^2)),$$

for $1 < \lambda < 2$, or, equivalently,

$$\delta(B^d) \leqq (\sin\varphi/2)^d M(d, \varphi)$$

for $\frac{\pi}{3} < \varphi < \pi$.

For large values of d, the best known upper bound for $M(d, \varphi)$ is

$$M(d, \varphi) \leqq (\sin\varphi/2)^{-d} 2^{-d \cdot 0.599 + o(d)} \qquad (\varphi \leqq 63^\circ),$$

due to Kabatjanskiĭ and Levenštein [7]. This yields the inequality

$$\delta(B^d) \leqq 2^{-d \cdot 0.599 + o(d)}$$

mentioned in the introduction. Using our theorem we obtain the same asymptotic upper bound for the packing density of infinite cylinders in E^{d+1}:

$$\delta(C_\infty^{d+1}) \leqq 2^{-d \cdot 0.599 + o(d)} .$$

Multiple packings. A collection of bodies is called a *k-fold packing* if each point of the space belongs to the interior of at most k bodies. The *k-fold packing density* $\delta^k(C)$ of C is defined as the supremum of the upper densities of all k-fold packings with congruent copies of C. Blichfeldt's original idea can be extended in a straightforward manner to obtain upper bounds for $\delta^k(C)$. Few [4, 5] obtained upper bounds for $\delta^k(B^d)$ in this very way. Just as well as the theorem of Blichfeldt, our theorem can be readily generalized for the case of multiple packings. Using Few's results, the density bounds of Sect. 3 can be extended to multiple packings of sausages and cylinders.

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