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Pure quasi-states and extremal quasi-measures

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1 Introduction

In this paper we continue the study of quasi-states and quasi-measures initiated in [1].

Throughout X will denote a compact Hausdorff space and $\mathbf{A} = C(X)$ is the space of real-valued continuous functions on X. For $a \in \mathbf{A}$ we let $\mathbf{A}(a)$ denote the smallest uniformly closed subalgebra of $\mathbf{A}$ containing a and 1. A function $\rho: \mathbf{A} \to \mathbb{R}$ satisfying $\rho(1) = 1$, $\rho(a) \geqq 0$ if $a \geqq 0$ and such that ρ is linear on $\mathbf{A}(a)$ for each $a \in \mathbf{A}$ is called a *quasi*-state.

Let $\mathscr{C}$ denote the collection of closed subsets of X, let $\mathscr{O}$ denote the collection of open subsets of X and put $\mathscr{A} = \mathscr{C} \cup \mathscr{O}$. A real-valued, non-negative function μ on $\mathscr{A}$ is called a *quasi-measure* in X if the following conditions are satisfied:

(1) $\mu(K) + \mu(X \backslash K) = \mu(X)$; $K \in \mathscr{C}$
(2) $K_1 \subseteq K_2 \Rightarrow \mu(K_1) \leqq \mu(K_2)$; $K_1, K_2 \in \mathscr{C}$
(3) $K_1 \cap K_2 = \emptyset \Rightarrow \mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$; $K_1, K_2 \in \mathscr{C}$
(4) $\mu(U) = \sup\{\mu(K): K \subseteq U; K \in \mathscr{C}\}$; $U \in \mathscr{O}$.

In [1] we established a 1-1 correspondance between quasi-measures and quasi-states (Theorem 4.1). We also showed that non-linear quasi-states really exist by exhibiting a quasi-measure which is not (the restriction of) a regular Borel-measure [1, Proposition 6.1]. In [2] and [6] more general procedures for the construction of quasi-measures are discussed. A quasi-measure is called *extremal* if it only takes the values 0 and 1. This paper is devoted to a close study of the properties of extremal quasi-measures and their corresponding quasi-states, which are called *simple*. The set of all quasi-states is a convex set denoted by Q, which is compact in the topology of pointwise convergence on $\mathbf{A}$. The set E of simple

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quasi-states is a proper subset of the set Q_e of extreme points in Q. The crucial property, however, is that a quasi-state is simple if and only if it is multiplicative on $\mathbf{A}(a)$ for each $a \in \mathbf{A}$. This enables us to show that E is closed in Q and therefore is a compact Hausdorff space. In turn this makes it possible to establish a "non-linear Gelfand-transform" Ψ of $\mathbf{A}$ into $C(E)$ which is discussed in Sect. 4 of the present paper. This transform enables us to show that each quasi-state ρ in the closed convex hull of E may be factored as $\rho = p \circ \Psi$, where p is an ordinary linear state on $C(E)$. In general this factorization is non-unique, as shown in an example towards the end of the last section. This non-uniqueness reflects that the order-structure of the positive cone generated by Q generally is quite complicated, and is closely bound up with the topological properties of the space X. These questions will be pursued in another paper. Our notation follows that of [1], where we also refer the reader for further background.

2 Pure quasi-states and extremal quasi-measures

A quasi-state ρ is *pure* if $0 \leqq \rho' \leqq \rho$ for any positive quasi-linear functional ρ' on $\mathbf{A}$ implies that $\rho' = r\rho$; $0 \leqq r \leqq 1, r \in \mathbb{R}$. In [3, Proposition 2.2] it was proved that a quasi-state is pure if and only if it is an extreme point of Q. For brevity let us say that a quasi-state is *simple* if its restriction to any singly generated subalgebra $\mathbf{A}(a)$ is multiplicative. In the linear case, i.e. if ρ is a pure state on $\mathbf{A}$, then ρ is multiplicative on $\mathbf{A}$ and a fortiori simple. In our situation, if ρ is a pure quasi-state on $\mathbf{A}$, it is therefore natural to ask whether ρ is simple. We shall provide an example towards the end of this section which shows that this is generally not so. We first want to characterize the simple quasi-states and their corresponding quasi-measures.

Let $\mathbf{A}'$ denote the algebra of all complex-valued continuous functions on X. A singly generated subalgebra of $\mathbf{A}'$ is a closed subalgebra generated by 1 and a single *real-valued* function $a \in \mathbf{A}'$. (We regard $\mathbf{A}$ as contained in $\mathbf{A}'$.) We denote this subalgebra by $\mathbf{A}'(a)$, and we clearly have $\mathbf{A}'(a) = \{b + ic : b, c \in \mathbf{A}(a)\}$. If ρ is a quasi-state on $\mathbf{A}$ we define ρ' on $\mathbf{A}'$ by $\rho'(c) = \rho(a) + i\rho(b)$, where $c = a + ib$; $a, b \in \mathbf{A}$ is the decomposition of an element $c \in \mathbf{A}'$ into its real and imaginary parts.

Theorem 2.1. *Let ρ be a quasi-state on $\mathbf{A}$. The following statements are equivalent:*

(1) *ρ is simple.*

(2) *$\rho'(c) \neq 0$ if c is an invertible element of $\mathbf{A}'$ belonging to some singly generated subalgebra.*

(3) *$\rho'(c) \in \operatorname{Sp} c$ for all $c \in \mathbf{A}'$ belonging to some singly generated subalgebra.*

(4) *If $a, b \in \mathbf{A}$ belong to the same singly generated subalgebra of $\mathbf{A}$, and satisfy $\rho(a) = \rho(b) = 0$, then $a^{-1}(\{0\}) \cap b^{-1}(\{0\}) \neq \varnothing$.*

(5) *If $a, b \in \mathbf{A}$ belong to the same singly generated subalgebra of $\mathbf{A}$, then $a^{-1}(\{\rho(a)\}) \cap b^{-1}(\{\rho(b)\}) \neq \varnothing$.*

Moreover, each of these conditions imply that ρ is pure.

Proof. (5) $\Rightarrow$ (1). Let $a \in \mathbf{A}$ be arbitrary, and let μ_a be the probability measure in $\operatorname{Sp} a$ corresponding to the state $\phi \to \rho(\phi(a))$ on $C(\operatorname{Sp} a)$ (cf. [1, Theorem 4.1]). To show that ρ is simple it suffices to show that μ_a is a point-measure. Suppose that this is not the case. Then there are functions $0 \leqq \phi, \psi \in C(\operatorname{Sp} a)$ such that $\phi \cdot \psi = 0$ and $\rho(\phi(a)) = \rho(\psi(a)) = 1$. However, if (5) holds there must then be an $x \in X$ such that $\phi(a(x)) = \psi(a(x)) = 1$. Since $\phi \cdot \psi = 0$ this is impossible. The assertion follows.

(4) $\Rightarrow$ (5). Suppose a, b belong to the same singly generated subalgebra $\mathbf{A}(c)$ of $\mathbf{A}$ and that $\rho(a) = r$, $\rho(b) = s$. Then $a' = r1 - a$ and $b' = s1 - b$ belong to $\mathbf{A}(c)$ and satisfy $\rho(a') = \rho(b') = 0$. Assuming (4) there is an $x \in X$ such that $a'(x) = b'(x) = 0$. But then $a(x) = \rho(a)$, $b(x) = \rho(b)$ which proves (5).

(1) $\Rightarrow$ (4). Suppose a, b belong to $\mathbf{A}(c)$ for some $c \in \mathbf{A}$ and that $\rho(a) = \rho(b) = 0$. If $a^{-1}(\{0\}) \cap b^{-1}(\{0\}) = \emptyset$ there is a real number $r > 0$ such that $a^2 + b^2 \geqq r$, and consequently $\rho(a^2 + b^2) \geqq r$. However, if ρ is multiplicative on $\mathbf{A}(c)$ it follows that $\rho(a^2 + b^2) = 0$, so we have a contradiction. Therefore (1) $\Rightarrow$ (4).

The equivalences (3) $\Leftrightarrow$ (5) and (2) $\Leftrightarrow$ (4) are almost immediate, and are left to the reader. We finally show that if ρ is simple then it is also pure. Let $a \in \mathbf{A}$ be arbitrary. If ρ is multiplicative on $\mathbf{A}(a)$ then $\rho | \mathbf{A}(a)$ is pure. Now let $0 \leqq \rho' \leqq \rho$, $r = \rho'(1)$. For all $b \in \mathbf{A}(a)$ we must have $\rho'(b) = k\rho(b)$ for some $k \in [0, 1]$. Now $1 \in \mathbf{A}(a)$ so $r = \rho'(1) = k\rho(1) = k$. Hence $\rho'(a) = r\rho(a) \Rightarrow \rho' = r\rho$ since a was arbitrary. The proof is complete. □

Remark. The implication (2) $\Rightarrow$ (1) will also follow from the Gleason-Kahane-Zelazko theorem (cf. Theorem 10.9 in [8]) applied to $\mathbf{A}$(a).

A quasi-measure is *extremal* if it only takes the values 0 and 1. To obtain a similar characterization of extremal quasi-measures we need some preliminary results.

Let I be a directed index set. A family of set $\{A_i\}_{i \in I}$ is *increasing* if $i \leqq j \Rightarrow A_i \subseteq A_j$, and we write $A_i \uparrow A$ if $A = \bigcup A_i$. The family $\{A_i\}_{i \in I}$ is *decreasing* if $i \leqq j \Rightarrow A_i \supseteq A_j$, and we write $A_i \downarrow A$ if $A = \bigcap A_i$.

Proposition 2.1. *Let μ be a quasi-measure in X.*
(a) *For any increasing family of open sets, if $U_i \uparrow U$ then $\mu(U_i) \uparrow \mu(U)$.*
(b) *For any decreasing family of closed sets, if $K_i \downarrow K$ then $\mu(K_i) \downarrow \mu(K)$.*

Proof. By property (1) in the definition of a quasi-measure it suffices to prove (a). With this in mind, first observe that $\mu(U_i) \leqq \mu(U)$ for all $i \in I$, so that $\lim_{i \in I} \mu(U_i) = \sup \mu(U_i)$ exists and is $\leqq \mu(U)$. Let $\varepsilon > 0$ be arbitrary. By (4) in the definition of a quasi-measure there is a compact set $K \subseteq U$ such that $\mu(K) > \mu(U) - \varepsilon$. Since $U = \bigcup U_i$, K is compact and the $\{U_i\}$ increasing, there is a $U_i \supseteq K$. But then $\mu(U_i) > \mu(U) - \varepsilon$ and (a) follows. The proof is complete. □

Corollary 2.1. *For any countable family of open, disjoint sets $\{U_n\}$, $n = 1, 2, \ldots$ we have*

$$\mu\left(\bigcup_{n=1}^{\infty} U_n\right) = \sum_{n=1}^{\infty} \mu(U_n) .$$

Remark 2.1. Property (b) of Proposition 2.1 means that any quasi-measure is a *capacity* (cf. [4]), when restricted to $\mathscr{A}$.

Now let $\mu(X)$ be a quasi-measure in X satisfying $\mu(X) = 1$. Employing the notation of [1], for any $a \in \mathbf{A}$:

$$K_\alpha^a = \{x : a(x) \geqq \alpha\};\ \hat{a}(\alpha) = \mu(K_\alpha^a);\ \alpha \in \mathbb{R} .$$

μ_a is the Borel-measure in $\mathbb{R}$ with compact support given by

$$\mu_a([\alpha, \beta)) = \hat{a}(\alpha) - \hat{a}(\beta)$$

[1, Sect. 3].

Lemma 2.1. *Let μ be a quasi-measure in X satisfying $\mu(X) = 1$. For any open or closed subset D of $\mathbb{R}$ we have, for all $a \in \mathbf{A}$:*

$$\mu_a(D) = \mu(a^{-1}(D)) . \tag{2.1}$$

Proof. It suffices to establish (2.2) for an arbitrary open subset D of $\mathbb{R}$. Any such set may be written as a countably infinite (or finite) disjoint union of open intervals. It is therefore, by Corollary 2.1 enough to show that (2.1) holds for open intervals. Let (α, β) be an arbitrary open interval and let $\alpha_n \downarrow \alpha$ so that $[\alpha_n, \beta) \uparrow (\alpha, \beta)$

$$\mu_a((\alpha, \beta)) = \lim_{n\to\infty} \mu_a([\alpha_n, \beta)) = \lim_{n\to\infty} \hat{a}(\alpha_n) - \hat{a}(\beta)$$

$$= \check{a}(\alpha) - \hat{a}(\beta) = \mu(V^a_\alpha) - \mu(K^a_\alpha) = \mu(V^a_\alpha - K^a_\alpha) = \mu(a^{-1}(\alpha, \beta)) .$$

Here

$$\check{a}(\alpha) = \mu(V^a_\alpha); \qquad V^a_\alpha = \{x : a(x) > \alpha\}$$

and we have also used Proposition 2.1(c) and Proposition 3.1 of [1]. The proof is complete. □

Theorem 2.2. *Let μ be a quasi-measure in X satisfying $\mu(X) = 1$. The following statements are equivalent:*

(6) *μ is extremal.*

(7) *For each $a \in \mathbf{A}$*, range $\hat{a} \subseteq \{0, 1\}$.

(8) *For each $a \in \mathbf{A}$, μ_a is a point-measure of mass* 1 *in* Sp *a*.

(9) *For each $a \in \mathbf{A}$ there is exactly one point $\alpha_0 \in \mathrm{Sp}\, a$ such that* $\mu(a^{-1}\{\alpha_0\}) = 1$.

(10) *For each $a \in \mathbf{A}$ there is exactly one point $\alpha_0 \in \mathbb{R}$ where $\hat{a}$ is discontinuous, $\alpha_0 \in \mathrm{Sp}\, a$, $\hat{a}(\alpha) = 1$ if $\alpha \leqq \alpha_0$, $\hat{a}(\alpha) = 0$ if $\alpha > \alpha_0$.*

Proof. (6) $\Rightarrow$ (7) by the definition of $\hat{a}$. (7) $\Rightarrow$ (10) by Proposition 3.1 in [1], and by the same proposition we also get that (10) $\Rightarrow$ (9). Using Lemma 2.1 above we see that (9) $\Rightarrow$ (8), μ_a is the point-measure with mass 1 at $\alpha_0 \in \mathrm{Sp}\, a$. It remains to prove (8) $\Rightarrow$ (6). If (8) is true then it follows from Lemma 2.1 that for all open or closed subsets D of $\mathbb{R}$ we have $\mu(a^{-1}(D)) \in \{0, 1\}$ for all $a \in \mathbf{A}$. Let K be an arbitrary compact subset of X and let $\varepsilon > 0$ be arbitrary. By Proposition 2.1(d) in [1] there is an open set $U \supseteq K$ such that $\mu(U) < \mu(K) + \varepsilon$. Choose $a \in \mathbf{A}$ such that $K \prec a \prec U$ and let $C = \{x : a(x) = 1\}$. Then $K \subseteq C \subseteq U$, such that if $\mu(K) > 0$, then $0 < \mu(C) = \mu(a^{-1}\{1\}) = 1 \Rightarrow \mu(U) = 1 \Rightarrow \mu(K) > 1 - \varepsilon$ which implies that $\mu(K) = 1$ since $\varepsilon > 0$ was arbitrary. The proof is complete □

In [1] we established that there is a 1-1 correspondance between the quasi-states on $\mathbf{A} = C(X)$ and the normalized quasi-measures on X. If ρ corresponds to μ, then for each $a \in \mathbf{A}$ and all $\phi \in C(\mathrm{Sp}\ a)$ we have

$$\rho(\phi(a)) = \int_{\mathrm{Sp}\, a} \phi(\lambda)\, d\mu_a(\lambda) \tag{2.2}$$

(Theorem 4.1 in [1]). Hence, if μ is extremal so that μ_a is concentrated at a point $\alpha_0 \in \mathrm{Sp}\, a$ then (2.2) yields

$$\rho(\phi(a)) = \phi(\alpha_0) \qquad (\phi \in C(\mathrm{Sp}\, a)) . \tag{2.3}$$

Since $\phi \to \phi(a)$ is an algebra-isomorphism of $C(\mathrm{Sp}\ a)$ onto $\mathbf{A}(a)$ it follows that $\rho | \mathbf{A}(a)$ is multiplicative. Conversely, if ρ is simple so that $\rho | \mathbf{A}(a)$ is multiplicative, then $\rho_a : \phi \to \rho(\phi(a))$ is a multiplicative linear functional on $C(\operatorname{Sp} a)$. By Theorem 4.1 in [1] μ_a is the measure associated with ρ_a and it is therefore concentrated in a point α_0. This establishes the equivalence of (1) in Theorem 2.1 and (8) in Theorem 2.2. We have proved:

Theorem 2.3. *A quasi-state ρ on* **A** *is simple if and only if the corresponding quasi-measure μ on X is extremal. Moreover μ_a is the point-measure of mass* 1 *located at the point $\rho(a) \in \operatorname{Sp} a$.*

We now return to the question when a pure quasi-state is also simple. The next result is in the positive direction:

Lemma 2.2. *Let ρ be a pure quasi-state on* **A**. *Then $\rho(e) \in \{0, 1\}$ for any idempotent $e \in$* **A**.

Proof. Let $e \neq 0$ be an idempotent in **A**, and define $\rho'(a) = \rho(ae)$; $a \in$ **A**. We claim that ρ' is a positive quasi-linear functional on **A**. Let $a \in$ **A** be arbitrary. We must show that ρ' is additive on **A**(a). Let $E = \{x \in X: e(x) = 1\}$ and take an arbitrary function $f \in C(\mathbb{R})$. Then

$$f(ae)(x) = f(a(x)e(x)) = \begin{cases} f(a(x)) & \text{if } x \in E \\ f(0) & \text{if } x \notin E \end{cases}$$

so that $f(ae) = f(a)e + f(0)(1 - e)$. It therefore follows from the additivity of quasi-states on orthogonal elements (cf. Lemma 3.3 in [1]) that $\rho(f(ae)) = \rho(f(a)e) + f(0)\rho(1 - e)$, or

$$\rho'(f(a)) = \rho(f(ae)) - f(0)\rho(1 - e)\,. \tag{2.4}$$

Now let $b = f(a)$, $c = g(a)$; $f, g \in C(\mathbb{R})$, be arbitrary elements of **A**(a). Then $b + c = (f + g)(a)$ so repeated use of (2.4) yields:

$$\begin{aligned} \rho'(b + c) &= \rho((f + g)(ae)) - (f + g)(0)\rho(1 - e) \\ &= \rho(f(ae) + g(ae)) - (f(0) + g(0))\rho(1 - e) \\ &= \rho(f(ae)) + \rho(g(ae)) - f(0)\rho(1 - e) - g(0)\rho(1 - e) \\ &= \rho'(b) + \rho'(c) \end{aligned}$$

since ρ is linear on **A**(ae). This shows that ρ' is additive on **A**(a). It is clearly positive, for if $a \geqq 0$ then $ae \geqq 0$; moreover $\rho'(ra) = \rho(rae) = r\rho'(a)$ for any $r \in \mathbb{R}$, so ρ' is a positive quasi-linear functional on **A**.

If $a \geqq 0$ then $ae \leqq a$ so $\rho'(a) = \rho(ae) \leqq \rho(a)$ (since ρ is monotone, Lemma 4.1(b) in [1]). Hence $0 \leqq \rho' \leqq \rho$ and consequently $\rho' = r\rho$ for some $r \in [0, 1]$ since ρ is pure. $\rho'(1) = \rho(e) = r$. On the other hand $\rho'(1 - e) = \rho((1 - e)e) = 0$ so that $0 = r\rho(1 - e) = r(1 - r) \Rightarrow r \in \{0, 1\}$ which proves the assertion. □

As a consequence of this result one may show that if **A** contains the spectral resolution of each of its elements, then each pure quasi-state is simple. To be precise, let $a \in$ **A** be arbitrary and assume that the sets $K_\alpha = \{x \in X: a(x) \geqq \alpha\}$ are open as well as closed for all $\alpha \in \mathbb{R}$. The characteristic functions e_α of these sets are then idempotents in **A**. If ρ is a quasi-state and μ is the corresponding quasi-measure we have $\rho(e_\alpha) = \mu(K_\alpha) = \hat{a}(\alpha)$. Therefore, if ρ is pure it follows from Lemma 2.2 above and Theorem 2.2.(7) that μ is extremal, and consequently that ρ is simple, by Theorem 2.3. At this point it must be remarked that the assumption on **A** implies that X will have a basis for the topology consisting of open and closed sets, and X is therefore totally disconnected. But then we know that each quasi-state is in fact linear, so the problem disappears and the apparent affirmative result tells us nothing new. We shall instead provide an example of a pure quasi-state which is not simple. We will do this by constructing a quasi-measure μ in $X = S^2$ such that $\mathrm{Sp}\ \mu = \{0, \frac{1}{2}, 1\}$ so that μ is not extremal, but its corresponding quasi-state is pure.

Example 2.1. Let $X = S^2$ and let $P = \{p_1, \ldots, p_5\} \subseteq X$ be a set of five distinct points in X. For any set $D \subseteq X$ we let $\# D$ denote the number of points in $P \cap D$. A subset D of X is *co-connected* if $X \backslash D$ is connected. D is *solid* if it is connected and co-connected. The family of all solid, closed (resp. open) subsets of X is denoted by $\mathscr{C}_s$ (resp. $\mathscr{O}_s$). Let $\mathscr{A}_s = \mathscr{C}_s \cup \mathscr{O}_s$, and make the following definition: For $A \in \mathscr{A}_s$ let

$$\mu(A) = \begin{cases} 0 & \text{if } \# A = 0 \text{ or } 1 \\ \frac{1}{2} & \text{if } \# A = 2 \text{ or } 3 \\ 1 & \text{if } \# A = 4 \text{ or } 5 \,. \end{cases}$$

The main problem is to extend μ to a quasi-measure in X. We sketch the argument. We first extend μ to the family of all closed, connected sets $\mathscr{C}_c$ as follows: If C is closed and connected, its complement $X \backslash C$ is the countable disjoint union of its connected components V_i, $i = 1, 2, \ldots$ Each set V_i is open and belongs to $\mathscr{O}_s$. We may therefore define $\mu(C) = 1 - \Sigma \mu(V_i)$. Next, let $\mathscr{C}_0$ denote the family of closed subsets of X which have only finitely many connected components. Each set K belonging to $\mathscr{C}_0$ may be written uniquely as a finite disjoint union $\bigcup \{C_k, k = 1, 2, \ldots, n\}$ with $C_k \in \mathscr{C}_c$. We define $\mu(K) = \Sigma \mu(C_k)$. One may now verify (somewhat laboriously) that μ so defined on $\mathscr{C}_0$ has all the properties of a quasi-measure. The extension theorem of [1, (Theorem 6.1)] may now be applied to obtain a unique extension of μ to the family $\mathscr{A}$ of all open or closed sets in X. We want to show that the quasi-state corresponding to μ is pure. So let us assume that there exist two quasi-measures μ_1 and μ_2 in X such that $\mu_1(X) = \mu_2(X) = 1$, and

$$\mu = \tfrac{1}{2}\mu_1 + \tfrac{1}{2}\mu_2 \,. \tag{1}$$

We are going to show that $\mu_1 = \mu_2$. For this, it will suffice to show that μ_1 and μ_2 coincide on $\mathscr{C}_s$, because of the uniqueness property of the extension process described above. First observe, however, that if for any closed set K we have $\mu(K) = 0$, then $\mu_1(K) = \mu_2(K) = 0$, and if $\mu(K) = 1$, then $\mu_1(K) = \mu_2(K) = 1$. It therefore only remains to verify that if $\# C = 2$ or 3 for $C \in \mathscr{C}_s$, then $\mu_1(C) = \mu_2(C) = \frac{1}{2}$.

Also, the case $\# C = 3$ will follow if we can show that this is true when $\# C = 2$. For if $\# C = 3$ then $\# (X \backslash C) = 2$, and $X \backslash C$ is open and connected and therefore contains a simple path C' connecting two of the points in P. $C' \in \mathscr{C}_s$ and $C \cap C' = \emptyset$ so $\mu(C \cup C') = 1$. Hence, if $\mu_i(C') = \frac{1}{2}$, then $\mu_i(C) = 1 - \mu_i(C') = \frac{1}{2}$ for $i = 1, 2$.

Let us now assume specifically that $C_1 \in \mathscr{C}_s$ and that $C_1 \cap P = \{p_1, p_2\}$. $X \backslash C_1$ is open and connected so there are simple paths in it; C_3 connecting p_3 with p_4, and C_4 connecting p_4 with p_5. We next choose simple paths C_5 and C_2 connecting p_5 with p_1 and p_2 with p_3 respectively, such that $C_3 \cap C_5 = C_5 \cap C_2 = C_2 \cap C_4 = \emptyset$. By construction $\mu(C_j) = \frac{1}{2}$ for $j = 1, \ldots, 5$ so that if

$C_j \cap C_k = \varnothing$, then $\mu(C_j \cup C_k) = 1$. Hence we have

$$\mu_i(C_j) + \mu_i(C_k) = 1 \text{ if } C_j \cap C_k = \varnothing, \quad i = 1, 2 . \tag{2}$$

Now suppose $\mu_1(C_1) = \alpha$; $\alpha \in [0, 1]$. By (2) we get

$$\mu_1(C_3) = \mu_1(C_4) = 1 - \alpha .$$

But then, again by (2) we must have

$$\mu_1(C_5) = \mu_1(C_2) = \alpha .$$

Since also $C_2 \cap C_5 = \varnothing$ this implies that $2\alpha = 1$, or $\alpha = \frac{1}{2}$. It follows that $\mu_2(C_1) = \frac{1}{2}$, and since C_1 was an arbitrary set in $\mathscr{C}_s$ satisfying $\# C_1 = 2$, we are finished.

Remark 2.2. The construction of a quasi-measure utilized above is a particular case of a general construction theorem, the proof of which may be found in [2].

3 Projective limits of compact spaces and simple quasi-states

To begin with in this section, let $\mathbf{A}$ just be a partially ordered set, not necessarily directed, and let $\{X_a : a \in \mathbf{A}\}$ be a family of compact Hausdorff spaces. We assume that if $a \succeq b$ then there is a surjective continuous map f_{ba}: $X_a \to X_b$ such that

$$f_{aa} = \mathrm{id}_{Xa}; f_{cb} \circ f_{ba} = f_{ca} \quad \text{if } a \succeq b \succeq c .$$

Let $\wp$ denote the projective limit of this system, i.e.:

$$\wp = \left\{ \rho \in \prod_{a \in \mathbf{A}} X_a : f_{ba}(\rho(a)) = \rho(b) \text{ if } a \succeq b \right\} .$$

We equip the product of the X_a with the product topology, making it into a compact Hausdorff space. By its definition $\wp$ is a closed subset, hence compact. We shall give an interpretation of $\wp$ in terms of simple quasi-states when $\mathbf{A}$ is taken to be $C(X)$. In this situation we introduce a partial ordering:

$$a \succeq b \text{ if } \mathbf{A}(a) \supseteq \mathbf{A}(b) . \tag{3.1}$$

For any $a \in \mathbf{A}$ let $X_a = \mathrm{Sp}\, a$. Then $a \succeq b$ if and only if $b \in \mathbf{A}(a)$, which by the Gelfand-theory is equivalent to the statement that there is a (unique) continuous function f of X_a onto X_b such that $b = f \circ a$. We write $f = f_{ba}$ if $a \succeq b$. f_{ba} is a homeomorphism if $a \succeq b$ and $b \succeq a$, i.e. if $\mathbf{A}(a) = \mathbf{A}(b)$. This makes $\{X_a; f_{ba}\}$ into a projective system of compact Hausdorff spaces, and we may form its projective limit $\wp$ as above.

Let E denote the set of simple quasi-states, equipped with the relative topology from Q. By definition it easily follows that E is closed in Q and is therefore compact.

Theorem 3.1. *$\wp$ coincides with the space E and contains X as a closed imbedded subspace.*

Proof. We first imbed X in $\wp$. For $x \in X$ define ρ_x: $\mathbf{A} \to \bigcup X_a$ by $\rho_x(a) = a(x)$; $a \in \mathbf{A}$. If $a \succeq b$ so $b = f_{ba} \circ a$ then $\rho_x(b) = b(x) = f_{ba}(a(x)) = f_{ba}(\rho_x(b))$ which shows that $\rho_x \in \wp$. The map $i: x \to \rho_x$ of X into $\wp$ is clearly injective since $\mathbf{A}$ distinguishes points. It is also continuous, and is therefore (by compactness of X) a homeomorphism of X onto its image $i(X)$ in $\wp$.

We next show why the spaces E and $\wp$ coincide. If $\rho \in E$ then $\rho(a) \in X_a$ for all $a \in \mathbf{A}$ (Theorem 2.1(5)). Moreover, $\rho|\mathbf{A}(a)$ is just evaluation at the point $\rho(a)$ via the Gelfand-transform $\phi \to \phi \circ a$ of $C(X_a)$ onto $\mathbf{A}(a)$. I.e. $\rho(\phi(a)) = \phi(\rho(a))$ (Theorem 2.3). So, if $a \succeq b$ then $\rho(b) = \rho(f_{ba}(a)) = f_{ba}(\rho(a))$ which shows that $\rho \in \wp$.

Conversely, let ρ be an element of $\wp$. Then $\rho(a) \in X_a$ for each $a \in \mathbf{A}$. In particular $1 \in \mathbf{A}$, $X_1 = 1(X) = \{1\}$ so $\rho(1) = 1$. If $a \geqq 0$ then $X_a \subseteq [0, \infty)$ so $\rho(a) \geqq 0$. Finally, if $b \in \mathbf{A}(a)$ then $b = f_{ba} \circ a$ and $\rho(b) = f_{ba}(\rho(a))$. Hence $\rho|\mathbf{A}(a)$ is just evaluation at the point $\rho(a)$ via the Gelfand-transform $\phi \to \phi(a)$. It follows that ρ is a multiplicative linear functional on $\mathbf{A}(a)$ i.e. is simple. That the topologies of E and $\wp$ are the same is obvious. The proof is complete.

4 The non-linear Gelfand-transform

In this section we introduce a "non-linear Gelfand-transform" Ψ of $\mathbf{A}$ into $C(E)$. We utilize this transform to show that each representable (to be defined below) quasi-state may be factored by Ψ and an ordinary (linear) state on $C(E)$.

Define, for $a \in \mathbf{A}$ the function $a^\sim$ on E by $a^\sim(\rho) = \rho(a)$; $\rho \in E$. By definition $a^\sim$ is continuous on E. The map $\Psi: a \to a^\sim$ of $C(X)$ into $C(E)$ is in general non-linear, for if $\rho \in E \setminus i(X)$, then for some $a, b \in \mathbf{A}$ we have $\rho(a + b) \neq \rho(a) + \rho(b)$ which means that $(a + b)^\sim(\rho) \neq a^\sim(\rho) + b^\sim(\rho)$. Ψ is therefore called the *non-linear Gelfand-transform on* $\mathbf{A}$. We list some properties of Ψ which will be needed later on, proving them as we go along:

(i) $a^\sim(\rho_x) = \rho_x(a) = a(x)$; $x \in X$.

Hence $a^\sim$ coincides with the usual Gelfand-transform on $i(X)$, i.e. $a^\sim|i(X) = \hat{a}$.

(ii) $a \geqq 0 \Rightarrow \Psi(a) \geqq 0$

(iii) $\Psi(0) = 0$; $\Psi(1_x) = 1_E$

(iv) $a \leqq b \Rightarrow \Psi(a) \leqq \Psi(b)$; $a, b \in \mathbf{A}$.

This follows from Lemma 4.1 of [1].

(v) $\|a^\sim\|_\infty = \sup\{|\rho(a)|: \rho \in E\} = \|a\|_\infty$

and more generally

(vi) $\|a^\sim - b^\sim\|_\infty = \|a - b\|_\infty$; $a, b \in \mathbf{A}$.

It suffices to prove (vi). By Lemma 4.1 of [1] we have $|\rho(a) - \rho(b)| \leqq \|a - b\|_\infty$ for all $\rho \in E$. Hence $\|a^\sim - b^\sim\|_\infty \leqq \|a - b\|_\infty$. On the other hand there is $x \in X$ such that $\|a - b\|_\infty = |(a - b)(x)| = |\rho_x(a) - \rho_x(b)| \leqq \|a^\sim - b^\sim\|_\infty$. Equality follows. We summarize in

Proposition 4.1. $\Psi: \mathbf{A} \to C(E)$ *is a positive, order-preserving and isometric map with a closed range* $\mathbf{B} = \Psi(\mathbf{A})$. *For any* $a \in \mathbf{A}$ Ψ *is an algebra-isomorphism of* $\mathbf{A}(a)$ *onto* $\mathbf{A}(a^\sim)$.

Proof. Since Ψ is an isometry and $\mathbf{A}$ is complete it follows that $\mathbf{B} = \Psi(\mathbf{A})$ is closed in $C(E)$. Let $a \in \mathbf{A}$ be arbitrary. For any $\phi \in C(\mathrm{Sp}\, a)$ we then have $\phi(a)^\sim(\rho) = \rho(\phi(a)) = \phi(\rho(a)) = \phi(a^\sim(\rho))$, so that $\phi(a)^\sim = \phi(a^\sim)$. By Theorem 2.1 we know that $\mathrm{Sp}\, a^\sim = \mathrm{Sp}\, a$, and it follows that Ψ is an algebra-isomorphism of $\mathbf{A}(a)$ onto $\mathbf{A}(a^\sim)$. The proof is complete. □

Let us say that a quasi-state is *representable* if it belongs to the closed convex hull of E in Q. Then we have:

Proposition 4.2. *Let* Ψ: $\mathbf{A} \to C(E)$ *be as above. The following are equivalent:*

(1) Ψ *is surjective.*
(2) $i(X) = E$.
(3) *Each representable quasi-state on* $\mathbf{A}$ *is linear.*
(4) Ψ *is linear.*
(5) $\Psi(\mathbf{A})$ *is a linear subspace of* $C(E)$.

Proof. (1) $\Rightarrow$ (2). If $i(X) \neq E$ there is a function $0 \neq h \in C(E)$ which vanishes on $i(X)$. Assuming $h = a\tilde{}$ for some $a \in \mathbf{A}$ we get $a(x) = \rho_x(a) = a\tilde{}(\rho_x) = h(\rho_x) = 0$ for all $x \in X$. Hence $a = 0$, but then $h = 0$, a contradiction.

(2) $\Rightarrow$ (3). This is immediate.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (1). Assuming (5) we immediately obtain that $\mathbf{B} = \Psi(\mathbf{A})$ is a closed linear subspace of $C(E)$ which contains 1 and separates points of E. Therefore, by the Stone-Weierstrass theorem it suffices to show that $\mathbf{B}$ is an algebra for (1) to be true. Let $f, g \in \mathbf{B}$. Since $fg = 1/4\,[(f+g)^2 - (f-g)^2]$ and $\mathbf{B}$ contains squares the assertion follows. The proof is complete. □

Now let p denote a *state* on $C(E)$, and define $\rho(a) = p(\Psi(a))$; $a \in \mathbf{A}$. By Proposition 4.1 it is immediately clear that ρ is a quasi-state on $\mathbf{A}$. Let R denote the set of representable quasi-states on $\mathbf{A}$, i.e. R is the closed convex hull of E in Q. It would be nice to have an *intrinsic* characterization of the elements of R. Presently, however, this is what we can say:

Theorem 4.1. *Let* ρ *be a quasi-state on* $\mathbf{A}$. *The following statements are equivalent:*

(1) $\rho \in R$.
(2) *There is a probability-measure* m *on* E *such that*

$$\rho(a) = \int_E \sigma(a)\, dm(\sigma); \quad a \in \mathbf{A}.$$

(3) *There is a state* p *on* $C(E)$ *such that*

$$\rho = p \circ \Psi. \tag{4.1}$$

Proof. Let $\mathbf{A}^{\#}$ be the real linear space generated by R, and equip it with the topology of pointwise convergence on elements of $\mathbf{A}$. We refer to this topology as the w^*-topology on $\mathbf{A}^{\#}$ even if the elements of $\mathbf{A}^{\#}$ are not linear on $\mathbf{A}$. Let H denote the linear space of w^*-continuous linear functionals on $\mathbf{A}^{\#}$. There is a natural injection Ψ' of $\mathbf{A}$ into H given by $\Psi'(a)(\rho) = \rho(a)$, $\rho \in \mathbf{A}^{\#}$. By a standard result the linear span of $\Psi'(\mathbf{A})$ equals H, so the $\sigma(\mathbf{A}^{\#}, H)$-topology on $\mathbf{A}^{\#}$ coincides with the w^*-topology. E is compact, and R, its closed convex hull, is compact since both sets are closed subsets of Q. Therefore, if $\rho \in R$ it follows by another stándard result (cf. Theorem 3.28 in [8]) that (2) holds. Next, if (2) holds, let p denote the state on $C(E)$ corresponding to m, i.e.

$$p(f) = \int_E f(\sigma)\, dm(\sigma); \quad f \in C(E).$$

For $f = \Psi(a)$ we therefore get

$$p(\Psi(a)) = \int_E a\tilde{}(\sigma)\, dm(\sigma) = \int_E \sigma(a)\, dm(\sigma) = \rho(a)$$

which establishes (4.1). Finally, if (3) is true, let m be the probability measure in E corresponding to p. Reasoning backwards we see that (2) is true, which then in turn implies (1) by the theorem quoted above. The proof is complete. □

The question of uniqueness of the above factorization will be discussed towards the end of the final section.

5 The space of extremal quasi-measures

In this section we introduce a *set-transform* corresponding to the transform Ψ of the last section. In the next section this new transform will allow us to obtain a factorization of quasi-measures by ordinary measures.

It will also enable us to give an alternative description of the topology of E which will be useful later on. Let X^* denote the set of extremal quasi-measures in X. (Of course, by Theorem 2.3 we know that we may identify X^* with E, but for the moment it is practical to distinguish between the two).

For $A \in \mathscr{A}$ let

$$A^* = \{\mu \in X^* : \mu(A) = 1\} = \Psi^*(A) .$$

The map $\Psi^* : \mathscr{A} \to \mathscr{P}(X^*)$ has the following properties:

(i) $\varnothing^* = \varnothing$; $\Psi^*(X) = X^*$
(ii) $A \subseteq B \Rightarrow A^* \subseteq B^*$

For $x \in X$ let $j(x) = \mu_x$ = the point-measure of mass one at $\{x\}$. With this notation we have

(iii) $A^* \cap j(X) = j(A)$; $A \in \mathscr{A}$
(iv) $A \cap B = \varnothing \Rightarrow A^* \cap B^* = \varnothing$; $A, B \in \mathscr{A}$
(v) $(X \backslash A)^* = X^* \backslash A^*$; $A \in \mathscr{A}$.

Indeed, $\mu \in (X \backslash A)^* \Leftrightarrow \mu(X \backslash A) = 1 \Leftrightarrow \mu(A) = 0 \Leftrightarrow \mu \notin A^* \Leftrightarrow \mu \in X^* \backslash A^*$.

(vi) $A \neq B \Rightarrow A^* \neq B^*$; $A, B \in \mathscr{A}$.

For if $A \neq B$ then there is $x \in A \backslash B$ (or conversely) $\Rightarrow \mu_x \in A^* \backslash B^* \Rightarrow A^* \neq B^*$.

(vii) Suppose $A, B \in \mathscr{A}$ and $A \cup B \in \mathscr{A}$. Then
(a) $(A \cup B)^* \supseteq A^* \cup B^*$
(b) $(A \cup B)^* = A^* \cup B^*$ if $A \cap B = \varnothing$.

(a) is obvious and (b) follows from Proposition 2.1(c) in [1].

(viii) Suppose $A, B \in \mathscr{A}$ and $A \cap B \in \mathscr{A}$
(a) $(A \cap B)^* \subseteq A^* \cap B^*$
(b) $(A \cap B)^* = A^* \cap B^*$ if $A \cup B = X$.

(a) is obvious and (b) may be deduced from (v) and (vii) (b) taking complements. Equality in (vii)(a) and (viii)(a) does not generally hold. We may have $\mu(A) = \mu(B) = 0$, while $\mu(A \cup B) = 1$.

Let $\mathscr{A}^* = \mathscr{C}^* \cup \mathscr{O}^*$ where $\mathscr{O}^*$ (resp. $\mathscr{C}^*$) is the family of open (resp. closed) subsets of X^* with respect to the topology it inherits from E. We shall see that $\Psi^*(\mathscr{A}) \subseteq \mathscr{A}^*$, but first we need to make the connection between E and X^* more explicit. If $\rho \in E$ and μ is the corresponding element of X^* then, for $K \in \mathscr{C}$:

$$\mu(K) = 1 \Leftrightarrow \rho(a) = 1 \text{ for all } a \succ K;\ a \in \mathbf{A} . \tag{5.1}$$

Conversely, if μ is given, then for $a \in \mathbf{A}$, $\rho(a)$ is the unique real number such that:

$$\mu(a^{-1}(\rho(a)) = 1 . \tag{5.2}$$

(The last statement follows from Theorem 2.2 and Theorem 2.3.)

Let $P\colon E \to X^*$ denote this identification map. Transfering the topology from E to X^* then makes X^* into a compact Hausdorff space, containing $j(X) = Pi(X)$ as a closed subspace.

Lemma 5.1. $\Psi^*(\mathscr{A}) \subseteq \mathscr{A}^*$.

Proof. By (v) above it suffices to show that if $K \subseteq X$ is closed, then K^* is closed in X^*, i.e. that $P^{-1}(K^*)$ is closed in E. By (5.1) we get that

$$P^{-1}(K^*) = \{\rho \in E : \rho(a) = 1; \forall a \succ K; a \in \mathbf{A}\} = \bigcap_{a \succ K} \{\rho \in E : a^{\tilde{}}(\rho) = 1\}\,.$$

Since $a^{\tilde{}}$ is continuous on E the assertion follows. □

Let D be an open or closed subset of $\mathbb{R}$ and let $a \in \mathbf{A}$ be arbitrary.

Lemma 5.2. $\{a^{-1}(D)\}^* = P(a^{\tilde{}\,-1}(D))$.

Proof. By Lemma 2.2 we have $\mu_a(D) = \mu(a^{-1}(D))$. Hence, if $\mu = P\rho$ then $\mu \in \{a^{-1}(D)\}^* \Leftrightarrow \mu_a(D) = 1 \Leftrightarrow \rho(a) \in D$ (since μ_a is concentrated at $\rho(a)) \Leftrightarrow a^{\tilde{}}(\rho) \in D \Leftrightarrow \rho \in a^{\tilde{}\,-1}(D)$. The proof is complete. □

By definition a basis for the topology of E is given by finite intersections of the sets $a^{\tilde{}\,-1}(I)$ where $a \in \mathbf{A}$ and I is an open interval in $\mathbb{R}$. By Lemma 5.2 it therefore follows that a basis for the topology of X^* may be given by finite intersections

$$U_1^* \cap U_2^* \cap \ldots \cap U_n^*$$

where the U_i are open in X. In particular U_i may be taken to be of the form $U_i = a_i^{-1}(I_i)$, where $a_i \in \mathbf{A}$ and I_i is an open interval in $\mathbb{R}$.

Corollary 5.1. *A net $\{\mu_i\}_{i \in J} \subseteq X^*$ converges to an element $\mu \in X^*$ if and only if for each $K \in \mathscr{C}$, satisfying $\mu(K) = 0$, there is $i_0 \in J$ such that $\mu_i(K) = 0$ if $i \geqq i_0$.*

Let us say that a quasi-measure is *representable* if its corresponding quasi-state is representable.

Proposition 5.1. *Let $\Psi^*\colon \mathscr{A} \to \mathscr{A}^*$ be as above. The following are equivalent:*

(1) *Ψ^* is surjective.*

(2) $j(X) = X^*$.

(3) *Each representable quasi-measure on X is the restriction of a regular Borel measure.*

(4) *If A, B and $A \cup B$ belong to $\mathscr{A}$ then $(A \cup B)^* = A^* \cup B^*$.*

(5) *If W_1 and W_2 are open subsets of X^* belonging to $\Psi^*(\mathscr{A})$ then $W_1 \cap W_2$ and $W_1 \cup W_2$ also belong to $\Psi^*(\mathscr{A})$.*

(6) *Each extremal quasi-measure μ is subadditive, i.e. if A, B and $A \cup B$ belong to $\mathscr{A}$ then $\mu(A \cup B) \leqq \mu(A) + \mu(B)$.*

Proof. (1) $\Rightarrow$ (2). If $j(X) \neq X^*$ then there is an open set $\varnothing \neq W \subseteq X^* \backslash j(X)$. Suppose $W = A^*$ for some $A \in \mathscr{A}$. Then by property (iii) above we have $j(A) = W \cap j(X) = \varnothing \Rightarrow A = \varnothing$, a contradiction.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (6). Obvious.

(6) $\Rightarrow$ (4). In general, if A, B and $A \cup B$ belong to $\mathscr{A}$, we have $A^* \cup B^* \subseteq (A \cup B)^*$ (Property (vii)(a) above). Assuming (6) and that $\mu \in (A \cup B)^*$ we get $\mu(A \cup B) = 1$ so that $\mu(A) = 1$ or $\mu(B) = 1$. Hence $\mu \in A^* \cup B^*$ which proves (4).

(4) $\Rightarrow$ (5). Obvious (using Property (v) above).

(5) $\Rightarrow$ (1). Assume (5) and let $W_i = V_i^*$; $i = 1, 2$, V_i open in X.

We have $(V_1 \cap V_2)^* \subseteq V_1^* \cap V_2^*$ in general, but by assumption there is now an open set $V \in \mathcal{O}$ such that $V^* = V_1^* \cap V_2^*$. We claim that $V = V_1 \cap V_2$. Clearly $V_1 \cap V_2 \subseteq V$, however, if there is an $x \in V$ such that $x \notin V_1$ (or V_2) then $\mu_x \in V^* \setminus V_1^*$ which is impossible. Hence $(V_1 \cap V_2)^* = V_1^* \cap V_2^*$ and similarly $(V_1 \cup V_2)^* = V_1^* \cup V_2^*$. To prove (1) it suffices to show that if $W \in \mathcal{O}^*$ then there is a $V \in \mathcal{O}$ such that $W = V^*$. Now let

$$\mathcal{O}_W = \{V \in \mathcal{O}: V^* \subseteq W\}$$

If $V_1, V_2 \in \mathcal{O}_W$ then $(V_1 \cup V_2)^* = V_1^* \cup V_2^* \subseteq W$ so $\mathcal{O}_W$ is a directed family of sets with respect to inclusion. Let

$$U = \bigcup \{V: V \in \mathcal{O}_W\}$$

If $\mu \in U^*$ then $\mu \in V^*$ for some $V \in \mathcal{O}_W$ by Proposition 2.1, hence $U^* \subseteq W$. Suppose there is an element $\mu \in W \setminus U^*$. W is open so there is a base neighborhood of μ: $U_1^* \cap \ldots \cap U_n^* \subseteq W$. However, by the first part of the argument it follows that $U_1^* \cap \ldots \cap U_n^* = U_0^*$ for some open set U_0 in X. Now $U_0 \in \mathcal{O}_W \Rightarrow U_0 \subseteq U \Rightarrow \mu \in U^*$, a contradiction. Hence $U^* = W$ and the proof is complete. □

6 Factorization of representable quasi-measures

Let m be a positive, regular Borel measure in X^* satisfying $m(X^*) = 1$. Define

$$\mu(A) = m(A^*); A \in \mathcal{A} . \tag{6.1}$$

Then, by the properties (i), (ii), (iv) and (v) given at the beginning of the preceding section it is more or less immediate that μ satisfies properties (1), (2) and (3) in the definition of a quasi-measure (see the Introduction). It only remains to verify the regularity condition (4). Let $U \subseteq X$ be open and take $\mathfrak{C}$ to be the family of open sets V satisfying $V^- \subseteq U$. $\mathfrak{C}$ is ordered by inclusion and is directed upwards: if $V_1, V_2 \in \mathfrak{C}$ then $V_1 \cup V_2 \in \mathfrak{C}$. Moreover

$$U^* = \bigcup \{V^*: V \in \mathfrak{C}\}$$

for if $\mu' \in U^*$ then $\mu'(V) = 1$ for some $V \in \mathfrak{C}$, by virtue of Proposition 2.1. Now m is a regular Borel measure so there is a compact set $C \subseteq U^*$ such that $m(C) > m(U^*) - \varepsilon$ for any given $\varepsilon > 0$. Since $\mathfrak{C}$ is directed it follows that there is a $V \in \mathfrak{C}$ such that $V^* \supseteq C$. Now $V^- \subseteq U \Rightarrow (V^-)^* \subseteq U^*$. Hence we get

$$\mu(V^-) = m((V^-)^*) \geqq m(C) > m(U^*) - \varepsilon = \mu(U) - \varepsilon$$

which proves (4) in the definition of quasi-measures. We can now formulate the factorization-theorem for representable quasi-measures.

Theorem 6.1. *For each positive, regular Borel measure m on X^* satisfying $m(X^*) = 1$ the set-function μ defined by*

$$\mu(A) = (m \circ \Psi^*)(A); A \in \mathcal{A} \tag{6.2}$$

is a representable quasi-measure μ on X. Conversely, for each representable quasi-measure μ on X there is a positive, regular Borel measure m on X^* satisfying

$m(X^*) = 1$, such that (6.2) holds. Moreover, if μ and m are related by (6.2) and p is the state associated with m, and we define

$$\rho(a) = (p \circ \Psi)(a); a \in \mathbf{A} , \tag{6.3}$$

then ρ is the quasi-state associated with μ.

Proof. The first part has already been established, except the representability of μ, which will follow from the last statement of the theorem. Now let μ' be a representable quasi-measure in X, and let ρ' be the quasi-state associated with μ'. By Theorem 4.1 there is a probability measure m' on E such that for each $a \in \mathbf{A}$:

$$\rho'(a) = \int_E \sigma(a)\, dm'(\sigma) .$$

Let m be the corresponding measure in X^*, i.e.

$$m(D) = m'(P^{-1}(D)) \tag{6.4}$$

for any Borel set $D \subseteq X^*$. We now define μ by (6.2) and observe that the proof will be finished if we can show that $\mu = \mu'$. We shall need:

Lemma 6.1. *Let $K \prec a \prec U$ where K is compact, U is open and $a \in \mathbf{A}$. Then*

$$K^* \prec \tilde{a} \circ P^{-1} \prec U^* . \tag{6.5}$$

Proof. By assumption we have $K \subseteq a^{-1}(\{1\})$ and $X \backslash U \subseteq a^{-1}(\{0\})$. Suppose $\mu = P\rho$ and that $\mu \in K^*$. Then $\mu(a^{-1}(\{1\}) = 1 \Leftrightarrow \rho(a) = 1 \Leftrightarrow \tilde{a}(\rho) = 1 \Leftrightarrow \tilde{a}(P^{-1}\mu) = 1$. Similarly, if $\mu \notin U^*$ then $\mu(X \backslash U) = 1$ so $\mu(a^{-1}(\{0\}) = 1 \Leftrightarrow \rho(a) = 0 \Leftrightarrow \tilde{a}(\rho) = 0 \Leftrightarrow \tilde{a}(P^{-1}\mu) = 0$. The proof is complete. □

Returning to the proof of the theorem, let p be the state on $C(E)$ associated with m', i.e.:

$$p(f) = \int_E f(\sigma)\, dm'(\sigma) = \int_{X^*} (f \circ P^{-1})(v)\, dm(v); f \in C(E), \sigma \in E, v \in X^* .$$

Let $K \in \mathscr{C}$. Then we have, since $\rho' = p \circ \Psi$:

$$\mu'(K) = \inf\{\rho'(a): K \prec a \in \mathbf{A}\} = \inf\{p(\tilde{a}): K \prec a \in \mathbf{A}\} .$$

On the other hand:

$$\mu(K) = m(K^*) = m'(P^{-1}(K^*)) = \inf\{p(f): K^* \prec f \circ p^{-1}; f \in C(E)\} .$$

Since $K \prec a \Rightarrow K^* \prec \tilde{a} \circ P^{-1}$ by the above lemma, it follows that $\mu(K) \leqq \mu'(K)$. Similarly, if $U \in \mathscr{O}$ we have:

$$\mu'(U) = \sup\{\rho'(a): a \prec U; a \in \mathbf{A}\} = \sup\{p(\tilde{a}): a \prec U; a \in \mathbf{A}\}$$

whereas:

$$\mu(U) = m(U^*) = m'(P^{-1}(U^*)) = \sup\{p(f): f \circ P^{-1} \prec U^*; f \in C(E)\} .$$

Again by the lemma $a \prec U \Rightarrow \tilde{a} \circ P^{-1} \prec U^*$ so that $\mu(U) \geqq \mu'(U)$.

Now μ is regular so for any $\varepsilon > 0$ there is $K \subseteq U$ such that $\mu(K) > \mu(U) - \varepsilon$. Hence we get:

$$\mu(K) \leqq \mu'(K) \leqq \mu'(U) \leqq \mu(U) < \mu(K) + \varepsilon .$$

Therefore, if we assume that $\mu'(U) < \mu(U)$ we arrive at a contradiction, since the above is true for all $\varepsilon > 0$. So $\mu(U) = \mu'(U)$ for all open sets $U \subseteq X$, but then $\mu = \mu'$ and the proof is complete. □

Remark 6.1. The factorizations (4.1) and (6.2) are generally not unique, reflecting that Q is not a Choquet-simplex. In the example below we shall see that non-trivial relations between elements of E may exist. However, we first need to say a few words about how to construct extremal quasi-measures in certain spaces.

Let X be either $X_1 = [0, 1] \times [0, 1]$ (= unit square) or $X_2 = S^2$ (= surface of the unit sphere), and let p_1, p_2, p_3 be three distinct points in X. If A is an open or closed subset of X with the property that both A and $X \backslash A$ are connected we define $\sigma(A) = 1$ if A contains at least two of the points p_i, and put $\sigma(A) = 0$ otherwise. Then σ extends to an extremal quasi-measure in X (for details we refer to [2] or [6]). The quasi-measure obtained this way is said to be *associated* with the set $\{p_i\}_{i=1,2,3}$.

Example 6.1. Let $X = X_1$ and let $p_0 = (0, 0)$, $p_1 = (1, 0)$, $p_2 = (1, 1)$ and $p_3 = (0, 1)$. Let σ_i be the quasi-measure associated with the set $\{p_j\}_{j \neq i}$, $i = 0, 1, 2, 3$. One may now show that

$$\sigma_0 + \sigma_2 = \sigma_1 + \sigma_3 . \tag{6.6}$$

From this relation it immediately follows that factorization of representable quasi-states on $C(X_1)$ is not unique. The relation (6.6) arises from the observation that if γ is a simple curve in X_1 connecting two diagonal points, say p_1 and p_3, and γ contains neither of the two other points, then $X_1 \backslash \gamma$ contains at least two distinct connected components. This contrasts the situation in X_2, where, if γ is a simple curve connecting two points, also $X_2 \backslash \gamma$ is connected. The relation (6.6) will therefore not hold in X_2. However, other relations involving "higher order" extremal quasi-measures will exist. These relations reflect properties of the order-structure of the positive cone of quasi-measures on X_2. A discussion of these questions will appear elsewhere.

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