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Absorption semigroups and Dirichlet boundary conditions

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1. Introduction

Let $V: \mathbf{R}^N \to [0, \infty]$ be a measurable function. There are various ways to define the Schrödinger semigroup $\{S_V(t): t \geqq 0\}$, with absorbing potential V, on $L^p(\mathbf{R}^N)$. One can consider the appropriate quadratic form on $L^2(\mathbf{R}^N)$ (whose domain may not be dense) in the case $p = 2$, and then interpolate for other values of p (see [36], etc.). Alternatively, one can use the Feynman-Kac formula to define S_V (see [26]). Both these methods are applicable without constraints on the size of V, provided that one does not require S_V to be a C_0-semigroup on $L^p(\mathbf{R}^N)$. In fact, S_V is always a semigroup of operators, S_V is strongly continuous for $t > 0$, and there is a subset Z_V of $\mathbf{R}^N$ such that S_V defines a C_0-semigroup on $L^p(Z_V)$ and vanishes on $L^p(\mathbf{R}^N \setminus Z_V)$. Various authors (see [38, 45], and the references cited therein) have considered criteria on V which ensure that $\frac{1}{2}\Delta - V$ is a densely-defined operator whose closure generates S_V, or, more generally, that S_V is a C_0-semigroup on $L^p(\mathbf{R}^N)$. Removing such constraints permits Dirichlet semigroups (the semigroups on $L^p(\Omega)$ generated by (half) the Laplacian on an open subset Ω of $\mathbf{R}^N$ with Dirichlet boundary conditions) to be brought within the ambit of Schrödinger semigroups. Indeed, if Ω is of class C^1, and $V = \infty$ on $\mathbf{R}^N \setminus \Omega$, $V = 0$ on Ω, then S_V is the Dirichlet semigroup.

A method commonly used to analyse Schrödinger semigroups is to approximate V by an increasing sequence of bounded functions and then to take a limit of the corresponding semigroups. Voigt [45, 46] has carried out this procedure in the abstract setting of positive semigroups on $L^p(X)$. In this paper, we perform the construction for arbitrary $V \geqq 0$. Apart from the increased generality, this provides information about Dirichlet semigroups, even in the case when Ω is not smooth provided that a second limiting procedure, in the opposite direction, is carried out. This method can also be applied to arbitrary absorbing potentials for Markov processes (as in [15]) and strongly elliptic operators on $\mathbf{R}^N$ (as in [14]). In particular, we study holomorphy of the semigroups, obtaining an abstract theorem

(Theorem 6.1), which can be applied to diffusion semigroups. Kato [22] showed that the Schrödinger semigroup S_V is holomorphic on $L^1(\mathbf{R}^N)$ whenever $V \in L^1_{\mathrm{loc}}(\mathbf{R}^N)_+$; we extend this to absorption-diffusion semigroups with arbitrary non-negative potentials.

Let A be a strongly elliptic operator of second order (with some smoothness of coefficients) with Dirichlet boundary conditions on an open subset Ω of $\mathbf{R}^N$. It is easy to establish that the associated semigroup on $L^p(\Omega)$ is holomorphic, for $1 < p < \infty$; the original method (see [29, Sect. 7.3]) depends on some estimates due to Agmon and involves some restrictions on Ω, but, for arbitrary Ω, one may use quadratic forms for $p = 2$ and the Stein interpolation theorem for $1 < p < \infty$. However, the situation is more delicate for the semigroups on $L^1(\Omega)$ and $C_0(\Omega)$. For Ω sufficiently smooth, Stewart [39] showed that the semigroup on $C_0(\Omega)$ is holomorphic (he extended this to other boundary conditions in [40]). Using duality arguments, Amann [4] deduced the corresponding result on $L^1(\Omega)$ for Ω smooth and bounded (the proof given in [29] is incomplete). Our method extends the result for $L^1(\Omega)$ to arbitrary open sets Ω, but it is confined to Dirichlet boundary conditions. This is analogous to a result of Lumer and Paquet [24] who showed that if the Laplacian generates a semigroup on $C_0(\Omega)$, then the semigroup is holomorphic.

The paper is organised broadly in decreasing order of generality. In Sect. 2, we consider pseudo-resolvents $R(\lambda)$ and convergence of $\lambda R(\lambda)$ as $\lambda \to \infty$; subsequently, we shall consider only pseudo-resolvents associated with degenerate semigroups, which are introduced in Sect. 3. From Sect. 4 onwards, we specialise to the case of degenerate semigroups on $L^p(X)$ arising from perturbing a positive C_0-semigroup T by a non-negative potential V (absorption semigroups). In Sect. 5, we specialise further to the case when V takes only the values 0 and ∞ (barred semigroups). In Sect. 6, we establish the results on holomorphic absorption semigroups on $L^1(X)$, and in Sect. 7, we reconcile our theory with quadratic forms on $L^2(X)$. Finally, in Sect. 8, we extend the results of [5, 6, 9] on asymptotic stability to semigroups generated by elliptic operators with Dirichlet boundary conditions and absorbing potentials. The discussion of Schrödinger semigroups (respectively, Dirichlet semigroups), and their generalisations to diffusion semigroups, is found in Sects. 4, 6, 7, and 8 (resp., Sects. 5, 6, 7, and 8), as special cases of the general theory. The reader who is interested only in these cases may therefore omit some of the earlier sections.

We mention here some conventions of terminology and notation which we shall adopt. The space of all bounded linear operators on a Banach space E will be denoted by $\mathscr{B}(E)$, and will be considered to have the strong operator topology unless otherwise stated. Thus $T_n \to T$ means that $\|T_n f - Tf\| \to 0$ for each f in E. Integrals of $\mathscr{B}(E)$-valued functions will be strongly convergent Bochner integrals. For a (densely-defined) operator A on E, $R(\lambda, A)$ will denote the resolvent of A: $R(\lambda, A) = (\lambda I - A)^{-1}(\lambda \in \rho(A))$. When E is a Banach lattice, we shall regard $\mathscr{B}(E)$ as being ordered by the cone of positive operators; $T_n \uparrow T$ will mean that $T_n f \leqq T_{n+1} f\ (n \geqq 1, f \in E_+)$ and $\|T_n f - Tf\| \to 0\ (f \in E)$. When (X, μ) is a measure space, and Y is a measurable subset of X, we shall identify $L^p(Y)$ with $\{f \in L^p(X) : f(x) = 0$ for almost all $x \in X \setminus Y\}$. If $V: X \to [0, \infty]$ is a measurable function, we shall also use the symbol V for the multiplication operator $D(V) \to L^p(X)$, where $D(V) = \{f \in L^p(X) : Vf \in L^p(X)\}$. We shall denote the constant function with value 1 by $\mathbf{1}$, and the indicator function of Y by $\mathbf{1}_Y$. Thus, $\mathbf{1}_Y$ will also denote the natural projection of $L^p(X)$ onto $L^p(Y)$.

2 Pseudo-resolvents

A *pseudo-resolvent* on a Banach space E is a family $(R(\lambda))_{\lambda \in U}$ (where U is as subset of $\mathbf{C}$) of bounded linear operators on X, satisfying the resolvent equation:

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \; (\lambda, \mu \in U) . \tag{2.1}$$

Let

$$K = \{f \in E : R(\lambda) f = 0\} ,$$

$$D = R(\lambda)E = \{R(\lambda)f : f \in E\}.$$

It is easy to verify from (2.1) that K and D are independent of λ. Moreover, $R(\lambda)$ is the resolvent of a densely-defined operator on E if and only if $K = \{0\}$ and D is dense in X [29, p. 36].

We shall consider pseudo-resolvents $(R(\lambda))_{\lambda > \omega}$ defined for $\lambda > \omega$ for some real ω, and satisfying the condition: $\sup_{\lambda > \omega} \| \lambda R(\lambda) \| < \infty$. A simple example is obtained by taking $R(\lambda) = \lambda^{-1} P$, where P is any bounded projection on E. The methods of the following proposition are rather standard [23, p. 84], [47, Sect. VIII.4], so we omit the proof.

Proposition 2.1 *Let $(R(\lambda))_{\lambda > \omega}$ be a pseudo-resolvent on E.*
(1) *If $\sup_{\lambda > \omega} \| \lambda R(\lambda) \| < \infty$, then $K \cap \bar{D} = \{0\}$.*
(2) *Suppose that, for each f in E, $\{\lambda R(\lambda) f : \lambda > \omega\}$ is relatively weakly compact. Then*
  (a) *$P := \lim_{\lambda \to \infty} \lambda R(\lambda)$ exists (in the strong operator topology);*
  (b) *$E = K \oplus \bar{D}$, and P is the projection of E onto $\bar{D}$ with kernel K;*
  (c) *$R(\lambda)|_{\bar{D}}$ is a resolvent on $\bar{D}$ with range D.*

Corollary 2.2. *Let $(R(\lambda))_{\lambda > \omega}$ be a pseudo-resolvent on a reflexive Banach space such that $\sup_{\lambda > \omega} \| \lambda R(\lambda) \| < \infty$. Then the conclusions of Proposition 2.1(2) hold.*

Corollary 2.3. *Let E be a Banach lattice with order-continuous norm, A be a closed operator on E such that $(\omega, \infty) \subseteq \rho(A)$ and $\lambda R(\lambda, A) \to I$ as $\lambda \to \infty$, and $(R(\lambda))_{\lambda > \omega}$ be a pseudo-resolvent on E such that $0 \leqq R(\lambda) \leqq R(\lambda, A)$ for all $\lambda > \omega$. Then the conclusions of Proposition 2.1(2) hold. Moreover, P is a band projection.*

Proof. Let $f \in E_+$. Then $0 \leqq \lambda R(\lambda) f \leqq \lambda R(\lambda, A) f$. Moreover, $\{\lambda R(\lambda, A) f : \lambda > \omega\}$ is relatively (norm) compact, so $\{\lambda R(\lambda) f : \lambda > \omega\}$ is relatively weakly compact [2, Theorem 13.8]. Hence $\{\lambda R(\lambda) f : \lambda > \omega\}$ is relatively weakly compact for all f in E, so the hypothesis of Proposition 2.1(2) is satisfied. Moreover, $0 \leqq P = \lim_{\lambda \to \infty} \lambda R \;\; (\lambda) \leqq \lim_{\lambda \to \infty} \lambda R(\lambda, A) = I$, so P is a band projection [2, Theorem 3.10]. □

Remark. Let $E = L^p(X)$ for some σ-finite measure space (X, μ), $1 \leqq p < \infty$. If P is any positive bounded projection on E, and $R(\lambda) = \lambda^{-1} P$, then $\bar{D} = D = PE$. Since any closed sublattice F of E is the range of a positive contractive projection [35, Chap. III, Theorem 11.4], F is the (closed) range space D of a positive pseudo-resolvent. However, it has been shown in [8] that if $(R(\lambda))_{\lambda > \omega}$ is a positive resolvent on E, then $\bar{D}$ is a band, so $\bar{D} = L^p(Y)$ for some measurable subset Y of X.

3 Degenerate semigroups

Let E be a Banach space. For the purposes of this paper, we shall use the term *semigroup* to mean a strongly continuous map $T:(0,\infty)\to\mathscr{B}(E)$, such that $T(s+t)=T(s)T(t)$. Further, we shall say that T is *continuous* if $T(0):=\lim_{t\downarrow 0}T(t)$ exists (in the strong operator topology).

For any semigroup T, there are constants M and ω such that $\|T(t)\|\leqq Me^{\omega t}$ for all $t\geqq 1$. If $\int_0^1\|T(t)\|\,dt<\infty$, then we can define $R(\lambda)=\int_0^\infty e^{-\lambda t}T(t)\,dt(\lambda>\omega)$. It is routine to verify that $(R(\lambda))_{\lambda>\omega}$ is a pseudo-resolvent (see [4, Theorem 3.1]).

Now suppose that T is continuous. Then $T(0)$ is a bounded projection, and $T(0)T(t)=T(t)T(0)=T(t)$ for all $t>0$ [19, Theorem 10.5.5]. Thus $E=E_0\oplus E_1$, where $E_0=(I-T(0))E$, $E_1=T(0)E$, $T|_{E_0}=0$ and $T|_{E_1}$ is a C_0-semigroup. Moreover, $R(\lambda)=R(\lambda,A_1)T(0)$, where A_1 is the generator of $T|_{E_1}$, $\sup_{\lambda>\omega+1}\|\lambda R(\lambda)\|<\infty$, and $\lambda R(\lambda)\to T(0)$ as $\lambda\to\infty$.

The following proposition and its proof are analogous to Proposition 2.1 (see [34, Lemma 1], [23, Sect. 7.1, Theorem 1.11]).

Proposition 3.1. *Let T be a semigroup on E, and suppose that, for each f in E, $\{T(t)f:0<t\leqq 1\}$ is relatively weakly compact. Then T is continuous.*

Corollary 3.2. *Let T be a semigroup on a reflexive space such that $\sup_{0<t\leqq 1}\|T(t)\|<\infty$. Then T is continuous.*

Corollary 3.3. *Let T be a semigroup on a Banach lattice E with order-continuous norm, and suppose that there is a C_0-semigroup S on E such that $0\leqq T(t)\leqq S(t)$ $(t>0)$. Then T is continuous, and $T(0)$ is a band projection.*

Proof. This follows from Proposition 3.1 in the same way as Corollary 2.3 followed from Proposition 2.1. □

We have remarked above that any continuous semigroup provides a pseudo-resolvent, via the Laplace transform. For converse results, one should assume the Hille-Yosida condition:

$$\sup\{\|(\lambda-\omega)^n R(\lambda)^n\|:n\geqq 1,\lambda>\omega\}<\infty\,. \tag{3.1}$$

It was shown in [4, Theorem 6.2] that if E has the Radon-Nikodym property, $(R(\lambda))_{\lambda>\omega}$ sastisfies (3.1), and $R(\lambda)=R(\lambda,A)$ for some operator $A:D(A)\to E$ (with $D(A)$ not necessarily dense), then there is a semigroup T on E with $\sup_{t>0}e^{-\omega t}\|T(t)\|<\infty$ and $R(\lambda)=\int_0^\infty e^{-\lambda t}T(t)\,dt$. However, T may not be continuous [4, Example 6.4].

Proposition 3.4. *Let $(R(\lambda))_{\lambda>\omega}$ be a pseudo-resolvent on E satisfying the Hille-Yosida condition* (3.1), *and suppose that either*

(1) *E is reflexive,*

or

(2) *E is a Banach lattice with order-continuous norm, and there is a closed operator A on E such that $(\omega',\infty)\subseteq\rho(A)$ for some $\omega'>\omega$, $0\leqq R(\lambda)\leqq R(\lambda,A)$ $(\lambda>\omega')$, and $\lambda R(\lambda,A)\to I$ as $\lambda\to\infty$.*

Then there is a continuous semigroup T on E such that $\sup_{t>0}e^{-\omega t}\|T(t)\|<\infty$ and $R(\lambda)=\int_0^\infty e^{-\lambda t}T(t)\,dt$ $(\lambda>\omega)$.

Proof. By Corollary 2.2 or 2.3, $\lambda R(\lambda)\to P$ as $\lambda\to\infty$, where P is a bounded projection, $R(\lambda)P=R(\lambda)$, and $R(\lambda)|_{E_1}$ is a resolvent on $E_1:=PE$. By the

Hille-Yosida-Phillips Theorem, there is a C_0-semigroup T_1 on E_1 such that $R(\lambda)|_{E_1} = \int_0^\infty e^{-\lambda t} T_1(t)\,dt$ and $\sup_{t>0} e^{-\omega t} \| T_1(t) \| < \infty$. Define $T(t)f = T_1(t)Pf\,(f \in E)$. Then T has the required properties. □

4 Absorption semigroups

From now on, our Banach space E will be $L^p(X)$ for some σ-finite measure space (X, μ) and some $1 \leqq p < \infty$, $T = \{T(t): t \geqq 0\}$ will be a given positive C_0-semigroup on $L^p(X)$, A will be the generator of T, and ω will be the type (growth bound) of T: $\omega = \lim_{t\to\infty} t^{-1} \log \| T(t) \|$. Occasionally, we may write e^{tA} for $T(t)$.

Let $V: X \to [0, \infty]$ be a measurable function. Our objective is to associate a semigroup T_V, known as an *absorption semigroup* [45, 46], with T and V, in such a way that T_V can loosely be regarded as being "generated" by $A - V$. For V in $L^\infty(X)$, there is no problem about this, as $A - V$ is defined as an operator with $D(A - V) = D(A)$, and it generates a C_0-semigroup T_V on $L^p(X)$, by the theory of bounded perturbations. We shall obtain our (degenerate) semigroups by approximating V pointwise by an increasing sequence (V_n) in $L^\infty(X)$, and taking the limit, as $n \to \infty$, of the C_0-semigroups generated by $A - V_n$. Thus our strategy is similar to [45, 46], but the context differs in that we consider only non-negative potentials V, we make no constraints on the size of V, and we do not expect our limiting semigroup to be a C_0-semigroup on $L^p(X)$.

Lemma 4.1. *Let $(V_n), (\tilde{V}_n)$ be two sequences in $L^\infty(X)$ such that $V_n \uparrow V$, $\tilde{V}_n \uparrow V$ a.e. as $n \to \infty$. For $\lambda > \omega$, $\lim_{n\to\infty} R(\lambda, A - V_n)$ and $\lim_{n\to\infty} R(\lambda, A - \tilde{V}_n)$ exist and are equal. Moreover, for $t > 0$, $\lim_{n\to\infty} T_{V_n}(t)$ and $\lim_{n\to\infty} T_{\tilde{V}_n}(t)$ exist and are equal.*

Proof. The Trotter product formula $T_{V_n}(t) = \lim_{r\to\infty} (T(t/r) e^{-tV_n/r})^r$ shows that $T_{V_n}(t) \geqq 0$. Since $A - V_n = (A - V_{n+1}) + (V_{n+1} - V_n)$, the Trotter product formula also shows that $T(t) \geqq T_{V_n}(t) \geqq T_{V_{n+1}}(t) \geqq 0$, and hence $R(\lambda, A) \geqq R(\lambda, A - V_n) \geqq R(\lambda, A - V_{n+1}) \geqq 0$ $(\lambda > \omega)$. By the monotone convergence theorem, $\lim_{n\to\infty} R(\lambda, A - V_n)$ and $\lim_{n\to\infty} T_{V_n}(t)$ exist. Similarly, $\lim_{n\to\infty} R(\lambda, A - \tilde{V}_n)$ and $\lim_{n\to\infty} T_{\tilde{V}_n}(t)$ exist.

For any fixed m, $\| V_n \wedge \tilde{V}_m \| \leqq \| \tilde{V}_m \|$ for all n, and $V_n \wedge \tilde{V}_m \uparrow \tilde{V}_m$ pointwise a.e. and hence as operators on $L^p(X)$ as $n \to \infty$. For large λ,

$$R(\lambda, A - (V_n \wedge \tilde{V}_m)) = R(\lambda, A) \sum_{r=0}^{\infty} (-(V_n \wedge \tilde{V}_m) R(\lambda, A))^r \to R(\lambda, A - \tilde{V}_m)$$

as $n \to \infty$, and $T_{V_n \wedge \tilde{V}_m}(t) \to T_{\tilde{V}_m}(t)$ by the Trotter-Kato Theorem. But $R(\lambda, A - V_n) \leqq R(\lambda, A - (V_n \wedge \tilde{V}_m))$, $T_{V_n}(t) \leqq T_{V_n \wedge \tilde{V}_m}(t)$, so

$$\lim_{n\to\infty} R(\lambda, A - V_n) \leqq R(\lambda, A - \tilde{V}_m), \quad \lim_{n\to\infty} T_{V_n}(t) \leqq T_{\tilde{V}_m}(t)\,.$$

Letting $m \to \infty$ and then interchanging V_n and $\tilde{V}_n$, the result follows. □

Now define

$$T_V(t) = \lim_{n\to\infty} T_{V_n}(t) \quad (t > 0)\,,$$

$$R(\lambda, A - V) = \lim_{n\to\infty} R(\lambda, A - V_n) \quad (\lambda > \omega)\,,$$

where (V_n) is any increasing sequence in $L^\infty(X)_+$ such that $V_n \uparrow V$. Lemma 4.1 shows that these definitions are independent of the choice of sequence (V_n), and it is easily checked that $0 \leqq T_V(t) \leqq T(t)$ and $T_V(s+t) = T_V(s)T_V(t)$ $(s, t > 0)$. Since T_V is strongly measurable, it is strongly continuous for $t > 0$ [19, Theorem 10.2.3], so T_V is a semigroup in our terminology. By Corollary 3.3, T_V is a continuous semigroup and $T_V(0)$ is a band projection, so there is a measurable subset X_V of X such that $T_V(0) = \mathbf{1}_{X_V}$. Writing $X_V^c = X \backslash X_V$, $T_V|_{L^p(X_V^c)} = 0$, $T_V|_{L^p(X_V)}$ is a C_0-semigroup on $L^p(X_V)$, and $R(\lambda, A - V)f = R(\lambda, A_V)\mathbf{1}_{X_V} f$, where A_V is the generator of this C_0-semigroup. We could alternatively use Corollary 2.3 and the results of [11] to derive these properties of T_V. The generator A_V is the graph limit of $(A - V_n)$ in the following sense [11]:

$$D(A_V) = \{f \in L^p(X_V): \text{there exist } (f_n) \text{ in } D(A),\ g \text{ in } L^p(X_V)$$
$$\text{such that } \|f_n - f\|_p \to 0,\ \|Af_n - V_n f_n - g\|_p \to 0\},$$
$$A_V f = g.$$

It follows from the construction that if $V = \tilde{V}$ a.e., then $T_V = T_{\tilde{V}}$, and if $V \leqq \tilde{V}$ (a.e.), then $T_V \geqq T_{\tilde{V}}$. The monotone convergence theorem shows that

$$R(\lambda, A - V) = \lim_{n\to\infty} R(\lambda, A - V_n) = \lim_{n\to\infty} \int_0^\infty e^{-\lambda t} T_{V_n}(t)\,dt$$
$$= \int_0^\infty e^{-\lambda t} T_V(t)\,dt \quad (\lambda > \omega).$$

The Dominated Convergence Theorem shows that

$$\lim_{n\to\infty} R(\lambda, A - V_n) = \int_0^\infty e^{-\lambda t} T_V(t)\,dt = R(\lambda, A_V)\mathbf{1}_{X_V} \tag{4.1}$$

whenever $\operatorname{Re}\lambda > \omega$.

The reader should be warned that $R(\lambda, A - V)$ is not necessarily the resolvent of an operator $A - V$. Moreover, the set X_V is determined only up to null sets. Thus an inequality of the form $T_V(0) \leqq \mathbf{1}_Y$ (where Y is a measurable subset of X) means that $X_V \backslash Y$ is null, or equivalently, $L^p(X_V) \subseteq L^p(Y)$.

Proposition 4.2. *Let $V_n: X \to [0, \infty]$ be measurable functions such that $V_n \uparrow V$ a.e. as $n \to \infty$. Then $T_{V_n}(t) \downarrow T_V(t)$ $(t > 0)$ and $R(\lambda, A - V_n) \downarrow R(\lambda, A - V)$ $(\lambda > \omega)$ as $n \to \infty$.*

Proof. Let $f \in L^p(X)_+$. Since $(T_{V_n}(t)f)$ is a decreasing sequence in $L^p(X)_+$, it converges in norm to its pointwise infimum, and

$$\lim_{n\to\infty} T_{V_n}(t)f = \inf_n \inf_k T_{V_n \wedge k1}(t)f = \inf_k \inf_n T_{V_n \wedge k1}(t)f = \inf_k T_{V \wedge k1}(t)f = T_V(t)f,$$

where Lemma 4.1 is used for the third equality. The other equality is similar. □

If F is a subspace of $L^p(X)$, the disjoint complement $F^\perp$ of F is $L^p(Y)$ where Y is the largest (up to null sets) measurable subset of X such that each f in F vanishes a.e. in Y. Similarly, $F^{\perp\perp}$, the band generated by F, is $L^p(Y)$ where Y is the smallest measurable subset such that each f in F vanishes a.e. in Y^c. The following simple result extends [45, Proposition 2.9].

Proposition 4.3. *For any f in $D(A) \cap D(V)$, $f = T_V(0)f \in D(A_V) \subseteq L^p(X_V)$ and $A_V f = T_V(0)Af - Vf$. Hence $(D(A) \cap D(V))^{\perp\perp} \subseteq L^p(X_V) = T_V(0)L^p(X)$. If $(D(A) \cap D(V))^{\perp} = \{0\}$, then T_V is a C_0-semigroup on $L^p(X)$.*

Proof. Let $\lambda > \omega$, $g = (\lambda I - (A - V))f$. Let (V_n) be a sequence in $L^\infty(X)$ increasing to V. Then $g = (\lambda I - A + V_n)f + (V - V_n)f$, so

$$R(\lambda, A - V_n)g = f + R(\lambda, A - V_n)(V - V_n)f .$$

But $\|R(\lambda, A - V_n)\| \leqq \|R(\lambda, A)\|$ and $\|(V - V_n)f\| \to 0$ as $n \to \infty$, so $f = R(\lambda, A - V)g = R(\lambda, A_V)T_V(0)g \in D(A_V) \subseteq L^p(X_V)$. Since $T_V(0)$ is a band projection, all the results follow. □

The converses of Proposition 4.3 do not hold. In Example 7.2, we will show that it is possible that $D(A) \cap D(V) = \{0\}$, but T_V is a C_0-semigroup. Nevertheless, the next proposition does establish an upper bound for $T_V(0)$.

Proposition 4.4. *Let $W_V = \{x \in X : V(x) < \infty\}$. Then $T_V(0) \leqq \mathbf{1}_{W_V}$.*

Proof. Let $M = W_V^c = \{x \in X : V(x) = \infty\}$, and put

$$\chi(x) = \begin{cases} \infty & (x \in M) \\ 0 & (x \in W_V) \end{cases} .$$

Then $V \geqq \chi$, *so* $T_V(0) \leqq T_\chi(0)$.

Take $\lambda > \omega$. For f in $L^p(X)$,

$$R(\lambda, A)f - R(\lambda, A - n\mathbf{1}_M)f = nR(\lambda, A)\mathbf{1}_M R(\lambda, A - n\mathbf{1}_M)f .$$

Thus, if $g_n = R(\lambda, A)\mathbf{1}_M R(\lambda, A - n\mathbf{1}_M)f$, then $ng_n \to R(\lambda, A)f - R(\lambda, A - \chi)f$, so $g_n \to 0$, and $R(\lambda, A)\mathbf{1}_M R(\lambda, A - \chi)f = 0$. Hence, $\mathbf{1}_M R(\lambda, A - \chi)f = 0$. Since $T_\chi(0)$ is the projection of $L^p(X)$ onto the closure of the range of $R(\lambda, A - \chi)$, it follows that $\mathbf{1}_M T_V(0) \leqq \mathbf{1}_M T_\chi(0) = 0$, so the result is proved. □

Proposition 4.4 establishes the inclusion: $L^p(X_V) = T_V(0)L^p(X) \subseteq L^p(EW_V)$. This inclusion may be strict (see Example 4.7), but equality does hold in the norm-continuous case. (By saying that T is norm-continuous, we mean that $t \mapsto T(t)$ is continuous in the operator norm on $[0, \infty)$.)

Corollary 4.5. *Suppose that T is norm-continuous. Then $L^p(X_V) = L^p(W_V)$, $D(A_V) = D(V)$ and $A_V f = \mathbf{1}_{W_V} Af - Vf$ $(f \in D(V))$.*

Proof. Since $D(A) = L^p(X)$ and $\overline{D(V)} = L^p(W_V)$, this follows from Propositions 4.3 and 4.4. □

Now let $V, \tilde{V} : X \to [0, \infty]$ be measurable functions. We can use the construction of this section to form the (degenerate) continuous semigroup T_V, which is a C_0-semigroup on $L^p(X_V)$ and vanishes on $L^p(X_V^c)$. We can then repeat the construction to form the continuous semigroup $(T_V)_{\tilde{V}}$, which will be a C_0-semigroup on part of $L^p(X_V)$ and vanish on $L^p(X_V^c)$. Similarly, we can construct

$$R(\lambda, (A - V) - \tilde{V}) := \left\{ \begin{matrix} R(\lambda, A_V - \tilde{V}_1) & \text{on } L^p(X_V) \\ 0 & \text{on } L^p(X_V^c) \end{matrix} \right\} = \int_0^\infty e^{-\lambda t} (T_V)_{\tilde{V}}(t)\, dt ,$$

where $\tilde{V}_1 = \tilde{V}|_{X_V}$.

Proposition 4.6. *Let $V, \tilde{V} : X \to [0, \infty]$ be measurable functions. Then $(T_V)_{\tilde{V}} = T_{V + \tilde{V}}$, and $R(\lambda, (A - V) - \tilde{V}) = R(\lambda, A - (V - \tilde{V}))$ $(\lambda > \omega)$.*

Proof. Firstly, suppose that $\tilde{V} \in L^\infty(X)$. Let (V_n) be a sequence in $L^\infty(X)$ increasing to V. Choose $\lambda_0 > \omega$ such that $\|\tilde{V}\| \; \|R(\lambda_0, A)\| := q < 1$. For $\lambda > \lambda_0$, $\|\tilde{V}\| \; \|R(\lambda, A - V_n)\| \leqq q$, so

$$R(\lambda, A - (V_n + \tilde{V})) = R(\lambda, A - V_n)(I - \tilde{V}R(\lambda, A - V_n))^{-1}$$

$$= R(\lambda, A - V_n) \sum_{r=0}^{\infty} (\tilde{V}R(\lambda, A - V_n))^r .$$

This series is uniformly convergent in n, so, letting $n \to \infty$,

$$R(\lambda, A - (V + \tilde{V})) = R(\lambda, A - V) \sum_{r=0}^{\infty} (\tilde{V}R(\lambda, A - V))^r .$$

For f in $L^p(X_V)$, $R(\lambda, A - V)f = R(\lambda, A_V)f$, so

$$R(\lambda, A - (V + \tilde{V}))f = R(\lambda, A_V) \sum_{r=0}^{\infty} (\tilde{V}R(\lambda, A_V))^r f$$

$$= R(\lambda, A_V - \tilde{V}_1)f = R(\lambda, (A - V) - \tilde{V})f .$$

For f in $L^p(X_V^c)_+$, $0 \leqq R(\lambda, A - (V + \tilde{V}))f \leqq R(\lambda, A - V)f = 0$, and $R(\lambda, (A - V) - \tilde{V})f = 0$, by definition. Thus, $R(\lambda, (A - V) - \tilde{V}) = R(\lambda, A - (V + \tilde{V}))$ whenever $\tilde{V} \in L^\infty(X)$. This holds for $\lambda > \lambda_0$, and hence for $\lambda > \omega$ by analytic continuation.

Now, consider a general $\tilde{V}$, and let $(\tilde{V}_n)$ be a sequence in $L^\infty(X)$ increasing to $\tilde{V}$. Then $R(\lambda, A - (V + \tilde{V}_n)) \downarrow R(\lambda, A - (V + \tilde{V}))$ by Proposition 4.2, and $R(\lambda, (A - V) - \tilde{V}_n) \downarrow R(\lambda, (A - V) - \tilde{V})$, from the definition. Hence

$$R(\lambda, A - (V + \tilde{V})) = \lim_{n \to \infty} R(\lambda, A - (V + \tilde{V}_n))$$

$$= \lim_{n \to \infty} R(\lambda, (A - V) - \tilde{V}_n) = R(\lambda, (A - V) - \tilde{V}) .$$

The fact that $(T_V)_{\tilde{V}} = T_{V + \tilde{V}}$ follows from uniqueness of Laplace transforms (or directly by taking suitable limits in the Dyson-Phylips formula). □

Example 4.7. Let $X = \mathbf{R}$ (with Lebesgue measure), and T be the translation semigroup:

$$(T(t)f)(x) = f(x + t) .$$

For V in $L^\infty(\mathbf{R})$, the perturbed semigroup is given by:

$$(T_V(t)f)(x) = \exp\left(- \int_x^{x+t} V(s)\, ds \right) f(x + t) .$$

It follows, on taking a limit, that this formula holds for arbitrary $V \geqq 0$. Thus

$$X_V = \left\{ x \in \mathbf{R} : \int_x^{x+\delta} V(s)\, ds < \infty \text{ for some } \delta > 0 \right\} .$$

It is not difficult to give a direct proof that $V(x) < \infty$ for almost all x in X_V, thus verifying Proposition 4.4.

Let $V:\mathbf{R}\to[0,\infty)$ be any measurable function with finite values which is not integrable over any open interval in $\mathbf{R}$. (For example, let $\{q_n : n \geqq 1\}$ be an enumeration of $\mathbf{Q}$, choose $\delta_n > 0$, $c_n > 0$ such that $\sum_{n=1}^{\infty}\delta_n < \infty$, $c_n\delta_n \to \infty$ as $n\to\infty$, and let $I_n = (q_n - \delta_n, q_n + \delta_n)$, $V = \sum_{n=1}^{\infty} c_n \mathbf{1}_{I_n}$; since $\limsup_{n\to\infty} I_n$ is nul, $V(x) < \infty$ a.e.) Then $T_V = 0$, and X_V is empty. This shows that the inclusion in Proposition 4.4 may be strict.

Example 4.8 (Feller semigroups). Let X be a second countable, locally compact, Hausdorff space, and μ be a stictly positive Radon measure on X. Recall that a *Feller semigroup* is a positive, contractive, C_0-semigroup T on $C_0(X)$. For any Feller semigroup T, there is a strong Markov process $Z = \{Z(t) : t \geqq 0\}$ with right-continuous paths in X such that

$$(T(t)f)(x) = \mathbf{E}^x[f(Z(t))] \tag{4.2}$$

[16, Chap. 4], [44, Chap. 2], [15, Sect. 2]. Here, $\mathbf{E}^x$ denotes expectation with respect to the probability measure $\mathbf{P}^x$ corresponding to the process starting at x. We shall assume that T interpolates to give C_0-semigroups on each $L^p(X)(1 \leqq p < \infty)$; this is automatic in the symmetric case [15, Proposition 2.5]. These semigroups will also be denoted by T; (4.2) remains valid for f in $L^p(X)$.

For V in $L^\infty(\mathbf{R}^N)$, the perturbed semigroup T_V is given by the Feynman-Kac formula:

$$(T_V(t)f)(x) = \mathbf{E}^x\left[\exp\left(-\int_0^t V(Z(s))\,ds\right)f(Z(t))\right]. \tag{4.3}$$

It follows, on taking a limit, that the Feynman-Kac formula is valid for arbitrary $V \geqq 0$. With the aid of Blumenthal's Zero-One Law, it is easily seen that

$$X_V = \left\{x\in X : \mathbf{P}^x\left[\int_0^t V(Z(s))\,ds < \infty \text{ for some } t>0\right] = 1\right\}$$

(see [26]). Thus Proposition 4.4 expresses the fact that $V(x) < \infty$ for almost all points x such that $\int_0^t V(Z(s))\,ds < \infty$ for small t if the process starts at x.

In the case when $X = \mathbf{R}^N$ (with Lebesgue measure), and T is the Gaussian semigroup: $(T(t)f)(x) = (f * p_t)(x)$, where p_t denotes the Gaussian kernel: $p_t(x) = (2\pi t)^{-N/2} e^{-|x|^2/2t}$, then Z is N-dimensional Brownian motion, and the generator A is given by: $A = \frac{1}{2}\Delta_p$, where Δ_p is the Laplacian on $L^p(\mathbf{R}^N)$ with appropriate domain. The Feynman-Kac formula (4.3) shows that it is consistent with the literature ([26, 36, 9, 45, 15] etc.) to call T_V a *Schrödinger semigroup* when T is the Gaussian semigroup, and a *generalized Schrödinger semigroup* when T is a Feller semigroup. Readers who prefer to define Schrödinger semigroups via quadratic forms should turn to Sect. 7.

Example 4.9 (Elliptic operators). Let $X = \mathbf{R}^N$ and μ be Lebesgue measure. Consider a real strongly elliptic operator

$$A = \sum_{i,j=1}^{N} a_{i,j}D_iD_j + \sum_{i=1}^{N} b_iD_i + c\,, \tag{4.4}$$

or

$$A = \sum_{i,j=1}^{N} D_i(a_{ij}D_j) + \sum_{i=1}^{N} b_i' D_i + c\,, \tag{4.5}$$

Here, D_i is the partial derivative: $D_i f = \dfrac{\partial f}{\partial x_i}$, the coefficients a_{ij}, b_i, b_i', c are real functions in $L^\infty(\mathbf{R}^N)$, and there is a constant $\gamma > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqq \gamma \sum_{i=1}^{N} \xi_i^2 \quad (x \in \mathbf{R}^N, \xi_i \in \mathbf{R})\,. \tag{4.6}$$

Without essential loss of generality, we assume that $a_{ij} = a_{ji}$. If the coefficients a_{ij} are Lipschitz continuous, then (4.4) and (4.5) are equivalent, but if a_{ij} are not differentiable, then (4.5) has only a formal sense. Nevertheless, there is a quadratic form $\mathbf{a}_0$ associated with A:

$$D(\mathbf{a}_0) = W^{1,2}(\mathbf{R}^N)\,,$$

$$\mathbf{a}_0(f) = \sum_{i,j=1}^{N} \int a_{ij}(D_i f)(D_j \bar{f}) - \sum_{i=1}^{N} \int b_i'(D_i f)\bar{f} - \int c|f|^2\,.$$

The (non-symmetric) Beurling-Deny criteria [25, 28, Sect. 4] can be applied to show that there is a positive semigroup T on $L^2(\mathbf{R}^N)$ which interpolates to provide positive semigroups on each $L^p(\mathbf{R}^N)$ for $2 \leqq p < \infty$. If the coefficients b_i' are Lipschitz continuous, then duality arguments show that T also interpolates to $L^p(\mathbf{R}^N)$ for $1 \leqq p \leqq 2$. This also occurs if $a_{ij} \in W^{3,\infty}(\mathbf{R}^N)$ and $b_i', c \in L^\infty(\mathbf{R}^N)$ [33, Chap. V, Theorem 2.7]. We shall write A_p for the generator of T on $L^p(\mathbf{R}^N)$ whenever this exists. If $a_{ij} \in W^{3,\infty}(\mathbf{R}^N)$, then A_p is the closure of the operator A defined by (4.4) with $D(A) = C_c^\infty(\mathbf{R}^N)$ [33, Chap. V, Theorem 2.7]. Further details of the quadratic form approach are given in Section 7, but here we relate this example to Example 4.8.

In many cases, A is associated with a diffusion process Z, a strong Markov process with continuous paths, and T is given by (4.2). This occurs in the symmetric case if $c \leqq 0$, or, more generally if each b_i' is Lipschitz continuous and c is sufficiently negative, as $\mathbf{a}_0$ is a regular Dirichlet form with the local property [17, 25]. Then the discussion of Example 4.8 is applicable, and T_V is given by (4.3), even though T may not define a Feller semigroup on $C_0(\mathbf{R}^N)$. Sufficient conditions on the coefficients for this last property are given in [16, Theorem 1.6, p. 370; 15, Example 2.12], etc. We shall call T the *diffusion semigroup* associated with A, and T_V an *absorption-diffusion semigroup*. By extension, we shall also use this terminology in all cases when T defines a semigroup on $L^p(\mathbf{R}^N)$.

When $A = \frac{1}{2}\Delta$, then T is the Gaussian semigroup, and T_V is the Schrödinger semigroup as in Example 4.8.

Remark. Example 4.9, and the forthcoming discussion of elliptic operators in Example 5.6, Proposition 5.7, Theorem 6.2, and Section 7, can be extended to the case when X is a second countable, connected, Lie group, μ is left Haar measure, and D_i is the left regular representation of some chosen basis of the Lie algebra. The necessary semigroup properties are given in [7, 33].

5 Barred semigroups

As in Sect. 4, we take as given a positive C_0-semigroup T on $L^p(X)(1 \leqq p < \infty)$ with generator A. Let Y be a measurable subset of X, $Y^c = X \setminus Y$, and

$$\chi_{Y^c}(x) = \begin{cases} \infty & (x \in Y^c) \\ 0 & (x \in Y) \,. \end{cases}$$

When $V = \chi_{Y^c}$, we shall write T_Y, A_Y, and Y^*, instead of T_V, A_V, and X_V, respectively. We shall call T_Y a *barred semigroup*, as the potential χ_{Y^c} represents a form of barrier to the semigroup (see Examples 5.4 and 5.5).

By construction and Proposition 4.4, $0 \leqq T_Y(t) \leqq T(t)$, $Y^* \setminus Y$ is null, and $T_Y(t)$ maps $L^p(X)$ into $L^p(Y)$ and vanishes on $L^p(Y^c)$. If $(D(A) \cap L^p(Y))^\perp = L^p(Y^c)$, then $\mathbf{1}_{Y^*} = T_Y(0) = \mathbf{1}_Y$, by Proposition 4.3.

Proposition 5.1. *With respect to the ordering of operators on $L^p(X)$, T_Y is the largest semigroup S such that (a) $0 \leqq S(t) \leqq T(t)$ and (b) $S(t)$ maps $L^p(X)$ into $L^p(Y)$ $(t > 0)$.*

Proof. Let S be any semigroup such that $0 \leqq S(t) \leqq T(t)$ and $S(t)$ maps $L^p(X)$ into $L^p(Y)$. For f in $L^p(X)_+$, $n \geqq 1$,

$$0 \leqq S(t)f = \mathbf{1}_Y S(t)f \leqq \mathbf{1}_Y T(t)f \leqq e^{-tn\mathbf{1}_{Y^c}} T(t)f \,.$$

Thus

$$S(t)f = S(t/m)^m f \leqq (e^{-(t/m)n\mathbf{1}_{Y^c}} T(t/m))^m f \,.$$

Letting $m \to \infty$, the Trotter product formula gives: $S(t)f \leqq T_{n\mathbf{1}_{Y^c}}(t)f$, and letting $n \to \infty$, we obtain $S(t)f \leqq T_Y(t)f$. □

A variant of Proposition 5.1 is that T_Y is the largest semigroup S such that (a) $0 \leqq S(t) \leqq T(t)$ and (b) $S(t)|_{L^p(Y^c)} = 0$ $(t > 0)$. It follows from Proposition 5.1, or its variant, that $T_Y|_{L^p(Y)}$ is the largest semigroup S on $L^p(Y)$ such that $0 \leqq S(t)f \leqq T(t)f$ $(t > 0, f \in L^p(Y)_+)$.

In the present context, Proposition 4.3 and Corollary 4.5 can be interpreted in the following way.

Proposition 5.2. *Let $f \in D(A) \cap L^p(Y)$. Then $f \in D(A_Y) \subseteq L^p(Y^*)$, and $A_Y f = \mathbf{1}_{Y^*} Af$. If T is norm-continuous, then $T_Y(0) = \mathbf{1}_Y$, $T_Y|_{L^p(Y)}$ is norm-continuous, and $A_Y f = \mathbf{1}_Y Af$ for all f in $L^p(Y)$.*

The multiplication operator $e^{-t\chi_{Y^c}}$ can naturally be interpreted as the projection $\mathbf{1}_Y$ of $Y^p(X)$ onto $L^p(Y)$. Thus the following result is a Trotter product formula for the perturbation $-\chi_{Y^c}$ of A. We do not know whether the corresponding formula holds for arbitrary absorptions V. However, if $p = 2$ and A is negative-definite and self-adjoint, then the formula does hold. This will follow from Proposition 7.1 and Kato's Theorem [13, Theorem 4.36; 21].

Theorem 5.3. *For any measurable subset Y of X,*

$$T_Y(t)f = \lim_{n\to\infty} (\mathbf{1}_Y T(t/n))^n f = \lim_{n\to\infty} (T(t/n)\mathbf{1}_Y)^n f \,(t > 0, f \in L^p(X)) \,.$$

Proof. We may suppose that $f \geqq 0$. Let $\varepsilon > 0$. There exists k such that $\| T_Y(t)f - T_k(t)f \| < \varepsilon$, where $T_k = T_{k\mathbf{1}_{Y^c}}$. By the Trotter product formula for bounded perturbations, there exists N such that $\|(e^{-\frac{t}{n}k1_{Y^c}}T(t/n))^n f - T_k(t)f\| < \varepsilon$ whenever $n \geqq N$. Since $T_Y(t) \leqq \mathbf{1}_Y T(t)$ for all t,

$$0 \leqq \left(\mathbf{1}_Y T\left(\frac{t}{n}\right)\right)^n f - T_Y(t)f \leqq \left(e^{-\frac{t}{n}k1_{Y^c}} T\left(\frac{t}{n}\right)\right)^n f - T_Y(t)f .$$

Hence

$$\left\|\left(\mathbf{1}_Y T\left(\frac{t}{n}\right)\right)^n f - T_Y(t)f\right\| \leqq \left\|\left(e^{-\frac{t}{n}k1_{Y^c}} T\left(\frac{t}{n}\right)\right)^n f - T_k(t)f\right\| + \|T_k(t)f - T_Y(t)f\|$$
$$< 2\varepsilon$$

whenever $n \geqq N$. This establishes one equality, and the other can be proved similarly. □

Example 5.4. Let $X = \mathbf{R}$ and T be the translation semigroup of Example 4.7. Then

$$Y^* = \{x \in \mathbf{R} : (x, x + \delta)\setminus Y \text{ is null for some } \delta > 0\} ,$$

$$(T_Y(t)f)(x) = \begin{cases} f(x+t) & \text{if } (x, x+t)\setminus Y \text{ is null} , \\ 0 & \text{otherwise} . \end{cases}$$

If $Y = \mathbf{R}\setminus\bigcup_n I_n$, where I_n is as in Example 4.7, then Y^c has finite measure, but $T_Y = 0$, Y^* is empty.

Example 5.5. Let T be a Feller semigroup as in Example 4.8. Then

$$Y^* = \{x \in X : \mathbf{P}^c \text{ [there exists } t > 0 \text{ such that } Z(s) \in Y \text{ for almost all } s \leqq t] = 1\} .$$

In the context of Brownian motion on $\mathbf{R}^N$, this set Y^* arises in the Kac approach to potential theory [12, 42, 18]. Our notation agrees with [18], but conflicts with [42]. In the language of [42], Y^* is the set of points which are *strongly exterior irregular* for Y^c. Proposition 4.4 includes the semigroup proof [18] that $Y^*\setminus Y$ is null (see also [12, Theorem 9.3, Corollary 9.6; 42, p. 833; 9, Proposition 5.1]). The semigroup T_Y is given by:

$$(T_Y(t)f)(x) = \mathbf{E}^x[f(Z(t))\mathbf{1}_{[Z(s)\in Y \text{ for almost all } s \leqq t]}] . \tag{5.1}$$

Now, take Y to be an open subset Ω of X. Then $\Omega \subseteq \Omega^*$, so $L^p(\Omega) = L^p(\Omega^*)$, and T_Ω is a C_0-semigroup on $L^p(\Omega)$. Moreover, there is a related C_0-semigroup T^d_Ω on $L^p(\Omega)$, given by the formula:

$$(T^d_\Omega(t)f)(x) = \mathbf{E}^x[f(Z(t))\mathbf{1}_{[Z(s)\in \Omega \text{ for all } s \leqq t]}] . \tag{5.2}$$

It is clear from (5.1) and (5.2) that $T^d_\Omega(t) \leqq T_\Omega(t)$ and $T_{\Omega'}(t) \leqq T_\Omega(t)$ and $T^d_{\Omega'}(t) \leqq T^d_\Omega(t)$ if $\Omega' \subseteq \Omega$. (The first two of these statements also follow from Proposition 5.1) If each point of $\partial\Omega$ is regular for $\bar{\Omega}^c$ in the sense of [10, Definition 11.1], then $\Omega^* = \Omega$. Following [42, 18], we shall say that Ω^c is *Kac-regular* for Z if $\mathbf{P}^x[Z(\tau_\Omega) \in \Omega^*] = 0$ for all x in Ω, where $\tau_\Omega = \inf\{t \geqq 0 : Z(t) \in \Omega^c\}$. If Ω^c is Kac-regular, then a straightforward application of the strong Markov property shows that $T^d_\Omega = T_\Omega$ (see [18]).

Now suppose that Z has continuous paths (for a sufficient condition, see [16, Proposition 2.9, p. 171]). Let (Ω_n) be an increasing sequence of open subsets of

Ω such that $\bar{\Omega}_n \subseteq \Omega$ and $\Omega = \bigcup_n \Omega_n$. Then it is clear from (5.1), (5.2) and the continuity of paths that

$$T_\Omega^d(t)f = \sup_n T_{\Omega_n}(t)f \; (f \in L^p(\Omega)_+, t > 0) . \tag{5.3}$$

Let $A_{\Omega,p}^d$ and $A_{\Omega,p}$ be the generators of T_Ω^d and T_Ω respectively. It follows from the monotone convergence theorem that

$$R(\lambda, A_{\Omega,p}^d)f = \int_0^\infty e^{-\lambda t} T_\Omega^d(t) f dt = \lim_{n\to\infty} \int_0^\infty e^{-\lambda t} T_{\Omega_n}(t) f dt = \lim_{n\to\infty} R(\lambda, A_{\Omega_n,p}) \mathbf{1}_{\Omega_n} f , \tag{5.4}$$

for $\lambda > 0$ and for all f in $L^p(\Omega)_+$ and hence for f in $L^p(\Omega)$.

We have shown that T_Ω^d is given by a double limiting procedure:

$$T_\Omega^d(t) = \lim_{n\to\infty} \lim_{k\to\infty} T_{k1_{\Omega_n^c}}(t) . \tag{5.5}$$

This has also been established in [15, Theorem 4.3]. If $V: \Omega \to [0, \infty]$ is measurable, we can apply the construction of Sect. 4 to obtain the absorption semigroup $T_{\Omega,V}^d = (T_\Omega^d)_V$; this is also given by a double limit:

$$T_{\Omega,V}^d(t)f = (T_\Omega^d)_V(t)f = \lim_{n\to\infty} \lim_{k\to\infty} T_{V_{nk}}(t)f$$

$$= \mathbf{E}\left[\exp\left(-\int_0^t V(Z(s))\,ds\right) f(Z(t)) \mathbf{1}_{[Z(s)\in\Omega \text{ for all } s \leqq t]}\right] . \tag{5.6}$$

where $V_{nk}(x) = k(x \in \Omega_n^c)$, $V_{nk}(x) = V(x) \wedge k(x \in \Omega_n)$ (see, [15, Theorem 4.4]).

Example 5.6. Let A be an elliptic operator as in Example 4.9, T be the associated diffusion semigroup on $L^p(\mathbf{R}^N)$, and Ω be an open subset of $\mathbf{R}^N$. If each b_i' is Lipschitz continuous and c is sufficiently negative, then A is associated with a diffusion process Z with continuous paths, T_Ω is defined by (5.1), and T_Ω^d by (5.2) or (5.5). Then T_Ω^d is the semigroup associated with the operator A with Dirichlet boundary conditions on Ω (see [36, Theorem 21.1; 29, Sect. 7.3; 16, Theorem 1.4, p. 368; 15, Sect. 4] and Sect. 7 of this paper). For arbitrary c in $L^\infty(\mathbf{R}^N)$, T_Ω^d may be defined by (5.5), or equivalently by letting $S(t) = e^{-\lambda t}T(t)$ for a large constant λ, forming S_Ω^d, and then putting $T_\Omega^d(t) = e^{\lambda t} S_\Omega^d(t)$. For arbitrary coefficients, T_Ω^d may be defined by (5.5). (Although it is not clear that this is independent of the choice of (Ω_n), we shall see in Sect. 7 that this is so if each Ω_n is bounded.) We shall call T_Ω^d and T_Ω the *Dirichlet diffusion semigroup* and the *pseudo-Dirichlet diffusion semigroup* associated with A on Ω, and we shall again denote their generators on $L^p(\Omega)$ by $A_{\Omega,p}^d$ and $A_{\Omega,p}$ respectively. When $V: \Omega \to [0, \infty]$ is measurable, we shall call $T_{\Omega,V}^d$, given by (5.6), the *Dirichlet absorption-diffusion semigroup* associated with A and V on Ω.

Suppose that A is given in the form (4.5) where a_{ij} and b_i' are Lipschitz continuous on Ω. Then the adjoint A^* is defined as an elliptic operator on $C_c^\infty(\Omega)$. Let $A_{\Omega,p,\max}$ be the maximal operator on $L^p(\Omega)$ associated with A, defined in the distributional sense:

$$D(A_{\Omega,p,\max}) = \{f \in L^p(\Omega): \text{there exists } g \text{ in } L^p(\Omega) \text{ such that } \langle A^*\varphi, f\rangle = \langle \varphi, g\rangle \, (\varphi \in C_c^\infty(\Omega))\} ,$$

$$A_{\Omega,p,\max} f = g ,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard duality.

If Ω is of class C^2, then each point of $\partial\Omega$ is regular for $\bar{\Omega}^c$ with respect to the diffusion process associated with A, so $T_\Omega = T_\Omega^d$. If, in addition, Ω is bounded and the coefficients of A are sufficiently smooth, then

$$D(A_{\Omega,1}^d) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\ (1 < p < \infty)\,,$$

$$D(A_{\Omega,1}) = W_0^{1,1}(\Omega) \cap D(A_{\Omega,p,\max})\,,$$

$$A_{\Omega,p}^d f = A_{\Omega,p,\max} f\ (f \in D(A_{\Omega,p}^d),\ 1 \leqq p < \infty)$$

[29, Sect. 7.3] (see also [43]). In the absence of any assumption on Ω, we can easily establish the following information about $A_{\Omega,p}$ and $A_{\Omega,p}^d$. Recall that if $a_{ij} \in W^{3,\infty}(\mathbf{R}^N)$, then A_p is the closure in $L^p(\mathbf{R}^N)$ of the operator A defined by (4.4) on $C_c^\infty(\mathbf{R}^N)$ [33, Chap. V, Theorem 2.7], so $W^{2,p}(\mathbf{R}^N) \subseteq D(A_p)$.

Proposition 5.7. *Let A be a real strongly elliptic operator of order 2 on $\mathbf{R}^N$ with coefficients a_{ij} in $W^{3,\infty}(\mathbf{R}^N)$, $b_i \in W^{1,\infty}(\mathbf{R}^N)$, $c \in L^\infty(\mathbf{R}^N)$, and T be the associated semigroup on $L^p(\mathbf{R}^N)$. Let Ω be any open subset of $\mathbf{R}^N$. Then*

(1) $D(A_p) \cap L^p(\Omega) \subseteq D(A_{\Omega,p}) \subseteq D(A_{\Omega,p,\max})$,

(2) $W_0^{2,p}(\Omega) \subseteq D(A_{\Omega,p}^d) \subseteq D(A_{\Omega,p,\max})$.

Moreover, $A_{\Omega,p}$ and $A_{\Omega,p}^d$ are the restrictions of $A_{\Omega,p,\max}$ to their respective domains.

Proof. (1) The first inclusion is given by Proposition 5.2.

Let $f \in D(A_{\Omega,p})$, $\lambda > \omega$, and $g = \lambda f - A_{\Omega,p} f$. Then $f = R(\lambda, A_p - \chi_{\Omega^c})g = \lim_{n\to\infty} f_n$, where $f_n = R(\lambda, A_p - n\mathbf{1}_{\Omega^c})g \in D(\Delta_p)$. Hence $\lambda f_n - A_p f_n + n\mathbf{1}_{\Omega^c} f_n = g$. Let $\varphi \in C_c^\infty(\Omega)$. Then

$$\lambda\langle \varphi, f_n\rangle - \langle A^*\varphi, f_n\rangle = \langle \varphi, g\rangle\,.$$

Letting $n \to \infty$,

$$\lambda\langle \varphi, f\rangle - \langle A^*\varphi, f\rangle = \langle \varphi, g\rangle\,.$$

Thus $f \in D(A_{\Omega,p,\max})$ and

$$A_{\Omega,p,\max} f = \ = \lambda f - g = A_{\Omega,p} f\,.$$

(2) Let (Ω_n) be an increasing sequence of open sets such that $\bar{\Omega}_n \subseteq \Omega$, $\bigcup_n \Omega_n = \Omega$. Let $\lambda > \omega$.

Let $f \in C_c^\infty(\Omega)$. For large n, $\operatorname{supp} f \subseteq \Omega_n$. Let $g = \lambda f - Af = \lambda f - A_{\Omega_n,p} f$, by (1). Hence $f = R(\lambda, A_{\Omega_n,p})g \to R(\lambda, A_{\Omega,p}^d)g$, by (5.4), so $f = R(\lambda, A_{\Omega,p}^d)g \in D(A_{\Omega,p}^d)$ and $A_{\Omega,p}^d f = \lambda f - g = Af$. It follows on taking closures that $W_0^{2,p}(\Omega) \subseteq D(A_{\Omega,p}^d)$.

Now, let $f \in D(A_{\Omega,p}^d)$, $g = \lambda f - A_{\Omega,p}^d f$. By (5.4), $f = R(\lambda, A_{\Omega,p}^d)g = \lim_{n\to\infty} f_n$, where $f_n = R(\lambda, A_{\Omega_n,p})\mathbf{1}_{\Omega_n} g$. Let $\varphi \in C_c^\infty(\Omega)$. For large n, (1) gives

$$\lambda\langle \varphi, f_n\rangle - \langle A^*\varphi, f_n\rangle = \langle \varphi, \mathbf{1}_{\Omega_n} g\rangle = \langle \varphi, g\rangle = \lambda\langle \varphi, f\rangle - \langle \varphi, A_{\Omega,p}^d f\rangle\,.$$

Letting $n \to \infty$, it follows that $f \in D(A_{\Omega,p,\max})$ and $A_{\Omega,p,\max} f = A_{\Omega,p}^d f$. □

Remark. It should be noted that the results of this section are valid in a more general framework. Let E be a Banach lattice with order-continuous norm, T be a positive C_0-semigroup on D with generator A, and P be a band projection on E. We can define the degenerate semigroup $T_p(t) = \lim_{n\to\infty} e^{t(A - n(I-P))}$. Then T_P is a continuous positive semigroup, $T_p(0)$ is a band projection with $T_p(0) \leqq P$, and $T_P(t) = \lim_{n\to\infty}(PT(t/n))^n$. If T is norm-continuous then T_P is also norm-continuous, and $T_P(t) = e^{tPA}P$.

6 Holomorphic semigroups

In this section, we suppose that $p = 1$, and that T is a positive, contractive, holomorphic, semigroup on $L^1(X)$. We shall show that the absorption semigroup T_V associated with a measurable function $V: X \to [0, \infty]$ is also holomorphic and we shall apply this to (pseudo) Dirichlet diffusion semigroups.

Theorem 6.1. *Suppose that T is a positive, contractive, holomorphic, semigroup on $L^1(X)$, and $V: X \to [0, \infty]$ is measurable. Then $T_V|_{L^1(X_V)}$ is also holomorphic.*

Proof. Since T is holomorphic, there are positive constants c, r such that

$$\|R(\lambda, A)\| \leqq \frac{c}{|\lambda|} \quad (\operatorname{Re}\lambda > 0, |\lambda| > r)\,, \tag{6.1}$$

where A is the generator of T [27, A-II, Theorem 1.14].

First, suppose that $V \in L^\infty(X)_+$. We adapt the argument of [22]. Let $f \in D(A)$, $\operatorname{Re}\lambda > 0$. Since T is contractive, A is dissipative, so

$$\langle \operatorname{Re}((\operatorname{sign}\bar{f})Af), \mathbf{1}\rangle \leqq 0\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $L^1(X)$ and $L^\infty(X)$ [27, A-II.2]. Thus

$$\begin{aligned}\|(\lambda I - A + V)f\| &\geqq \langle \operatorname{Re}((\operatorname{sign}\bar{f})(\lambda I - A + V)f), \mathbf{1}\rangle \\ &= \langle \operatorname{Re}\lambda|f| + V|f|, \mathbf{1}\rangle - \langle \operatorname{Re}((\operatorname{sign}\bar{f})Af), \mathbf{1}\rangle \\ &\geqq (\operatorname{Re}\lambda)\|f\| + \|Vf\| \\ &\geqq \|Vf\|\,.\end{aligned}$$

By (6.1), if $\operatorname{Re}\lambda > 0$ and $|\lambda| > r$, then

$$|\lambda|\,\|f\| \leqq c\|(\lambda I - A)f\| \leqq c(\|\lambda I - A + V)f\| + \|Vf\|) \leqq 2c\|(\lambda I - A + V)f\|\,,$$

so

$$\|R(\lambda, A - V)g\| \leqq \frac{2c}{|\lambda|}\|g\| \quad (g \in L^1(X), \operatorname{Re}\lambda > 0, |\lambda| > r)\,. \tag{6.2}$$

Now, consider a general $V \geqq 0$. We complete the argument with a variant of resolvent convergence for holomorphic semigroups (see [11]). Taking a sequence (V_n) in $L^\infty(X)$ increasing to V, applying (6.2) to V_n, and taking the limit as $n \to \infty$, it follows from (4.1) that (6.2) is valid for all $V \geqq 0$. Restricting to g in $L^1(X_V)$, it follows that

$$\|R(\lambda, A_V)\| \leqq \frac{2c}{|\lambda|} \quad (\operatorname{Re}\lambda > 0, |\lambda| > r)\,. \tag{6.3}$$

Hence A_V generates a holomorphic semigroup T_V [27, C-II, Theorem 1.14]. □

The fact that the right-hand side of (6.3) is independent of V implies that there is an angle $\alpha > 0$, independent of V, such that T_V generates a holomorphic semigroup of angle (at least) α, with bounds independent of V. (This is easily seen by examining the proof of [27, C-II, Theorem 1.14], but the *maximal* angle α will depend on V in general.) This fact will also be relevant to the discussion of the Dirichlet diffusion semigroups in the following result. For V in $L^1_{\text{loc}}(\mathbf{R}^N)$, Kato [22] proved that the Schrödinger semigroup S_V is holomorphic on $L^1(\mathbf{R}^N)$. For Ω of class C^2, the Dirichlet diffusion semigroup T^d_Ω associated with a strongly elliptic operator is known to be holomorphic [3]. Our result removes these constraints on V and Ω.

Theorem 6.2. *Let A be a real strongly elliptic operator of order 2 on $\mathbf{R}^N$ with $a_{ij} \in W^{3,\infty}(\mathbf{R}^N)$, $b_i \in W^{1,\infty}(\mathbf{R}^N)$, $c \in L^\infty(\mathbf{R}^N)$, and T be the associated diffusion semigroup on $L^p(\mathbf{R}^N)(1 \leqq p < \infty)$. For any potential $V \geqq 0$, the asbsorption-diffusion semigroup T_V is holomorphic on $L^p(X_V)$. For any open subset Ω of $\mathbf{R}^N$, the pseudo-Dirichlet diffusion semigroup T_Ω and the Dirichlet diffusion semigroup T^d_Ω are holomorphic on $L^p(\Omega)$.*

Proof. By adding a constant to A, we may assume that T is contractive on $L^1(\mathbf{R}^N)$.

For $p = 2$, the results are standard because in the symmetric case the semigroups are associated with positive forms (see Section 7), and in the general case the first-order terms are a small perturbation [7, p. 385]. Alternatively, holomorphy may be derived from an appropriate sector condition for quadratic forms (see [1, 11, 25]). Since the semigroups interpolate on the L^p-spaces, it follows from the Stein Interpolation Theorem as in [31, Theorem X.55] that the semigroups are holomorphic for $1 < p < \infty$. Alternatively, once holomorphy is established for $p = 1$, it follows for other p by duality and interpolation, if each b_i is Lipschitz continuous.

Consider the case $p = 1$. The diffusion semigroup T is holomorphic on $L^1(\mathbf{R}^N)$ [33, Chap. 5, Theorem 2.7]. It follows from Example 4.9 and Theorem 6.1 that T_V is holomorphic on $L^1(X_V)$. In particular, T_Ω is holomorphic. Moreover, (6.2) shows that

$$\| R(\lambda, A_1 - \chi_{\Omega^c})g \| \leqq \frac{2c}{|\lambda|} \quad (g \in L^1(\mathbf{R}^N), |\lambda| > r, \operatorname{Re}\lambda > 0)\,, \tag{6.4}$$

where c, r are constants independent of Ω. Now, if we take Ω fixed, and (Ω_n) to be an increasing sequence of open sets with $\bar{\Omega}_n \subseteq \Omega$ and $\bigcup_n \Omega_n = \Omega$, then it follows from (5.4) and (6.4) that, for g in $L^1(\Omega)$, $|\lambda| > r$, and $\operatorname{Re}\lambda > 0$,

$$\| R(\lambda, A^d_{\Omega,1})g \| = \lim_{n\to\infty} \| R(\lambda, A_{\Omega_n,1})g \| = \lim_{n\to\infty} \| R(\lambda, A_1 - \chi_{\Omega^c_n})g \| \leqq \frac{2c}{|\lambda|} \| g \|\,,$$

so the semigroup T^d_Ω, generated by $A_{\Omega,1}$, is holomorphic.

Corollary 6.3. *Let S be the Gaussian semigroup on $L^p(\mathbf{R}^N)(1 \leqq p < \infty)$. For any potential $V \geqq 0$, the Schrödinger semigroup S_V is holomorphic on $L^p(X_V)$. For any open subset Ω* of $\mathbf{R}^N$, *the pseudo-Dirichlet semigroup S_Ω and the Dirichlet semigroup S^d_Ω are holomorphic on $L^p(\mathbf{R}^N)$.*

7 Quadratic forms

Let A be an elliptic operator as in Example 4.9, and T be the associated diffusion semigroup. We saw that we could define the absorption-diffusion semigroup by the limiting procedure of Sect. 4, and we reconciled this with the Feynman-Kac formula (4.3). In Example 5.6, we defined Dirichlet diffusion semigroups by a second limiting procedure, and we reconciled this with the formula (5.2). An accepted alternative method of defining absorption-diffusion semigroups and Dirichlet diffusion semigroups is to consider the appropriate quadratic form on $L^2(\mathbf{R}^N)$ or $L^2(\Omega)$, verify that it satisfies the Beurling-Deny criteria, and deduce that it is associated with a positive semigroup which interpolates to each L^p-space. In this section, we show that this method is consistent with the previous definitions. This will then complete proofs of Theorem 6.2 and other properties, which are completely free of probabilistic considerations. Since both approaches to absorption-diffusion and Dirichlet diffusion semigroups provide interpolating semigroups, it suffices to establish consistency for $p = 2$, where we use the convergence theory of quadratic forms. For general absorption semigroups, the lack of a suitable theorem for convergence "from below" for non-symmetric forms confines us to the symmetric case.

We show first that the absorption semigroups associated with a C_0-semigroup arising from a quadratic form are themselves associated with quadratic forms.

Suppose then that $p = 2$. Let $\mathbf{a}$ be a closed non-negative form on $L^2(X)$ with domain $D(\mathbf{a})$, and let $-A$ be the associated self-adjoint operator on $\overline{D(\mathbf{a})}$. We adopt the convention that $\mathbf{a}(f) = \infty$ for f in $L^2(X)\backslash D(\mathbf{a})$. Define linear operator on $L^2(X)$ by:

$$J(\lambda, \mathbf{a})f = (\lambda - A)^{-1} P_{\mathbf{a}} f, \quad U_{\mathbf{a}}(t)f = e^{tA} P_{\mathbf{a}} f (f \in L^2(X), \lambda > 0, t > 0),$$

where $P_{\mathbf{a}}$ is the orthogonal projection of $L^2(X)$ onto $\overline{D(\mathbf{a})}$. These form a pseudo-resolvent and a continuous semigroup on $L^2(X)$. The Beurling-Deny criteria [32, Sect. XIII.12, Appendix 1] (applied to the restriction of $\mathbf{a}$ to $\overline{D(\mathbf{a})}$) show that $U_{\mathbf{a}} \geqq 0$ if and only if $\mathbf{a}(|f|) \leqq \mathbf{a}(f)$ for all f in $L^2(X)$; under these conditions, $U_{\mathbf{a}}$ interpolates to form positive contraction semigroups on each $L^p(X)$ if and only if $\mathbf{a}(f \wedge 1) \leqq \mathbf{a}(f)$ for all f in $L^2(X)_+$.

Now suppose that $\mathbf{a}_0$ is a given closed non-negative form with dense domain in $L^2(X)$ such that $U_{\mathbf{a}_0} \geqq 0$. We shall write $-A$ for the self-adjoint operator associated with $\mathbf{a}_0$, and T for the corresponding semigroup: $T(t) = e^{tA} = U_{\mathbf{a}_0}(t)$. Let $V: X \to [0, \infty]$ be measurable, and define a closed non-negative form $\mathbf{b}_V$ by: $\mathbf{b}_V(f) = \int V|f|^2$. Then $\mathbf{a}_V := \mathbf{a}_0 + \mathbf{b}_V$ is a closed non-negative form with $D(\mathbf{a}_V) = D(\mathbf{a}_0) \cap D(\mathbf{b}_V) = \{f \in D(\mathbf{a}_0) : \int V|f|^2 < \infty\}$. We can therefore associate with V a pseudo-resolvent $J(\lambda, \mathbf{a}_V)$ and a degenerate semigroup $U_{\mathbf{a}_V}$.

Proposition 7.1. *In the notation above,* $U_{\mathbf{a}_V}(t) = T_V(t)$ $(t > 0)$ *and* $J(\lambda, \mathbf{a}_V) = R(\lambda, A - V)$ $(\lambda > 0)$. *Hence* $L^2(X_V) = \overline{D(\mathbf{a}_V)}$.

Proof. Let (V_n) be a sequence in $L^\infty(X)_+$ increasing to V. Then $U_{\mathbf{a}_{V_n}}(t) = e^{t(A - V_n)} = T_{V_n}(t)$, by the theory of quadratic forms and bounded perturbations. Moreover, $\mathbf{a}_{V_1} \leqq \mathbf{a}_{V_2} \leqq \ldots$, and $\mathbf{a}_V(f) = \lim_{n\to\infty} \mathbf{a}_{V_n}(f)$. The result therefore follows from the theory of convergence of forms [13, Theorem 4.32; 30, Theorem S14, p. 373]. □

If $\mathbf{a}$ is a closed sectorial form and $V = \chi_{\Omega^c}$ for some measurable subset Y of X, then Proposition 7.1 remains valid. The application of the convergence theory of forms is replaced by a comparison of the Kato-Simon version of the Trotter product formula for sectorial forms [21, p. 194] with the version of the Trotter product formula given in Theorem 5.3 above.

Now suppose that A is a symmetric strongly elliptic operator on $\mathbf{R}^N$. Formally,

$$A = \sum_{i,j=1}^{N} D_i(a_{ij}D_j),$$

where a_{ij} are real functions in $L^\infty(\mathbf{R}^N)$, $a_{ij} = a_{ji}$, and (4.6) holds. (We can ignore any term of order 0, as it forms a bounded perturbation.) The associated quadratic form $\mathbf{a}_0$ is:

$$D(\mathbf{a}_0) = W^{1,2}(\mathbf{R}^N), \quad \mathbf{a}_0(f) = \sum_{i,j=1}^{N} \int a_{ij}(D_i f)(D_j \bar{f}). \tag{7.1}$$

Proposition 7.1 shows that our definition of absorption-diffusion semigroups in Example 4.9 is consistent with their definition by means of the quadratic form $\mathbf{a}_V$ for $p = 2$ and interpolation for other values of p. Moreover,

$$L^2(X_V) = \overline{W^{1,2}(\mathbf{R}^N) \cap D(\mathbf{b}_V)}.$$

In particular, if Ω is open and $\int_\Omega V(x)\,dx < \infty$, then $L^2(\Omega) \subseteq L^2(X_V)$. This should be compared with Proposition 4.3, which shows that $L^2(X_V) \subseteq L^2(W_V)$.

Example 7.2. Let $N \geqq 2$, and $A = \frac{1}{2}\Delta$. Then there exists $V: \mathbf{R}^N \to [0, \infty)$ such that V is nowhere locally integrable, but $T_V(0) = I$ [41]. In the case $N = 2$, $D(A_2) = D(\Delta_2) = W^{2,2}(\mathbf{R}^N) \subseteq C_0(\mathbf{R}^2)$. Hence $D(A_2) \cap D(V) = \{0\}$, but T_V is a C_0-semigroup on $L^2(\mathbf{R}^2)$. Thus the inclusion in Proposition 4.4 may be strict.

Let Ω be an open subset of $\mathbf{R}^N$, A be a symmetric strongly elliptic operator, and $V = \chi_{\Omega^c}$. Proposition 7.1 shows that the pseudo-Dirichlet diffusion semigroup T_Ω on $L^2(\Omega)$ defined in Example 5.6 is associated with the form $\mathbf{a}_\Omega$ where $D(\mathbf{a}_\Omega) = W^{1,2}(\mathbf{R}^N) \cap L^2(\Omega)$, $\mathbf{a}_\Omega(f) = \mathbf{a}_0(f)\,(f \in D(\mathbf{a}_\Omega))$. Let (Ω_n) be an increasing sequence of bounded open sets such that $\bar{\Omega}_n \subseteq \Omega$, $\bigcup_n \Omega_n = \Omega$. The decreasing sequence of forms $(\mathbf{a}_{\Omega_n})$ has limit $\mathbf{b}$:

$$D(\mathbf{b}) = \bigcup_n D(\mathbf{a}_{\Omega_n}) = W^{1,2}(\mathbf{R}^N) \cap \bigcup_n L^2(\Omega_n), \quad \mathbf{b}(f) = \mathbf{a}_0(f)\;(f \in D(\mathbf{b})),$$

and the closure of $\mathbf{b}$ is the form $\mathbf{a}^d_\Omega$ associated with A with Dirichlet boundary conditions:

$$D(\mathbf{a}^d_\Omega) = W^{1,2}_0(\Omega), \quad \mathbf{a}^d_\Omega(f) = \mathbf{a}_0(f)\;(f \in D(\mathbf{a}^d_\Omega))$$

(see [14, Theorem 2.1.6]). The convergence theory of forms (a routine extension of [30, Theorem S16, p. 373] to forms which are not densely-defined) shows that $U_{\mathbf{a}^d_\Omega}(t) = \lim_{n\to\infty} T_{\Omega_n}(t)$. Comparing this with (5.3) shows that our definition of the Dirichlet diffusion semigroups T^d_Ω in Example 5.6 is consistent with the alternative definition by means of the form $\mathbf{a}^d_\Omega$ for $p = 2$ and by interpolation for other values of p. Moreover, knowing (by interpolation) that (5.3) holds for all p is sufficient for the proof of Theorem 6.2, so we have completed a non-probabilistic proof of the holomorphy of absorption-diffusion semigroups and Dirichlet diffusion semigroups defined on $L^p(\Omega)$ by means of symmetric forms. For Dirichlet diffusion semigroups, the argument may be extended to the non-symmetric case, using the remark following Proposition 7.1 and the theorem for convergence "from above" for sectorial forms (whose domains need not be dense) [20, Theorem 3.6, p. 455]).

Note that if Ω^c is Kac-regular, then $T^d_\Omega = T_\Omega$, so $\mathbf{a}^d_\Omega = \mathbf{a}_\Omega$ and $W^{1,2}(\mathbf{R}^N) \cap L^2(\Omega) = W^{1,2}(\Omega)$ (see [18, Theorem 2.1]).

8 Asymptotic stability

Let $A = \sum_{i,j=1}^{N} D_i(a_{ij}D_j)$ be a symmetric strongly elliptic operator on $\mathbf{R}^N$, and T be the corresponding diffusion semigroup on $L^p(\mathbf{R}^N)$, so that T is associated with the quadratic form $\mathbf{a}_0$ given by (7.1). Let Ω be an open subset of $\mathbf{R}^N$, $V:\Omega \to [0, \infty]$ be measurable, and $T^d_{\Omega,V} = (T^d_\Omega)_V$ be the corresponding Dirichlet absorption-diffusion semigroup, associated with the form $\mathbf{a}^d_{\Omega,V}$:

$$D(\mathbf{a}^d_{\Omega,V}) = \{f \in W_0^{1,2}(\Omega) : \int V|f|^2 < \infty\} ,$$

$$\mathbf{a}^d_{\Omega,V}(f) = \mathbf{a}_0(f) + \int V|f|^2.$$

We shall discuss the asymptotic behaviour of $T^d_{\Omega,V}$, firstly whether it is exponentially stable on $L^p(\Omega)$, that is: $\|T^d_{\Omega,V}(t)\| \to 0$ as $t \to \infty$. Since T, and hence $T^d_{\Omega,V}$, is ultracontractive, and its kernel p_t satisfies a Gaussian upper bound [14], an argument of Simon [37] shows that this condition is independent of p. Hence $T^d_{\Omega,V}$ is exponentially stable if and only if there is a constant $c > 0$ such that

$$\mathbf{a}^d_{\Omega,V}(f) \geqq c \int |f|^2 \; (f \in W_0^{1,2}(\Omega)) .$$

The strong ellipticity condition (4.6) implies that this condition on Ω is independent of A. In the case when A is the Laplacian and $\Omega = \mathbf{R}^N$, exponential stability was characterised in [5; 9, Sect. 4], and it is routine to modify the arguments for general Ω. We recall the following notions.

Let $\mathscr{F}$ be the class of all Borel sets F in $\mathbf{R}^N$ such that

$$\sup_{x \in \mathbf{R}^N} \mathbf{P}^x[B(s) \in F \text{ for all } s \leqq t] = 1 ,$$

where B is Brownian motion on $\mathbf{R}^N$. This condition is independent of $t\,(0 < t < \infty)$ [9, Proposition 4.2]. An open set Ω' belongs to $\mathscr{F}$ if and only the Dirichlet semigroup $S^d_{\Omega'}$ is not exponentially stable on $L^p(\Omega')$ (independent of p), or equivalently, Poincaré's inequality fails, that is,

$$\inf\{\int |\nabla f|^2 : f \in C_c^\infty(\Omega'), \int |f|^2 = 1\} = 0 .$$

The methods of [5, 9] establish the following. Many other variants of condition (2) may also be read off from [9].

Theorem 8.1. *Let $A = \sum_{i,j=1}^{N} D_i(a_{ij}D_j)$ be a symmetric strongly elliptic opetator on $\mathbf{R}^N$ with bounded coefficients, Ω be an open subset of $\mathbf{R}^N$, and $V:\Omega \to [0, \infty]$ be measurable. The following are equivalent:*

(1) *$T^d_{\Omega,V}$ is exponentially stable on $L^p(\Omega)$;*
(2) *$\int_F V(x)\,dx = \infty$ for all (closed) sets F in $\mathscr{F}$ contained in Ω.*

If $V \in L^1_{\text{loc}}(\Omega)$, then these conditions are equivalent to:

(3) *$\int_{\Omega'} V(x)\,dx = \infty$ for all open sets Ω' in $\mathscr{F}$ contained in Ω.*

Now let Z be a diffusion process associated with A, with continuous paths. Define classes $\mathscr{F}_A$ and $\tilde{\mathscr{F}}_A$ of Borel sets by:

$$F \in \mathscr{F}_A \Leftrightarrow \sup_{x \in \mathbf{R}^N} \mathbf{P}^x[Z(s) \in F \text{ for all } s \leqq t] = 1 ,$$

$$F \in \tilde{\mathscr{F}}_A \Leftrightarrow \sup_{x \in \mathbf{R}^N} \mathbf{P}^x[Z(s) \in F \text{ for almost all } s \leqq t] = 1 .$$

Similar considerations to those above show that

$$F \in \tilde{\mathscr{F}}_A \Leftrightarrow T_F \text{ is not exponentially stable on } L^\infty(F)$$
$$\Leftrightarrow T_F \text{ is not exponentially stable on } L^2(F)$$
$$\Leftrightarrow \inf\{\mathbf{a}_0(f): f \in W^{1,2}(\mathbf{R}^N) \cap L^2(F), \textstyle\int |f|^2 = 1\} = 0 ,$$

and that this last condition is independent of A. Since the kernel of T has Gaussian bounds, the argument of [9, Proposition 5.1] (see also Example 5.5) shows that $F \in \tilde{\mathscr{F}}_A \Leftrightarrow F \cup N \in \mathscr{F}_A$ for some null set N. Thus in Theorem 8.1, it is possible to replace $\mathscr{F}$ by $\mathscr{F}_A$ or $\tilde{\mathscr{F}}_A$.

Now suppose that A is not symmetric, but is associated with a diffusion process Z with continuous paths. We also suppose that the coefficients are sufficiently smooth to ensure that the kernel satisfies Gaussian upper and lower bounds (see [33, Chap. V]). Quadratic form methods are no longer suitable; in particular, it is not clear that exponential stability is independent of p, nor that $\tilde{\mathscr{F}}_A$ is independent of A. Nevertheless, it is possible to follow the arguments of [9], replacing Brownian motion by Z, and explicit formulae for p_t by Gaussian bounds, to obtain the following.

Theorem 8.2. *Let A be a real strongly elliptic operator on $\mathbf{R}^N$ associated with a diffusion process, and with sufficiently smooth bounded coefficients. Let Ω be an open subset of $\mathbf{R}^N$, and $V: \Omega \to [0, \infty]$ be measurable. The following are equivalent:*
(1) *$T^d_{\Omega,V}$ is exponentially stable on $L^\infty(\Omega)$;*
(2) *$\int_F V(x)\,dx = \infty$ for all (closed) sets F in $\mathscr{F}_A$ contained in Ω.*
If $V \in L^1_{\text{loc}}(\Omega)$, then these conditions are equivalent to:
(3) *$\int_{\Omega'} V(x)\,dx = \infty$ for all open sets Ω' in $\mathscr{F}_A$ contained in Ω.*

If the adjoint operator A^* is associated with a diffusion process, then exponential stability of $T^d_{\Omega,V}$ is characterised by conditions (2) and (3) of Theorem 8.2 with $\mathscr{F}_A$ replaced by $\mathscr{F}_{A^*}$.

We can also consider the question of strong stability of $T^d_{\Omega,V}$ on $L^1(\Omega)$, as in [6], [9, Sect. 3]. Although quadratic form methods are not applicable, there are technical complications concerning the Gaussian bounds in the non-symmetric case, so we confine our statement to the symmetric case. Let $\mathscr{E}_A$ be the class of all Borel subsets of $\mathbf{R}^N$ whose complements are transient for Z, so

$$F \in \mathscr{E}_A \Leftrightarrow \sup_{x \in \mathbf{R}^N} \mathbf{P}^x[Z(s) \in F \text{ for all } s \geqq 0] = 1 .$$

For an open set Ω',

$$\Omega' \notin \mathscr{E}_A \Leftrightarrow T^d_{\Omega'}(t)f \to 0 \text{ a.e. } (t \to \infty, f \in L^\infty(\Omega'))$$
$$\Leftrightarrow \| T^d_{\Omega'}(t)f \|_1 \to 0 \ \ (t \to \infty, f \in L^1(\Omega')) ,$$

where the last line depends on the symmetry of A. Following the method of [9, Sect. 3], using the Gaussian upper and lower bounds, we obtain the following.

Theorem 8.3. *Let $A = \sum_{i,j=1}^N D_i(a_{ij}D_j)$ be a symmetric strongly elliptic operator on $\mathbf{R}^N$ with bounded coefficients, Ω be an open subset of $\mathbf{R}^N$, and $V: \Omega \to [0, \infty]$ be measurable.*

(a) *If $N = 1$ or 2 and $V \not\equiv 0$ (on a set of positive measure), then*

$$\| T^d_{\Omega,V}(t)f \|_1 \to 0 \quad (t \to \infty, f \in L^1(\Omega)) . \tag{8.1}$$

(*b*) *If* $N \geqq 3$, *then* (8.1) *holds if and only if* $\int_F \frac{V(x)}{|x|^{N-2}}\,dx = \infty$ *for all (closed) sets* F *in* $\mathscr{E}_A$ *contained in* Ω. *If* $V \in L^1_{\mathrm{loc}}(\Omega)$, *then it suffices that* $\int_F \frac{V(x)}{|x|^{N-2}}\,dx = \infty$ *for all open sets* F *in* $\mathscr{E}_A$ *contained in* Ω.

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Note added in proof. Non-linear absorption semigroups have been studied recently by Ph. Bénilan and P. Wittbold: Absorption non-liñeaire. J. Funct. Anal., to appear.