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Riemannian manifolds whose Laplacians have purely continuous spectrum

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1 Introduction

Suppose M^n is a complete, noncompact, Riemannian manifold. The Laplacian Δ on M^n is given in local coordinates by

$$\Delta = \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g_{ij})}\, g^{ij} \frac{\partial}{\partial x_j} \right).$$

Here g_{ij} are the components of the metric tensor on M^n. Δ is essentially self-adjoint on $C_0^\infty(M^n)$. In general, the unbounded operator Δ on $L^2(M^n)$ may have both point and continuous spectrum. The purpose of this paper is to establish conditions, on the manifold M^n, which ensure that the spectrum is purely continuous.

In a seminal paper, Rellich [R] proved the absence of positive eigenvalues for the Laplace operator, in unbounded domains in $\mathscr{R}^n$. His approach relies on an integral identity which plays an important role in many different contexts. We state a general version of Rellich's indentity in Lemma 2.1. As a consequence of it, we obtain for $u \in L^2(M^n)$, with $\Delta u = -\lambda u$, the following general formula (see Theorem 2.6 below)

$$\int_{M^n} (X_{i,j} + X_{j,i}) u_i u_j = \int_{M^n} (|\nabla u|^2 - \lambda u^2) \operatorname{div} X \tag{1.1}$$

Here X is a C^1 vector field, on M^n, with bounded covariant derivatives $X_{i,j}$. In Corollary 2.8, we apply (1.1) to give criteria for the absence of L^2 eigenfunctions with positive eigenvalues. *Assume that* $X_{i,j} + X_{j,i} \geqq 0$. *Then* $-\Delta$ *has no* L^2 *eigenfunction, with positive eigenvalue, if one of the following holds:*

(i) *There exists* $p \in M^n$ *such that* $X_{i,j} + X_{j,i} > 0$ *at* p *and* $\operatorname{div} X \equiv n$ *on* M^n;

(ii) *For some* $b > 0$ *there exists* $\varepsilon = \varepsilon(M^n, b) > 0$ *sufficiently small such that* $X_{i,j} + X_{j,i} \geqq 2(b - \varepsilon) g_{ij}$ *and* $|\operatorname{div} X - n| < \varepsilon$ *on* M^n;

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(iii) $X_{i,j} + X_{j,i} \geqq (\operatorname{div} X - n) g_{ij}$, $0 \leqq \operatorname{div} X - n$ *and either inequality holds strictly at least at one point* $p \in M^n$.

Section 2 concludes with some extensions to exterior domains and eigenfunctions of Laplacians of conformally related metrics.

Sections 3 and 4 are concerned with some specific applications of Corollary 2.8. Here we take X to be the gradient of f, where f is a convex function. For rotationally symmetric metrics we construct convex functions f of constant Laplacian. We show that such f exists if and only if: (i) *The mean curvature of* $\partial B(r)$ *with respect to an outward pointing normal is* $\geqq 0$ *for* $r > 0$; (ii) *The ratio* $\dfrac{\operatorname{vol} \partial B(r)}{\operatorname{vol} B(r)}$ *is decreasing for* $r > 0$. In particular, see Theorem 3.9, if $K(r)$ is the radial sectional curvature of M^n, and either (i) $K(r) \geqq 0$, or (ii) $K(r) \leqq 0$ and $K'(r) \geqq 0$, then $-\Delta$ has no L^2 eigenfunction with positive eigenvalue. We note that part (i) of Theorem 3.9 was proved earlier, by different methods, in Escobar's thesis [E].

In Sect. 4 we study perturbations of rotationally symmetric metrics. For manifolds with a pole, curvature conditions are formulated which guarantee the existence of convex functions with Laplacian close to a constant. In particular, for perturbations of $\mathbb{R}^n$, see Proposition 4.15, we have the following: Suppose that the radial sectional curvature satisfies

$$\left| K\left(\frac{\partial}{\partial r}, v_\omega\right) \right| \leqq \frac{\delta}{(1+r)^2}, \qquad r \geqq 0,$$

uniformly in $v_\omega \in TS^{n-1}$, where (r, ω) denote geodesic polar coordinates. If $\delta > 0$ is sufficiently small, then $-\Delta$ has no L^2 eigenfunction with positive eigenvalue. We prove an analogous result for perturbations of generalized paraboloids and the hyperbolic space $\mathbb{H}^n$.

Section 5 has been inspired by a paper of Tayoshi [T]. This author used Rellich type identities to establish absence of L^2 eigenfunctions for surfaces of revolution in $\mathbb{R}^3$. In general, let X be a conformal vector field on a manifold M^n, and $|X| \leqq C(1+r)$. Then, for a solution $u \in L^2(M^n)$ to $\Delta u = -\lambda u$, we prove (Proposition 5.1), for a suitable exhaustion D_k of M,

$$\lim_{k\to\infty} \left[(n-2) \int_{D_k} |\nabla u|^2 \operatorname{div} X - n\lambda \int_{D_k} u^2 \operatorname{div} X \right] = 0 . \tag{1.2}$$

When $n = 2$ we give some applications of (1.2) which, for rotationally symmetric manifolds, recover Tayoshi's result. Moreover, our results apply to metrics in the same conformal class of a given rotationally symmetric metric.

In addition to the papers of Rellich, Tayoshi, and Escobar, there are some other works on absence of positive eigenvalues of the Laplacian on a manifold. Karp [K] studied complete surfaces with nonpositive curvature $K(r, \omega)$ with K and K_ω suitably decaying to zero at infinity. One of us investigated manifolds whose curvature decays to a nonpositive constant [D1,D2]. Xavier [X] considered the more general problem of proving that the spectrum of Δ is purely absolutely continuous. His criterion requires the existence of convex functions f with $|\nabla f| \leqq C$ and $\Delta^2 f \leqq 0$. We have heard that Escobar and Freire have obtained results related to ours.

Finally, the methods in this paper generalize to Schrödinger operators $H = -\Delta + V$, where the potential V is allowed to be singular. We plan to study these operators in a future publication. In a different direction, the current results can be extended from Δ acting on functions to Δ acting on differential forms.

2 Rellich identities and eigenvalues

Let M^n be a complete noncompact Riemannian manifold and let Δ be the Laplacian on M^n. When $M^n = \mathbb{R}^n$, then it is well known that Δ has no point spectrum. For general M^n, Δ might admit square integrable eigenfunctions [D1]. The purpose of this section is to prove that under suitable conditions there exist no positive eigenvalues of $-\Delta$. Since $-\Delta$ is positive semi-definite, there are never any negative eigenvalues. Moreover, the eigenvalue zero occurs only for constant eigenfunctions on manifolds of finite volume [K].

The following important identity, originally due to Rellich for the case $M^n = \mathbb{R}^n$, is well known:

Lemma 2.1. *Let $D \subset M^n$ be a bounded, C^1 domain and let X be a C^1 vector field on M^n. For any $u \in C^2(D) \cap C^1(\bar{D})$ we have*

$$\int_D (X_{i,j} + X_{j,i}) u_i u_j - \int_D \operatorname{div} X |\nabla u|^2 + 2 \int_D X u \Delta u$$
$$= 2 \int_{\partial D} X u \frac{\partial u}{\partial \eta} - \int_{\partial D} |\nabla u|^2 X \cdot \eta \, .$$

In Lemma 2.1, $X_{i,j}$ denote the components of the covariant derivatives of X, whereas η denotes the outward pointing unit normal to ∂D.

Proof. The next identity follows by a direct computation

$$2X_{i,j} u_i u_j = \operatorname{div}(2Xu\nabla u - |\nabla u|^2 X) - 2Xu\Delta u + \operatorname{div} X |\nabla u|^2 .$$

Integrating the above over D finishes the proof of Lemma 2.1. □

We now assume that u is a solution to the equation $\Delta u = -\lambda u$. Then one has

$$2 \int_D X u \Delta u = \lambda \int_D \operatorname{div} X u^2 - \lambda \int_{\partial D} u^2 (X \cdot \eta) \, .$$

Substitution in Lemma 2.1 gives

$$\int_D (X_{i,j} + X_{j,i}) u_i u_j + \int_D (\lambda u^2 - |\nabla u|^2) \operatorname{div} X$$
$$= 2 \int_{\partial D} X u \frac{\partial u}{\partial \eta} + \int_{\partial D} (\lambda u^2 - |\nabla u|^2) X \cdot \eta \, . \tag{2.2}$$

Let now $p \in M^n$ be a fixed basepoint and denote by $r(x)$ the geodesic distance of x from p. In general, $r(x)$ is only a Lipschitz function of x. On any complete manifold, there exists a C^∞ regularization ϱ of r [GW1], satisfying for every $x \in M^n \backslash \{p\}$:

$$|\nabla \varrho(x)| \leqq C \qquad |\varrho(x) - r(x)| \leqq C \, . \tag{2.3}$$

For $t \in \mathbb{R}$ we let $D_t = \{x \in M^n | \varrho(x) < t\}$. By Sard's theorem, for a.e. $t \in \mathbb{R}$, the set D_t is a C^∞ domain in M^n. We plan to choose a sequence $t_k \uparrow \infty$ of regular values of ϱ such that $D_k = D_{t_k}$ exhausts M^n. We will apply (2.2) to the sequence D_k and let $k \to \infty$.

To control this limiting process we impose further restrictions on u and on the vector field X. We assume that $u \in L^2(M^n)$. Then $\Delta u = -\lambda u \in L^2(M^n)$, and therefore $|\nabla u| \in L^2(M^n)$ [K]. Concerning X we assume the norm of ∇X is bounded by a constant, i.e., $|\nabla X| \leqq C$. Consequently, one has

Lemma 2.4. (a) $|\operatorname{div} X| \leqq C$; (b) $|X| \leqq C_1 r + C_2$.

Proof. (a) Since $\operatorname{div} X = \operatorname{trace} \nabla X$, Schwarz's inequality implies (a). (b) Let $x \in M^n \backslash \{p\}$. Since M^n is geodesically complete there exists a geodesic $\gamma(s)$, where s is arc-length, joining p to x and with length $r(x)$. We have along γ

$$\left|\frac{\partial}{\partial s}|X|^2\right| = 2\left|X \cdot \left(\nabla_{\frac{\partial}{\partial s}} X\right)\right| \leqq 2|X|\,|\nabla X| \leqq C|X|.$$

If X never vanishes along γ, let $s_0 = 0$, otherwise let

$$s_0 = \sup\{s \in [0, r(x)] | X(\gamma(s)) = 0\}.$$

On $(s_0, r(x))$ the function $s \mapsto |X(\gamma(s))|$ is differentiable and $\frac{\partial}{\partial s}(|X|) \leqq C$. This gives (b). □

Since u, $|\nabla u| \in L^2(M^n)$, from (2.3) and the co-area formula [C] we have

$$\int_{M^n} (u^2 + |\nabla u|^2)|\nabla \varrho| = \int_0^\infty dt \int_{\partial D_t} (u^2 + |\nabla u|^2) < \infty.$$

This implies there exists a sequence $t_k \uparrow \infty$ of regular values of ϱ such that

$$\lim_{k \to \infty} t_k \int_{\partial D_k} (u^2 + |\nabla u|^2) = 0 \tag{2.5}$$

Applying (2.2) to $D = D_k$, and using Lemma 2.4, we conclude

$$\left|\int_{D_k} (X_{i,j} + X_{j,i}) u_i u_j - \int_{D_k} (|\nabla u|^2 - \lambda u^2) \operatorname{div} X\right| \leqq C t_k \int_{\partial D_k} (u^2 + |\nabla u|^2).$$

Letting $k \to \infty$, from (2.5) we finally obtain

Theorem 2.6. *Let $u \in L^2(M^n)$ be a solution to $\Delta u = -\lambda u$. Suppose that X is a C^1 vector field on M^n with $|\nabla X| \leqq C$, then*

$$\int_{M^n} (X_{i,j} + X_{j,i}) u_i u_j = \int_{M^n} (|\nabla u|^2 - \lambda u^2) \operatorname{div} X. \tag{2.7}$$

Remark. More generally, if X is C^1 and we only assume $|X| \leqq c_1 r + c_2$, then for a sequence of D_k's as above we can still conclude

$$\lim_{k \to \infty} \left[\int_{D_k} (X_{i,j} + X_{j,i}) u_i u_j - \int_{D_k} (|\nabla u|^2 - \lambda u^2) \operatorname{div} X\right] = 0.$$

If X satisfies additional conditions, then Theorem 2.6 may be applied to deduce nonexistence of L^2 eigenfunctions. In this regard we have the following:

Corollary 2.8. *Suppose that X is a C^1 vector field on M^n with $|\nabla X| \leqq C$, and that $X_{i,j} + X_{j,i} \geqq 0$. Assume that one of the following conditions is satisfied:*

(i) *There exists $p \in M^n$ at which $X_{i,j} + X_{j,i} > 0$ and $\operatorname{div} X \equiv n$ on M^n;*

(ii) *For some $b > 0$, there exists $\varepsilon = \varepsilon(M^n, b)$ sufficiently small such that $X_{i,j} + X_{j,i} \geqq 2(b-\varepsilon)g_{ij}$ and $|\operatorname{div} X - n| \leqq \varepsilon$ on M^n. Here g_{ij} is the metric tensor on M^n;*

(iii) *$X_{i,j} + X_{j,i} \geqq (\operatorname{div} X - n)g_{ij}$, $0 \leqq \operatorname{div} X - n$ on M^n, and either inequality holds strictly at least at one point.*

Then $-\Delta$ has no L^2 eigenfunction with eigenvalue $\lambda > 0$.

Proof. (i) By [K] we have for any L^2 eigenfunction

$$\int_{M^n} |\nabla u|^2 = \int_{M^n} u(-\Delta u) = \lambda \int_{M^n} u^2.$$

Since $\operatorname{div} X \equiv n$ (2.7) yields

$$\int_{M^n} (X_{i,j} + X_{j,i}) u_i u_j = 0\,.$$

On the other hand $(X_{i,j} + X_{j,i}) u_i u_j \geqq 0$. Since, by assumption, there exists a point at which $X_{i,j} + X_{j,i} > 0$, we infer that must $|\nabla u| \equiv 0$ in a neighborhood of that point. By unique continuation [A] we conclude that u is constant on M^n. This contradicts $\lambda > 0$.

(ii) By (2.7) and the assumption we have

$$2(b-\varepsilon)\int_{M^n} |\nabla u|^2 \leqq \int_{M^n} (X_{i,j} + X_{j,i}) u_i u_j = \int_{M^n} (\operatorname{div} X - n)(|\nabla u|^2 - \lambda u^2)$$
$$\leqq \varepsilon \int_{M^n} (|\nabla u|^2 + \lambda u^2) = 2\varepsilon \int_{M^n} |\nabla u|^2.$$

If $\varepsilon < \frac{b}{2}$ the above contradicts $\lambda > 0$.

(iii) Invoking (2.7) and the hypothesis we have

$$\int_{M^n} (X_{i,j} + X_{j,i}) u_i u_j = \int_{M^n} (|\nabla u|^2 - \lambda u^2)(\operatorname{div} X - n)$$
$$\leqq \int_{M^n} |\nabla u|^2 (\operatorname{div} X - n)\,.$$

Using the assumption on strict inequality at one point, and arguing as above, we again reach a contradiction. □

An important special case of Corollary 2.8 is that in which $X = \nabla f$. The hypothesis then requires that f should be convex with Hessf bounded. In Sects. 3 and 4 we will construct convex functions which also satisfy the additional requirements in parts (i)–(iii) of Corollary 2.8.

We now proceed to give two extensions of Theorem 2.6. First of all, we consider exterior domains in a Riemannian manifold M^n. Let $\Omega \subset M^n$ be a bounded C^1 domain. Given a vector field X on M^n we say that Ω is *X-starshaped* if $X \cdot \eta \geqq 0$

on $\partial\Omega$, where η denotes the exterior unit normal to $\partial\Omega$. If $M^n = \mathbb{R}^n$ and $X = \frac{\partial}{\partial r}$, then X-starshaped coincides with the standard notion of starshaped with respect to the origin. In what follows we assume that Ω is X-starshaped for a suitable choice of the vector field X. Moreover, suppose that the eigenfunction $u \in L^2(M^n \backslash \Omega)$ and satisfies the Dirichlet boundary condition $u|_{\partial\Omega} = 0$. If D_k is the exhaustion of M^n previously introduced, we apply formula (2.2) to $D = D_k \backslash \Omega$ obtaining

$$\int_{D_k\backslash\Omega} (X_{i,j} + X_{j,i}) u_i u_j + \int_{D_k\backslash\Omega} (\lambda u^2 - |\nabla u|^2) \operatorname{div} X$$
$$= 2 \int_{\partial D_k} Xu \frac{\partial u}{\partial \eta} - 2 \int_{\partial\Omega} Xu \frac{\partial u}{\partial \eta} + \int_{\partial D_k} (\lambda u^2 - |\nabla u|^2) X \cdot \eta + \int_{\partial\Omega} |\nabla u|^2 X \cdot \eta .$$

Using the condition $u|_{\partial\Omega} = 0$ we have $Xu = \frac{\partial u}{\partial \eta} X \cdot \eta$ and $|\nabla u|^2 = \left(\frac{\partial u}{\partial \eta}\right)^2$ on $\partial\Omega$. Therefore, the above identity becomes

$$\int_{D_k\backslash\Omega} (X_{i,j} + X_{j,i}) u_i u_j + \int_{D_k\backslash\Omega} (\lambda u^2 - |\nabla u|^2) \operatorname{div} X$$
$$= 2 \int_{\partial D_k} Xu \frac{\partial u}{\partial \eta} + \int_{\partial D_k} (\lambda u^2 - |\nabla u|^2) X \cdot \eta - \int_{\partial\Omega} \left(\frac{\partial u}{\partial \eta}\right)^2 X \cdot \eta .$$

Since Ω is X-starshaped by assumption, we obtain

$$\int_{D_k\backslash\Omega} (X_{i,j} + X_{j,i}) u_i u_j + \int_{D_k\backslash\Omega} (\lambda u^2 - |\nabla u|^2) \operatorname{div} X$$
$$\leqq 2 \int_{\partial D_k} Xu \frac{\partial u}{\partial \eta} + \int_{\partial D_k} (\lambda u^2 - |\nabla u|^2) X \cdot \eta .$$

Letting $k \to \infty$ in the above inequality, we deduce the following.

Proposition 2.9. *Let X be a C^1 vector field on M^n with $|\nabla X| \leqq C$. Suppose that $\Omega \subset M^n$ is X-starshaped and $u \in L^2(M^n \backslash \Omega)$ satisfies $\Delta u = -\lambda u$, $u|_{\partial\Omega} = 0$. Then*

$$\int_{M^n\backslash\Omega} (X_{i,j} + X_{j,i}) u_i u_j \leqq \int_{M^n\backslash\Omega} (|\nabla u|^2 - \lambda u^2) \operatorname{div} X .$$

Starting from Proposition 2.9, and arguing as before, we can obtain a nonexistence result similar to Corollary 2.8.

Now we turn to some special results involving conformal structures. Let M^2 be a complete, two-dimensional Riemannian manifold with metric g_{ij}. Suppose that $\hat{g}_{ij} = \phi g_{ij}$, where ϕ is a C^∞ function on M^2 satisfying $0 < c_1 \leqq \phi \leqq c_2$, fur suitable constants c_1, c_2. Therefore, $u \in L^2(M^2, \hat{g})$ if and only if $u \in L^2(M^2, g)$. Since M^2 is two-dimensional, $\Delta_{\hat{g}} = \phi^{-1} \Delta$, hence if $u \in L^2(M^2, \hat{g})$ is a solution of $\Delta_{\hat{g}} u = -\lambda u$, then $\Delta u = -\lambda \phi u$. In addition to the above assumptions on ϕ, we also suppose that for a suitably chosen C^1 vector field X on M^2, we have $X\phi \geqq 0$.

For any C^1 bounded domain $D \subset M^2$, one has

$$2\int_D Xu\Delta u = \lambda \int_D \phi u^2 \operatorname{div} X + \lambda \int_D u^2 X\phi - \lambda \int_{\partial D} \phi u^2 X \cdot \eta\,,$$

where u is a solution to $\Delta_{\hat{g}} u = -\lambda u$ in M^2. Substitution in Lemma 2.1, with $D = D_k$, gives

$$\int_{D_k} (X_{i,j} + X_{j,i}) u_i u_j + \int_{D_k} \operatorname{div} X(\lambda \phi u^2 - |\nabla u|^2) + \lambda \int_{D_k} u^2 X\phi$$
$$= 2\int_{\partial D_k} Xu \frac{\partial u}{\partial \eta} + \int_{\partial D_k} (\lambda \phi u^2 - |\nabla u|^2) X \cdot \eta\,.$$

For $X\phi \geqq 0$ and $\lambda > 0$, this yields

$$\int_{D_k} (X_{i,j} + X_{j,i}) u_i u_j + \int_{D_k} \operatorname{div} X(\lambda \phi u^2 - |\nabla u|^2)$$
$$\leqq 2\int_{\partial D_k} Xu \frac{\partial u}{\partial \eta} + \int_{\partial D_k} (\lambda \phi u^2 - |\nabla u|^2) X \cdot \eta\,.$$

Letting $k \to \infty$, we obtain as before

Proposition 2.10. *Let X be a C^1 vector field on M^2 with $|\nabla X| \leqq C$. Suppose that two conformally metrics on M^2 are given satisfying $\hat{g}_{ij} = \phi g_{ij}$, $0 < c_1 \leqq \phi \leqq c_2$. Moreover, assume that $X\phi \geqq 0$. Let $u \in L^2(M^2, \hat{g})$ be a solution to $\Delta_{\hat{g}} u = -\lambda u$, $\lambda > 0$. Then*

$$\int_{M^2} (X_{i,j} + X_{j,i}) u_i u_j + \int_{M^2} \operatorname{div} X(|\nabla u|^2 - \lambda \phi u^2)\,,$$

where $X_{i,j}$ denote covariant derivatives of X with respect to g_{ij}, $\operatorname{div} X = g^{ij} X_{i,j}$, *and the above integrals are taken with respect to the measure induced by g_{ij}.*

Under additional hypotheses, similar to those of Corollary 2.8, Proposition 2.10 forces the vanishing of square integrable eigenfunctions of $\Delta_{\hat{g}}$. In dimension $n \geqq 3$, one employs the conformal Laplacian to establish results analogous to Proposition 2.10.

3 Rotationally symmetric metrics

Suppose M^n is a complete Riemannian manifold, with a basepoint p. We assume that the exponential map $\exp : T_pM \to M$ is a global diffeomorphism. Moreover, let the metric be given in geodesic spherical coordinates by $ds^2 = dr^2 + \gamma^2(r)\, d\omega^2$, with the associated volume element $d\,\mathrm{vol} = \theta\, dr\, d\,\mathrm{vol}(\omega) = \gamma^{n-1}\, dr\, d\,\mathrm{vol}(\omega)$.

In this section we plan to construct convex functions f whose gradients $X = \nabla f$ satisfy the assumptions in Corollary 2.8(i). This requires certain constraints on the rotationally symmetric metric. For $f = f(r)$ the equation $\Delta f = \operatorname{div} X = n$ leads to the ordinary differential equation $\theta^{-1} \frac{d}{dr}\left(\theta \frac{df}{dr}\right) = n$. Integration yields the solution

$$f(r) = n \int_0^r \theta^{-1}(t) \int_0^t \theta(s)\, ds\, dt, \qquad r > 0. \tag{3.1}$$

Since $\theta(r) \sim r^{n-1}$ as $r \to 0$ [BGM], the function $f(r)$ is bounded in $U \backslash \{p\}$, where U is a neighborhood of the basepoint p. Since f satisfies $\Delta f = n$ in $U \backslash \{p\}$ the removable singularities theorem implies f regular in U.

We now compute the Hessian of the function f in (3.1). Since f is radial, we have [GW2]

$$\begin{aligned} \operatorname{Hess} f(r) &= \nabla df(r) = \nabla(f'(r)\, dr) = f''(r)\, dr \otimes dr + f'(r) \nabla dr \\ &= f''(r)\, dr \otimes dr + f'(r) \frac{\gamma'(r)}{\gamma(r)} [g - dr \otimes dr]. \end{aligned} \tag{3.2}$$

Consequently, (3.1) gives

$$\begin{aligned} \operatorname{Hess} f(r) = n &\left[1 - \theta'(r)\theta^{-2}(r) \int_0^r \theta(s)\, ds\right] dr \otimes dr \\ &+ \frac{n}{n-1} \theta'(r)\theta^{-2}(r) \int_0^r \theta(s)\, ds [g - dr \otimes dr]. \end{aligned} \tag{3.3}$$

Proposition 3.4. *Let f be as in* (3.1). *Then f has bounded nonnegative Hessian if and only if*

$$0 \leqq \theta'(r)\theta^{-2}(r) \int_0^r \theta(s)\, ds \leqq 1$$

for all $r > 0$.

Proof. Obvious consequence of (3.3). □

Proposition 3.4 has an interesting geometric interpretation. Let $B(r)$ denote the geodesic ball with radius r centered at the basepoint p.

Proposition 3.5. *The function f in* (3.1) *has bounded nonnegative Hessian if and only if:* (i) *The mean curvature of $\partial B(r)$ with respect to an outward pointing normal is $\geqq 0$ for $r > 0$;* (ii) *The ratio $\frac{\operatorname{vol} \partial B(r)}{\operatorname{vol} B(r)}$ is decreasing for $r > 0$.*

Proof. Let X and Y be tangent vectors to the sphere $\partial B(r)$. If $B(\cdot,\cdot)$ denotes the second fundamental form of $\partial B(r)$ we have: $B(X,Y) = \left\langle \nabla_X \frac{\partial}{\partial r}, Y \right\rangle = \langle \nabla_X dr, Y \rangle = \operatorname{Hess} r(X,Y)$. Letting $f(r) = r$ in (3.2) we have that $B(\cdot,\cdot)$ is positive semidefinite if and only if $\theta' \geqq 0$. Since M^n is rotationally symmetric $B(\cdot,\cdot) \geqq 0$ if and only if the mean curvature is $\geqq 0$. This proves (i). For (ii) we first observe that $\dfrac{\operatorname{vol}\partial B(r)}{\operatorname{vol}B(r)} = \dfrac{\theta(r)}{\int_0^r \theta(s)\,ds}$. An elementary calculation gives

$$\frac{d}{dr}\frac{\theta(r)}{\int_0^r \theta(s)\,ds} = \frac{\theta'(r)\int_0^r \theta(s)\,ds - \theta^2(r)}{\left(\int_0^r \theta(s)\,ds\right)^2},$$

from which (ii) obviously follows. □

The characterization given in Proposition 3.4 is sometimes difficult to verify. The next proposition constitutes a useful sufficient criterion.

Proposition 3.6. *Let* $\theta'(r) \geqq 0$ *and* $\dfrac{\theta'(r)}{\theta(r)}$ *decreasing for* $r > 0$. *Then* f *in* 3.1 *is convex with bounded Hessian.*

Proof. We use Proposition 3.5. Recall that $\theta'(r) \geqq 0$ is equivalent to (i). Since $\dfrac{\theta'(r)}{\theta(r)}$ is decreasing, an easy real variable lemma [CGT] shows that

$$\frac{\int_0^r \theta'(s)\,ds}{\int_0^r \theta(s)\,ds} = \frac{\theta(r)}{\int_0^r \theta(s)\,ds} = \frac{\operatorname{vol}\partial B(r)}{\operatorname{vol}B(r)}$$

is decreasing. □

To illustrate the utility of Proposition 3.6 we check two significant examples.

1. $M^n = \mathbb{R}^n$. In this case $\theta(r) = r^{n-1}$ and $f(r) = \dfrac{r^2}{2}$. One obviously has $\theta'(r) \geqq 0$, $\dfrac{\theta'(r)}{\theta(r)} = \dfrac{n-1}{r}$. Proposition 3.6 implies that f is convex, and, in fact, $\operatorname{Hess} f = \operatorname{Id}$.

2. $M^n = \mathbb{H}^n$, the simply-connected complete space of constant curvature $-K_0$. Now $\theta(r) = \left(\dfrac{\sinh\sqrt{K_0}r}{\sqrt{K_0}}\right)^{n-1}$ so that

$$\frac{\theta'(r)}{\theta(r)} = (n-1)\sqrt{K_0}\,\frac{\cosh\sqrt{K_0}r}{\sinh\sqrt{K_0}r}\,.$$

Then,

$$\frac{d}{dr}\left(\frac{\theta'}{\theta}\right)(r) = -\frac{(n-1)K_0}{\left(\sinh\sqrt{K_0}r\right)^2},$$

which according to Proposition 3.6 implies that f given by (3.1) has bounded non-negative Hessian.

In addition to the above specific examples, the conditions in Proposition 3.6 also hold for two more general classes of rotationally symmetric manifolds. First of all, one has

Corollary 3.7. *Suppose that the radial sectional curvatures of* M^n *are* $\geqq 0$. *Then,* f *in* (3.1) *has bounded non-negative Hessian.*

Proof. Consider a radial geodesic starting at p. Along it the Jacobi equation reduces to $\gamma'' + K(r)\gamma = 0$, due to rotational symmetry. This gives $(\gamma')' = -K\gamma \leqq 0$. If $\gamma'(r_0) < 0$ for some r_0, then $\gamma'(r) \leqq \gamma'(r_0)$ for all $r > r_0$, and thus $\gamma(r_1) = 0$ for some $r_1 > r_0$. This contradicts completeness. So $\theta'(r) \geqq 0$ for all r. To establish the second hypothesis of Proposition 3.6, we compute

$$\frac{1}{n-1}\frac{d}{dr}\left(\frac{\theta'}{\theta}\right) = \frac{d}{dr}\left(\frac{\gamma'}{\gamma}\right) = \frac{\gamma''\gamma - (\gamma')^2}{\gamma 2} = -K - \left(\frac{\gamma'}{\gamma}\right)^2 \leqq 0 . \qquad \square$$

Secondly, we have

Corollary 3.8. *Assume that the radial sectional curvatures of M^n are $\leqq 0$ and increasing. Then, f in* (3.1) *has bounded non-negative Hessian.*

Proof. Since $K(r) \leqq 0$, the Rauch's comparison theorem [C] gives $\dfrac{\theta'}{\theta} \geqq \dfrac{n-1}{r}$. More work is required to check the second condition in Proposition 3.6. Consider a fixed value r_0 of r. Let M_0^n be the model space of constant curvature $K_0 = K(r_0)$. By Rauch's comparison theorem, and the fact that $K(r)$ is increasing, we have $\dfrac{\theta'(r_0)}{\theta(r_0)} \geqq \dfrac{\theta_0'(r_0)}{\theta_0(r_0)}$, where $\theta_0(r) = \left(\dfrac{\sinh\sqrt{K_0}r}{\sqrt{K_0}}\right)^{n-1}$. Since $\dfrac{\theta_0'}{\theta_0}$ is decreasing (see example 2 above), there exists $s_0 \leqq r_0$, with $\dfrac{\theta'(r_0)}{\theta(r_0)} = \dfrac{\theta_0'(s_0)}{\theta_0(s_0)}$. By Heintze-Karcher's comparison theorem [HK], we have for $t > 0$

$$\frac{\theta'(r_0+t)}{\theta(r_0+t)} \leqq \frac{\theta_0'(s_0+t)}{\theta_0(s_0+t)} \leqq \frac{\theta_0'(s_0)}{\theta_0(s_0)} = \frac{\theta'(r_0)}{\theta(r_0)},$$

where in the first inequality we have used the fact that $K(r)$ is increasing, whereas in the second that $\dfrac{\theta_0'}{\theta_0}$ is decreasing. By Proposition 3.6 the conclusion follows. $\square$

Using the previous results and Corollary 2.8 we now establish a nonexistence theorem for L^2 eigenfunctions of the Laplacian.

Theorem 3.9. *Let M^n be a complete, rotationally symmetric manifold. Assume that the radial sectional curvatures satisfy either* (i) *$K(r) \geqq 0$, or* (ii) *$K(r) \leqq 0$ and $K(r)$ increasing. Then, $-\Delta$ has no L^2 eigenfunction with eigenvalue $\lambda > 0$.*

Proof. In the present situation we apply Corollary 2.8, part (i), with $X = \nabla f$ and f given by (3.1). By Corollaries 3.7, 3.8, we have $|\nabla X| \leqq C$, $X_{i,j} + X_{j,i} \geqq 0$, and $\operatorname{div} X = \Delta f = n$, by construction. We only need to check $\operatorname{Hess} f > 0$ at some point. In fact, we will prove that $\operatorname{Hess} f(p) = g$, where g is the metric tensor. Given $v \in T_pM^n$, $|v| = 1$, consider a geodesic $\gamma(r)$ with $\gamma(0) = p$ and $\gamma'(0) = v$. Then, (3.3) gives

$$\operatorname{Hess} f(v,v) = \lim_{r\to 0} \operatorname{Hess} f\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \lim_{r\to 0} n\left(1 - \frac{\theta'(r)}{\theta^2(r)}\int_0^r \theta(s)\,ds\right).$$

It suffices to show that

$$\lim_{r\to 0} \frac{\theta'(r)}{\theta^2(r)}\int_0^r \theta(s)\,ds = \frac{n-1}{n}. \tag{3.10}$$

In order to accomplish this we argue as follows. Near $r = 0$, $|K| \leqq c$. Then, by Rauch's comparison theorem [C] we have

$$(n-1)\sqrt{c}\,\frac{\cos\sqrt{c}r}{\sin\sqrt{c}r} \leqq \frac{\theta'(r)}{\theta(r)} \leqq (n-1)\sqrt{c}\,\frac{\cosh\sqrt{c}r}{\sinh\sqrt{c}r},$$

$$\left(\frac{\sin\sqrt{c}r}{\sqrt{c}}\right)^{n-1} \leqq \theta(r) \leqq \left(\frac{\sinh\sqrt{c}r}{\sqrt{c}}\right)^{n-1}.$$

This implies the asymptotic relations as $r \to 0$

$$\frac{\theta'(r)}{\theta(r)} = \frac{n-1}{r}(1+o(1)), \theta(r) = r^{n-1}(1+o(1)).$$

From the latter, (3.10) easily follows. □

Theorem 3.9, part (i), gives a new proof of earlier results by Escobar [E]. Concerning part (ii), examples constructed in [D1] show that the condition that K increases is required.

4 Perturbations of rotationally symmetric metrics

In this section we extend the class of manifolds for which the results of Sect. 2 are applicable. A manifold with a rotationally symmetric metric, satisfying the assumption in Proposition 3.4, supports convex functions f of constant Laplacian. In certain cases one may perturb the metric so that f remains convex and its Laplacian is close to a constant. For instance, this is always possible when $M^n = \mathbb{R}^n$ with the flat metric. However, we intend to measure the perturbation in terms of curvature, rather than in C^2 norm. For this purpose we make use of the comparison theory of [HK].

We start with a rotationally symmetric metric $g = dr^2 + \gamma^2(r)\,d\omega^2$, and we define

$$h = dr^2 + \beta^2(r)\,d\omega^2,$$

with

$$\beta(r) = \gamma(r)\exp\left(\int_0^r \Phi(s)\,ds\right), \tag{4.1}$$

where Φ will be specified below. We plan to choose Φ so that the difference of the radial curvature functions $K_h(r) - K_g(r)$ has constant sign. We have the following

Lemma 4.2.

$$K_h - K_g = -2\frac{\gamma'}{\gamma}\Phi - \Phi' - \Phi^2.$$

Proof. By the Jacobi's equation $K_h = -\dfrac{\beta''}{\beta}$, $K_g = -\dfrac{\gamma''}{\gamma}$. A computation using (4.1) yields the conclusion. □

Moreover, with f as in (3.1), with $\theta(r) = \gamma^{n-1}(r)$, we recall that in Sect. 3 conditions were given under which $\Delta_g f = n$ and $\mathrm{Hess}_g f \geqq 0$. Now we investigate the analogous entities with respect to the metric h.

Lemma 4.3. (i) $\mathrm{Hess}_h f = f''\,dr\otimes dr + f'\left[\dfrac{\gamma'}{\gamma} + \Phi\right][h - dr\otimes dr]$,
(ii) $\Delta_h f = n + (n-1)\Phi f'$.

Proof. Easily follows from differentiating (4.1) and substituting in (3.2), where γ has to be replaced by β.

The metric g on M^n is said to represent a *strong model* if there exist constants $0 < a_1 < a_2 < 1$, such that for $r > 0$

$$a_1 \leqq \theta' \theta^{-2} \int_0^r \theta(s)\, ds \leqq a_2 . \tag{4.4}$$

In this case formula (3.3) implies that $\operatorname{Hess} f \geqq a_3 g$, for all $r \geqq 0$, and some $a_3 > 0$. Our goal is to apply Corollary 2.8, part (ii) to perturbations of certain strong models. We begin by giving some specific examples.

1. $M^n = \mathbb{R}^n$. This is the simplest example of a strong model. Now $\theta(r) = r^{n-1}$, so that

$$\theta'(r)\theta^{-2}(r) \int_0^r \theta(s)\, ds = \frac{n-1}{n}. \tag{4.5}$$

2. $M^n =$ generalized paraboloid. We assume $K_g(r) \geqq 0$ for all $r \geqq 0$. On θ we assume that $\theta(r) \sim r^{\alpha(n-1)}$ as $r \to \infty$, with $0 < \alpha < 1$, and also $\theta'(r) \sim \alpha(n-1) r^{\alpha(n-1)-1}$ as $r \to \infty$. We notice that the above assumption on θ yields $\int_0^r \theta(s)\, ds \sim \dfrac{r^{\alpha(n-1)+1}}{\alpha(n-1)+1}$ as $r \to \infty$. We then have

$$\lim_{r \to \infty} \theta'(r)\theta^{-2}(r) \int_0^r \theta(s)\, ds = \frac{\alpha(n-1)}{\alpha(n-1)+1}. \tag{4.6}$$

On the other hand formula (3.10) reads

$$\lim_{r \to 0} \theta'(r)\theta^{-2}(r) \int_0^r \theta(s)\, ds = \frac{n-1}{n}. \tag{4.7}$$

In order to achieve (4.4) it suffices to show: $\theta'(r) > 0$, $\theta'(r)\theta^{-2}(r) \int_0^r \theta(s)\, ds < 1$. In the proof of Corollary 3.7 we showed that γ' is decreasing and $\gamma' \geqq 0$. If there exists $r_0 > 0$ such that $\theta'(r_0) = 0$, then $\gamma'(r_0) = 0$, and therefore $\gamma \equiv 0$ on $r \geqq r_0$. This implies $\dfrac{\theta'}{\theta} \equiv 0$, for $r \geqq r_0$, and this violates the asymptotic assumptions on $\dfrac{\theta'}{\theta}$. We are left with proving $\theta'(r)\theta^{-2}(r) \int_0^r \theta(s)\, ds < 1$. The following lemma is an adaptation of an argument in [CGT].

Lemma 4.8. *Let ϕ, ψ be positive functions for $r > 0$, continuous for $r \geqq 0$. Suppose that $h \overset{\text{def}}{=} \dfrac{\phi}{\psi}$ is strictly decreasing. Then $\dfrac{d}{dr}\left(\int_0^r \phi \big/ \int_0^r \psi \right) < 0$ for every $r > 0$.*

Proof. One computes

$$\frac{d}{dr}\left(\int_0^r \phi \Big/ \int_0^r \psi\right) = \frac{\psi(r)}{\left(\int_0^r \psi\right)^2}\left[\frac{\phi(r)}{\psi(r)}\int_0^r \psi - \int_0^r \phi\right]$$

$$= \frac{\psi(r)}{\left(\int_0^r \psi\right)^2}\left[\int_0^r \psi(x)[h(r)-h(x)]\,dx\right] < 0$$

by the assumption on h.

We now apply Lemma 4.8 with $\phi = \theta'$, $\psi = \theta$, and $h = \frac{\theta'}{\theta} = \frac{1}{n-1}\frac{\gamma'}{\gamma}$. Then,

$$(n-1)h' = \frac{\gamma''}{\gamma} - \left(\frac{\gamma'}{\gamma}\right)^2 = -K_g - \left(\frac{\gamma'}{\gamma}\right)^2 < 0,$$

since, as we showed, $\gamma' > 0$. This gives

$$0 > \frac{d}{dr}\left(\frac{\theta}{\int_0^r \theta(s)\,ds}\right) = \theta^2\,\frac{\theta'\theta^{-2}\int_0^r \theta(s)\,ds - 1}{\left(\int_0^r \theta(s)\,ds\right)^2}.$$

This shows $\theta'\theta^{-2}\int_0^r \theta(s)\,ds < 1$, hence the proof of (4.4) is complete. □

We now establish nonexistence of L^2 eigenfunctions for perturbations of the generalized paraboloid in example 2. We accomplish this in two steps. First we control the change in the Hessian of $f(r)$ by a curvature decay condition, then we apply Corollary 2.8. In the sequel, M^n denotes a manifold with a pole whose metric tensor is denoted by $\hat{g}_{ij}$. Let (r,ω) denote geodesic polar coordinates with respect to $\hat{g}_{ij}$. Suppose $\frac{\partial}{\partial r}$ is the radial unit vector, and v_ω is a tangent unit vector to a level sphere of r, both based at (r,ω). One has

Lemma 4.9. *Let $g = dr^2 + \gamma^2(r)\,d\omega^2$ be the metric of the generalized paraboloid of example 2, with $\alpha \geqq \frac{1}{2}$. Suppose that $K_h(r)$ is a smooth function on M^n satisfying, for $r \geqq 0$, $\eta > 0$:*

$$|K_h(r) - K_g(r)| \leqq \begin{cases} \delta(1+r)^{\alpha-3} & \alpha > \frac{1}{2} \\ \delta(1+r)^{\alpha-3-\eta} & \alpha = \frac{1}{2} \end{cases} \tag{4.10}$$

and $K_g(r) \geqq K_h(r) \geqq 0$. Consider a perturbation $\hat{g}_{ij}$ of g whose radial sectional curvatures satisfy

$$K_g(r) \geqq K_{\hat{g}}\left(\frac{\partial}{\partial r}, v_\omega\right) \geqq K_h(r), \qquad r \geqq 0, \tag{4.11}$$

uniformly in v_ω and ω. Then, given $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon) > 0$ in (4.10) above such that

$$\mathrm{Hess}_{\hat{g}} f \geqq (a_3 - \varepsilon)\hat{g}_{ij} \quad \text{and} \quad |\Delta_{\hat{g}} f - n| \leqq \varepsilon, \tag{4.12}$$

where $a_3 > 0$ is the number for which $\mathrm{Hess}_g f \geqq a_3 g$.

Proof. We solve Jacobi's equation $\beta''(r) + K_h(r)\beta(r) = 0$, $r \geqq 0$, with initial conditions $\beta(0) = 0$, $\beta'(0) = 1$. By the Sturm's comparison theorem [H], since $K_g \geqq K_h$, and $\gamma(r) > 0$ for $r > 0$, we infer $\beta(r) > 0$ for $r > 0$. Therefore, $h = dr^2 + \beta^2(r)\,d\omega^2$ is a complete metric with curvature $K_h(r)$. As in (4.1) let $\Phi = \frac{\beta'}{\beta} - \frac{\gamma'}{\gamma}$. We claim $\Phi(0) = 0$. This can be shown as follows. $\beta''(0) = -K_h(0)\beta(0) = 0$. Since β is a smooth function, we have $\beta(r) = r(1 + 0(r^2))$ as $r \to 0$. Also, $\beta'(r) = 1 + 0(r^2)$ as $r \to 0$. Therefore, $\frac{\beta'(r)}{\beta(r)} = \frac{1}{r}(1 + 0(r^2))$ as $r \to 0$. The same applies to γ, hence the claim follows. By Lemma 4.2, we have $\Phi' = K_g - K_h - \Phi w$, where $w = \frac{2\gamma'}{\gamma} + \Phi = \frac{\gamma'}{\gamma} + \frac{\beta'}{\beta}$. By integration, we obtain

$$\Phi(r) = \exp\left(-\int_1^r w\right)\int_0^r (K_g - K_h)(s)\exp\left(\int_1^s w\right)ds\,, \tag{4.13}$$

since $\Phi(0) = 0$. By Rauch's comparison theorem [C] and $K_g \geqq K_h \geqq 0$, we conclude $\frac{\gamma'}{\gamma} \leqq \frac{\beta'}{\beta} \leqq \frac{1}{r}$, and thus

$$2\frac{\gamma'}{\gamma} \leqq w \leqq \frac{\gamma'}{\gamma} + \frac{1}{r}, \qquad r > 0\,.$$

This estimate and (4.13) yield

$$|\Phi(r)| \leqq C\gamma^{-2}(r)\int_0^r |K_h(s) - K_g(s)|\, s\gamma(s)\,ds\,.$$

Using (4.10), we conclude for $\alpha \geqq \frac{1}{2}$,

$$|\Phi(r)| \leqq C'\delta(1 + r)^{-1}\,, \qquad r \geqq 0\,.$$

Recalling (3.1), we have $f'(r) = n\theta^{-1}(r)\int_0^r \theta(s)\,ds$, with $\theta(r) \sim r^{(n-1)\alpha}$ as $r \to \infty$. Then, $|f'(r)| \leqq C(1 + r)$, for $r \geqq 0$. This and the above estimate finally imply $|f'(r)\Phi(r)| \leqq C''\delta$ for $r \geqq 0$. Combining the latter with Lemma 4.3 and the fact that g is a strong model we see that (4.12) holds for the metric h. (4.11) and the Hessian comparison theorem [GW2] allow us to complete the proof. □

Remark. If the strong model satisfies $\gamma''(r) \sim \alpha(\alpha - 1)r^{\alpha-2}$ as $r \to \infty$, then $K_g(r) \sim \frac{\alpha(1-\alpha)}{r^2}$ as $r \to \infty$. Since $0 < \alpha < 1$, there exist curvature functions K_h satisfying the hypotheses of Lemma 4.9.

Combining Lemma 4.9 with Corollary 2.8, part (ii), we arrive at

Proposition 4.14. *Suppose that $\hat{g}_{ij}$ satisfies the assumptions in Lemma* 4.9. *Then, there exists no L^2 eigenfunction of $-\Delta_{\hat{g}}$ with positive eigenvalue.*

A more explicit approach is available to study perturbations of the strong model $\mathbb{R}^n$. In this case we have

Proposition 4.15. *Let M^n be a complete manifold with a pole with metric tensor $\hat{g}_{ij}$. Suppose that the radial sectional curvature function satisfies*

$$\left|K_{\hat{g}}\left(\frac{\partial}{\partial r}, v_\omega\right)\right| \leqq \frac{\delta}{(1+r)^2}, \qquad r \geqq 0, \tag{4.16}$$

uniformly in v_ω and ω. Then, given $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon) > 0$ in (4.16) such that

$$\operatorname{Hess}_{\hat{g}} f \geqq (1-\varepsilon)\hat{g}_{ij} \quad \text{and} \quad |\Delta_{\hat{g}} f - n| \leqq \varepsilon. \tag{4.17}$$

Moreover, there exist no L^2 eigenfunctions of $-\Delta_{\hat{g}}$ with positive eigenvalue.

Proof. For the standard metric on $\mathbb{R}^n$, one has $g = dr^2 + \gamma^2(r)d\omega^2$ with $\gamma(r) = r$. We now let $\Phi(r) = -cr(1+r^2)^{-1}$, where $c \neq 0$ will be suitably chosen. Using (4.1), we define a function $\beta(r)$, and we let $h = dr^2 + \beta^2(r)d\omega^2$, $r > 0$. An easy calculation shows that h extends from $\mathbb{R}^n \backslash \{0\}$ to a smooth metric on $\mathbb{R}^n$. The point is that $\frac{\beta(r)}{r} = (1+r^2)^{-c/2}$ is a smooth function of r^2. By rotational symmetry it suffices to verify the smoothness of h when $n = 2$. In this case $x = r\cos\omega$, $y = r\sin\omega$. A computation gives: $h(dx, dx) = 1 + \left(\frac{r^2}{\beta^2(r)} - 1\right)\frac{y^2}{r^2}$, $h(dx, dy) = \frac{xy}{r^2}\left(1 - \frac{r^2}{\beta^2(r)}\right)$. Moreover, $\frac{\beta(r)}{r} = 1 + O(r^2)$. At this point, we use Lemma 4.2 to compute the curvature. We have

$$\begin{aligned} K_h &= -\Phi\left(\frac{2}{r} + \frac{\Phi'}{\Phi} + \Phi\right) \\ &= \frac{c}{1+r^2}\left(2 + \frac{1-r^2}{1+r^2} - \frac{cr^2}{1+r^2}\right) \\ &= c\left[\frac{3+(1-c)r^2}{(1+r^2)^2}\right]. \end{aligned}$$

This shows that, for c sufficiently small, the sign of K_h is the same as the sign of c. We now consider two rotationally symmetric models, one with $c > 0$, the other with $c < 0$. Since (3.1) gives $f(r) = \frac{r^2}{2}$ on $\mathbb{R}^n$, we have $|\Phi f'| \leqq |c|$. By Lemma 4.3, (4.17) holds for both models. By the Hessian comparison theorem [GW2] and (4.16), we conclude that (4.17) holds for $\hat{g}_{ij}$.

Finally, Corollary 2.8, part (ii), implies that $-\Delta_{\hat{g}}$ has no L^2 eigenfunction with positive eigenvalue. □

A rotationally symmetric metric is said to be a *weak model* if

$$0 \leqq \theta'(r)\theta^{-2}(r)\int_0^r \theta(s)\,ds \leqq 1, \qquad r \geqq 0,$$

but (4.4) fails to hold. Here are two specific examples.

1. M^n = capped cylinder. We assume $K_g \geqq 0$ for all $r \geqq 0$. Furthermore, we want $\theta(r) \sim 1$ as $r \to \infty$, $\theta'(r) = 0$ for $r \geqq r_0$, for some $r_0 > 0$. (3.2) gives $\operatorname{Hess} f(r) = f''(r)dr \otimes dr$ for $r \geqq r_0$. Thus, $\operatorname{Hess} f(r)$ has $(n-1)$ zero eigenvalues for $r \geqq r_0$. This means that Corollary 2.8 is not applicable to perturbations of the capped cylinder.

2. $M^n = \mathbb{H}^n$. Now $\gamma(r) = \sinh r$, $\theta(r) = (\sinh r)^{n-1}$, so that formula (3.1) gives

$$f'(r) = n(\sinh r)^{1-n} \int_0^r (\sinh s)^{n-1}\, ds\,.$$

Also,

$$f''(r) = n\left[1 - (n-1)\cosh r(\sinh r)^{-n} \int_0^r (\sinh s)^{n-1}\, ds\right].$$

Since $\lim_{r\to\infty} f''(r) = 0$, from (3.2) we conclude that $\mathbb{H}^n$ is a weak model. Next, we show that Hess $f(r)$ has no zero eigenvalue, for any $r \geqq 0$. To begin with, we remark that $f''(r) \geqq ce^{-2r}$ for $r \geqq r_0$ and all n. When $n = 2$ this may be improved to $f''(r) \geqq ce^{-r}$, $r \geqq r_0$, and similarly $f''(r) \geqq cre^{-2r}$, for $r \geqq r_0$, when $n = 3$. Moreover $\lim_{r\to\infty} f'(r)\dfrac{\gamma'(r)}{\gamma(r)} = \dfrac{n}{n-1}$. By (3.2), Hess $f(r)$ has no zero eigenvalue for $r \geqq r_0$. On the other hand, Hess $f = g_{ij}$ at the pole $r = 0$. Since $\dfrac{d}{dr}\left(\dfrac{\gamma'}{\gamma}\right) = -\dfrac{1}{(\sinh r)^2} > 0$, then $\dfrac{\theta'}{\theta}$ is strictly decreasing. By Lemma 4.8, $\dfrac{d}{dr}\left(\theta(r)/\int_0^r \theta(s)ds\right) < 0$, which implies, as in the discussion of the generalized paraboloid, that $f''(r) > 0$ for $r \geqq 0$. Also, $\gamma'(r) = \cosh r > 0$, implies $f'(r)\dfrac{\gamma'(r)}{\gamma(r)} > 0$, for $r \geqq 0$. In summary, Hess $f(r)$ never has a zero eigenvalue.

We now study perturbations of $\mathbb{H}^n$. Because of the rapid decay of the smallest eigenvalue of Hess f, the allowable perturbations are very restricted. We have

Proposition 4.18. *Let M^n be a complete manifold with a pole and metric tensor $\hat{g}_{ij}$. Suppose that the radial curvature function satisfied*

$$-\delta(\cosh(2+\eta)r)^{-1} \leqq K_{\hat{g}}\left(\frac{\partial}{\partial r}, v_\omega\right) + 1 \leqq 0, \qquad r \geqq 0\,. \tag{4.19}$$

Then, given $\eta > 0$ one can choose $\delta = \delta(\eta) > 0$ in (4.19) *such that*

$$2\,\mathrm{Hess}_{\hat{g}} f \geqq (\Delta_{\hat{g}} f - n)\hat{g}_{ij} \quad \textit{and} \quad \Delta_{\hat{g}} f - n \geqq 0\,. \tag{4.20}$$

Furthermore, there exists no L^2 eigenfunction of $-\Delta_{\hat{g}}$ with positive eigenvalue.

Proof. We let $K_h(r) = -1 - \delta(\cosh(2+\eta)r)^{-1}$ and notice that this is a smooth function on M^n. We suitably modify the proof of Lemma 4.9. The first step is to solve Jacobi's equation $\beta''(r) + K_h(r)\beta(r) = 0$, with initial conditions $\beta(0) = 0$, $\beta'(0) = 1$. By Rauch's comparison theorem [C], $\beta(r) \geqq r$ and therefore $h = dr^2 + \beta^2(r)d\omega^2$ is a complete metric with curvature $K_h(r)$. Let $\gamma(r) = \sinh r$, and $\Phi = \dfrac{\beta'}{\beta} - \dfrac{\gamma'}{\gamma}$. We recall $\Phi(0) = 0$. By Lemma 4.2, we have $\Phi' = -1 - K_h - \Phi w$, with $w = \dfrac{\gamma'}{\gamma} + \dfrac{\beta'}{\beta}$, so that (4.13) holds with $K_g = -1$. By Rauch's comparison theorem, we infer for $r \geqq 0$:

$$2\frac{\cosh r}{\sinh r} \leqq w(r) \leqq \frac{\cosh r}{\sinh r} + \sqrt{1+\delta}\,\frac{\cosh\left(\sqrt{1+\delta}r\right)}{\sinh(\sqrt{1+\delta}r)}\,.$$

This estimate and (4.13) yield

$$|\Phi(r)| \leqq C(\sinh r)^{-2} \int_0^r |K_h(s)+1| \sinh s \sinh\left(\sqrt{1+\delta}s\right) ds\,.$$

Using (4.19) we conclude for $0 < \delta < (1+\eta)^2 - 1$, $|\Phi(r)| \leqq C\delta e^{-2r}$ for $r \geqq 0$. Since $f'(r) = n(\sinh r)^{1-n} \int_0^r (\sinh s)^{n-1} ds$, we deduce that $|f'(r)\Phi(r)| \leqq C'\delta e^{-2r}$ for $r \geqq 0$. Using Lemma 4.3 and the discussion in example 2 above we have, for $\delta > 0$ sufficiently small, $2\,\mathrm{Hess}_h f \geqq (\Delta_h f - n)h$. Moreover, $K_h \leqq -1$ and the Laplacian comparison theorem [GW2] give $\Delta_h f \geqq n$. This establishes (4.20) for the metric h. By (4.19), and the Hessian comparison theorem, (4.20) holds for the metric $\hat{g}_{ij}$. Finally, applying Corollary 2.8, part (iii), we conclude the nonexistence of L^2 eigenfunctions for $-\Delta_{\hat{g}}$. □

Remark. When $n = 2$, we can improve the decay condition in (4.19) to: $-\delta(\cosh(1+\eta)r)^{-1} \leqq K_{\hat{g}}\left(\frac{\partial}{\partial r}, v_\omega\right) + 1 \leqq 0$. This follows from the better decay condition on the smallest eigenvalue of $\mathrm{Hess}_g f$, $f''(r) \geqq Ce^{-r}$, from example 2.

5 Conformal vector fields, surfaces, and eigenvalues

Conformal vector fields play an important role in differential geometry, especially for two-dimensional manifolds. We recall that a C^1 vector field X on a Riemannian manifold M^n is said to be *conformal* if

$$X_{i,j} + X_{j,i} = \frac{2}{n}\,\mathrm{div}\,X\, g_{ij}\,, \tag{5.1}$$

where g_{ij} is the metric tensor of M^n. In this section we develop some consequences of Theorem 2.6 when X is a conformal vector field. Our first result is

Proposition 5.1. *Let X be conformal and suppose $|X| \leqq c_1 r + c_2$. Let $u \in L^2(M^n)$ be a solution to $\Delta u = -\lambda u$. Then, there exists a sequence $D_k \uparrow M^n$ for which*

$$\lim_{k\to\infty}\left[(n-2)\int_{D_k} |\nabla u|^2\, \mathrm{div}\,X - n\lambda \int_{D_k} u^2\, \mathrm{div}\,X\right] = 0\,. \tag{5.2}$$

Proof. Follows immediately from (5.1) and the remark after Theorem 2.6. □

In the case in which $n = 2$, Proposition 5.1 has some particularly interesting consequences. We immediately note that (5.2) reduces to

$$\lim_{k\to\infty} \int_{D_k} u^2\, \mathrm{div}\,X = 0 \tag{5.3}$$

for $\lambda > 0$. This leads to the following.

Corollary 5.4. *Let X be a conformal vector field on a complete surface M^2 satisfying $|X| \leqq c_1 r + c_2$. If* $\mathrm{div}\,X \geqq 0$ *and* $\mathrm{div}\,X > 0$ *at some point $p \in M$, then $-\Delta$ has no L^2 eigenfunctions with positive eigenvalues.*

Proof. Follows from (5.3) and the unique continuation theorem [A]. □

Suppose that M^n is a rotationally symmetric manifold with metric tensor $ds^2 = dr^2 + \gamma^2(r)\,d\omega^2$. We will construct a radial conformal vector field.

Proposition 5.5 *Let $X = \gamma\,\dfrac{\partial}{\partial r}$. Then, $\operatorname{div} X = n\gamma'$ and X is conformal.*

Proof. Let $f(r) = \int_0^r \gamma(s)\,ds$, so that $X = \nabla f$. Since $f'(r) = \gamma(r)$, $f''(r) = \gamma'(r)$, from (3.2) we obtain $\operatorname{Hess} f = \gamma'\,dr \otimes dr + \gamma'[g - dr \otimes dr] = \gamma' g$. This yields $X_{i,j} + X_{j,i} = 2\gamma' g_{ij}$ and $\operatorname{div} X = n\gamma'$. □

Corollary 5.6. *Let M^2 be a complete, rotationally symmetric manifold with metric tensor $ds^2 = dr^2 + \gamma^2(r)\,d\omega^2$. Assume that $\gamma(r) \leqq c_1 r + c_2$ and that $\gamma'(r) \geqq 0$ for all $r > 0$. Then $-\Delta$ has no L^2 eigenfunctions with positive eigenvalues.*

Proof. By Proposition 5.5 the vector field $X = \gamma\,\dfrac{\partial}{\partial r}$ is conformal and $\operatorname{div} X = 2\gamma' \geqq 0$, by the assumption on γ. Moreover, at the basepoint $p \in M^2$, we have $\gamma'(0) = 1$. The conclusion then follows from Corollary 5.4. □

Tayoshi [T] proved Corollary 5.6 for surfaces of revolution in R^3 satisfying $\gamma'(r) \geqq 0$. In that case, one automatically has $\gamma(r) \leqq r$.

Let M^2 be a differentiable surface endowed with two conformally related complete metrics $\hat{g}_{ij} = \phi g_{ij}$. A given vector field X on M^2 is conformal with respect to g_{ij} if and only if it is so for $\hat{g}_{ij}$. We have the following

Proposition 5.7. *Suppose that the conformal vector field X satisfies $|X|_g \leqq c_1 r + c_2$, $r \geqq 0$, with respect to the metric g_{ij}. Moreover, assume that there exist constants a_1, $a_2 > 0$ such that $a_1 \leqq \phi \leqq a_2$. If $-\phi^{-1}X\phi \leqq \operatorname{div}_g X$, with strict inequality at some point $p \in M^2$, then $-\Delta_{\hat{g}}$ has no L^2 eigenfunctions with positive eigenvalues.*

Proof. Since the two metrics are quasi-isometric, we have $|X|_{\hat{g}} \leqq c_3\hat{r} + c_4$, for suitable c_3, $c_4 > 0$. Moreover, $\operatorname{div}_{\hat{g}} X = \operatorname{div}_g X + \phi^{-1}X\phi \geqq 0$, with strict inequality at some point $p \in M$. Invoking Corollary 5.4, we reach the conclusion. □

Combining Proposition 5.7 with Proposition 5.5, we deduce

Corollary 5.8. *Let $g = dr^2 + \gamma^2(r)\,d\omega^2$ be a complete, rotationally symmetric metric tensor on M^2, with $\gamma(r)$ satisfying the assumption $\gamma(r) \leqq c_1 r + c_2$, $r \geqq 0$. Suppose that $\hat{g}_{ij} = \phi g_{ij}$, with $a_1 \leqq \phi \leqq a_2$, for some a_1, $a_2 > 0$. If $-\phi^{-1}\dfrac{\partial\phi}{\partial r} \leqq 2\gamma^{-1}\dfrac{\partial\gamma}{\partial r}$, with strict inequality at some point $p \in M^2$, then there exist no L^2 eigenfunctions of $-\Delta_{\hat{g}}$ with positive eigenvalues.*

Proof. Let $X = \gamma\,\dfrac{\partial}{\partial r}$. From Proposition 5.5 we know that X is conformal and $\operatorname{div}_g X = 2\gamma'$. Thus one has $-\phi^{-1}X\phi \leqq \operatorname{div}_g X$, with strict inequality at some point $p \in M^2$. The conclusion follows from Proposition 5.7. □

We conclude this section with an example of how Corollary 5.8 can be applied. Let g be the standard metric on $\mathbb{R}^2$, $g = dr^2 + r^2\,d\omega^2$. Suppose $\hat{g}_{ij} = \phi g_{ij}$ with $0 < a_1 \leqq \phi \leqq a_2$. If $-\dfrac{\partial\phi}{\partial r} \leqq 2\,\dfrac{\phi}{r}$, then $-\Delta_{\hat{g}}$ has no L^2 eigenfunctions with positive eigenvalue. In terms of $h = \log\phi$, the above conditions read: $b_1 \leqq h \leqq b_2$, $\dfrac{\partial h}{\partial r} \geqq -\dfrac{2}{r}$.

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