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Global solutions to a class of strongly coupled parabolic systems

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Introduction and main results

In this paper we want to study some class of strongly coupled quasilinear parabolic systems. Our main interest is to show that under certain structure conditions it is possible to prove the existence of classical solutions, which exist globally in time.

To be precise, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $T > 0$ and $\Omega_T = \Omega \times (0, T)$ the space-time cylinder. The evolution of the vector function

$$u: \overline{\Omega_T} \to \mathbb{R}^N$$

is supposed to be governed by the reaction-diffusion system

$$(1) \qquad u_t^k + \operatorname{div}(j^k) = f^k(x, t, u, \nabla u) \text{ on } \Omega_T, \quad 1 \leqq k \leqq N$$

where the so-called flux-vectors j^k are affine in ∇u:

$$j_\beta^k = - A_{\alpha\beta}^{ik}(x, t, u) u_{x_\alpha}^i + B_\beta^k(x, t, u) .$$

Here and in the sequel we use the summation convention, where the greek indices run from 1 to n, latin indices from 1 to N, and v_t resp. v_{x_α} denote partial derivatives. We shall assume, that $A_{\alpha\beta}^{ik} \in C_2(\overline{\Omega_T} \times \mathbb{R}^N, \mathbb{R})$ and the strong Legendre-condition to hold:

There is a $\lambda_0 > 0$, such that

$$(2) \qquad A_{\alpha\beta}^{ik}(x, t, v) \xi_i^\alpha \xi_k^\beta \geqq \lambda_0 |\xi|^2 \quad \text{for } \xi \in \mathbb{R}^{nN}, (x, t, v) \in \overline{\Omega_T} \times \mathbb{R}^N .$$

Further

$$(3) \qquad B_\beta^k \in C_2(\overline{\Omega_T} \times \mathbb{R}^N, \mathbb{R}), \quad f^k \in C_2(\overline{\Omega_T} \times \mathbb{R}^N \times \mathbb{R}^{nN}, \mathbb{R}) \text{ with}$$

$$|f(x, t, v, p)| \leqq a(|v|)(1 + |p|^2) .$$

The system is complemented by initial boundary conditions

(4) $u^k = 0$ ("Dirichlet-condition") or
$j^k \cdot v = 0$ ("no-flux-condition") on $\partial\Omega \times (0, T)$
for each k, $1 \leqq k \leqq N$, where v denotes the
outer normal of $\partial\Omega$.

(5) $u(x, 0) = u_0(x) \in C_2(\bar{\Omega})$
which is supposed to be compatible with (4)
in case of Dirichlet-condition .

In fact the smoothness assumptions and the growth restriction may be relaxed in order to state the *local* existence of a unique classical solution. This might be proven either via fixpoint arguments in Hölder spaces [GM] or via analytic semigroups in L_p-spaces [A2].

The problem to answer is the question, whether this solution can be continued to a *global* solution. It cannot be expected that this is possible in all circumstances, as certain counterexamples seem to indicate, that solutions may start smoothly and even remain bounded, but develop a singularity after finite time (cf. [SJM]). Though the coefficients of those counterexamples are not smooth, but only continuous, this makes it plausible that some additional information about the structure of the system is needed in order to guarantee the existence of global classical solutions. These informations have to open the possibility to control some lower-order norms a-priori.

Now, in the semilinear case with smooth coefficients, independent of u, it is enough to control the L_∞-norm of all possible classical solutions (see e.g. Redlinger [R]). In contrast things are more complicated in the general quasilinear case. Fortunately there is now an elaborated existence theory developed by Amann [A1, A2, A3], suitable for the case of only Hölder continuous coefficients via analytic semigroups in "weak" extrapolation spaces. From this we get the following.

Proposition. *Let* $T > 0$ *be arbitrary,* (2), (3) *hold and assume that* u *solves* (1), (4), (5) *classically on* $[0, T]$. *If one has an a-priori bound for* u *in* $C_{\alpha,\alpha/2}(\overline{\Omega_T})^N$ *with* $\alpha > n/(n+1)$, *which may depend on* T *and the data, then the* (*unique*) *solution exists globally in time.*

Proof. The proof is a consequence of [A2, Theorem 4.3] and [A3, Theorem 2].

Remark 1. Note, that $C_{\alpha,\alpha/2}(\overline{\Omega_T}) \subset C_\varepsilon([0, T], C_\theta(\bar{\Omega}))$ with a continuous embedding for $\alpha = \theta + 2\varepsilon < 1$. The space $C_{\alpha,\alpha/2}(\overline{\Omega_T})$ consists of those continuous functions defined on $\overline{\Omega_T}$, which are α-Hölder continuous with respect to the parabolic distance $d((x, t), (y, s)) = |x - y| + |t - s|^{1/2}$.

Remark 2. The bound for α may be relaxed in certain situations – e.g. if the boundary conditions (4) are only of Dirichlet type. As we derive estimates for arbitrary $\alpha < 1$, this is of no importance in our situation.

Thus we are faced with in fact two problems, which may be treated separately. On the one hand, we still have to estimate the maximum of an assumed classical solution. On the other hand we have to prove Hölder estimates for solutions u, for which the quantity $\sup_{\overline{\Omega_T}} |u(x, t)|$ is already known. To the author's knowledge there are only few situations where the second task has been accomplished – the case of *diagonal systems*, meaning $A^{ik}_{\alpha\beta} = \delta^{ik} a_{\alpha\beta}$ (see [GS]), and the case of certain *triagonal* operators [A4].

We shall show in this paper that the same is possible for the following strongly coupled system:

(6) There are matrices a, c^k of class C_2 with a symmetric ,
such that the fluxvectors j^k are given by

$$j^k = -a(x, t, u)\nabla u^k + r^k(x, t, u) - c^k(x, t, u)\nabla H .$$

Here $H = H(x, t, u)$ is some C_3-function ,
such that with some $\gamma_0, \gamma_1 > 0$:

$$\gamma_0 |\eta|^2 \leqq \frac{\partial^2 H}{\partial u^i \partial u^k}(x, t, u)\eta_i \eta_k \leqq \gamma_1 |\eta|^2$$

for $\eta \in \mathbb{R}^N$, $(x, t, u) \in \overline{\Omega_T} \times K$.

(7) The coeffcients $A^{ik}_{\alpha\beta} := a_{\alpha\beta}\delta^{ik} + c^k_{\alpha\beta}\dfrac{\partial H}{\partial u^i}$

fulfill (2), and (3) holds with $\sup\{a(|u|) | u \in K\}$ small .

(8) $$\left.\frac{\partial H}{\partial u^i}\right|_{\partial\Omega \times (0, T)} = 0, \text{ if the Dirichlet-condition}$$

is supposed to hold for u^i in (4)
(This technical condition should be superfluous ,
but we were unable to remove it) .

Then we have

Theorem 1. *If u is a solution of the diffusion system of type* (6), (7) *and* $u(\overline{\Omega_T}) \subset K$, *then for $\alpha < 1$ the norm of u in $C_{\alpha, \alpha/2}(\Omega_T)$ is bounded in terms of K, T and the data, where K is some compact set.*

In order to apply the above proposition, we have to give an estimate for K, too. Naturally, we have to impose some growth condition. The following theorem holds:

Theorem 2. *Assume* (6), (8), (2) (*with* $K = \mathbb{R}^N$) *and for all* $(x, t, v) \in \overline{\Omega_T} \times \mathbb{R}^N$

(9) $$\frac{\partial H}{\partial u}(x, t, v) \cdot f(x, t, v, p) \leqq \varepsilon_0 |p|^2 + c(1 + |v|^2), \quad \varepsilon_0 < \lambda_0 \gamma_0$$

(10) $$\left|\frac{\partial H}{\partial t}(x, t, v)\right| + \left|\frac{\partial H}{\partial x_\beta}(x, t, v)\right| \leqq c(1 + |v|^2)$$

(11) $$|r_k| + \left|\frac{\partial H}{\partial u_i}(x, t, v)\right| + \left|\frac{\partial^2 H}{\partial u_i \partial x_\beta}(x, t, v)\right| \leqq c(1 + |v|)$$

(12) $$|a_{\alpha\beta}(x, t, v)| \leqq c$$

(13) $$|c^k_{\alpha\beta}(x, t, v)| \leqq c(1 + |v|)^{-1} .$$

Then $M_0 := \sup_{\overline{\Omega_T}} |u(x, t)|$ *is bounded in terms of* T *and the data.*

Let us remark, that this is a general theorem. In concrete cases there might be different methods to get a-priori information about K; e.g. in biological or chemical systems, K belongs usually to the positive cone, and more sophisticated methods can be applied.

The consequence of the theorems above and the proposition is

Theorem 3. *Systems of type* (6) *together with the above assumptions have global classical solutions.*

As already mentioned systems of this type occur e.g. as biological models, with u representing some (sub-)species, and one may think of H as a function describing environmental influences. Naturally this influence might depend on the species itself (via overpopulation, production of waste, pollution etc.). Each fluxvector in our type of system consists of two parts – one diffusive part according to Fick's law and another in reaction on H, which may be different for each species. The case where the environmental influence H does not depend on u (but with cross diffusion, $N = 2$) was discussed by Shigesada et al. [SKT] (see also [O, p. 88]).

As a special example modelling cross diffusion of two species, consider the system

(14) $$u_t - \operatorname{div}(\alpha \nabla u + c_1(u) d_2(v) \nabla v) = f_1(u, v)$$

$$v_t - \operatorname{div}(\alpha \nabla v + c_2(v) d_1(u) \nabla u) = f_2(u, v)$$

together with initial boundary conditions.

Systems of this type have been studied e.g. by Shigesada et al. [SKT], Deuring [D], Matano and Mimura [MM], and Schnadt [Sch]. Our results imply the

Corollary. *Assume that if* $T > 0$ *and if* (u, v) *is a classical solution of* (14), *then* $0 \leqq u, v \leqq C(u_0, v_0, T)$ *and the uniform Legrende condition holds along the solution. If further* $0 \leqq d_i(t)/c_i(t)$ *and* $0 < \delta_0 \leqq \frac{d}{dt}(\sqrt{d_i(t)/c_i(t)})$ *are bounded on* $[0, C(u_0, v_0, T)]$ *for* $i = 1, 2$, *the solution exists globally.*

Proof. Let $H = \int_0^u (d_1(t)/c_1(t))^{1/2}\, dt + \int_0^v (d_2(t)/c_i(t))^{1/2}\, dt$. Then it is easy to see, that (14) is of the type considered in Theorem 1.

We think that the technique to derive $C_{\alpha, \alpha/2}$-estimates presented below may be modified and adjusted to some other concrete (two-species) models. This would reduce the problem of existence of global solution to the derivation of L_∞-bounds like in the semilinear case.

Hölder-estimates

The weak form of the system may be written as follows:

$$\iint_{\Omega_T} \frac{\partial u^i}{\partial t}\psi^i + \iint_{\Omega_T}\left(a_{\alpha\beta}u^i_{x_\beta} + c^i_{\alpha\beta}\frac{dH}{dx_\beta} + r^i_\alpha\right)\psi^i_{x_\alpha} = \iint_{\Omega_T} f^i\psi^i$$

if $\psi^i = 0$ on $\partial\Omega\times[0, T]$ in case of Dirichlet-condition for the i-th component.

Then take $\psi^i = \dfrac{\partial H}{\partial u^i}\phi$ with $\phi \geqq 0$ and sum the resulting equations to get

$$(15)\qquad \iint_{\Omega_T}\frac{dH}{dt}\phi + \iint_{\Omega_T} d_{\alpha\beta}\frac{dH}{dx_\beta}\phi_{x_\alpha} + \iint_{\Omega_T} a_{\alpha\beta}u^i_{x_\beta}\frac{\partial^2 H}{\partial u^i\partial u^k}u^k_{x_\alpha}\phi$$
$$= \iint_{\Omega_T}\left\{-a_{\alpha\beta}u^i_{x_\beta}g^i_\alpha - \left(c^i_{\alpha\beta}\frac{dH}{dx_\beta} + r^i_\alpha\right)\left(\frac{\partial^2 H}{\partial u^i\partial u^k}u^k_{x_\alpha} + g^i_\alpha\right) + f^i\frac{\partial H}{\partial u^i} + e_0\right\}\phi$$
$$+ \iint_{\Omega_T}\left\{a_{\alpha\beta}e_\beta - r^i_\alpha\frac{\partial H}{\partial u^i}\right\}\phi_{x_\alpha}$$

with $e_0 = \dfrac{\partial H}{\partial t}$, $e_\beta = \dfrac{\partial H}{\partial x_\beta}$, $g^i_\alpha = \dfrac{\partial^2 H}{\partial u^i\partial x_\alpha}$ and $d_{\alpha\beta} := a_{\alpha\beta} + c^i_{\alpha\beta}\dfrac{\partial H}{\partial u^i}$.

Note, that $d_{\alpha\beta}\eta^\alpha\eta^\beta \geqq \lambda_0|\eta|^2$ (let $\xi^\alpha_i = \eta_\alpha\dfrac{\partial H}{\partial u^i}$ in (2)) and also that $a_{\alpha\beta}\eta^\alpha\eta^\beta \geqq \lambda_0|\eta|^2$ $\left(\text{let } \xi^\alpha_i = \eta_\alpha\omega^i,\ \omega\perp\dfrac{\partial H}{\partial u} \text{ in (2)}\right)$. Hence the third term on the left is bounded from below by $\iint_{\Omega_T}\lambda_0\gamma_0|\nabla u|^2\phi$.

Due to (6)–(13) standard estimations imply

$$(16)\qquad \iint_{\Omega_T}\frac{dH}{dt}\phi + \iint_{\Omega_T} d_{\alpha\beta}\frac{dH}{dx_\beta}\phi_{x_\alpha} + \varepsilon_1\iint_{\Omega_T}|\nabla u|^2\phi \leqq \iint_{\Omega_T}\left(R\phi + Q^\alpha\phi_{x_\alpha} + \frac{c}{(|u|^2+1)}|\nabla H|^2\phi\right)$$

where $\varepsilon_1 > 0$ and $|R| + |Q^\alpha| \leqq c(|u|^2 + 1)$.

If we are in the situation of Theorem 1, assuming that $u(\overline{\Omega_T}) \subset K$, we get similarly – just assuming (6), (7), (8) –

$$(17)\qquad \iint_{\Omega_T}\frac{dH}{dt}\phi + \iint_{\Omega_T} d_{\alpha\beta}\frac{dH}{dx_\beta}\phi_{x_\alpha} + \varepsilon_2\iint_{\Omega_T}|\nabla u|^2\phi \leqq c\iint_{\Omega_T}(1 + |\nabla H|^2)\phi + |\nabla\phi|\ .$$

(16) resp. (17) are the starting points for our considerations.

Proof of Theorem 1. Suppose $|u| \leqq M_0$ on $\overline{\Omega_T}$. By adding a suitable constant (depending on M_0), we may assume, that $\inf_{\overline{\Omega_T}} H = 1$. Due to [G-M], we may assume, that the α-Höldernorm of u on a small slice $0 \leqq t \leqq t_1$ is estimated in terms of the data, so that we may restrict ourselves to times larger than $\frac{1}{2}t_1 > 0$. Take $(x_0, t_0) \in \overline{\Omega_T}$ with $t_0 \geqq \frac{1}{2}t_1$ and let $Q_R := [t_0 - R^2, t_0] \times (B_R(x_0) \cap \Omega)$ for R with $R^2 \leqq \frac{1}{2}t_1$. Let $\mu_R := \sup\limits_{Q_R} H$.

We are going to show the following

Alternative. Let $\varepsilon_0 > 0$. Then there are $\delta = \delta(\varepsilon_0) > 0$, $\beta = \beta(\varepsilon_0) \in (0, 1)$ and $c(\varepsilon_0)$, $R(\varepsilon_0)$ such that for all $R \leqq R(\varepsilon_0)$

$$\text{either(A)} \quad \mu_{\delta 2R} \leqq (1 - \beta)\, \mu_{2R} + c(\varepsilon_0) R$$

$$\text{or (B)} \quad \fint\!\!\fint_{Q_R} |u - u_R|^2 \leqq \varepsilon_0 \text{ with } u_R = \fint\!\!\fint_{Q_R} u$$

holds. Here $\fint\!\!\fint_{Q_R} v := \frac{1}{|Q_R|} \iint_{Q_R} v(x, t)\, dx\, dt$ and

$$|Q_R| = \text{meas}\, Q_R = \iint_{Q_R} dx\, dt\ .$$

Once this is shown, the claim of Theorem 1 follows easily: Considering the sequence $R_k = \delta^k 2R(\varepsilon_0)$, we see, that (B) must occur for some R_k with $k \leqq k_0 = k_0(M_0, \varepsilon_0)$ – otherwise, iterating (A) would imply

$$\mu_{R_{k_0}} \leqq (1 - \beta)^{k_0} \left(\mu_{R_0} + \frac{c(\varepsilon_0) R(\varepsilon_0)}{1 - \beta - \delta} \right) < 1$$

for suitable chosen k_0, contradicting $\mu_R \geqq 1$.

Now, provided ε_0 is taken sufficiently small depending on the data, Theorem 3.1 of [G-S] implies for all $\alpha < 1$, that $\fint\!\!\fint_{Q_R} |u - u_R|^2 \leqq c_1 R^\alpha$ for all R with $c_1 = c(M_0, \varepsilon_0, \alpha)$ uniformly in (x_0, t_0). As is well known, this implies an estimate for the Hölderconstant in terms of c_1.

Proof of the alternative. Let $\mu = \mu_{2R}$. We start with (B) and assume, that

$$\text{meas}\{(x, t) \in Q_{(1+\sigma)R} \,|\, H \leqq (1 - \rho)\mu\} < \rho\, |Q_{(1+\sigma)R}|$$

where $\sigma, \rho \in (0, \frac{1}{4})$ will be chosen below.

Put $\phi = (H - k)_+ \eta^2$ into (17), with $k \in \mathbb{R}$ and η a cutoff-function on Q_{2R} with respect to $Q_{\frac{3}{2}R}$ – meaning a smooth function with support in $[t_0 - 4R^2, t_0] \times B_{2R}(x_0)$, $0 \leqq \eta \leqq 1$, $\eta \equiv 1$ on $[t_0 - (\frac{3}{2}R^2), t_0] \times B_{\frac{3}{2}R}(x_0)$ and $|\nabla \eta|^2 + |\eta_t| \leqq cR^{-2}$. Then

$$\begin{aligned} &\varepsilon_2 \iint_{Q_{\frac{3}{2}R}} |\nabla u|^2 (H - k)_+ + \lambda_0 \iint_{H \geqq k} |\nabla H|^2 \eta^2 \\ &\quad \leqq c \iint \{(H - k)_+^2 (|\nabla \eta|^2 + |\eta_t| + \eta^2) + |\nabla H|^2 (H - k)_+ \eta^2\} + cR^{n+2}\ . \end{aligned}$$

Chose $k := (1 - 2\rho)\mu$, hence $(H - k)_+ \leqq 2\rho\mu$ on Q_{2R}, and assume ρ so small, that $2\rho\mu c \leqq \lambda_0$. Then

$$\iint_{Q_{\frac{3}{2}R}} |\nabla u|^2 (H - k)_+ \leqq c(\rho\mu)^2 R^n + cR^{n+2}\,.$$

Let $A_0 := \{(x, t) \in Q_{(1+\sigma)R} | H \geqq (1 - \rho)\mu\}$. Then $(H - k)_+ \geqq \rho\mu$ on A_0 and by assumption

$$|Q_{(1+\sigma)R} \setminus A_0| < \rho |Q_{(1+\sigma)R}|\,. \tag{18}$$

Therefore

$$\iint_{A_0} |\nabla u|^2 \leqq cR^n \left(\rho\mu + \frac{R^2}{\rho\mu}\right) \leqq c\rho\mu R^n \tag{19}$$

as we may assume $R \leqq \rho\mu$, because the opposite inequality would imply that (A) holds and nothing is left to show.

Next, we need the standard inequality

$$\iint_{Q_{(1+\sigma)R}} |\nabla u|^2 \leqq cR^n\,. \tag{20}$$

This follows from (17) by letting $\phi = e^{sH} \cdot \eta^2$ with s large enough to absorb the terms on the right hand side by standard estimations.

Now we need the following Sobolev-Poincaré-Inequality (see [LU, p. 45, (2.10)]):

$$\int_{B_\rho \cap \Omega} |v|^2 \leqq c \int_{B_\rho \cap \Omega} |\nabla v|^{\frac{2n}{n+1}} \cdot \left(\int_{B_\rho \cap \Omega} |v| \right)^{\frac{2}{n+1}}$$

for functions with $\displaystyle\fint_{B_\rho \cap \Omega} v = 0$.

Let $v = u - \displaystyle\fint_{B_{(1+\sigma)_R} \cap \Omega} u$; we get after integration with respect to time:

$$\iint_{Q_{(1+\sigma)R}} \left| u - \fint_{B_{(1+\sigma)R} \cap \Omega} u \right|^2 \leqq c \cdot \iint_{Q_{(1+\sigma)R}} |\nabla u||^{\frac{2n}{n+1}} \cdot R^{\frac{2n}{n+1}}\,. \tag{21}$$

The righthandside of (21) will be estimated by (18), (19), and (20). First we have

$$\begin{aligned} \iint_{A_0} |\nabla u|^{\frac{2n}{n+1}} &\leqq \left(\iint_{A_0} |\nabla u|^2 \right)^{\frac{n}{n+1}} |Q_{(1+\sigma)R}|^{\frac{1}{n+1}} \\ &\leqq c\rho^{\frac{n}{n+1}} R^{\frac{n^2}{n+1}} R^{\frac{n+2}{n+1}} \end{aligned}$$

and then we note

$$\begin{aligned} \iint_{Q_{(1+\sigma)R} \setminus A_0} |\nabla u|^{\frac{2n}{n+1}} &\leqq \left(\iint_{Q_{(1+\sigma)R}} |\nabla u|^2 \right)^{\frac{n}{n+1}} |Q_{(1+\sigma)R} \setminus A_0|^{\frac{1}{n+1}} \\ &\leqq cR^{\frac{n^2}{n+1}} \rho^{\frac{1}{n+1}} R^{\frac{n+2}{n+1}}\,; \end{aligned}$$

by adding these estimates we get

$$\iint\limits_{Q_{(1+\sigma)R}} \left| u - \fint\limits_{B_{(1+\sigma)R}\cap\Omega} u \right|^2 \leqq c\rho^{\frac{1}{n+1}} R^{n+2}, \tag{22}$$

as $\dfrac{2n}{n+1} + \dfrac{n^2}{n+1} + \dfrac{n+2}{n+1} = n+2.$

As $\fint\limits_{B_R\cap\Omega} \left| u - \int\limits_{B_R\cap\Omega} u \right|^2 \leqq \int\limits_{B_R\cap\Omega} |u-c|^2 \leqq \int\limits_{B_{R(1+\sigma)}\cap\Omega} |u-c|^2$ for all $c \in \mathbb{R}^N$ we also have

$$\iint\limits_{Q_R} \left| u - \fint\limits_{B_R\cap\Omega} u \right|^2 \leqq c\rho^{\frac{1}{n+1}} R^{n+2}. \tag{23}$$

The next task is to estimate the quantity

$$D := \iint\limits_{Q_R} \left| \fint\limits_{B_R\cap\Omega} u - u_R \right|^2, \quad u_R = \fint\!\!\fint\limits_{Q_R} u .$$

We have

$$\begin{aligned} D &\leqq cR^{n+2} \sup_t \left| \fint\limits_{B_R\cap\Omega} u(x,t)\,dx - \frac{1}{R^2} \int\limits_{t_0-R^2}^{t_0} \fint\limits_{B_R\cap\Omega} u(x,s)\,dx\,ds \right|^2 \\ &\leqq cR^{-n+2} \sup_{s,t} \left| \int\limits_{B_R\cap\Omega} (u(x,t)-u(x,s))\,dx \right|^2 . \end{aligned}$$

In order to proceed, we integrate the system from s to t, multiply by $\varphi \cdot d$ and integrate, where φ is a cutoff-function on $B_{(1+\sigma)R}$ with resp. to B_R and $d \equiv 0$ on $\partial\Omega$, $d \equiv 1$ if $\operatorname{dist}(x, \partial\Omega) \geqq d_0$, $|\nabla d| \leqq c d_0^{-1}$. From

$$\begin{aligned} &\int\limits_{B_{(1+\sigma)R}\cap\Omega} \varphi d(u^i(x,t) - u^i(x,s))\,dx \\ &+ \int\limits_s^t \int\limits_{B_{(1+\sigma)R}\cap\Omega} (A_{ij}^{\alpha\beta} u_{x_\beta}^j + r_\alpha^i)(\varphi d)_{x_\alpha} = \int\limits_s^t \int\limits_{B_{(1+\sigma)R}\cap\Omega} f^i \varphi d \end{aligned}$$

it follows

$$\begin{aligned} \left| \int\limits_{B_{(1+\sigma)R}\cap\Omega} \varphi d(u(x,t)-u(x,s))\,dx \right| &\leqq c \iint\limits_{Q_{(1+\sigma)R}} (|\nabla u| + 1)\left(\frac{1}{\sigma R} + \frac{1}{d_0}\right) \\ &+ ca(M_0) \iint\limits_{Q_{(1+\sigma)R}} |\nabla u|^2 + cR^{n+2} \end{aligned}$$

with $a(M_0)$ small.

Choose $d_0 = \sigma R$. Then

$$\left| \int\limits_{B_R\cap\Omega} (1-d) \right| \leqq c\sigma R^n$$

and

$$\begin{aligned}E :&= \left| \int_{B_R \cap \Omega} (u(x,t) - u(x,s))\,dx \right| \\ &= \left| \int_{B_R \cap \Omega} (u(x,t) - u(x,s))\,\varphi(x)\,dx \right| \\ &\leqq \left| \int_{B_R \cap \Omega} (u(x,t) - u(x,s))\,\varphi\,dx \right| + c\sigma R^n \\ &\leqq \left| \int_{B_{R(1+\sigma)} \cap \Omega} (u(x,t) - u(x,s))\,\varphi\,d\,dx \right| + c\sigma R^n + c|B_{(1+\sigma)R} \setminus B_R| \\ &\leqq \frac{c}{\sigma R} \int_{Q_{(1+\sigma)R}} |\nabla u| + cR^n\left(\sigma + \frac{R^2}{\sigma R}\right) + ca(M_0)R^n .\end{aligned}$$

This integral is estimated as above by splitting the region of integration into A_0 and $Q_{(1+\sigma)R} \setminus A_0$, the result is

$$\int_{Q_{(1+\sigma)R}} |\nabla u| \leqq c\rho^{\frac{1}{2}} R^{n+1} .$$

This implies

$$E \leqq cR^n\left(\sigma + \frac{R}{\sigma} + \frac{\rho^{\frac{1}{2}}}{\sigma}\right) \leqq cR^n \rho^{\frac{1}{4}}$$

if $R \leqq \rho^{\frac{1}{2}}$, $\sigma = \rho^{\frac{1}{4}}$.

Going back to the estimation of D, we see that

$$D \leqq cR^{n+2}\rho^{\frac{1}{2}} .$$

Combined with (22), we end up with

$$\fint\!\!\fint_{Q_R} |u - u_R|^2 \leqq c\rho^{\frac{1}{n+1}} < \varepsilon_0$$

if $\rho = \rho(\varepsilon_0)$, $\sigma = \rho(\varepsilon_0)^{\frac{1}{4}}$, $R \leqq R(\varepsilon_0)$.

Let us now consider the other part of the alternative, this is (A). In this case we may assume that

$$\text{(24)} \qquad \operatorname{meas}\{(x,t) \in Q_{(1+\sigma)R} \,|\, H \leqq (1-\rho)\mu\} \geqq \rho |Q_{R(1+\sigma)}|$$

with σ and ρ given by the first part of the proof. Let $0 < \varepsilon < \rho e^{-1}$ and define

$$V := -\ln\left(1 - \frac{H}{\mu} + \varepsilon\right) + \ln \rho .$$

Note that on Q_{2R} we have the trivial estimate

$$V \leqq \ln\left(\frac{\rho}{\varepsilon}\right)$$

and $V \geqq 0 \Leftrightarrow 1 - \frac{H}{\mu} + \varepsilon \leqq \rho$; hence (24) implies

$$\text{(25)} \qquad \operatorname{meas}\{(x,t) \in Q_{(1+\sigma)R} \mid V \geqq 0\} \leqq (1-\rho)|Q_{(1+\sigma)R}| .$$

Let now $g := \left(1 - \frac{H}{\mu} + \varepsilon\right)^{-1}$, hence

$$\frac{dV}{dx_\alpha} = \frac{g}{\mu}\frac{dH}{dx_\alpha} \quad \text{and} \quad \frac{dV}{dt} = \frac{g}{\mu}\frac{dH}{dt} .$$

Therefore let $\phi = \frac{g}{\mu}\varphi$, $\varphi \geqq 0$ in (17) to get

$$\iint \frac{dV}{dt}\varphi + \iint d_{\alpha\beta}\frac{dV}{dx_\beta}\varphi_{x_\alpha} + \iint d_{\alpha\beta}\frac{dV}{dx_\beta}\frac{dV}{dx_\alpha}\varphi + c_0\iint |\nabla u|^2 \frac{g}{\mu}\varphi$$

$$\leqq c\iint \frac{g}{\mu}(1 + |\nabla V|)\varphi + \iint \frac{g}{\mu}Q_\alpha \varphi_{x_\alpha} + c\iint \frac{\mu}{g}|\nabla V|^2 \varphi .$$

Assume, that $\operatorname{supp}\varphi \subset \{(x,t) \mid V \geqq 0\}$, hence $g^{-1} \leqq \rho$ on $\operatorname{supp}\varphi$ and we see that for $\rho \leqq \rho_0$ the last term is absorbed by the third term on the left side.

This gives (with $g \leqq \varepsilon^{-1}$)

$$\text{(26)} \qquad \iint \frac{dV}{dt}\varphi + c_1\iint |\nabla V|^2\varphi + \iint d_{\alpha\beta}\frac{dV}{dx_\beta}\varphi_{x_\alpha} \leqq c(\mu\varepsilon)^{-2}\iint \varphi + c(\mu\varepsilon)^{-1}\iint |\nabla\varphi| .$$

Our aim is to show, that (25) and (26) imply a better estimate for V than the trivial one on a smaller cylinder.

Choose first $\varphi = V_+\eta^2$, where η is a standard cutoff-function with $\eta \equiv 1$ on $Q_{\frac{3}{2}R}$. From (26) we get

$$\iint_{Q_{\frac{3}{2}R}} |\nabla V_+|^2 V_+ + |\nabla V_+|^2) \leqq c \iint_{Q_{2R}} V_+^2(|\nabla\eta|^2 + |\eta_t|) + (V_+ + 1)(\mu\varepsilon)^{-2}$$

and therefore (assuming $R \leqq \mu\varepsilon$)

$$\text{(27)} \qquad \iint_{Q_{\frac{3}{2}R}} |\nabla V_+|^2 V_+ \leqq cR^n\left(1 + \left(\ln\frac{\rho}{\varepsilon}\right)^2\right) .$$

Next we take $\varphi = 2V_+\eta^2(x)\chi_{[t_1,t_2]}(t)$ with $t_i \in [t_0 - R^2, t_0]$ and η a cutoff-function on $B_{R(1+\sigma)(1+\tilde\delta)}$ with respect to $B_{R(1+\sigma)}$; $\tilde\delta \leqq \frac{1}{4}$ will be chosen later. Note that $(1+\sigma)(1+\tilde\delta) \leqq (\frac{5}{4})^2 < \frac{3}{2}$. Then from (26) we get (with $B_2 = B_{R(1+\sigma)} \cap \Omega$, $B_1 = B_{R(1+\sigma)(1+\tilde\delta)} \cap \Omega$)

$$\int_{B_2} V_+^2(x,t_2)\,dx + c_0\iint |\nabla V_+|^2(1 + V_+)\eta^2 \leqq \int_{B_1} V_+^2(x,t_1)\,dx$$

$$+ c\iint(|\nabla V_+|V_+\eta|\nabla\eta| + (\mu\varepsilon)^{-2}V_+\eta^2)$$

$$+ c\iint(|\nabla V_+|(\mu\varepsilon)^{-1}\eta^2 + V_+\eta|\nabla\eta|)$$

where the double integrals are taken over $[t_1, t_2] \times B_1$. Using Hölder's inequality and the trivial estimate for V_+, we get

$$(28)\quad \int_{B_2} V_+^2(x, t_2)\,dx \leqq \int_{B_1} V_+^2(x, t_1)\,dx + c\int_{t_1}^{t_2}\int_{B_1} ((1 + V_+)|\nabla\eta|^2 + (\mu\varepsilon)^{-2}\eta^2)$$
$$\leqq \int_{B_1} V_+^2(x, t_1)\,dx + cR^n\left(1 + \ln\left(\frac{\rho}{\varepsilon}\right)\right)\tilde{\delta}^{-2}.$$

Next we have to choose t_1 appropriately.

Let $I' = [t_0 - (1 + \sigma)^2R^2, t_0 - \frac{\rho}{2}(1 + \sigma)^2R^2]$ and

$$A_{k,r}(t) = \{x \in B_r \cap \Omega \,|\, V(x, t) \geqq k\}.$$

Then there is some $t_1 \in I'$ with $|A_{0,R(1+\sigma)}(t_1)| \leqq \left(1 - \frac{\rho}{2}\right)|B_2|$ because otherwise

$$\begin{aligned}\operatorname{meas}\{(x, t) \in Q_{R(1+\sigma)} \,|\, V_+ \geqq 0\} &\geqq \int_{I'} |A_{0,R(1+\sigma)}(t)|\,dt\\ &\geqq \left(1 - \frac{\rho}{2}\right)^2(1 + \sigma^2)R^2|B_2| > (1 - \rho)|Q_{R(1+\sigma)}|,\end{aligned}$$

contradicting (25).

Therefore, for all t_2 with $t_2 \in \left[t_0 - \frac{\rho}{2}(1 + \sigma)^2R^2, t_0\right]$,

$$(29)\quad \int_{B_2} V_+^2(x, t_2)\,dx \leqq \left(\ln\left(\frac{\rho}{\varepsilon}\right)\right)^2\left\{\left(1 - \frac{\rho}{2}\right) + c\tilde{\delta}\right\}|B_2| + cR^n\left(1 + \ln\left(\frac{\rho}{\varepsilon}\right)\right)\tilde{\delta}^{-2}$$

Let $\gamma \in (0, 1)$ and $M > 0$ be arbitrary – from (29) we get

$$(30)\quad (\gamma M)^2|A_{\gamma M, R(1+\sigma)(1+\delta)}(t_2)| \leqq \left(\ln\left(\frac{\rho}{\varepsilon}\right)\right)^2\left\{1 - \frac{\rho}{4} + \tilde{c}\tilde{\delta}\right\}|B_2| + c_0|B_1|\tilde{\delta}^{-4}.$$

Now choose $\tilde{\delta}$, such that $\tilde{c}\tilde{\delta} \leqq \frac{\rho}{8}$ and assume, that

$$(31)\quad \text{(a)}\quad \left(\ln\left(\frac{\rho}{\varepsilon}\right)\right)^2 \leqq M^2\left(1 - \frac{\rho}{16}\right)^{-1}$$

$$\text{(b)}\quad c_0\tilde{\delta}^{-4} \leqq \frac{\rho}{32}M^2.$$

Then

$$\gamma^2|A_{\gamma M, R(1+\sigma)(1+\tilde{\delta})}(t_2)| \leqq \left(1 - \frac{\rho}{32}\right)|B_1|$$

hence

$$|A_{\gamma M, R(1+\sigma)(1+\tilde{\delta})}(t_2)| \leqq \left(1 - \frac{\rho}{50}\right)|B_1|$$

if $\gamma = \gamma(\rho)$ close to 1. Therefore, for fixed t_2, Poincare's inequality is valid and implies

$$\begin{aligned}\int_{B_1} (V-\gamma M)_+^2\,dx &\leqq cR^2 \int_{V \geqq \gamma M} |\nabla V|^2\,dx \\ &\leqq \frac{R^2}{\gamma M} \int_{V \geqq \gamma M} |\nabla V|^2 V\,dx \\ &\leqq \frac{cR^2}{\gamma M} \int_{V \geqq 0} |\nabla V_+|^2 V_+\,dx\end{aligned}$$

and integrating with respect to t_2, we get with (27)

$$\iint_{Q_{R_1}} (V-\gamma M)_+^2 \leqq c\frac{R^{n+2}}{M}\left(1 + \ln\left(\frac{\rho}{\varepsilon}\right)^2\right) \tag{32}$$

for $R_1 = \left(\frac{\rho}{2}\right)^{\frac{1}{2}}(1+\sigma)R$, and using (31.a) we end with

$$\fint\!\!\fint_{Q_{R_1}} (V-\gamma M)_+^2 \leqq c\left(1 + \ln\left(\frac{\rho}{\varepsilon}\right)\right). \tag{33}$$

As a third choice we take $\phi = (V-k)_+\eta^2$ in (26) for $k \geqq \gamma M > 0$, where η now denotes a cutoff-function on Q_{ρ_1} with respect to Q_{ρ_2}, $\frac{3}{4}R_1 \leqq \rho_2 < \rho_1 \leqq R_1$, with $|\nabla\eta|^2 + |\eta_+| \leqq c(\rho_1-\rho_2)^{-2}$. Then one gets with $I_\rho = [t_0 - \rho^2, t_0]$, assuming $R \leqq \mu\varepsilon$,

$$\begin{aligned}&\sup_{I_{\rho_2}} \int_{B_{\rho_2}\cap\Omega} (V-k)_+^2\,dx + \iint_{Q_{\rho_2}} |\nabla(V-k)_+|^2 \\ &\leqq c(\rho_1-\rho_2)^{-2}\iint_{Q_{\rho_1}}(V-k)_+^2 + cR^{-2}\int_{I_{\rho_1}} |A_{k,\rho_1}(t)|\,dt\,.\end{aligned} \tag{34}$$

Define now $M := \sup_{Q_{\frac{3}{4}R_1}} V$; we claim that

$$M \leqq c\left(\fint\!\!\fint_{Q_{R_1}} (V-\gamma M)_+^2\right)^{\frac{1}{2}} + 1 \tag{35}$$

which follows from Lemma 1 below.

It is in fact a variant of a well known technique (see e.g. [LUS, Lemma 7.3, p. 116]).

Then by (33) and (35)

$$M^2 \leqq c_1\left(1 + \ln\left(\frac{\rho}{\varepsilon}\right)\right).$$

Now choose $\varepsilon < \rho e^{-1}$ so small that

$$c_1\left(1 + \left(\ln\frac{\rho}{\varepsilon}\right)\right) \leqq \left(\ln\left(\frac{\rho}{\varepsilon}\right)^2\left(1 - \frac{\rho}{16}\right)\right)$$

and $\frac{32}{\rho} c_0 \tilde{\delta}^{-4} \leqq \left(\ln \frac{\rho}{\varepsilon}\right)^2 \left(1 - \frac{\rho}{16}\right)$, c_0 from (31.b). Then $M^2 \leqq \left(\ln \frac{\rho}{\varepsilon}\right)^2 \left(1 - \frac{\rho}{32}\right)^2$ and this obviously holds also, if the assumptions (31) are not valid, provided only that $R \leqq \mu\varepsilon$. Therefore in all cases

$$\text{either } \sup_{Q_{\frac{3}{4}R_1}} V \leqq \left(1 - \frac{\rho}{32}\right)\left(\ln \frac{\rho}{\varepsilon}\right) \text{ or } \mu \leqq \varepsilon^{-1} R .$$

In terms of H, this means

$$\mu_{\delta 2R} \leqq (1 - \beta)\mu_{2R} + cR$$

with $\delta = \frac{3}{8}\left(\frac{\rho}{2}\right)^{\frac{1}{2}}$, $\beta = \rho\left(\frac{\rho}{\varepsilon}\right)^{-(1 - \frac{\rho}{32})} - \varepsilon > 0$. This finishes the proof of the alternative and the theorem.

We close this part with the proof of

Lemma 1. *Let* $V \in H^1_2 \cap L_\infty$, $B_r := B_r(x_0) \cap \Omega$, $I_r = [t_0 - r^2, t_0]$, $Q_r = I_r \times B_r$, $A_{k,r}(t) = \{x \in B_r \mid V(x,t) \geqq k\}$, $R > 0$ *and suppose that*

$$\sup_{I_{\rho 2}} \int_{B_{\rho 2}} (V - k)_+^2 \, dx + \iint_{Q_{\rho 2}} |\nabla (V - k)_+|^2$$
$$\leqq c_1(\rho_1 - \rho_2)^{-2} \iint_{Q_{\rho 1}} (V - k)_+^2 + c_1 R^{-2} \int_{I_{\rho 1}} |A_{k,\rho 1}(t)| \, dt$$

for $\frac{3}{4}R \leqq \rho_2 \leqq \rho_1 \leqq R$, $k \geqq \gamma M$, *with* $\gamma < 1$ *and* $M := \sup_{Q_{\frac{3}{4}R_1}} V$. *Then*

$$M \leqq c(c_1, \gamma, n)\left(\iint_{Q_{R_1}} (V - \gamma M)_+^2\right)^{\frac{1}{2}} + 1 .$$

Proof. Let $\tau := \frac{1}{2}\left(1 + \frac{1}{\gamma}\right) > 1$; $k_0 \geqq \max\{\gamma M, \tau^{-1}\}$ will be choosen later. Define monotone sequences

$$k_s := \tau k_0 - k_0(\tau - 1)\tau^{-s} \nearrow \tau k_0$$
$$R \geqq \rho_s := (\tfrac{3}{4} + \tfrac{1}{4}\cdot 4^{-s})R_1 \searrow \tfrac{3}{4}R_1, \text{ and let}$$
$$I_s := \iint_{Q_{\rho s}} (V - k_s)_+^2 .$$

Let $k_s^* = \frac{1}{2}(k_{s+1} + k_s)$, $\rho_s^* = \frac{1}{2}(\rho_{s+1} + \rho_s)$, $\rho_s^{**} = \frac{1}{2}(\rho_s^* + \rho_s)$ and φ_s a cutoff-function on $B_{\rho_s^*}$ with respect to $B_{\rho_{s+1}}$. Then by Sobolev's imbedding theorem (assume $n \geqq 3$)

$$I_{s+1} \leqq \iint_{Q_{\rho_s^*}} (V - k_{s+1})_+^2 \varphi_s^2$$
$$\leqq c \int_{I_{\rho_s^*}} |A_{k_{s+1},\rho_s^*}(t)|^{\frac{2}{n}} \int_{B_{\rho_s^*}} |\nabla (V - k_{s+1})_+|^2 \varphi_s^2 + (V - k_{s+1})_+^2 |\nabla \varphi_s|^2 \, dx\, dt$$
$$\leqq c c_1 \sup_{I_{\rho_s^*}} |A_{k_{s+1},\rho_s^*}(t)|^{\frac{2}{n}}$$
$$\times \left((\rho_s^{**} - \rho_s^*)^{-2} \iint_{Q_{\rho_s^{**}}} (V - k_{s+1})_+^2 + R^{-2} \int_{I_{\rho_s^{**}}} |A_{k_{s+1},\rho_s^{**}}(t)| \, dt\right) .$$

As by assumption also

$$(k_{s+1} - k_s^*)^2 \sup_{I_{\rho_s^*}} |A_{k_{s+1},\rho_s^*}(t)| \leqq \sup_{I_{\rho_s^*}} \int_{B_{\rho_s^*}} (V - k_s^*)_+^2$$
$$\leqq c_1(\rho_s^* - \rho_s^{**})^{-2} \iint_{Q_{\rho_s^{**}}} (V - k_s^*)_+^2 + c_1 R^{-2} \int_{I_{\rho_s^{**}}} |A_{k_s^*,\rho_s^{**}}(t)|\, dt$$

we get

$$I_{s+1} \leqq c c_1^{1+\frac{2}{n}} (k_{s+1} - k_s)^{-\frac{4}{n}}$$
$$\times \left((\rho_s - \rho_{s+1})^{-2} I_s + R^{-2} \int_{I_{\rho_s^{**}}} |A_{k_s^*,\rho_s^{**}}(t)|\, dt \right)^{1+\frac{2}{n}}.$$

On the other hand

$$I_s \geqq \int_{Q_{\rho_s^{**}}} (V - k_s)_k^2 \geqq (k_s^* - k_s)^2 \int_{I_{\rho_s^{**}}} |A_{k_s^*,\rho_s^{**}}(t)|\, dt$$

and therefore

$$R^{-2}(\rho_s - \rho_{s+1})^2 \int_{I_{\rho_s^{**}}} |A_{k_s^*,\rho_s^{**}}(t)|\, dt \leqq c(\tau)\left(\frac{\tau}{4}\right)^{2s} k_0^{-2} I_s \leqq \tau^2 c(\tau)\left(\frac{\tau}{4}\right)^{2s} I_s .$$

This gives

$$I_{s+1} \leqq c(\tau) c_1^{1+\frac{2}{n}} k_0^{-\frac{4}{n}} R^{-2-\frac{4}{n}} b^s I_s^{1+\frac{2}{n}} := A b^s I_s^{1+\frac{2}{n}}$$

with $b := \tau^{\frac{4}{n}} 4^{2+\frac{4}{n}} \max\left\{1, \left(\frac{\tau}{4}\right)^2\right\} > 1.$

As it is easy to see, $I_s \to 0$, provided

$$I_0 \leqq A^{-\frac{n}{2}} b^{-\frac{n^2}{4}}, \text{ hence if } \fint\!\!\fint_{Q_{R_1}} (V - \gamma M)_+^2 \leqq c(\tau, c_1, n) k_0^2 .$$

Therefore choose

$$k_0 := \max\left\{\gamma M, \tau^{-1}, c(\tau, c_1, n)^{-\frac{1}{2}} \left(\fint\!\!\fint_{Q_{R_1}} (V - \gamma M)_+^2 \right)^{\frac{1}{2}} \right\}$$

and get $M := \sup_{Q_{\frac{3}{4}R_1}} V \leqq \tau k_0$. Due to $\tau\gamma < 1$, the claimed estimate follows.

A-priori bounds in C_0

We intend to give an a-priori estimate for $M_0 := \sup_{\overline{\Omega_T}} |u(x, t)|$ under the natural growth conditions (9)–(13) of Theorem 2, together with (6), (7), (8), $T > 0$ fixed. The starting point is the inequality (16)

$$\iint_{\Omega_T} \frac{dH}{dt}\phi + \iint_{\Omega_T} d_{\alpha\beta}\frac{dH}{dx_\beta}\phi_{x_\alpha} \leqq \iint_{\Omega_T} \left(R\phi + Q^\alpha \phi_{x_\alpha} + \frac{c}{(|u|^2+1)} |\nabla H|^2 \phi \right) \quad \text{for } \phi \geqq 0$$

where $|R| + |Q^\alpha| \leqq c(|u|^2 + 1)$.

By assumption $\frac{\gamma_0}{4}|u|^2 - c_2 \leqq H \leqq c_0 + c_1|u|^2$ for certain constants $c_i \geqq 1$. Let

$$k_0 \geqq \max\left\{2c_0, \max_{\bar{\Omega}} H(x, 0, u(x, 0))\right\} \tag{36}$$

and take some testfunction ϕ with $\operatorname{supp}\phi \subset \{(x,t)\in\overline{\Omega_T}|H \geqq k_0\}$. Then $|u|^2 \geqq \frac{k_0}{2c_1}$ on $\operatorname{supp}\phi$, hence the last term of (16) is bounded by $\frac{c_3}{k_0}\iint_{\Omega_T}|\nabla H|^2\phi$. Now let $s := \frac{c_3}{\lambda_0 k_0}$, $\tilde{H} := e^{sH}$, $\phi = \tilde{H}\cdot\psi$ with $\operatorname{supp}\psi \subset \{(x,t)\in\overline{\Omega_T}|H \geqq k_0\}$. Then the last term can be absorbed on the left side; we get

$$\frac{1}{s}\iint_{\Omega_T}\frac{d\tilde{H}}{dt}\psi + \frac{1}{s}\iint_{\Omega_T} d_{\alpha\beta}\frac{d\tilde{H}}{dx_\beta}\psi_{x_\alpha} \leqq \iint_{\Omega_T} R\tilde{H}\psi + Q^\alpha(\tilde{H}\psi)_{x_\alpha}.$$

Now let $\tilde{k} \geqq \tilde{k}_0 := e^{sk_0}$, $\psi := (\tilde{H} - \tilde{k})_+ \cdot \chi_{[0,t]}$. By definition of k_0, $\psi(x, 0) = 0$, hence for $t \leqq T$

$$\tfrac{1}{2}\int_\Omega (\tilde{H} - \tilde{k})_+^2(t) + \lambda_0\iint_{\Omega_t}|\nabla(\tilde{H} - \tilde{k})_+|^2$$
$$\leqq s\iint_{\Omega_t}\left(R\tilde{H}(\tilde{H} - \tilde{k})_+ + Q^\alpha\left(\tilde{H}\frac{dH}{dx_\alpha} + (\tilde{H} - \tilde{k})\frac{dH}{dx_\alpha}\right)\chi_{\tilde{H}\geqq\tilde{k}}\right).$$

Let $B := \max_{\overline{\Omega_{t_0}}}(|u(x,t)|^2 + 1)$ with t_0 to be chosen. Then the right hand side is bounded by

$$c(s^2B^2 + 1)\iint_{\Omega_t}((\tilde{H} - \tilde{k})_+^2 + \tilde{k}^2\chi_{\tilde{H}\geqq\tilde{k}}) + \lambda_0\iint_{\Omega_t}|\nabla(\tilde{H} - \tilde{k})_+|^2$$

and we end with

$$\int_\Omega (\tilde{H} - \tilde{k})_+^2(t) + \iint_{\Omega_t}|\nabla(\tilde{H} - \tilde{k})_+|^2 \leqq A\iint_{\Omega_t}((\tilde{H} - \tilde{k})_+^2 + \tilde{k}^2\chi_{\tilde{H}\geqq\tilde{k}}) \tag{37}$$

for $t \leqq t_0$, $\tilde{k} \geqq \tilde{k}_0$, $A = c(s^2B^2 + 1)$.

Let first $\tilde{k} = \tilde{k}_0$, $y(t) := \int_\Omega (\tilde{H} - \tilde{k}_0)_+^2(t)$. Then by (37) $y(t) \leqq A\int_0^t y(s)\,ds + A\tilde{k}_0^2|\Omega|t$. Now Gronwall's inequality implies $y(t) \leqq \tilde{k}_0^2|\Omega|e^{At}$. Therefore $I_0 := \iint_{\Omega_{t_0}}(\tilde{H} - \tilde{k}_1)_+^2 \leqq \iint_{\Omega_{t_0}}(\tilde{H} - \tilde{k}_1)_+^2 \leqq \tilde{k}_0^2\frac{|\Omega|}{A}e^{At_0}$ for $\tilde{k}_1 \geqq \tilde{k}_0$. By the following Lemma 2, there is some $c_4 = c_4(\Omega, n)$, *independent of* t_0, such that (37) implies $\tilde{H} \leqq 2k_1$, provided $I_0 \leqq c_4\tilde{k}_1^2A^{-1-\frac{n}{2}}$. The last obviously holds for $\tilde{k}_1^2 := c_4^{-1}A^{\frac{n}{2}}\tilde{k}_0^2e^{At_0}$. As $\tilde{k}_0 = e^{c_3/\lambda_0}$, we arrive at the estimate

$$\tilde{H} \leqq c\left(\Omega, n, \frac{c_3}{\lambda_0}\right)A^{\frac{n}{4}}e^{At_0/2}.$$

Going back to H, this means

$$H \leqq k_0c_5(1 + \ln A) + \frac{A}{2s}t_0$$

hence

$$B \leqq \frac{4}{\gamma_0} k_0 c_5 (1 + \ln A) + \frac{2}{\gamma_0} \frac{A}{s} t_0 + c_6 .$$

Let $k_0 := \varepsilon \max_{\overline{\Omega_{t_0}}} (|u(x,t)|^2 + 1)$. Then $B = \frac{k_0}{\varepsilon} = \frac{c}{s\varepsilon}$, $A = c(1 + \varepsilon^{-2})$ and we get

$$B \leqq B\{\varepsilon c_7 (1 + |\ln \varepsilon|) + c_7 (1 + \varepsilon^{-1}) t_0\} + c_8 .$$

Choose $\varepsilon = \varepsilon_1$ so small, that $\varepsilon_1 c_7 (1 + |\ln \varepsilon_1|) \leqq \frac{1}{4}$ and then t_0 with $c_7 (1 + \varepsilon_1^{-1}) t_0 \leqq \frac{1}{4}$ – the result is $B \leqq 2c_8$. Note, that this conclusion holds only, if for this ε_1 the corresponding k_0 fulfills (36). But otherwise a bound for B is immediate; therefore

$$\max_{\overline{\Omega_{t_0}}} |u(x,t)|^2 \leqq \max \left\{ 2c_0, 2c_8 \varepsilon_1^{-1}, \varepsilon_1^{-1} \max_{\bar{\Omega}} H(x, 0, u(x,0)) \right\} .$$

As ε_1 and t_0 depend only on structure constants, the same procedure works on all slices $\bar{\Omega} \times [(k-1)t_0, kt_0]$, which combines to a bound for M_0. This proves Theorem 2.

We still have to prove

Lemma 2. *Assume*

$$\sup_{t \leqq T} \int_\Omega (v-k)_+^2 + \iint_{\Omega_T} |\nabla (v-k)_+|^2 \leqq A \iint_{\Omega_T} ((v-k)_+^2 + k^2 \chi_{v \geqq k})$$

for $k \geqq k_0$ *with some* $A \geqq 1$. *Then there is some* $c_1 = c(\Omega, n) > 0$, *independent of* T, *such that*

$$\iint_{\Omega_T} (v - k_0)_+^2 \leqq c_1 k_0^2 A^{-1-\frac{n}{2}}$$

implies

$$\sup_{\Omega_T} v \leqq 2k_0 .$$

Proof. Let $k_s := 2k_0 - 2^{-s} k_0 \nearrow 2k_0$, $k_s^* = \frac{1}{2}(k_{s+1} + k_s)$

$$I_s := \iint_{\Omega_T} (v - k_s)_+^2 \qquad \text{and} \qquad A_k(t) = \{x \in \Omega \mid v(x,t) \geqq k\} .$$

By Sobolev's Imbedding

$$\begin{aligned} I_{s+1} &\leqq c \sup_t |A_{k_{s+1}}(t)|^{\frac{2}{n}} \left(\iint_{\Omega_T} |\nabla (v - k_{s+1})_+|^2 + (v - k_{s+1})_+^2 \right) \\ &\leqq 2cA \sup_t |A_{k_{s+1}}(t)|^{\frac{2}{n}} \left(I_s + k_{s+1}^2 \int_0^t |A_{k_{s+1}}(t)| \, dt \right) \\ &\leqq 2cA \sup_t |A_{k_{s+1}}(t)|^{\frac{2}{n}} \left(I_s + k_{s+1}^2 (k_{s+1} - k_s)^{-2} I_s \right) . \end{aligned}$$

As $\sup_t |A_{k_{s+1}}(t)| (k_{s+1} - k_s^*)^2 \leqq \sup_t \int_\Omega (v - k_s^*)_+^2$

$$\leqq A(I_s + k_s^{*2} (k_s^* - k_s)^{-2} I_s)$$

we have

$$I_{s+1} \leqq cA^{1+\frac{2}{n}} I_s^{1+\frac{2}{n}} (4^{1+\frac{2}{n}})^s k_0^{-\frac{4}{n}}.$$

Now it is well known, that $y_{k+1} \leqq cb^k y_k^{(1+\varepsilon)}$ implies

$$y_k \leqq c^{-\frac{1}{\varepsilon}} b^{\frac{k}{\varepsilon}} (c^{\frac{1}{\varepsilon}} b^{\frac{1}{\varepsilon^2}} y_0)^{(1+\varepsilon)^k},$$

hence $y_k \to 0$, if $\varepsilon > 0$ and $c^{\frac{1}{\varepsilon}} b^{\frac{1}{\varepsilon^2}} y_0 < 1$. Therefore $I_s \to 0$, if $I_0 \leqq c(A^{1+\frac{2}{n}} k_0^{-\frac{4}{n}})^{-\frac{n}{2}} (4^{1+\frac{2}{n}})^{-\frac{n^2}{4}}$ hence the claim.

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