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Existence of threedimensional, steady, inviscid, incompressible flows with nonvanishing vorticity

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1 Introduction and statement of results

In this paper we study steady flow of an inviscid, incompressible medium through a bounded, simply connected domain $\Omega \subseteqq \mathbb{R}^3$. Our goal is to construct solutions with nonvanishing vorticity of the boundary value problem

$$(v(x) \cdot \nabla) v(x) + \nabla p(x) = 0, \qquad x \in \Omega, \tag{1.1}$$

$$\operatorname{div} v(x) = 0, \qquad x \in \Omega, \tag{1.2}$$

$$n(x) \cdot v(x) = f(x), \qquad x \in \partial\Omega, \tag{1.3}$$

where $v(x) \in \mathbb{R}^3$ denotes the velocity and $p(x) > 0$ the pressure of the flow. $n(x)$ denotes the exterior unit normal to the boundary $\partial\Omega$ at $x \in \partial\Omega$. Of course, the given function f must satisfy

$$\int_{\partial\Omega} f(x) dS_x = \int_{\partial\Omega} n(x) \cdot v(x) dS_x = \int_{\Omega} \operatorname{div} v(x) dx = 0. \tag{1.4}$$

It is well known that for simply connected domains Ω the problem (1.1)–(1.4) has an irrotational solution (v, p), which is unique up to addition of constants to the pressure. Namely, in such domains any velocity field v with

$$\operatorname{curl} v(x) = 0 \tag{1.5}$$

for all $x \in \Omega$ is a gradient field

$$v(x) = \nabla \varphi(x), \tag{1.6}$$

and from (1.2) and (1.3) it follows that

$$\Delta \varphi(x) = 0, \qquad x \in \Omega, \tag{1.7}$$

$$\frac{\partial}{\partial n} \varphi(x) = f(x), \qquad x \in \partial\Omega. \tag{1.8}$$

The Neumann boundary value problem (1.4), (1.7), (1.8) has a solution φ, which is unique up to constants. Therefore the velocity field v given by (1.6) is unique. To construct the pressure p note that with the relation

$$(v \cdot \nabla)v = \nabla(\tfrac{1}{2}|v|^2) - v \times \operatorname{curl} v$$

the Eq. (1.1) can be written as

$$-v \times \operatorname{curl} v + \nabla(\tfrac{1}{2}|v|^2 + p) = 0. \tag{1.9}$$

From (1.5) we thus obtain that

$$\tfrac{1}{2}|v|^2 + p = \text{const}.$$

It is clear that the functions v, p thus constructed satisfy (1.1)–(1.3).

It is much less obvious whether the problem (1.1)–(1.3) has solutions representing flows with nonvanishing vorticity. On physical grounds one expects many such flows to exist. On the other hand, the conventional expectation is that these flows are unstable and that in physical reality a steady flow governed by (1.1)–(1.3) would switch immediately into a turbulent flow because of the absence of viscosity. A general experience in mathematical physics seems to be that lack of stability of the objects under consideration introduces difficulties into the existence proof for these objects, and in many cases these difficulties have not yet been overcome. The fact that the existence of irrotational solutions of (1.1)–(1.3) can easily be proved would then be attributed to the introduction of artificial stability by the requirement $\operatorname{curl} v(x) = 0$, which excludes turbulent motion.

We shall prove, however, that if (v_0, p_0) is a solution of (1.1)–(1.3) satisfying $v_0(x) \neq 0$ for all $x \in \bar{\Omega}$ and has sufficiently small vorticity, then there exist a neighborhood of this solution and flows with nonvanishing vorticity in this neighborhood, which satisfy (1.1)–(1.3) and two additional boundary conditions. These additional boundary conditions hold only on that part of the boundary, through which liquid is entering the domain Ω, and prescribe the vorticity of the flow on this part of the boundary. They are necessary because the requirement $\operatorname{curl} v = 0$ used in the construction above is dropped. Moreover, we show that any such flow is stable in the sense that in the neighborhood mentioned above it is the unique flow satisfying (1.1)–(1.3) and the additional boundary conditions, and that it depends continuously on the boundary data. In particular, such flows with nonvanishing vorticity exist in a neighborhood of the irrotational solution of (1.1)–(1.3) constructed above.

To see what these additional boundary conditions should be, apply the operator curl to (1.9). This yields

$$\operatorname{curl}(v \times \operatorname{curl} v) = 0. \tag{1.10}$$

From the relation

$$\operatorname{curl}(v \times z) = v \operatorname{div} z + (z \cdot \nabla)v - z \operatorname{div} v - (v \cdot \nabla)z \tag{1.11}$$

and from $\operatorname{div} v = 0$ we conclude that (1.10) is equivalent to

$$(v \cdot \nabla) \operatorname{curl} v = [(\operatorname{curl} v) \cdot \nabla]v, \tag{1.12}$$

the Vorticity Transport Theorem, also called Helmholtz' equation, cf. [10]. The Eqs. (1.2), (1.3), (1.12) constitute the velocity-vorticity formulation of the boundary value problem (1.1)–(1.3). In simply connected domains both formulations are equivalent.

If the velocity field v is known, then (1.12) can be considered to be a system of first order partial differential equations for $\operatorname{curl} v$, the characteristics of which coincide with the stream lines of v. This means that (1.12) can also be considered to be a system of linear ordinary differential equations for $\operatorname{curl} v$ along stream lines. It follows that if $\operatorname{curl} v$ is known at one point of a stream line, it can be computed along the whole stream line from (1.12). By definition, stream lines $\tau \mapsto \omega(\tau)$ are solutions of the system

$$\frac{d}{d\tau}\,\omega(\tau)=v(\omega(\tau))$$

of ordinary differential equations. From this definition it immediately follows that any stream line contains at most one point $x\in\partial\Omega$ with

$$n(x)\cdot v(x)=f(x)<0,$$

and therefore it is possible to prescribe $\operatorname{curl} v(x)$ at any point $x\in\partial\Omega$ with $f(x)<0$. On the other hand, in all the solutions with nonvanishing vorticity we construct, the domain Ω is covered by stream lines starting at such points. This follows from the following properties of these solutions v: They are continuously differentiable, satisfy $v(x)\neq 0$ for all $x\in\bar{\Omega}$, and do not have closed stream lines. Moreover, the length of all stream lines is uniformly bounded, and any stream line that is tangential to the boundary at one point is completely contained in the boundary. To assure that v has these properties it is necessary to make special assumptions for the unperturbed flow v_0. In particular, v_0 must have these properties, but since the last property is not necessarily stable against perturbations, we must add another technical condition, which is precisely formulated in the theorem stated below.

It follows that $\operatorname{curl} v(y)$ is uniquely determined for all $y\in\Omega$ if we prescribe $\operatorname{curl} v(x)$ for all $x\in\partial\Omega$ with $f(x)<0$. Observe however, that it is not possible to prescribe all three components of $\operatorname{curl} v$ independently, because (1.10) yields

$$n(x)\cdot\operatorname{curl}(v(x)\times\operatorname{curl} v(x))=0 \tag{1.13}$$

for all $x\in\partial\Omega$, which implies by Stokes' theorem that the component $(v\times\operatorname{curl} v)_T$ of $(v\times\operatorname{curl} v)_{|\partial\Omega}$ tangential to $\partial\Omega$ is equal to the tangential gradient $\nabla_T g$ of a function $g:\partial\Omega\to\mathbb{R}$. Here and in the following we mean by $(v\times\operatorname{curl} v)_T$ and $\nabla_T g$ vectors in $\mathbb{R}^3$ tangential to $\partial\Omega$. As boundary conditions for $\operatorname{curl} v$ we therefore choose

$$n(x)\cdot\operatorname{curl} v(x)=h(x),\qquad (v(x)\times\operatorname{curl} v(x))_T=\nabla_T g(x) \tag{1.14}$$

for all $x\in\partial\Omega$ with $f(x)<0$. h and g are given functions. From (1.9) it follows that (1.14) is equivalent to the requirement that there exists a constant c with

$$\tfrac{1}{2}|v(x)|^2+p(x)=g(x)+c$$

for all $x\in\partial\Omega$ with $f(x)<0$, and in the following we use this form of the boundary condition (1.14), where we also normalize p such that $c=0$.

We remark that for any vector field z satisfying

$$(v\cdot\nabla)z=(z\cdot\nabla)v$$

the relations $n(x)\cdot\operatorname{curl}(v(x)\times z(x))=0$ and $\operatorname{div} z(x)=0$ are equivalent on the set of all $x\in\partial\Omega$ with $f(x)<0$. To see this note that (1.11) yields

$$n\cdot\operatorname{curl}(v\times z)=n\cdot v\operatorname{div} z=f\operatorname{div} z.$$

Therefore the condition imposed by (1.13) on the boundary values of the vector field $z=\operatorname{curl} v$ can be considered to be a consequence of the relation $\operatorname{div} z=\operatorname{div}\operatorname{curl} v=0$.

To state the main result of this paper we need some spaces of functions defined on the part of the boundary $\partial\Omega$ where f is negative. To introduce the norms of these spaces we now state several definitions and notations, some of them are standard.

For an open set $\Gamma\subseteqq\mathbb{R}^\ell$ and for any nonnegative integer k let $H_k(\Gamma)=H_k(\Gamma,\mathbb{R}^m)$ denote the usual Sobolev space of functions from Γ into $\mathbb{R}^m$ with norm

$$\|u\|_{k,\Gamma}=\Big(\sum_{|\beta|\leqq k}\int_\Gamma |D^\beta u(x)|^2 dx\Big)^{1/2}.$$

Here $\beta=(\beta_1,\dots,\beta_\ell)$ is a multi-index. We assume that the bounded domain $\Omega\subseteqq\mathbb{R}^3$ is of class C^∞. As usual, this means that there exist open subsets $U_1,\dots,U_\mu$ of $\mathbb{R}^3$ with $\partial\Omega\subseteqq\bigcup_{i=1}^{\mu} U_i$, and diffeomorphisms $\Phi_i: D_3\to U_i$, where

$$D_\ell=\{y\in\mathbb{R}^\ell: |y|<1\},$$

such that

$$U_i\cap\partial\Omega=\Phi_i(D_3\cap\{x_3=0\})$$

and

$$U_i\cap\Omega=\Phi_i(D_3\cap\{x_3>0\}).$$

$H_k(\partial\Omega,\mathbb{R}^m)$ denotes the usual trace space. The functions $\psi_i: D_2\to\partial\Omega$ with

$$\psi_i(\xi_1,\xi_2)=\Phi(\xi_1,\xi_2,0)$$

define coordinate systems on $\partial\Omega$. Let $\alpha_1,\dots,\alpha_\mu:\partial\Omega\to\mathbb{R}$ be a partition of unity on $\partial\Omega$ with $0\leqq\alpha_i\leqq 1$, $\operatorname{sup}\alpha_i\subseteqq\psi_i(D_2)$, and with $\alpha_i\circ\psi_i\in C_0^\infty(D_2)$. As norm of $H_k(\partial\Omega,\mathbb{R}^m)$ we use

$$\|q\|_{k,\partial\Omega}=\sum_{i=1}^{\mu}\sum_{|\beta|\leqq k}\|(\alpha_i\circ\psi_i)D^\beta(q\circ\psi_i)\|_{0,D_2}. \tag{1.15}$$

For $f\in H_2(\partial\Omega,\mathbb{R})$ let

$$\begin{aligned}\partial\Omega_-&=\partial\Omega_-(f)=\{x\in\partial\Omega: f(x)<0\},\\ \partial\Omega_+&=\partial\Omega_+(f)=\{x\in\partial\Omega: f(x)>0\}.\end{aligned} \tag{1.16}$$

$\partial\Omega_-$, $\partial\Omega_+$ are open subsets of the C^∞-manifold $\partial\Omega$, because f is continuous. Therefore they are themselves C^∞-manifold. The boundary of $\partial\Omega_\pm$ in $\partial\Omega$ is denoted by

$$\partial\partial\Omega_\pm=\overline{\partial\Omega_\pm}\cap\overline{(\partial\Omega\backslash\partial\Omega_\pm)}.$$

We say that a bounded domain $G\subseteqq\mathbb{R}^2$ has Lipschitz boundary, if the following two properties are satisfied.

a) About every $x_0\in\partial G$ there is an open neighborhood $U\subseteqq\mathbb{R}^2$ and $i\in\{1,2\}$, such that the set $\partial G\cap U$ has the representation

$$x_j=g(x_i),\qquad x_i\in U',$$

where $j \in \{1,2\}$ and $j \neq i$, where U' is the projection of U on $\{x_j=0\}$, and where $g: U' \to \mathbb{R}$ is a Lipschitz continuous function.

b) The set $U \cap G$ is either contained in the half cylinder $\{x_j > g(x_i)\}$ or in the half cylinder $\{x_j < g(x_i)\}$.

We say that $\partial\Omega_-(f)$ has Lipschitz boundary, if the function $\Phi_1, \dots, \Phi_\mu$ can be chosen such that for every $i=1,\dots,\mu$ the domain

$$D_2^i = \psi_i^{-1}(\partial\Omega_-)$$

is empty or has Lipschitz boundary.

The norms for the functions with domain $\partial\Omega_-$ are defined as follows. For $q: \partial\Omega_- \to \mathbb{R}^m$ and $k \leqq 2$ let

$$\|q\|_{k,\partial\Omega_-} = \sum_{i=1}^{\mu} \sum_{|\beta| \leqq k} \|(\alpha_i \circ \psi_i) D^\beta (q \circ \psi_i)\|_{0,D_2^i}, \tag{1.17}$$

$$|q|_{k,\partial\Omega_-} = \sum_{i=1}^{\mu} \sum_{|\beta|+|\gamma| \leqq k} \left\|(\alpha_i \circ \psi_i) D^\beta \left(\frac{1}{f \circ \psi_i}\right) D^\gamma (q \circ \psi_i)\right\|_{0,D_2^i}, \tag{1.18}$$

$$|||q|||_{k,\partial\Omega_-} = \sum_{i=1}^{\mu} \sum_{|\beta|+|\beta'|+|\gamma| \leqq k} \left\|(\alpha_i \circ \psi_i) D^\beta \left(\frac{1}{f \circ \psi_i}\right) D^{\beta'} \left(\frac{1}{f \circ \psi_i}\right) D^\gamma (q \circ \psi_i)\right\|_{0,D_2^i}, \tag{1.19}$$

if these expressions are finite. The last two norms are finite only if q and its derivatives vanish sufficiently rapidly at the boundary $\partial\partial\Omega_-$. Note that there exist constants $c_1, c_2 > 0$, depending on f, with

$$\|q\|_{k,\partial\Omega_-} \leqq c_1 |q|_{k,\partial\Omega_-} \leqq c_2 |||q|||_{k,\partial\Omega_-}. \tag{1.20}$$

Our main result is

Theorem 1.1. *Let the bounded, simply connected domain Ω be of class C^∞. Assume that $f \in H_2(\partial\Omega, \mathbb{R})$ satisfies* (1.4) *and is such that $\partial\Omega_- = \partial\Omega_-(f)$ is a manifold with Lipschitz boundary.*

Let $(v_0, p_0) \in H_3(\Omega)$ be a solution of (1.1)–(1.3) *satisfying* $\operatorname{curl} v_0 \in H_3(\Omega)$ *and*

$$\underline{v}_0 = \inf_{x \in \Omega} |v_0(x)| > 0. \tag{2.21}$$

Moreover, assume that v_0 does not have closed stream lines and that the least upper bound L_0 of the length of all stream lines of v_0 in Ω is finite. Finally, assume that there exist constants $\hat{c} > 0$, $\hat{t} > 0$ such that

$$\operatorname{dist}(\partial\Omega_-(f), x + t v_0(x)) \geqq \hat{c} t \tag{1.22}$$

for all $x \in \partial\partial\Omega_-(f)$ and for all $0 \leqq t \leqq \hat{t}$, and

$$\operatorname{dist}(\partial\Omega_+(f), x - t v_0(x)) \geqq \hat{c} t \tag{1.23}$$

for all $x \in \partial\partial\Omega_+(f)$ and for all $0 \leqq t \leqq \hat{t}$.

Then there exist constants

$$\hat{\gamma} = \hat{\gamma}(v_0, \Omega) > 0,$$

$$\hat{K}_i = \hat{K}_i(L_0, \underline{v}_0, \|v_0\|_{3,\Omega}, f, \hat{\gamma}, \Omega) > 0, \qquad i = 1, \dots, 3$$

with the following properties:

Let $g \in H_3(\partial\Omega_-, \mathbb{R})$, $h \in H_2(\partial\Omega_-, \mathbb{R})$ and v_0 satisfy

$$I(g, h, \operatorname{curl} v_0) \leqq \hat{K}_1 \tag{1.24}$$

with

$$I(g,h,\operatorname{curl} v_0)=\left\|\frac{h}{f}\right\|_{2,\partial\Omega_-}+\left\|\left|\frac{1}{f}\nabla_T g\right|\right\|_{2,\partial\Omega_-}+|D^2\operatorname{curl} v_0|_{0,\partial\Omega_-}$$
$$+\sum_{m=0}^{1}|||D^m\operatorname{curl} v_0|||_{2-m,\partial\Omega_-}$$
$$+\left\|\left|\frac{1}{f}(n\cdot\operatorname{curl} v_0)\right|\right\|_{2,\partial\Omega_-}+\|\operatorname{curl} v_0\|_{3,\Omega}.$$

Here $D^m\operatorname{curl} v_0$ *denotes the vector*

$$D^m\operatorname{curl} v_0=(D^\beta(\operatorname{curl} v_0)_j)_{\substack{j=1,2,3\\|\beta|\leq m}}.$$

$(\operatorname{curl} v_0)_j$ *are the components of* $\operatorname{curl} v_0$, *and* $\beta=(\beta_1,\beta_2,\beta_3)$ *is a multi-index. Then there exists a solution* $(v,p)\in H_3(\Omega,\mathbb{R}^3\times\mathbb{R})$ *of* (1.1)–(1.3) *with*

$$n(x)\cdot\operatorname{curl} v(x)=h(x)+n(x)\cdot\operatorname{curl} v_0(x) \tag{1.25}$$

$$\tfrac{1}{2}|v(x)|^2+p(x)=g(x)+\tfrac{1}{2}|v_0(x)|^2+p_0(x) \tag{1.26}$$

for all $x\in\partial\Omega_-$.

v satisfies

$$\|v-v_0\|_{3,\Omega}\leq\hat{\gamma}, \tag{1.27}$$

and (v,p) *is the only solution of* (1.1)–(1.3), (1.25), (1.26) *from* $H_3(\Omega,\mathbb{R}^3\times\mathbb{R})$ *satisfying this estimate.*

If $(g^{(1)},h^{(1)})$ *and* $(g^{(2)},h^{(2)})$ *are two sets of boundary data on* $\partial\Omega_-(f)$ *both satisfying* (1.24), *and if* $(v^{(1)},p^{(1)})$, $(v^{(2)},p^{(2)})$ *are solutions of* (1.1)–(1.3), (1.25), (1.26) *to the boundary data* $(g^{(1)},h^{(1)})$ *and* $(g^{(2)},h^{(2)})$, *respectively, both satisfying* (1.27), *then*

$$\|v^{(1)}-v^{(2)}\|_{1,\Omega}\leq\hat{K}_2(|h^{(1)}-h^{(2)}|_{0,\partial\Omega_-}+|\nabla_T(g^{(1)}-g^{(2)})|_{0,\partial\Omega_-}), \tag{1.28}$$

$$\|p^{(1)}-p^{(2)}\|_{1,\Omega}\leq\hat{K}_3(|h^{(1)}-h^{(2)}|_{0,\partial\Omega_-}+|\nabla_T(g^{(1)}-g^{(2)})|_{0,\partial\Omega_-}+\|g^{(1)}-g^{(2)}\|_{0,\partial\Omega_-}). \tag{1.29}$$

We comment on some points in this theorem:

The estimates (1.28) and (1.29) can be improved. It is possible to estimate the difference of the solutions in the H_3-norm, but the calculations are technical.

Condition (1.24) implies that h, $\nabla_T g$ and $\operatorname{curl} v_0$ must vanish sufficiently rapidely at the boundary $\partial\partial\Omega_-(f)$. This condition can be compared to the compatibility conditions needed in initial-boundary value problems for hyperbolic equations.

It is assumed that $f\in H_2(\partial\Omega)$, but because of (1.3) the condition $v_0\in H_3(\Omega)$ implicitly requires more regularity of f. It is on the other hand assumed that $g\in H_3(\partial\Omega_-)$ and $h\in H_2(\partial\Omega_-)$, which is more than the trace theorem would require. Namely, (1.25) shows that h is the normal component of the trace of $\operatorname{curl}(v-v_0)\in H_2(\Omega)$, and (1.26) shows that g is the trace of $\frac{1}{2}|v|^2+p-\frac{1}{2}|v_0|^2-p_0\in H_3(\Omega)$. Therefore the trace theorem indicates that either it would suffice to assume $g\in H_{5/2}(\partial\Omega_-)$ and $h\in H_{3/2}(\partial\Omega_-)$, or else that the solution (v,p) is of higher regularity than $H_3(\Omega)$. We believe that it is possible to prove such results by a refined analysis, but we do not investigate this question here.

The conditions (1.22) and (1.23), which are stable with respect to perturbations of v_0, are needed to show that not only the unperturbed flow v_0 but also every flow v which satisfies (1.3) and is close to v_0 has the property that any stream line which is tangential to the boundary at one point is completely contained in the boundary. From (1.3) it follows that $v(x)$ is tangential to $\partial\Omega$ for all $x \in \partial\partial\Omega_-(f) \cup \partial\partial\Omega_+(f)$. Therefore (1.22) means that the flow is directed outward of $\partial\Omega_-(f)$ at the boundary, and it is not possible that particles move tangentially along the boundary until they reach $\partial\Omega_-(f)$, where they would be transported into Ω by the flow. (1.23) has a similar meaning for the set $\partial\Omega_+(f)$, where the flow leaves Ω.

As a simple example for Ω and v_0 satisfying the hypotheses of the theorem consider the cylinder

$$Z = \{x_1^2 + x_2^2 < a^2,\ -b < x_3 < b\}.$$

Close this cylinder by two half balls

$$S_+ = \{x_1^2 + x_2^2 + (x_3 - b)^2 < a^2,\ x_3 \geqq b\}$$

$$S_- = \{x_1^2 + x_2^2 + (x_3 + b)^2 < a^2,\ x_3 \leqq -b\}$$

and set $\Omega = Z \cup S_+ \cup S_-$. For v_0 take the constant flow $v_0(x) = (0, 0, 1)$. In this example $\partial\Omega_-(f)$ and $\partial\Omega_+(f)$ coincide with the spherical parts of the surfaces of S_- and S_+.

The assumption that Ω be simply connected is needed in Theorem 2.4 and therefore also in Theorem 1.1.

There exists a larger literature dealing with the nonstationary version of the problem (1.1)–(1.3). The early investigations concern the case when the liquid is confined to Ω. In the two-dimensional case existence global-in-time was proved by Wolibner [14] and Hölder [3], whereas in the three-dimensional case existence local-in-time was proved by Kato [4], Ebin and Marsden [2], Bourguignon and Brezis [1], and Temam [11]. In the case when the liquid can pass through the boundary of Ω the nonstationary problem was treated in Kochin [7], Kazhikhov [5], Kazhikhov and Ragulin [6], Yudovich [15], and Zajaczkowski [16–22]. Also in the nonstationary problem it is necessary to prescribe additional boundary conditions on the inflow part of the boundary.

This short account on the existing literature is not complete. More references can be found in the cited literature.

Theorem 1.1 is proved in the remainder of this paper. The proof is based on a contraction argument and on Banach's fixed point theorem. In Sect. 2 the main lines of the proof are given and results and estimates needed in the proof are stated in a sequence of lemmas and theorems. Some of these lemmas and theorems are proved in Sect. 2, the rest is proved in Sects. 3–5.

We conclude this introduction by discussing some directions, into which the results of Theorem 1.1 might be extended. The estimates (2.27) indicate that the constant in (1.24) satisfies

$$\hat{K}_1 \sim \frac{1}{L_0^{1/2}}, \tag{1.30}$$

for $L_0 \to 0$, where L_0 is the least upper bound for the length of the stream lines of v_0. This is because the constant $\tilde{M}$ in (2.27) does not explicitly depend on L_y, and since the constants $\tilde{K}_2$, $\tilde{K}_3$ in (2.27) remain bounded for $L_y \to 0$, as noted in Theorem 2.3. It would follow that we could construct solutions with nonvanishing vorticity for large values of g, h, and $\operatorname{curl} v_0$, if the domain Ω is "short". However, $\tilde{M}$, $\tilde{K}_2$, and $\tilde{K}_3$ all depend on the shape of Ω. The reason is that in the derivation of (2.13) and (2.14) in Sects. 4 and 5 at several places Sobolev's inequality and embedding theorems for

Sobolev spaces are used. To prove (1.30) would therefore require to show that the constants $\tilde{M}, \tilde{K}_2, \tilde{K}_3$ remain bounded for all sufficiently "short" domains Ω. We do not study this question here.

Along the same lines of thought one could try to proceed as in hyperbolic problems and continue the solution into a second short domain after it has been constructed in a first short domain. This procedure is not immediately possible, however, because (1.25) and (1.26) are initial conditions, but (1.3) is a boundary condition, which must be satisfied on the whole boundary.

As a final remark we note that the fact that the flows v_0 and v must be different from zero everywhere might indicate that a steady state flow is unstable at points where the velocity vanishes.

2 Outline of the proof

In this section we lay out the main lines of the proof of Theorem 1.1. The basic idea is to construct an operator B in a subspace V of $H_3(\Omega, \mathbb{R}^3)$ with the property that for a fixed point u of B the function $v = v_0 + u$ is the velocity field of a solution of (1.1)–(1.3), (1.25), (1.26). We start with the definition of V and B.

Let V be the space of all functions $w \in H_3(\Omega, \mathbb{R}^3)$ satisfying

$$\operatorname{div} w(x) = 0, \qquad x \in \Omega \tag{2.1}$$

$$n(x) \cdot w(x) = 0, \qquad x \in \partial\Omega. \tag{2.2}$$

V is a closed subspace of $H_3(\Omega, \mathbb{R}^3)$ and therefore also a Hilbert space with the scalar product $(u, w)_{3,\Omega}$. For $\gamma > 0$ let V_γ be the closed ball of all $w \in V$ with $\|w\|_{3,\Omega} \leqq \gamma$. To define the operator $B: V_\gamma \to V$ let $u \in V_\gamma$, let $W \in H_3(\Omega, \mathbb{R}^3)$ with $\operatorname{div} W = 0$ in Ω, and let $z: \Omega \to \mathbb{R}^3$ be the solution of

$$[(v_0 + u) \cdot \nabla] z = (z \cdot \nabla)(v_0 + u) - (u \cdot \nabla) W + (W \cdot \nabla) u \tag{2.3}$$

$$z_{|\partial\Omega_-} = \eta, \tag{2.4}$$

where the components of $\eta: \partial\Omega_- \to \mathbb{R}^3$ are defined by the equations

$$n(x) \cdot \eta(x) = h(x) \tag{2.5}$$

$$\eta_T(x) = \frac{h}{f}(v_0 + u)_T(x) + \frac{1}{f}(n \cdot W) u_T(x) - \frac{1}{f} n(x) \times \nabla_T g(x) \tag{2.6}$$

with $x \in \partial\Omega_-$ and with the functions f, g, and h from the conditions (1.3), (1.25), (1.26). The vector field W in (2.3) and (2.6) will later be replaced by $\operatorname{curl} v_0$. For later use we note that if $W = \operatorname{curl} v_0$ and if (2.5) is satisfied, then (2.3) is equivalent to

$$[(v_0 + u) \cdot \nabla](z + \operatorname{curl} v_0) = [(z + \operatorname{curl} v_0) \cdot \nabla](v_0 + u), \tag{2.7}$$

and (2.6) is equivalent to

$$[(v_0 + u)(x) \times (\eta + \operatorname{curl} v_0)(x)]_T = \nabla_T (g(x) + \tfrac{1}{2}|v_0(x)|^2 + p_0(x)) \tag{2.8}$$

for $x \in \partial\Omega_-$. The equivalence of (2.3) and (2.7) is seen if one expands (2.7) and uses that v_0 satisfies (1.12), since (v_0, p_0) is a solution of (1.1), (1.2). To see that (2.6) and (2.8) are equivalent multiply (2.6) by f and use (1.3), (2.2), (2.5) to obtain

$$(n \cdot \eta)(v_0 + u)_T + (n \cdot W) u_T - n \cdot (v_0 + u) \eta_T - n \cdot u W_T = n \times \nabla_T g.$$

The left hand side of this equation is not changed if the tangential components are replaced by the vectors themselves. Therefore the last equation is equivalent to

$$n \times [(v_0+u)\times \eta] + n \times [u \times W] = n \times \nabla_T g$$

or

$$[(v_0+u)\times \eta]_T + [u \times W]_T = \nabla_T g. \tag{2.9}$$

If one replaces W by $\operatorname{curl} v_0$ then this equation is equivalent to (2.8) since

$$[v_0 \times \operatorname{curl} v_0]_T = \nabla_T(\tfrac{1}{2}|v_0|^2 + p_0).$$

This equation holds since (v_0, p_0) solves (1.1) and therefore also (1.9).

Note that (2.3) is an inhomogeneous linear system of ordinary differential equations for z along integral curves of v_0+u. Therefore (2.4) and (2.3) determine z on the subset of Ω covered by integral curves starting at $\partial\Omega_-(f)$. Below it will be shown that this set is equal to Ω, if γ and therefore also $\|u\|_{3,\Omega}$ is chosen sufficiently small. In Sect. 3 we show that the solution satisfies $\operatorname{div} z = 0$. In Sects. 4 and 5 we prove that $z \in H_2(\Omega)$.

From these properties and from Theorem 2.4 we deduce that there exists a unique $w \in V$ with

$$\operatorname{curl} w = z. \tag{2.10}$$

We define

$$B(u) = w. \tag{2.11}$$

This completes the definition of $B: V_\gamma \to V$.

Note that B depends on the functions g, h, W, and v_0, hence $B = B[g,h,W,v_0]$, and from (2.3)–(2.6), (2.10) it follows that the mapping $(g,h,W) \mapsto B[g,h,W,v_0](u) \in V$ is linear. We set

$$B[g,h,v_0] = B[g,h,\operatorname{curl} v_0, v_0].$$

We state now a sequence of lemmas and theorems which show that B is well defined and has a fixed point. They also establish the correspondence between fixed points of B and solutions of (1.1)–(1.3), (1.25), (1.26). Some of the assertions are proved in this section, the remaining proofs are postponed to the following sections.

Lemma 2.1. *Let $v_0 \in H_3(\Omega, \mathbb{R}^3)$ satisfy the hypotheses of Theorem 1.1. Then there exist constants $\hat{C} > 0$ and $\gamma_0 > 0$ with the following three properties*
(P1) *The vector field $v = v_0 + u$ with $u \in V_{\gamma_0}$ satisfies*

$$\underline{v} = \inf_{x\in\Omega} |v(x)| \geqq \underline{v}_0 - \hat{C}\|u\|_{3,\Omega} \geqq \underline{v}_0 - \hat{C}\gamma_0 > 0.$$

(P2) *No vector field $v \in v_0 + V_{\gamma_0}$ has closed integral curves. For $0 < \gamma \leqq \gamma_0$ let L_γ denote the least upper bound of the length of all integral curves of all the vector fields $v \in v_0 + V_\gamma$. Then $L_\gamma < \infty$ and*

$$\lim_{\gamma\to 0} L_\gamma = L_0.$$

(P3) *If an integral curve of $v \in v_0 + V_{\gamma_0}$ is tangential to the boundary $\partial\Omega$ at one point, then it is completely contained in the boundary.*

This lemma is proved in Sect. 3. Remember that an integral curve $\omega(t)$ is a solution of $\frac{d}{dt}\omega(t)=v(\omega(t))$. Since $v\in v_0+V_{\gamma_0}$ satisfies (1.3), the statements of this lemma together imply that every integral curve of v that passes over a point $x\in\Omega$ meets the boundary in exactly one point from $\partial\Omega_-(f)$, the starting point of the integral curve, and in exactly one point from $\partial\Omega_+(f)$, the endpoint of this integral curve. Therefore Ω is completely covered by integral curves of v starting at $\partial\Omega_-(f)$. It also follows that every integral curve that starts at $\partial\Omega_-$ has its endpoint in $\partial\Omega_+$ and does not meet the boundary in a third point.

From now on $\gamma_0=\gamma_0(v_0)$ always means the constant from the preceding lemma. Also the following lemma is proved in Sect. 3.

Lemma 2.2. *For every $u\in V_\gamma$ with $\gamma\leqq\gamma_0$ and every $W\in H_3(\Omega,\mathbb{R}^3)$ with* div $W=0$ *the unique solution z of* (2.3)–(2.6) *exists in all of Ω and satisfies* div $z=0$.

Of course, this solution depends on g, h, W, v_0, and $u\in V_\gamma$, hence $z=z[g,h,W,v_0,u]$. We set $z[g,h,v_0,u]=z[g,h,\operatorname{curl}v_0,v_0,u]$.

Theorem 2.3. *There exists a constant $\tilde{M}=\tilde{M}(\Omega)>0$, and to any $\gamma\leqq\gamma_0$ constants $\tilde{K}_i=\tilde{K}_i(L_\gamma,v_0,\|v_0\|_{3,\Omega},f,\gamma,\Omega)>0, 1=1,\dots,3$, which remain bounded for $L_\gamma\to 0$, such that for all $u,w\in V_\gamma$*

$$\|z[g,h,W,v_0,u]\|_{0,\Omega}\leqq L_\gamma^{1/2}\tilde{K}_1[|h|_{0,\partial\Omega_-}+|n\cdot W|_{0,\partial\Omega_-}+|\nabla_T g|_{0,\partial\Omega_-}+\|W\|_{3,\Omega}] \tag{2.12}$$

$$\|z[g,h,v_0,u]\|_{2,\Omega}\leqq L_\gamma^{1/2}\tilde{K}_2 I(g,h,\operatorname{curl}v_0) \tag{2.13}$$

$$\|z[g,h,v_0,u]-z[g,h,v_0,w]\|_{0,\Omega}\leqq L_\gamma^{1/2}\tilde{K}_3 I(g,h,\operatorname{curl}v_0)\|u-w\|_{1,\Omega} \tag{2.14}$$

and,

$$\|B[g,h,W,v_0](u)\|_{1,\Omega}\leqq\tilde{M}L_\gamma^{1/2}\tilde{K}_1[|h|_{0,\partial\Omega_-}+|n\cdot W|_{0,\partial\Omega_-}+|\nabla_T g|_{0,\partial\Omega_-}+\|W\|_{3,\Omega}] \tag{2.15}$$

$$\|B[g,h,v_0](u)\|_{3,\Omega}\leqq\tilde{M}L_\gamma^{1/2}\tilde{K}_2 I(g,h,\operatorname{curl}v_0) \tag{2.16}$$

$$\|B[g,h,v_0](u)-B[g,h,v_0](w)\|_{1,\Omega}\leqq\tilde{M}L_\gamma^{1/2}\tilde{K}_3 I(g,h,\operatorname{curl}v_0)\|u-w\|_{1,\Omega} \tag{2.17}$$

with

$$I(g,h,\operatorname{curl}v_0)=\left|\left|\left|\frac{h}{f}\right|\right|\right|_{2,\partial\Omega_-}+\left|\left|\left|\frac{1}{f}\nabla_T g\right|\right|\right|_{2,\partial\Omega_-}+|D^2\operatorname{curl}v_0|_{0,\partial\Omega_-}+\sum_{m=0}^{1}|||D^m\operatorname{curl}v_0|||_{2-m,\partial\Omega_-}+\left|\left|\left|\frac{1}{f}(n\cdot\operatorname{curl}v_0)\right|\right|\right|_{2,\partial\Omega_-}+\|\operatorname{curl}v_0\|_{3,\Omega}, \tag{2.18}$$

where $D^m\operatorname{curl}v_0$ denotes the vector

$$D^m\operatorname{curl}v_0=(D^\beta(\operatorname{curl}v_0)_j)_{\substack{j=1,2,3\\|\beta|\leqq m}}.$$

Here $(\operatorname{curl}v_0)_j$ are the components of $\operatorname{curl}v_0$, and $\beta=(\beta_1,\beta_2,\beta_3)$ is a multi-index.

The norms in the expression for $I(g, h, \operatorname{curl} v_0)$ are defined in (1.18), (1.19). It is clear that the estimates (2.15)–(2.17) are immediate consequences of (2.12)–(2.14), of (2.10), (2.11), and of the following theorem. It therefore remains to verify the estimates (2.12)–(2.14), the proof of which is given in Sects. 4 and 5.

Theorem 2.4. *Let $z \in H_2(\Omega, \mathbb{R}^3)$ satisfy $\operatorname{div} z = 0$ and let $\Omega \in C^\infty$ be a bounded, simply connected domain. Then there exists a unique function $w \in H_3(\Omega, \mathbb{R}^3)$ with*

$$\operatorname{curl} w(x) = z(x), \quad x \in \Omega, \tag{2.19}$$

$$\operatorname{div} w(x) = 0, \quad x \in \Omega, \tag{2.20}$$

$$n(x) \cdot w(x) = 0, \quad x \in \partial\Omega. \tag{2.21}$$

Moreover, there exists a constant $\tilde{M}$, only depending on Ω, such that

$$\|w\|_{3,\Omega} \leqq \tilde{M} \|z\|_{2,\Omega}. \tag{2.22}$$

A proof that the solution w exists and is unique can be found in [9], and (2.22) is proved in [12].

If Ω is not simply connected and has genus ν, then the solvability of (2.19)–(2.21) is guaranteed only if z satisfies ν additional conditions, cf. [9, 13]. This is why in Theorem 1.1 we need the assumption that Ω be simply connected.

Corollary 2.5. *For every γ with $0 < \gamma \leqq \gamma_0(v_0)$ the operator $B[g, h, v_0]$ maps V_γ into itself if*

$$I(g, h, \operatorname{curl} v_0) \leqq \frac{\gamma}{\tilde{M} L_\gamma^{1/2} \tilde{K}_2}. \tag{2.23}$$

The operator $B[g, h, v_0]$ has a unique fixed point in V_γ if (2.23) is satisfied and if

$$I(g, h, \operatorname{curl} v_0) < \frac{1}{\tilde{M} L_\gamma^{1/2} \tilde{K}_3}. \tag{2.24}$$

If $g^{(1)}$, $h^{(1)}$ and $g^{(2)}$, $h^{(2)}$ are two sets of boundary data on $\partial\Omega_-(f)$, both satisfying (2.24), and if $u^{(1)}, u^{(2)} \in V_\gamma$ are fixed points of $B[g^{(1)}, h^{(1)}, v_0]$ and $B[g^{(2)}, h^{(2)}, v_0]$, respectively, then

$$\|u^{(1)} - u^{(2)}\|_{1,\Omega} \leqq \frac{\tilde{M} L_\gamma^{1/2} \tilde{K}_1}{1 - \tilde{M} L_\gamma^{1/2} \tilde{K}_3 I(g^{(1)}, h^{(1)}, \operatorname{curl} v_0)} \times (|h^{(1)} - h^{(2)}|_{0,\partial\Omega_-} + |\nabla_T (g^{(1)} - g^{(2)})|_{0,\partial\Omega_-}). \tag{2.25}$$

Proof. The inequalities (2.16) and (2.23) together imply that $B[g, h, v_0]$ maps V_γ into itself. To see that B has a fixed point if (2.23) and (2.24) are satisfied, note that V_γ is a closed subset of $H_1(\Omega, \mathbb{R}^3)$. For, let $\{u_n\}_{n=1}^\infty \subseteqq V_\gamma$ converge to $u \in H_1(\Omega, \mathbb{R}^3)$ in the norm of this space. Since $\|u_n\|_{3,\Omega} \leqq \gamma$, this sequence is bounded in $H_3(\Omega, \mathbb{R}^3)$ and therefore has a subsequence which converges weakly in $H_3(\Omega, \mathbb{R}^3)$ to w. V_γ is closed and convex, hence weakly closed, which implies $w \in V_\gamma$. But this subsequence converges also weakly in $H_1(\Omega, \mathbb{R}^3)$ to w, since any continuous linear functional on $H_1(\Omega, \mathbb{R}^3)$ is also continuous in the norm of $H_3(\Omega, \mathbb{R}^3)$, if we restrict it to this space. Since limits with respect to the norm are also weak limits, it follows that $u = w \in V_\gamma$. This shows that V_γ is a closed subset of $H_1(\Omega, \mathbb{R}^3)$. Since (2.17) and (2.24) imply that $B: V_\gamma \subseteqq H_1(\Omega) \to H_1(\Omega)$ is a contraction mapping, it follows from Banach's fixed point theorem that B has a unique fixed point in V_γ.

To prove (2.25), note that (2.17) and (2.15) yield

$$\begin{aligned}\|u^{(1)}-u^{(2)}\|_{1,\Omega} &= \|B[g^{(1)},h^{(1)},v_0](u^{(1)})-B[g^{(2)},h^{(2)},v_0](u^{(2)})\|_{1,\Omega}\\ &\leqq \|B[g^{(1)},h^{(1)},v_0](u^{(1)})-B[g^{(1)},h^{(1)},v_0](u^{(2)})\|_{1,\Omega}\\ &\quad+\|B[g^{(1)},h^{(1)},\operatorname{curl} v_0,v_0](u^{(2)})-B[g^{(2)},h^{(2)},\operatorname{curl} v_0,v_0](u^{(2)})\|_{1,\Omega}\\ &\leqq \tilde{M}L_\gamma^{1/2}\tilde{K}_3 I(g^{(1)},h^{(1)},\operatorname{curl} v_0)\,\|u^{(1)}-u^{(2)}\|_{1,\Omega}\\ &\quad+\|B[g^{(1)}-g^{(2)},h^{(1)}-h^{(2)},0,v_0](u^{(2)})\|_{1,\Omega}\\ &\leqq \tilde{M}L_\gamma^{1/2}\tilde{K}_3 I(g^{(1)},h^{(1)},\operatorname{curl} v_0)\,\|u^{(1)}-u^{(2)}\|_{1,\Omega}\\ &\quad+\tilde{M}L_\gamma^{1/2}\tilde{K}_1[|h^{(1)}-h^{(2)}|_{0,\partial\Omega_-}+|\nabla_T(g^{(1)}-g^{(2)})|_{0,\partial\Omega_-}]\,.\end{aligned}$$

Here we use the linearity of $(g,h,W)\to B[g,h,W,v_0](u)$. (2.25) follows from this estimate.

Lemma 2.6. (i) *Let* $u\in V_\gamma$ *with* $0<\gamma\leqq\gamma_0$. *Then* u *is a fixed point of* $B=B[g,h,v_0]:V_\gamma\to V$ *if and only if* $v=v_0+u$ *is the velocity field of a solution* $(v,p)\in H_3(\Omega,\mathbb{R}^3\times\mathbb{R})$ *of* (1.1)–(1.3), (1.25), (1.26).

(ii) *If* (v,p), $(\tilde{v},\tilde{p})\in H_3(\Omega,\mathbb{R}^3\times\mathbb{R})$ *are solutions of* (1.1)–(1.3), (1.25), (1.26) *with* $v=\tilde{v}$, *then also* $p=\tilde{p}$.

Proof. Assume that u is a fixed point of B, and let $v=v_0+u$. Then $\operatorname{div} v=0$ and

$$n\cdot v_{|\partial\Omega}=n\cdot v_{0_{|\partial\Omega}}+n\cdot u_{|\partial\Omega}=f\,,$$

so (1.2) and (1.3) are satisfied. (2.10) and (2.11) imply

$$\operatorname{curl} v=\operatorname{curl} v_0+\operatorname{curl} u=\operatorname{curl} v_0+\operatorname{curl} B(u)=\operatorname{curl} v_0+z\,.$$

Therefore (2.7), (2.4) yield

$$(v\cdot\nabla)\operatorname{curl} v=[\operatorname{curl} v\cdot\nabla]v\,,\qquad \operatorname{curl} v_{|\partial\Omega_-}=\eta+\operatorname{curl} v_{0_{|\partial\Omega_-}}\,. \tag{2.26}$$

This shows that the Vorticity Transport Theorem (1.12) and therefore also (1.10) is satisfied. We now show that $p\in H_3(\Omega)$ can be constructed satisfying (1.9) and (1.26). To construct this p, define first p by (1.26) on $\partial\Omega_-(f)$, and continue it to all of Ω by setting $\frac{1}{2}|v|^2+p$ equal to a constant along the integral curves of v. From the properties of the integral curves summarized after Lemma 2.1 it follows that p is defined in all of Ω in this way. p is continuously differentiable. To see this, let $x(y)\in\partial\Omega_-(f)$ be the starting point of the integral curve of v passing over $y\in\Omega$. From Sobolev's embedding theorem it follows that the vector field v is continuously differentiable, because $v\in H_3(\Omega)$. Since integral curves $\omega(\tau)$ are solutions of the system

$$\frac{d}{d\tau}\,\omega(\tau)=v(\omega(\tau))$$

of ordinary differential equations, and since integral curves meet $\partial\Omega_-$ transversally, it follows from the theory of ordinary differential equations that the mapping $x(y)$ is continuously differentiable. But then also

$$p(y)=\tfrac{1}{2}|v(x(y))|^2+p(x(y))-\tfrac{1}{2}|v(y)|^2$$

is continuously differentiable, since by definition p is continuously differentiable on $\partial\Omega_-(f)$.

From (1.26), (2.8), and (2.26) it follows that

$$\tau(x)\cdot(v(x)\times \operatorname{curl} v(x))=\tau(x)\cdot\nabla(\tfrac{1}{2}|v(x)|^2+p(x))$$

for all $x\in\partial\Omega_-(f)$ and for every unit vector $\tau(x)$ tangential to $\partial\Omega$ at x. Thus,

$$\begin{aligned}\tfrac{1}{2}|v(x)|^2+p(x)&=\int_\omega \tau(y)\cdot\nabla(\tfrac{1}{2}|v(y)|^2+p(y))ds_y+C\\&=\int_\omega \tau(y)\cdot(v(y)\times \operatorname{curl} v(y))ds_y+C\end{aligned}$$

for all $x\in\partial\Omega_-(f)$, connected to a fixed point x_0 by an arc ω in $\partial\Omega_-(f)$. $\tau(y)$ is a unit tangent vector to this arc. Since

$$\tau(x)\cdot(v(x)\times \operatorname{curl} v(x))=0$$

if $\tau(x)$ is a unit vector parallel to $v(x)$, it follows

$$\tfrac{1}{2}|v(x)|^2+p(x)=\int_\omega (v(y)\times \operatorname{curl} v(y))\cdot\tau(y)ds_y+C$$

for all $x\in\Omega$ connected to x_0 by an arc ω in $\bar{\Omega}$, if ω only consists of arcs in $\partial\Omega_-(f)$ and of integral curves of v. From (1.10) and from Stokes' theorem we conclude that

$$\tfrac{1}{2}|v(x)|^2+p(x)=\int_{\omega'} (v(y)\times \operatorname{curl} v(y))\cdot\tau(y)ds_y+C$$

for any curve ω' in $\bar{\Omega}$ connecting x_0 to x, hence

$$\nabla(\tfrac{1}{2}|v(x)|^2+p(x))=v(x)\times \operatorname{curl} v(x)$$

for all $x\in\Omega$, which is (1.9). Because $v\in H_3(\Omega,\mathbb{R}^3)$, it also follows from this equation and from $H_k(\Omega)H_m(\Omega)\subseteqq H_\nu(\Omega)$ for $\nu=\min\{k,m,k+m-2\}$, which is a well known consequence of Sobolev's embedding theorem [8, p. 72] and of Hölder's inequality, that $\nabla p\in H_2(\Omega,\mathbb{R}^2)$, hence $p\in H_3(\Omega)$. Summing up, (v,p) satisfies (1.26), (1.2), (1.3), and (1.9), hence also (1.1). From (2.26) and (2.5) it follows that

$$n(x)\cdot \operatorname{curl} v(x)=n(x)\cdot\eta(x)+n(x)\cdot \operatorname{curl} v_0(x)=h(x)+n(x)\cdot \operatorname{curl} v_0(x)$$

for all $x\in\partial\Omega_-(f)$, which is (1.25). Thus (v,p) is a solution.

On the other hand, assume that $u\in V_\gamma$ and that $v=v_0+u$ is the velocity field of the solution (v,p). We show that u is a fixed point of $B[g,h,v_0]$. v satisfies (1.12), which can be written as

$$[(v_0+u)\cdot\nabla](\operatorname{curl} u+\operatorname{curl} v_0)=[(\operatorname{curl} u+\operatorname{curl} v_0)\cdot\nabla](v_0+u)\,.$$

Comparing this with (2.7) we see that $\operatorname{curl} u$ and the function $z[g,h,v_0,u]$ used in the definition of $B[g,h,v_0](u)$ satisfy the same differential equation. From (1.9) and (1.26) we obtain

$$[(v_0+u)(x)\times(\operatorname{curl} u+\operatorname{curl} v_0)(x)]_T=\nabla_T(g(x)+\tfrac{1}{2}|v_0(x)|^2+p_0(x))$$

for all $x\in\partial\Omega_-(f)$. Moreover, from (1.25) we obtain

$$n(x)\cdot \operatorname{curl} u(x)=h(x)$$

for all $x\in\partial\Omega_-(f)$. Comparing the last two equations with (2.4), (2.5), (2.8), we see that $\operatorname{curl} u$ also satisfies the same initial conditions as $z[g,h,v_0,u]$, hence $z=\operatorname{curl} u$. By definition of $Bu\in V$ in (2.10), (2.11) we obtain for the function $Bu-u\in H_3(\Omega,\mathbb{R}^3)$

$$\operatorname{curl}(Bu-u)=\operatorname{curl} Bu-\operatorname{curl} u=z-\operatorname{curl} u=0\,,$$

$$\operatorname{div}(Bu-u)=\operatorname{div} Bu-\operatorname{div} u=0\,,$$

$$n\cdot(Bu-u)_{|\partial\Omega}=n\cdot Bu_{|\partial\Omega}-n\cdot u_{|\partial\Omega}=0\,.$$

By Theorem 2.4 there exists exactly one function satisfying these equations, hence $Bu-u=0$, and u is a fixed point of B. This proves (i).

To prove (ii), note that (1.1) implies $\tilde{p}=p+\text{const}$, and (1.26) yields $\text{const}=0$.

Proof of Theorem 1.1. Let $\gamma_0=\gamma_0(v_0)$ be the constant from Lemma 2.1 and choose for $\hat{\gamma}$ any constant with $0<\hat{\gamma}\leqq\gamma_0$. With the constants

$$\tilde{K}_i=\tilde{K}_i(L_\gamma, \underline{v}_0, \|v_0\|_{3,\Omega}, f, \hat{\gamma}, \Omega), \qquad i=2,3$$

from Theorem 2.3 and $\tilde{M}$ from Theorem 2.4 choose for $\hat{K}_1>0$ any constant with

$$\hat{K}_1\leqq\frac{\hat{\gamma}}{\tilde{M}L_\gamma^{1/2}\tilde{K}_2}, \qquad \hat{K}_1<\frac{1}{\tilde{M}L_\gamma^{1/2}\tilde{K}_3}. \tag{2.27}$$

From (1.24) it then follows that the assumptions (2.23) and (2.24) of Corollary 2.5 are satisfied, whence $B: V_{\hat{\gamma}}\to V$ has a unique fixed point $u\in V_{\hat{\gamma}}$, and Lemma 2.6 implies that a solution $(v,p)\in H_3(\Omega,\mathbb{R}^3\times\mathbb{R})$ of (1.1)–(1.3), (1.25), (1.26) exists with $v=v_0+u$, hence

$$\|v-v_0\|_{3,\Omega}=\|u\|_{3,\Omega}\leqq\hat{\gamma},$$

which is (1.27). If $(\tilde{v},\tilde{p})\in H_3(\Omega,\mathbb{R}^3\times\mathbb{R})$ is any solution of (1.1)–(1.3), (1.25), (1.26) satisfying (1.27), then Lemma 2.6(i) implies that $u=\tilde{v}-v_0\in V_{\hat{\gamma}}$ is the unique fixed point of B, hence $\tilde{v}=v$, and therefore $\tilde{p}=p$, by Lemma 2.6(ii). This shows that (v,p) is the only solution satisfying (1.27).

To prove (1.28), note that $u^{(1)}=v^{(1)}-v_0$ and $u^{(2)}=v^{(2)}-v_0$ are fixed points of $B[g^{(i)},h^{(i)},v_0]: V_{\hat{\gamma}}\to V$, $i=1,2$. The inequality (1.24) and thus also the inequality (2.24) is satisfied for $(g^{(1)},h^{(1)})$ and $(g^{(2)},h^{(2)})$. Therefore the assumptions of Corollary 2.5 are satisfied, and (2.25), (1.24) yield

$$\|v^{(1)}-v^{(2)}\|_{1,\Omega}\leqq\hat{K}_2(|h^{(1)}-h^{(2)}|_{0,\partial\Omega_-}+|\nabla_T(g^{(1)}-g^{(2)})|_{0,\partial\Omega_-}), \tag{2.28}$$

where

$$\hat{K}_2=\frac{\tilde{M}L_\gamma^{1/2}\tilde{K}_1}{1-\tilde{M}L_\gamma^{1/2}\tilde{K}_3\hat{K}_1},$$

and where $\tilde{K}_1=\tilde{K}_1(L_\gamma, \underline{v}_0, \|v_0\|_{3,\Omega}, f, \hat{\gamma}, \Omega)$ is the constant from Theorem 2.3. Note that our choice of $\hat{K}_1$ in (2.27) yields $1-\tilde{M}L_\gamma^{1/2}\tilde{K}_3\hat{K}_1>0$. This proves (1.28).

To prove (1.29) we use (1.1) and (2.28) to obtain

$$\begin{aligned}
&\|\nabla p^{(1)}-\nabla p^{(2)}\|_{0,\Omega}\\
&=\|(v^{(2)}\cdot\nabla)v^{(2)}-(v^{(1)}\cdot\nabla)v^{(1)}\|_{0,\Omega}\\
&\leqq\|(v^{(2)}\cdot\nabla)(v^{(2)}-v^{(1)})\|_{0,\Omega}+\|[(v^{(2)}-v^{(1)})\cdot\nabla]v^{(1)}\|_{0,\Omega}\\
&\leqq\|v^{(2)}\|_{L^\infty(\Omega)}\|v^{(2)}-v^{(1)}\|_{1,\Omega}+\|\nabla v^{(1)}\|_{L^\infty(\Omega)}\|v^{(2)}-v^{(1)}\|_{0,\Omega}\\
&\leqq C_1(\|v^{(1)}\|_{3,\Omega}+\|v^{(2)}\|_{3,\Omega})\|v^{(2)}-v^{(1)}\|_{1,\Omega}\\
&\leqq C_2\|v^{(2)}-v^{(1)}\|_{1,\Omega},
\end{aligned} \tag{2.29}$$

where

$$\|v^{(2)}\|^2_{L^\infty(\Omega)}=\sup_{x\in\Omega}|v^{(2)}(x)|^2, \qquad \|\nabla v^{(1)}\|^2_{L^\infty(\Omega)}=\sup_{x\in\Omega}\sum_{|\beta|=1}|D^\beta v^{(1)}(x)|^2,$$

and

$$C_2=2C_1(\|v_0\|_{3,\Omega}+\hat{\gamma}).$$

We also used that

$$\|v^{(2)}\|_{L^\infty(\Omega)} \leqq C_1 \|v^{(2)}\|_{3,\Omega}, \qquad \|\nabla v^{(1)}\|_{L^\infty(\Omega)} \leqq C_1 \|v^{(1)}\|_{3,\Omega},$$

with the constant C_1 only depending on Ω. This is a consequence of Sobolev's inequality. To complete the proof we need the following lemma, which is proved at the end of the appendix.

Lemma 2.7. *There exists a constant $\tilde{K}_4 = \tilde{K}_4(L_\gamma, \underline{v}_0, \|v_0\|_{3,\Omega}, f, \hat{\gamma}) > 0$ with*

$$\|q\|_{0,\Omega} \leqq \tilde{K}_4(\|q\|_{0,\partial\Omega_-} + \|\nabla q\|_{0,\Omega})$$

for all $q \in H_1(\Omega)$.

We apply this estimate to $p^{(1)} - p^{(2)}$ and use (2.29) to obtain

$$\|p^{(1)} - p^{(2)}\|_{1,\Omega} \leqq (1 + \tilde{K}_4) C_2 \|v^{(2)} - v^{(1)}\|_{1,\Omega} + \tilde{K}_4 \|p^{(1)} - p^{(2)}\|_{0,\partial\Omega_-}. \tag{2.30}$$

From (1.26) we obtain as in the derivation of (2.29) that

$$\begin{aligned}
&\|p^{(1)} - p^{(2)}\|_{0,\partial\Omega_-} \\
&\quad = \|g^{(1)} - \tfrac{1}{2}|v^{(1)}|^2 - g^{(2)} + \tfrac{1}{2}|v^{(2)}|^2\|_{0,\partial\Omega_-} \\
&\quad \leqq \|g^{(1)} - g^{(2)}\|_{0,\partial\Omega_-} + \tfrac{1}{2}(\|v^{(1)}\|_{3,\Omega} + \|v^{(2)}\|_{3,\Omega}) \|v^{(1)} - v^{(2)}\|_{0,\partial\Omega_-} \\
&\quad \leqq \|g^{(1)} - g^{(2)}\|_{0,\partial\Omega_-} + (\|v_0\|_{3,\Omega} + \hat{\gamma}) \|v^{(1)} - v^{(2)}\|_{0,\partial\Omega_-} \\
&\quad \leqq \|g^{(1)} - g^{(2)}\|_{0,\partial\Omega_-} + (\|v_0\|_{3,\Omega} + \hat{\gamma}) C_3 \|v^{(1)} - v^{(2)}\|_{1,\Omega}.
\end{aligned}$$

In the last step we used the trace theorem. Combination of the last estimate with (2.28) and (2.30) yields (1.29).

To complete the proof of Theorem 1.1 it remains to prove Lemma 2.1, Lemma 2.2, the first three estimates of Theorem 2.3, and Lemma 2.7.

3 The integral curves

In this section we prove Lemma 2.1 and Lemma 2.2.

Proof of Lemma 2.1. Sobolev's inequality implies for $v - v_0 = u \in V_\gamma$ and all $x \in \bar{\Omega}$

$$\underline{v} = \inf_{x \in \Omega} |v(x)| \geqq \underline{v}_0 - \sup_{x \in \Omega} |u(x)| \geqq \underline{v}_0 - C_1 \|u\|_{3,\Omega}, \tag{3.1}$$

which proves (P1).

To prove (P2) and (P3) we need some definitions and notations. For $x \in \bar{\Omega}$ and $u \in V_\gamma$ with $\gamma \leqq \gamma_1$ let $t \mapsto \omega(t, x, u) \in \bar{\Omega}$ be the integral curve of v with $\omega(0, x, u) = x$. The function ω is the solution of

$$\frac{d}{dt} \omega(t, x, u) = v(\omega(t, x, u)) \tag{3.2}$$

It is defined on a maximal closed interval containing 0. By assumption $t \mapsto \omega(t, x, 0)$ is defined on an interval of length not larger than $L_0/\underline{v}_0$.

The integral curves ω can be extended to functions $t \mapsto \hat{\omega}(t, x, u)$ defined for all $t \in \mathbb{R}$ as follows: By Calderón's extension theorem [8, p. 80] there exists a constant C_2 and to every vector field $w \in H_3(\Omega, \mathbb{R}^3)$ an extension to $H_3(\mathbb{R}^3, \mathbb{R}^3)$, also denoted by w, such that

$$\|w\|_{3,\mathbb{R}^3} \leqq C_2 \|w\|_{3,\Omega}. \tag{3.3}$$

We apply this theorem to the vector fields v_0 and $u \in V_\gamma$ and consider these functions now to belong to $H_3(\mathbb{R}^3, \mathbb{R}^3)$. It is clear that with the extended function $v = v_0 + u \in H_3(\mathbb{R}^3, \mathbb{R}^3)$ the solution $\hat{\omega}(t, x, u)$ of (3.2) now exists for all $t \in \mathbb{R}$ and defines the extension sought. Of course, ω is the restriction of $\hat{\omega}$ to the largest interval I that contains 0 and satisfies $\hat{\omega}(t, x, u) \in \bar{\Omega}$ for all $t \in I$. By $\ell(\omega(\cdot, x, u))$ we denote the arc length of ω, which we take to be infinite if ω is closed.

To prove (P2) we first note that the mapping

$$u \mapsto \ell(\omega(\cdot, x, u)) : V_{\gamma_1} \to [0, \infty]$$

is upper semi-continuous at $0 \in V$, uniformly with respect to $x \in \bar{\Omega}$. By this we mean that to all $\varepsilon > 0$ there exists $\gamma > 0$ with

$$\ell(\omega(\cdot, x, u)) \leqq \ell(\omega(\cdot, x, 0)) + \varepsilon \tag{3.4}$$

for all $(x, u) \in \bar{\Omega} \times V_\gamma$.

The proof follows by standard methods from the continuity of the mapping $(x, u) \mapsto \hat{\omega}(t, x, u)$ and uses the compactness of $\bar{\Omega}$. We leave it to the reader. From (3.4) we obtain

$$L_\gamma = \sup_{x \in \bar{\Omega}} \sup_{u \in V_\gamma} \ell(\omega(\cdot, x, u)) \leqq \sup_{x \in \bar{\Omega}} \ell(\omega(\cdot, x, 0)) + \varepsilon = L_0 + \varepsilon,$$

hence $\limsup_{\gamma \to 0} L_\gamma \leqq L_0$, and therefore $\lim_{\gamma \to 0} L_\gamma = L_0$, since $L_\gamma \geqq L_0$. This proves (P2).

To prove (P3) let the integral curve $\omega(t) = \omega(t, y, u)$ be tangential to the boundary at the point $x_0 = \omega(t_0) \in \partial\Omega$. We must show that it is completely contained in the boundary.

Let $[t_1, t_2]$ be the largest interval containing t_0 such that $\omega(t) \in \partial\Omega$ for $t \in [t_1, t_2]$. This definition implies that the vector $v(\omega(t))$ is tangential to $\partial\Omega$ for all $t \in [t_1, t_2]$, since

$$v(\omega(t)) = \frac{d}{dt}\omega(t).$$

The domain of definition of ω is a bounded interval containing $[t_1, t_2]$; we must show that the domain of definition is equal to $[t_1, t_2]$. With the extension $\hat{\omega}$ of ω defined above let

$$D(t) = \begin{cases} \inf_{y \in \partial\Omega} |\hat{\omega}(t) - y|, & \hat{\omega}(t) \in \bar{\Omega} \\ -\inf_{y \in \partial\Omega} |\hat{\omega}(t) - y|, & \hat{\omega}(t) \in \mathbb{R}^3 \setminus \bar{\Omega}. \end{cases}$$

We show that there exist δ_1, $\delta_2 > 0$ with

$$D(t) \leqq 0 \tag{3.5}$$

for all $t \in (-\delta_1 + t_1, t_1] \cup [t_2, t_2 + \delta_2)$, which means that the extended integral curve $\hat{\omega}$ leaves $\bar{\Omega}$ at t_1 and t_2 and therefore proves that $[t_1, t_2]$ is the domain of definition of ω. Consequently, to finish the proof of (P3) it suffices to verify (3.5).

To prove this estimate we derive now a differential inequality for $D(t)$. Note that if $\hat{\omega}(t)$ is sufficiently close to $\partial\Omega$ then there exists a unique $x(t) \in \partial\Omega$ with

$$D(t) = \pm \inf_{y \in \partial\Omega} |\hat{\omega}(t) - y| = \pm |\hat{\omega}(t) - x(t)|. \tag{3.6}$$

Of course, $x(t) \in \partial\Omega$ is the solution of

$$(\hat{\omega}(t) - x(t)) \cdot \tau_i(x(t)) = 0, \quad i = 1, 2,$$

where $\tau_1, \tau_2 : \partial\Omega \to \mathbb{R}^3$ are linearly independent tangential vector fields of $\partial\Omega$. From this equation and the implicit function theorem it follows that x is a continuously differentiable function of t, since ω is continuously differentiable. Moreover,

$$\hat{\omega}(t) - x(t) = \begin{cases} -|\hat{\omega}(t) - x(t)|\, n(x(t)), & \hat{\omega}(t) \in \bar{\Omega} \\ |\hat{\omega}(t) - x(t)|\, n(x(t)), & \hat{\omega}(t) \in \mathbb{R}^3 \setminus \bar{\Omega}. \end{cases}$$

Together with (3.6) this equation yields

$$\begin{aligned} \frac{d}{dt} D(t) &= \pm \frac{\hat{\omega}(t) - x(t)}{|\hat{\omega}(t) - x(t)|} \cdot \left(\frac{d}{dt} \hat{\omega}(t) - \frac{d}{dt} x(t) \right) \\ &= -n(x(t)) \cdot \left(\frac{d}{dt} \hat{\omega}(t) - \frac{d}{dt} x(t) \right) = -n(x(t)) \cdot \frac{d}{dt} \hat{\omega}(t) \\ &= -n(x(t)) \cdot v(\hat{\omega}(t)) \\ &= -n(x(t)) \cdot [v(\hat{\omega}(t)) - v(x(t))] - f(x(t)), \end{aligned} \tag{3.7}$$

because $\frac{d}{dt} x(t)$ is tangential to the boundary. In the last step we used (1.3).

To prove that (3.5) holds for $t \in [t_2, t_2 + \delta_2]$ with a suitable δ_2, note that $D(t_2) = 0$, so that (3.7) is valid in a suitable interval $[t_2, t_2 + \delta_2')$. In this interval we thus obtain from (3.6) and (3.7)

$$\begin{aligned} \frac{d}{dt} D(t) &\leqq |v(\hat{\omega}(t)) - v(x(t))| - f(x(t)) \\ &\leqq \sup_{y \in \mathbb{R}^3} |\nabla v(y)|\, |\hat{\omega}(t) - x(t)| - f(x(t)) \\ &= a(t) D(t) - f(x(t)) \end{aligned}$$

with

$$a(t) = (\operatorname{sign} D(t)) \sup_{y \in \mathbb{R}^3} |\nabla v(y)|.$$

This is the differential inequality for $D(t)$. Integration yields

$$\begin{aligned} D(t) &\leqq e^{\int_{t_2}^{t} a(\eta) d\eta} D(t_2) - \int_{t_2}^{t} e^{\int_{\tau}^{t} a(\eta) d\eta} f(x(\tau)) d\tau \\ &= -\int_{t_2}^{t} e^{\int_{\tau}^{t} a(\eta) d\eta} f(x(\tau)) d\tau \end{aligned} \tag{3.8}$$

for $t_2 \leqq t \leqq t_2 + \delta_2'$, because $D(t_2) = 0$.

It is clear that (3.5) is a consequence of this inequality if there exists δ_2 with $0 < \delta_2 \leqq \delta_2'$ such that

$$f(x(t)) \geqq 0 \tag{3.9}$$

for all $t \in [t_2, t_2 + \delta_2)$.

It thus remains to prove (3.9). Observe first that $x(t_2) \notin \partial\Omega_-(f)$, since $x(t_2) = \omega(t_2)$ and since $v(\omega(t_2))$ is tangential to $\partial\Omega$, as we noted above, hence

$$f(x(t_2)) = f(\omega(t_2)) = n(\omega(t_2)) \cdot v(\omega(t_2)) = 0.$$

Therefore it remains to distinguish the two cases $x(t_2) \in \partial\Omega \setminus \overline{\partial\Omega_-(f)}$ and $x(t_2) \in \partial\partial\Omega_-(f)$. In the first case (3.9) clearly holds, because $t \mapsto x(t) : \mathbb{R} \to \partial\Omega$ is a continuous function of t, and since $\partial\Omega \setminus \overline{\partial\Omega_-(f)}$ is an open subset of $\partial\Omega$ with $f \geqq 0$ in this set. To prove (3.9) in the second case, note that the fact that $\omega_t(t_2)$ is tangential to the boundary at t_2 implies

$$\frac{d}{dt}x(t_2)=\frac{d}{dt}\omega(t_2)=v(\omega(t_2)).$$

From the hypothesis (1.22) we thus obtain

$$\begin{aligned}\operatorname{dist}(\partial\Omega_-(f),x(t)) &\geqq \operatorname{dist}(\partial\Omega_-(f),\omega(t_2)+(t-t_2)v(\omega(t_2)))\\ &\quad -|x(t)-\omega(t_2)-v(\omega(t_2))(t-t_2)|\\ &\geqq \operatorname{dist}(\partial\Omega_-(f),\omega(t_2)+(t-t_2)v_0(\omega(t_2)))\\ &\quad -|(t-t_2)[v(\omega(t_2))-v_0(\omega(t_2))]|\\ &\quad -|x(t)-\omega(t_2)-v(\omega(t_2))(t-t_2)|\\ &\geqq \hat{c}(t-t_2)-|u(\omega(t_2))|\,|t-t_2|-C|t-t_2|^2>0\end{aligned}$$

for all sufficiently small, positive $t-t_2$ and for all $u\in V_{\gamma_0}$, if $\gamma_0>0$ is chosen so small that

$$|u(\omega(t_2))|\leqq C_1\|u\|_{3,\Omega}\leqq C_1\gamma_0\leqq\frac{\hat{c}}{2}.$$

Here we used Sobolev's inequality. For these t we thus have $x(t)\in\partial\Omega\backslash\partial\Omega_-(f)$, which implies (3.9) and thus proves (3.5) in the case $t\in[t_2,t_2+\delta_2)$.

To prove (3.5) in the case $t\in(-\delta_1+t_1,t_1]$ we use (3.7) to derive the estimate

$$D(t)\leqq\int_t^{t_1}e^{\int_t^\tau a(\eta)d\eta}f(x(\tau))d\tau,\qquad -\delta_1'+t_1\leqq t\leqq t_1,$$

analogous to (3.8), and use the hypothesis (1.23) to conclude that $f(x(t))\leqq 0$ in an interval $(-\delta_1+t_1,t_1]$. We leave the obvious modifications to the reader. The proof of Lemma 2.1 is complete.

Proof of Lemma 2.2. Since the integral curves of v are the characteristic curves of the first order partial differential equation (2.3), we can solve this partial differential equation as usual by integrating along the integral curves of v. As noted after Lemma 2.1, Ω is covered by integral curves starting at $\partial\Omega_-$, where the initial data for z are prescribed by (2.4)–(2.6). We recall the fact that every integral curve starting at $\partial\Omega_-$ ends at $\partial\Omega_+$ and does not meet the boundary $\partial\Omega$ in a third point. Therefore the solution z of (2.3)–(2.6) is uniquely determined in all of Ω.

From our assumptions v_0, u, $W\in H_3(\Omega)$ we cannot conclude that $z(x)$ has classical derivatives, but the estimate (2.13) proved in Sects. 4 and 5 shows that $z\in H_2(\Omega)$. Here we assume that this is true and prove that $\operatorname{div}z=0$ under this assumption. Since v_0, u, $W\in H_3(\Omega)\subseteqq C_1(\bar{\Omega})$ we can differentiate (2.3) and obtain

$$\begin{aligned}&(v\cdot\nabla)\operatorname{div}z+\sum_{i=1}^3(\partial_{x_i}v\cdot\nabla)z_i\\ &\quad=(z\cdot\nabla)\operatorname{div}v-(u\cdot\nabla)\operatorname{div}W+(W\cdot\nabla)\operatorname{div}u\\ &\qquad+\sum_{i=1}^3(\partial_{x_i}z\cdot\nabla)v_i-\sum_{i=1}^3(\partial_{x_i}u\cdot\nabla)W_i+\sum_{i=1}^3(\partial_{x_i}W\cdot\nabla)u_i.\end{aligned}$$

But $\operatorname{div} v = \operatorname{div} W = \operatorname{div} u = 0$,

$$\sum_{i=1}^{3} (\partial_{x_i} z \cdot \nabla) v_i = \sum_{i,j=1}^{3} (\partial_{x_i} z_j)(\partial_{x_j} v_i) = \sum_{j=1}^{3} (\partial_{x_j} v \cdot \nabla) z_j ,$$

and

$$\sum_{i=1}^{3} (\partial_{x_i} u \cdot \nabla) W_i = \sum_{i=1}^{3} (\partial_{x_i} W \cdot \nabla) u_i ,$$

whence

$$(v \cdot \nabla) \operatorname{div} z = 0 .$$

This means that $\operatorname{div} z$ is constant along integral curves of v and therefore vanishes identically if $(\operatorname{div} z)_{|\partial\Omega_-} = 0$, which we prove now. (1.11) and (2.3) yield

$$\operatorname{curl}(v \times z) + \operatorname{curl}(u \times W) = v \operatorname{div} z$$

and (2.4), (2.9) imply

$$n(x) \cdot [\operatorname{curl}(v(x) \times z(x) + u(x) \times W(x))] = 0$$

for $x \in \partial\Omega_-$, which can be seen for example by application of Stokes' theorem. Combination of these two equations and of (1.3) yields

$$f(x) \operatorname{div} z(x) = (n(x) \cdot v(x)) \operatorname{div} z(x) = 0$$

for all $x \in \partial\Omega_-$, whence $\operatorname{div} z(x) = 0$.

4 Estimates for the solutions of the Vorticity Transport Theorem

This section and the following are devoted to the proof of the estimates (2.12)–(2.14) in Theorem 2.3. The proof is given in a sequence of lemmas. The results proved in these lemmas are collected at the end of Sect. 5 to prove Theorem 2.3. To see the purpose of every lemma proved in this and the next section the reader is therefore advised to first look at the proof of Theorem 2.3 at the end of Sect. 5.

As in the preceding section, for $u \in V_\gamma$ and $y \in \partial\Omega_-(f)$ let $t \mapsto \omega(t, y, u)$ be the integral curve of $v = v_0 + u$ with $\omega(0, y, u) = y$. By $s \mapsto \omega(s, y, u)$ we denote the arc length parametrization of this integral curve. This means that $\omega(s, y, u)$ is the solution of

$$\frac{d}{ds} \omega(s, y, u) = \frac{1}{|v(\omega(s, y, u))|} v(\omega(s, y, u)) , \qquad \omega(0, y, u) = y \in \partial\Omega_-(f) .$$

For convenience, if u is understood, then we write for the arc length

$$\ell(y) = \ell(\omega(\cdot, y, u)) ,$$

and we drop the index γ and write

$$L = L_\gamma = \sup_{u \in V_\gamma} \sup_{y \in \partial\Omega_-} \ell(\omega(\cdot, y, u)) .$$

For $x \in \bar{\Omega}$ let $y = y(x) \in \partial\Omega_-(f)$ and $s = s(x) \in [0, \ell(y)]$ be the points with $x = \omega(s, y, u)$. $(s(x), y(x))$ are the "integral curve coordinates" of x. If q is a function with domain contained in $\bar{\Omega}$, then we write for brevity

$$q(s, y) = q(\omega(s, y, u)) ,$$

and if y is understood, $q(s)=q(s,y)$. For a function $q=(q_1,\dots,q_m)\colon \Gamma\subseteqq\mathbb{R}^2\to\mathbb{R}^m$ we denote by $\nabla q(x)$ the matrix of first derivatives of q, and for $k\geqq 0$ let

$$|q|_k(x)=|q|_k(s,y)=\left(\sum_{i=1}^{m}\sum_{|\beta|=k}|D^\beta q_i(x)|^2\right)^{1/2}.$$

Finally, for $u\in V_\gamma$ and $W\in H_3(\Omega)$ let

$$E(x)=E(u,W,x)=(W(x)\cdot\nabla)u(x)-(u(x)\cdot\nabla)W(x), \tag{4.1}$$

which belongs to $H_2(\Omega)$ since $H_k(\Omega)H_m(\Omega)\subseteqq H_\nu\Omega$ for $\nu=\min\{k,m,k+m-2\}$.

We investigate now the solution z of (2.3)–(2.6) and derive estimates for the L_2-norms of this solution and its first and second derivatives. As mentioned in the proof of Lemma 2.2, we cannot conclude from the assumptions that z has classical derivatives. But by formal differentiation of (2.3)–(2.6) we derive in this and the following section a-priori estimates for the first and second derivatives of z, and one can use these estimates to show by standard considerations that z has weak L_2-derivatives up to second order. But since these considerations are technical, we omit them here.

Lemma 4.1. *The solution z of* (2.3)–(2.6) *satisfies*

$$|z(s)|=|z(s,y)|\leqq e^{\int_0^s\frac{|v|_1(\tau)}{|v(\tau)|}d\tau}\left(|z(0)|+\int_0^s\frac{|E(\tau)|}{|v(\tau)|}\,d\tau\right). \tag{4.2}$$

Proof. From (2.3) we obtain

$$\begin{aligned}\frac{d}{ds}z(s)&=\frac{d}{ds}z(\omega(s))=\nabla z(s)\cdot\frac{d}{ds}\omega(s)=\frac{1}{|v(s)|}(v(s)\cdot\nabla)z(s)\\&=\frac{1}{|v(s)|}[(z(s)\cdot\nabla)v(s)+E(s)].\end{aligned}$$

This implies

$$\begin{aligned}\frac{d}{ds}|z(s)|&=\frac{d}{ds}(|z(s)|^2)^{1/2}=\frac{1}{2|z(s)|}2z(s)\cdot\frac{d}{ds}z(s)\\&\leqq\left|\frac{d}{ds}z(s)\right|=\left|\frac{1}{|v(s)|}[(z(s)\cdot\nabla)v(s)+E(s)]\right|\\&\leqq\frac{1}{|v(s)|}(|z(s)|\,|v|_1(s)+|E(s)|).\end{aligned} \tag{4.3}$$

Integration of this differential inequality yields (4.2).

Lemma 4.2. *The solution z of* (2.3)–(2.6) *satisfies*

$$\begin{aligned}|z|_1(s)&=|z|_1(s,y)\\&\leqq e^{2\int_0^s\frac{|v|_1(\tau)}{|v(\tau)|}d\tau}\left[|z|_1(0)+\int_0^s\frac{|E|_1(\tau)}{|v(\tau)|}\,d\tau\right.\\&\qquad\left.+\left(|z(0)|+\int_0^s\frac{|E(\tau)|}{|v(\tau)|}\,d\tau\right)\int_0^s\frac{|v|_2(\tau)}{|v(\tau)|}\,d\tau\right]\end{aligned} \tag{4.4}$$

for almost all (s,y).

Proof. We use the notation $q_{|i}=\frac{\partial}{\partial x_i}q$ and obtain by formal differentiation of (2.3)

$$(v\cdot\nabla)z_{|i}+(v_{|i}\cdot\nabla)z=(z_{|i}\cdot\nabla)v+(z\cdot\nabla)v_{|i}+E_{|i}, \tag{4.5}$$

whence

$$\frac{d}{ds}z_{|i}=\frac{1}{|v|}(v\cdot\nabla)z_{|i}=\left(\frac{z_{|i}}{|v|}\cdot\nabla\right)v-\left(\frac{v_{|i}}{|v|}\cdot\nabla\right)z+\left(\frac{z}{|v|}\cdot\nabla\right)v_{|i}+\frac{E_{|i}}{|v|}$$

for $i=1,2,3$. We apply the triangle inequality and obtain

$$\begin{aligned}\left[\sum_{i,\ell=1}^{3}\left(\frac{d}{ds}z_{\ell|i}\right)^2\right]^{1/2} &=\left\{\sum_{i,\ell=1}^{3}\left[\left(\frac{z_{|i}}{|v|}\cdot\nabla\right)v_\ell-\left(\frac{v_{|i}}{|v|}\cdot\nabla\right)z_\ell\right.\right.\\ &\qquad\left.\left.+\left(\frac{z}{|v|}\cdot\nabla\right)v_{\ell|i}+\frac{E_{\ell|i}}{|v|}\right]^2\right\}^{1/2}\\ &\leqq\left\{\sum_{i,\ell=1}^{3}\left[\left(\frac{z_{|i}}{|v|}\cdot\nabla\right)v_\ell\right]^2\right\}^{1/2}+\left\{\sum_{i,\ell=1}^{3}\left[\left(\frac{v_{|i}}{|v|}\cdot\nabla\right)z_\ell\right]^2\right\}^{1/2}\\ &\qquad+\left\{\sum_{i,\ell=1}^{3}\left[\left(\frac{z}{|v|}\cdot\nabla\right)v_{\ell|i}\right]^2\right\}^{1/2}+\left\{\sum_{i,\ell=1}^{3}\left(\frac{E_{\ell|i}}{|v|}\right)^2\right\}^{1/2}\\ &\leqq\frac{1}{|v|}\left\{\sum_{i,\ell=1}^{3}|z_{|i}|^2|v_\ell|_1^2\right\}^{1/2}+\frac{1}{|v|}\left\{\sum_{i,\ell=1}^{3}|v_{|i}|^2|z_\ell|_1^2\right\}^{1/2}\\ &\qquad+\frac{1}{|v|}\left\{\sum_{i,\ell=1}^{3}|z|^2|v_{\ell|i}|_1^2\right\}^{1/2}+\frac{1}{|v|}|E|_1\\ &=\frac{1}{|v|}(|z|_1|v|_1+|z|_1|v|_1+|z|\,|v|_2+|E|_1).\end{aligned}$$

As in (4.3) we obtain from this inequality

$$\frac{d}{ds}|z|_1\leqq\left[\sum_{i,\ell=1}^{3}\left(\frac{d}{ds}z_{\ell|i}\right)^2\right]^{1/2}\leqq\frac{1}{|v|}(2|z|_1|v|_1+|z|\,|v|_2+|E|_1).$$

Integration of this differential inequality yields

$$\begin{aligned}|z|_1(s)&\leqq e^{\int_0^s 2\frac{|v|_1}{|v|}d\tau}|z|_1(0)+\int_0^s e^{\int_\tau^s 2\frac{|v|_1}{|v|}d\sigma}\left(\frac{|v|_2(\tau)}{|v(\tau)|}|z(\tau)|+\frac{|E|_1(\tau)}{|v(\tau)|}\right)d\tau\\ &\leqq e^{\int_0^s 2\frac{|v|_1}{|v|}d\tau}\left[|z|_1(0)+\int_0^s\frac{|v|_2(\tau)}{|v(\tau)|}\right.\\ &\qquad\left.\times\left(|z(0)|+\int_0^\tau\frac{|E(\sigma)|}{|v(\sigma)|}d\sigma\right)d\tau+\int_0^s\frac{|E|_1(\tau)}{|v(\tau)|}d\tau\right],\end{aligned}$$

where we applied Lemma 4.1 to estimate $|z(\tau)|$. From this estimate we immediately obtain (4.4).

Lemma 4.3. *The solution z of* (2.3)–(2.6) *satisfies*

$$\begin{aligned}|z|_2(s)&=|z|_2(s,y)\\ &\leqq e^{3\int_0^s\frac{|v|_1}{|v|}d\tau}\left\{|z|_2(0)+\int_0^s\frac{|E|_2}{|v|}d\tau+\left(|z|_1(0)+\int_0^s\frac{|E|_1}{|v|}d\tau\right)3\int_0^s\frac{|v|_2}{|v|}d\tau\right.\\ &\qquad\left.+\left(|z(0)|+\int_0^s\frac{|E|}{|v|}d\tau\right)\left[3\left(\int_0^s\frac{|v|_2}{|v|}d\tau\right)^2+\int_0^s\frac{|v|_3}{|v|}d\tau\right]\right\}\end{aligned} \tag{4.6}$$

for almost all (s,y).

Proof. With the notation $q_{|ij}=\dfrac{\partial^2}{\partial x_i \partial x_j} q$ we obtain by formal differentiation of (4.5)

$$(v\cdot\nabla)z_{|ij}+(v_{|j}\cdot\nabla)z_{|i}+(v_{|i}\cdot\nabla)z_{|j}+(v_{|ij}\cdot\nabla)z$$
$$=(z_{|i}\cdot\nabla)v_{|j}+(z_{|ij}\cdot\nabla)v+(z\cdot\nabla)v_{|ij}+(z_{|j}\cdot\nabla)v_{|i}+E_{|ij},$$

hence

$$\begin{aligned}\frac{d}{ds} z_{\ell|ij} &= \left(\frac{v}{|v|}\cdot\nabla\right) z_{\ell|ij}\\ &= \left(\frac{z_{|ij}}{|v|}\cdot\nabla\right) v_\ell - \left(\frac{v_{|j}}{|v|}\cdot\nabla\right) z_{\ell|i} - \left(\frac{v_{|i}}{|v|}\cdot\nabla\right) z_{\ell|j}\\ &\quad + \left(\frac{z_{|i}}{|v|}\cdot\nabla\right) v_{\ell|j} + \left(\frac{z_{|j}}{|v|}\cdot\nabla\right) v_{\ell|i} - \left(\frac{v_{|ij}}{|v|}\cdot\nabla\right) z_\ell\\ &\quad + \left(\frac{z}{|v|}\cdot\nabla\right) v_{\ell|ij} + \frac{1}{|v|} E_{\ell|ij}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\frac{d}{ds} z_{\ell|ij}\right| \leqq \frac{1}{|v|} [&|z_{|ij}|\,|v_\ell|_1 + |v_{|j}|\,|z_{\ell|i}|_1 + |v_{|i}|\,|z_{\ell|j}|_1\\ &+|z_{|i}|\,|v_{\ell|j}|_1 + |z_{|j}|\,|v_{\ell|i}|_1 + |v_{|ij}|\,|z_\ell|_1 + |z|\,|v_{\ell|ij}|_1 + |E_{\ell|ij}|].\end{aligned}$$

The triangle inequality yields

$$\begin{aligned}\left(\sum_{\ell,i,j=1}^{3}\left|\frac{d}{ds} z_{\ell|ij}\right|^2\right)^{1/2} &\leqq \frac{1}{|v|}\left\{\left[\sum_{\ell,i,j=1}^{3}(|z_{|ij}|\,|v_\ell|_1)^2\right]^{1/2} + \left[\sum_{\ell,i,j=1}^{3}(|v_{|j}|\,|z_{\ell|i}|_1)^2\right]^{1/2}\right.\\ &\quad + \left[\sum_{\ell,i,j=1}^{3}(|v_{|i}|\,|z_{\ell|j}|_1)^2\right]^{1/2} + \left[\sum_{\ell,i,j=1}^{3}(|z_{|i}|\,|v_{\ell|j}|_1)^2\right]^{1/2}\\ &\quad + \left[\sum_{\ell,i,j=1}^{3}(|z_{|j}|\,|v_{\ell|i}|_1)^2\right]^{1/2} + \left[\sum_{\ell,i,j=1}^{3}(|v_{|ij}|\,|z_\ell|_1)^2\right]^{1/2}\\ &\quad \left. + \left[\sum_{\ell,i,j=1}^{3}(|z|\,|v_{\ell|ij}|_1)^2\right]^{1/2} + \left[\sum_{\ell,i,j=1}^{3}|E_{\ell|ij}|^2\right]^{1/2}\right\}\\ &= \frac{1}{|v|}[|z|_2|v|_1 + |v|_1|z|_2 + |v|_1|z|_2 + |z|_1|v|_2\\ &\quad + |z|_1|v|_2 + |v|_2|z|_1 + |z|\,|v|_3 + |E|_2],\end{aligned}$$

so, as in (4.3),

$$\begin{aligned}\frac{d}{ds}|z|_2 &\leqq \left(\sum_{\ell,i,j=1}^{3}\left|\frac{d}{ds} z_{\ell|ij}\right|^2\right)^{1/2}\\ &\leqq \frac{1}{|v|}[3|v|_1|z|_2 + 3|v|_2|z|_1 + |v|_3|z| + |E|_2].\end{aligned}$$

Integration yields

$$\begin{aligned}|z|_2(s) &\leqq e^{3\int_0^s \frac{|v|_1}{|v|}d\tau}|z|_2(0)\\ &\quad + \int_0^s e^{3\int_\tau^s \frac{|v|_1}{|v|}d\sigma}\left(3\frac{|v|_2}{|v|}|z|_1 + \frac{|v|_3}{|v|}|z| + \frac{|E|_2}{|v|}\right)d\tau.\end{aligned}$$

We use Lemma 4.1 and Lemma 4.2 and obtain

$$|z|_2(s) \leqq e^{3\int_0^s \frac{|v|_1}{|v|} d\tau} \Big\{ |z|_2(0) + \int_0^s 3 \frac{|v|_2}{|v|} \times \left[|z|_1(0) + \left(|z(0)| + \int_0^\tau \frac{|E|}{|v|} d\sigma \right) \int_0^\tau \frac{|v|_2}{|v|} d\sigma + \int_0^\tau \frac{|E|_1}{|v|} d\sigma \right] d\tau + \int_0^s \frac{|v|_3}{|v|} \left(|z(0)| + \int_0^\tau \frac{|E|}{|v|} d\sigma \right) d\tau + \int_0^s \frac{|E|_2}{|v|} d\tau \Big\}.$$

(4.6) follows from this inequality.

We now use the following notation: Let $\mu \geqq 1$ and let k be a nonnegative integer. If Γ is an open subset of Ω or of $\partial\Omega$, and if $q:\Gamma \to \mathbb{R}^m$, then we write

$$|q|_{m,\mu,\Gamma} = \left[\int_\Gamma (|q|_m(x))^\mu d\lambda \right]^{1/\mu}, \tag{4.7}$$

where λ is the Lebesgue measure of Ω or where $d\lambda$ is the surface element of $\partial\Omega$. For brevity, we set

$$|q|_{m,\Gamma} = |q|_{m,2,\Gamma}.$$

Lemma 4.4. *Let $\bar{v}$, $\underline{v}$, $\bar{f}$ be defined as in Corollary* A.2 *in the appendix. Then there exist constants $C, K > 0$, only depending on Ω, such that the solution z of* (2.3)–(2.6) *satisfies*

$$\begin{aligned} |z|_{2,\Omega} &= \left(\int_\Omega (|z|_2(x))^2 dx \right)^{1/2} \\ &\leqq \exp\{3KL\underline{v}^{-1} \|v\|_{3,\Omega}\} \Bigg[\left(\frac{L\bar{f}}{\underline{v}} \right)^{1/2} |z|_{2,\partial\Omega_-} + \frac{\bar{v}^{1/2}}{\underline{v}^{3/2}} L|E|_{2,\Omega} \\ &\quad + 3\underline{v}^{-3/2}(CL^5\bar{v})^{1/4} \|v\|_{3,\Omega}(\bar{f}^{1/4}|z|_{1,4,\partial\Omega_-} + \underline{v}^{-1}(L^3\bar{v}C)^{1/4} \|E\|_{2,\Omega}) \\ &\quad + \underline{v}^{-3/2} KL\bar{v}^{1/2}(|v|_{3,\Omega} + 3\underline{v}^{-1}LC^{1/2}\|v\|_{3,\Omega}^2)(\|z\|_{2,\partial\Omega_-} + \underline{v}^{-1}L\|E\|_{2,\Omega}) \Bigg]. \end{aligned}$$

The norm $\|\cdot\|_{2,\partial\Omega_-}$ *is defined in* (1.15).

Proof. We estimate the norms of the terms on the right hand side of (4.6). First, note that (A.3) in the appendix implies

$$\left\| \int_0^{s(\cdot)} \frac{|v|_3(\tau, y(\cdot))}{|v(\tau, y(\cdot))|} d\tau \right\|_{0,\Omega} \leqq \left(\frac{\bar{v}}{\underline{v}} \right)^{1/2} L \left\| \frac{|v|_3}{|v|} \right\|_{0,\Omega} \leqq \frac{\bar{v}^{1/2}}{\underline{v}^{3/2}} L|v|_{3,\Omega}. \tag{4.8}$$

Next, Cauchy-Schwarz inequality yields

$$\left(\int_0^s \frac{|v|_2}{|v|} (\tau, y) d\tau \right)^2 \leqq \int_0^s \frac{1}{|v|^2} d\tau \int_0^s |v|_2^2 d\tau \leqq \frac{L}{\underline{v}^2} \int_0^s |v|_2^2 d\tau.$$

With (A.3) we thus obtain

$$\begin{aligned} \left\| \left(\int_0^s \frac{|v|_2}{|v|} (\tau, y(\cdot)) d\tau \right)^2 \right\|_{0,\Omega} &\leqq \left(\frac{\bar{v}}{\underline{v}^5} \right)^{1/2} L^2 \| |v|_2^2 \|_{0,\Omega} \\ &= \left(\frac{\bar{v}}{\underline{v}^5} \right)^{1/2} L^2 \left(\int_\Omega |v|_2^4(x) dx \right)^{1/2} \\ &\leqq \left(\frac{\bar{v}C}{\underline{v}^5} \right)^{1/2} L^2 \|v\|_{3,\Omega}^2, \end{aligned} \tag{4.9}$$

since $H_1(\Omega) \subseteqq L_\mu(\Omega)$ and

$$\left(\int_\Omega |q(x)|^\mu dx \right)^{1/\mu} \leqq \tilde{C} \|q\|_{1,\Omega}$$

for all μ with $\frac{1}{6} \leqq \frac{1}{\mu} \leqq \frac{1}{2}$ and $\tilde{C} = \tilde{C}(\mu, \Omega)$, cf. [8, p. 69]. Cauchy-Schwarz' inequality and (A.4), (4.9) imply

$$\begin{aligned} &\left\| |z|_1(0, y(\cdot)) \int_0^{s(\cdot)} \frac{|v|_2}{|v|}(\tau, y(\cdot))d\tau \right\|_{0,\Omega} \\ &\leqq \|(|z|_1(0, y(\cdot)))^2\|_{0,\Omega}^{1/2} \left\| \left(\int_0^{s(\cdot)} \frac{|v|_2}{|v|}(\tau, y(\cdot))d\tau \right)^2 \right\|_{0,\Omega}^{1/2} \\ &\leqq \frac{L^{5/4}\bar{v}^{1/4}}{\underline{v}^{3/2}} C^{1/4} \bar{J}^{1/4} |z|_{1,4,\partial\Omega_-} \|v\|_{3,\Omega}, \end{aligned} \tag{4.10}$$

because $\| |z|_1^2 \|_{0,\partial\Omega_-}^{1/2} = |z|_{1,4,\partial\Omega_-}$. Similarly,

$$\begin{aligned} &\left\| \int_0^{s(\cdot)} \frac{|E|_1}{|v|}(\tau, y(\cdot))d\tau \int_0^{s(\cdot)} \frac{|v|_2}{|v|}(\tau, y(\cdot))d\tau \right\|_{0,\Omega} \\ &\leqq \left\| \left(\int_0^{s(\cdot)} \frac{|E|_1}{|v|} d\tau \right)^2 \right\|_{0,\Omega}^{1/2} \left\| \left(\int_0^{s(\cdot)} \frac{|v|_2}{|v|} d\tau \right)^2 \right\|_{0,\Omega}^{1/2} \\ &\leqq \left(\frac{\bar{v}C}{\underline{v}^5} \right)^{1/2} L^2 \|E\|_{2,\Omega} \|v\|_{3,\Omega}, \end{aligned} \tag{4.11}$$

where we applied (4.9) in the last step and estimated the term containing $|E|_1$ just as in (4.9). Further, (A.4) yields

$$\| |z|_2(0, y(\cdot)) \|_{0,\Omega} = \left(\frac{\bar{J}}{\underline{v}} L \right)^{1/2} |z|_{2,\partial\Omega_-} \tag{4.12}$$

and (A.3) implies

$$\left\| \int_0^{s(\cdot)} \frac{|E|_2}{|v|}(\tau, y(\cdot))d\tau \right\|_{0,\Omega} \leqq \left(\frac{\bar{v}}{\underline{v}} \right)^{1/2} L \left\| \frac{|E|_2}{|v|} \right\|_{0,\Omega} \leqq \frac{\bar{v}^{1/2}}{\underline{v}^{3/2}} L |E|_{2,\Omega}. \tag{4.13}$$

We also need the estimates

$$|z(0, y)| \leqq K \|z\|_{2,\partial\Omega_-}, \tag{4.14}$$

$$\int_0^s \frac{|E|}{|v|}(\tau, y)d\tau \leqq K \frac{L}{\underline{v}} \|E\|_{2,\Omega}, \tag{4.15}$$

$$\int_0^s \frac{|v|_1}{|v|}(\tau, y)d\tau \leqq \int_0^s \frac{1}{\underline{v}} K \|v\|_{3,\Omega} d\tau \leqq \frac{L}{\underline{v}} K \|v\|_{3,\Omega}, \tag{4.16}$$

which are direct consequences of Sobolev's inequality. We use (4.8)–(4.16) to estimate the L_2-norm of the terms on the right hand side of (4.6) and obtain the statement of Lemma 4.4.

Lemma 4.5. *There exist constants $C, C_1, K > 0$, only depending on Ω, such that the solution z of* (2.3)–(2.6) *satisfies*

$$\begin{aligned} |z|_{1,\Omega} \leqq \exp\{2KL\underline{v}^{-1}\|v\|_{3,\Omega}\} &\left[\left(\frac{L\bar{J}}{\underline{v}} \right)^{1/2} |z|_{1,\partial\Omega_-} + \left(\frac{\bar{v}}{\underline{v}^3} \right)^{1/2} L|E|_{1,\Omega} \right. \\ &\left. + \|v\|_{3,\Omega} \left(\underline{v}^{-3/2} (L^5 \bar{v} C \bar{J})^{1/4} C_1 \|z\|_{1,\partial\Omega_-} + \left(\frac{C\bar{v}}{\underline{v}^5} \right)^{1/2} L^2 \|E\|_{1,\Omega} \right) \right]. \end{aligned}$$

Proof. To prove this lemma we estimate the L_2-norms of the terms on the right hand side of (4.4). Just as in (4.10) we obtain

$$\left\| z(0, y(\cdot)) \int_0^{s(\cdot)} \frac{|v|_2}{|v|}(\tau, y(\cdot))d\tau \right\|_{0,\Omega} \leqq \underline{v}^{-3/2}(L^5\bar{v}C\bar{f})^{1/4}|z|_{0,4,\partial\Omega_-}\|v\|_{3,\Omega}$$
$$\leqq \underline{v}^{-3/2}(L^5\bar{v}C\bar{f})^{1/4}C_1\|z\|_{1,\partial\Omega_-}\|v\|_{3,\Omega}, \quad (4.17)$$

where in the second step we used that

$$H_1(\partial\Omega_-) \subsetneqq L_\mu(\partial\Omega_-)$$

and

$$\left(\int_{\partial\Omega_-} |q(y)|^\mu dS_y\right)^{1/\mu} \leqq C_1\|q\|_{1,\partial\Omega_-}$$

for all $\mu \geqq 2$. This result is an easy consequence of the definition of the norms $\|\cdot\|_{1,\partial\Omega_-}$, $|\cdot|_{0,\mu,\partial\Omega_-}$ in (1.17), (4.7), of our assumption that $\partial\Omega_-(f)$ has Lipschitz boundary, and of the corresponding result for plane domains with Lipschitz boundary proved in [8, p. 72]. As in (4.11) we get

$$\left\| \int_0^{s(\cdot)} \frac{|E|}{|v|}(\tau, y(\cdot))d\tau \int_0^{s(\cdot)} \frac{|v|_2}{|v|}(\tau, y(\cdot))d\tau \right\|_{0,\Omega} \leqq \left(\frac{\bar{v}C}{\underline{v}^5}\right)^{1/2} L^2\|E\|_{1,\Omega}\|v\|_{3,\Omega}. \quad (4.18)$$

Finally, as in (4.12) and (4.8) we obtain

$$\| |z|_1(0, y(\cdot)) \|_{0,\Omega} \leqq \left(L\frac{\bar{f}}{\underline{v}}\right)^{1/2} |z|_{1,\partial\Omega_-} \quad (4.19)$$

$$\left\| \int_0^{s(\cdot)} \frac{|E|_1}{|v|}(\tau, y(\cdot))d\tau \right\|_{0,\Omega} \leqq \left(\frac{\bar{v}}{\underline{v}^3}\right)^{1/2} L|E|_{1,\Omega}. \quad (4.20)$$

We use (4.17)–(4.20) and (4.16) to estimate the L_2-norms of the terms on the right hand side of (4.4) and obtain the statement of the lemma.

Lemma 4.6. *There exists a constant* $K>0$, *only depending on* Ω, *such that the solution of* (2.3)–(2.6) *satisfies*

$$\|z\|_{0,\Omega} \leqq \exp\{KL\underline{v}^{-1}\|v\|_{3,\Omega}\}\left[\left(\frac{L\bar{f}}{\underline{v}}\right)^{1/2}\|z\|_{0,\partial\Omega_-} + \left(\frac{\bar{v}}{\underline{v}^3}\right)^{1/2} L\|E\|_{0,\Omega}\right].$$

Proof. As in (4.12) and (4.8) we obtain

$$\|z(0, y(\cdot))\|_{0,\Omega} \leqq \left(L\frac{\bar{f}}{\underline{v}}\right)^{1/2}\|z\|_{0,\partial\Omega_-}$$

$$\left\| \int_0^{s(\cdot)} \frac{|E|}{|v|}(\tau, y(\cdot))d\tau \right\|_{0,\Omega} \leqq \left(\frac{\bar{v}}{\underline{v}^3}\right)^{1/2} L\|E\|_{0,\Omega}.$$

We use these inequalities and (4.16) to estimate the L_2-norms of the terms on the right hand side of (4.2) and obtain the statement.

Lemma 4.4, 4.5, and 4.6 show that $\|z\|_{0,\Omega}$, $|z|_{1,\Omega}$, and $|z|_{2,\Omega}$ can be controlled by norms of E and by norms of the values of z and its derivatives on $\partial\Omega_-(f)$. To complete the proof of (2.12) and (2.13) we therefore need estimates for the boundary values of the derivatives of z. These estimates are derived in the next section.

The estimates stated in the following two lemmas are necessary to prove (2.14). For $i=1,2$ let $u^{(i)}\in V_y$, $v^{(i)}=v_0+u^{(i)}$, and with the notation introduced before Theorem 2.3 let $z^{(i)}=z[g,h,v_0,u^{(i)}]$ be the solutions of (2.3)–(2.6). In the following we use the "integral curve coordinates" belonging to the vector field $v^{(1)}$ and write

$$q(s,y)=q(\omega(s,y,u^{(1)}))\,.$$

Moreover, we use the notation $[z]=z^{(2)}-z^{(1)}$, $[u]=[v]=u^{(2)}-u^{(1)}$.

Lemma 4.7. *The solutions* $z^{(1)},z^{(2)}$ *satisfy*

$$|[z](s)|=|[z](s,y)|\leqq e^{\int_0^s \frac{|v^{(1)}|_1}{|v^{(1)}|}d\tau}\Bigg\{|[z](0)|+\int_0^s \frac{1}{|v^{(1)}|}\big((|z^{(2)}|_1+|\mathrm{curl}\,v_0|_1)\,|[u]| \\ +(|z^{(2)}|+|\mathrm{curl}\,v_0|)\,|[u]|_1\big)d\tau\Bigg\}\,. \tag{4.21}$$

Proof. From (2.3) we obtain

$$\begin{aligned}(v^{(1)}\cdot\nabla)[z]&=(v^{(2)}\cdot\nabla)z^{(2)}-([v]\cdot\nabla)z^{(2)}-(v^{(1)}\cdot\nabla)z^{(1)}\\&=(z^{(2)}\cdot\nabla)v^{(2)}-(z^{(1)}\cdot\nabla)v^{(1)}+(\mathrm{curl}\,v_0\cdot\nabla)[u]\\&\quad-([u]\cdot\nabla)\,\mathrm{curl}\,v_0-([u]\cdot\nabla)z^{(2)}\\&=([z]\cdot\nabla)v^{(1)}+(z^{(2)}\cdot\nabla)[u]-([u]\cdot\nabla)z^{(2)}\\&\quad+(\mathrm{curl}\,v_0\cdot\nabla)[u]-([u]\cdot\nabla)\,\mathrm{curl}\,v_0\,.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{ds}|[z]|\leqq\left|\frac{d}{ds}[z]\right|&=\frac{1}{|v^{(1)}|}|(v^{(1)}\cdot\nabla)[z]|\\&\leqq\frac{1}{|v^{(1)}|}\Big\{|v^{(1)}|_1|[z]|+|[u]|_1|z^{(2)}|+|[u]|\,|z^{(2)}|_1\\&\quad+|\mathrm{curl}\,v_0|\,|[u]|_1+|\mathrm{curl}\,v_0|_1|[u]|\Big\}\,.\end{aligned}$$

Integration of this differential inequality yields (4.21).

Lemma 4.8. *There exist constants* C_2,C_3, $K>0$, *only depending on* Ω, *such that*

$$\begin{aligned}&\|z[g,h,v_0,u^{(2)}]-z[g,h,v_0,u^{(1)}]\|_{0,\Omega}\\&\leqq\exp\{\underline{v}^{-1}LK\|v\|_{3,\Omega}\}\Bigg\{C_2\left(L\frac{\bar{f}}{\underline{v}^{(1)}}\right)^{1/2}(|h|_{2,\partial\Omega_-}+|n\cdot\mathrm{curl}\,v_0|_{2,\partial\Omega_-})\\&\quad+2(\underline{v}^{(1)})^{-3/2}(\bar{v}^{(1)})^{1/2}LC_3(\|z^{(2)}\|_{2,\Omega}+\|\mathrm{curl}\,v_0\|_{2,\Omega})\Bigg\}\|u^{(2)}-u^{(1)}\|_{1,\Omega}\,.\end{aligned}$$

Proof. Note that (2.4)–(2.6) imply

$$[z]_{\partial\Omega_-}=\frac{1}{f}(h+n\cdot\mathrm{curl}\,v_0)\,[u_{|\partial\Omega_-}]_T\,.$$

We use this equation to estimate the L_2-norms of the terms on the right hand side of (4.21). (A.4) in Corollary A.2 in the appendix yields

$$\|[z](0,y(\cdot))\|_{0,\Omega} \leqq \left(L\frac{\bar{f}}{\underline{v}^{(1)}}\right)^{1/2} \|[z]\|_{0,\partial\Omega_-}$$
$$\leqq \left(L\frac{\bar{f}}{\underline{v}^{(1)}}\right)^{1/2} \sup_{\partial\Omega_-}\left|\frac{h}{f}+\frac{1}{f}n\cdot\operatorname{curl} v_0\right| \|[u]\|_{0,\partial\Omega_-}$$
$$\leqq C_2\left(L\frac{\bar{f}}{\underline{v}^{(1)}}\right)^{1/2} (|h|_{2,\partial\Omega_-}+|n\cdot\operatorname{curl} v_0|_{2,\partial\Omega_-})\,\|[u]\|_{1,\Omega}, \quad (4.22)$$

with the norms defined in (1.18). Here we used two times Sobolev's inequality, which yields

$$\|[u]\|_{0,\partial\Omega_-} \leqq C_2'\|[u]\|_{1,\Omega}$$

and

$$\sup_{\partial\Omega_-}\left|\frac{h}{f}+\frac{1}{f}n\cdot\operatorname{curl} v_0\right| \leqq C_2''\left(\left\|\frac{h}{f}\right\|_{2,\partial\Omega_-}+\left\|\frac{1}{f}n\cdot\operatorname{curl} v_0\right\|_{2,\partial\Omega_-}\right)$$
$$\leqq cC_2''(|h|_{2,\partial\Omega_-}+|n\cdot\operatorname{curl} v_0|_{2,\partial\Omega_-}).$$

From (A.3) we conclude that

$$\left\|\int_0^{s(\cdot)} \frac{1}{|v^{(1)}|}(|z^{(2)}|_1+|\operatorname{curl} v_0|_1)\,|[u]|\,d\tau\right\|_{0,\Omega}$$
$$\leqq \left(\frac{\bar{v}^{(1)}}{\underline{v}^{(1)}}\right)^{1/2} L\left\|\frac{1}{|v^{(1)}|}(|z^{(2)}|_1+|\operatorname{curl} v_0|_1)\,|[u]|\right\|_{0,\Omega}$$
$$\leqq (\underline{v}^{(1)})^{-3/2}(\bar{v}^{(1)})^{1/2}LC_3\|(|z^{(2)}|_1+|\operatorname{curl} v_0|_1)\|_{1,\Omega}\|[u]\|_{1,\Omega}$$
$$\leqq (\underline{v}^{(1)})^{-3/2}(\bar{v}^{(1)})^{1/2}LC_3(\|z^{(2)}\|_{2,\Omega}+\|\operatorname{curl} v_0\|_{2,\Omega})\,\|[u]\|_{1,\Omega}, \quad (4.23)$$

where we used that $H_k(\Omega)H_m(\Omega)\subseteqq H_v(\Omega)$ and

$$\|q_1q_2\|_{v,\Omega} \leqq C_3\|q_1\|_{k,\Omega}\|q_2\|_{m,\Omega}, \quad (4.24)$$

for $v=\min\{k,m,k+m-2\}$, which we used with $k=m=1$, $v=0$. As mentioned earlier, (4.24) follows from Sobolev's inequality [8, p. 72] and from Hölder's inequality.

In the next estimate we use (4.24) with $k=2$, $m=0$, and $v=0$ and obtain

$$\left\|\int_0^{s(\cdot)} \frac{1}{|v^{(1)}|}(|z^{(2)}|+|\operatorname{curl} v_0|)\,|[u]|_1\,d\tau\right\|_{0,\Omega}$$
$$\leqq (\underline{v}^{(1)})^{-3/2}(\bar{v}^{(1)})^{1/2}L\|(|z^{(2)}|+|\operatorname{curl} v_0|)\,|[u]|_1\|_{0,\Omega}$$
$$\leqq (\underline{v}^{(1)})^{-3/2}(\bar{v}^{(1)})^{1/2}LC_3(\|z^{(2)}\|_{2,\Omega}+\|\operatorname{curl} v_0\|_{2,\Omega})\,|[u]|_{1,\Omega}. \quad (4.25)$$

We use (4.22), (4.23), (4.25), and (4.16) to estimate the L_2-norms of the terms on the right hand side of (4.21) and obtain the statement.

5 Boundary estimates

In this section we derive estimates for $\|z\|_{0,\partial\Omega_-}$, $|z|_{1,\partial\Omega_-}$, and $|z|_{2,\partial\Omega_-}$ and combine them with the results of Sect. 4 to prove Theorem 2.3. The estimates follow easily from the preparatory results proved in the next two lemmas, which concern products and tangential derivatives of functions defined on $\partial\Omega_-$. The norms are defined in (1.17)–(1.19).

Lemma 5.1. *There exist constants $C_1, \ldots, C_6 > 0$, only depending on $\partial\Omega_-(f)$, with*

$$\|q_1 q_2\|_{0,\partial\Omega_-} \leqq C_1 \|q_1\|_{1,\partial\Omega_-} \|q_2\|_{1,\partial\Omega_-}, \tag{5.1}$$

$$\|q_1 q_2\|_{m,\partial\Omega_-} \leqq C_2 \|q_1\|_{m,\partial\Omega_-} \|q_2\|_{2,\partial\Omega_-}, \tag{5.2}$$

$$|q_1 q_2|_{m,\partial\Omega_-} \leqq \begin{cases} C_3 |q_1|_{m,\partial\Omega_-} \|q_2\|_{2,\partial\Omega_-} \\ C_4 |q_1|_{2,\partial\Omega_-} \|q_2\|_{m,\partial\Omega_-} \end{cases}, \tag{5.3}$$

$$|||q_1 q_2|||_{m,\partial\Omega_-} \leqq \begin{cases} C_5 |||q_1|||_{m,\partial\Omega_-} \|q_2\|_{2,\partial\Omega_-} \\ C_6 |||q_1|||_{2,\partial\Omega_-} \|q_2\|_{m,\partial\Omega_-} \end{cases}. \tag{5.4}$$

for $m = 0, 1, 2$.

Proof. We use the notation of (1.17) and (1.18). For $i = 1, \ldots, \mu$ choose functions $\alpha_i' \in C_0^\infty(D_2)$ with $\alpha_i' \geqq 0$ and with $\alpha_i'(y) = 1$ for $y \in \operatorname{supp}(\alpha_i \circ \psi_i)$. Then there exist constants C, C', C'' with

$$\|\alpha_i' D^\beta (q \circ \psi_i)\|_{m, D_2^i} \leqq 2 \sum_{|\gamma + \gamma'| \leqq m + |\beta|} \|D^\gamma \alpha_i' D^{\gamma'} (q \circ \psi_i)\|_{0, D_2^i} \leqq 2C \|q\|_{|\beta| + m, \partial\Omega_-}, \tag{5.5}$$

$$\left\| \alpha_i' D^\beta \left(\frac{1}{f \circ \psi_i} \right) D^\gamma (q \circ \psi_i) \right\|_{m, D_2^i} \leqq C' |q|_{|\beta| + |\gamma| + m, \partial\Omega_-}, \tag{5.6}$$

$$\left\| \alpha_i' D^\beta \left(\frac{1}{f \circ \psi_i} \right) D^{\beta'} \left(\frac{1}{f \circ \psi_i} \right) D^\gamma (q \circ \psi_i) \right\|_{m, D_2^i} \leqq C'' |||q|||_{|\beta| + |\beta'| + |\gamma| + m, \partial\Omega_-} \tag{5.7}$$

for all q, for which the right hand side of these inequalities is finite, and for all multi-indices β, β', γ and non-negative integers m with $|\beta| + m \leqq 2$ in (5.5), $|\beta| + |\gamma| + m \leqq 2$ in (5.6) and $|\beta| + |\beta'| + |\gamma| + m \leqq 2$ in (5.7), respectively. We leave the proof to the reader.

The estimate (4.24) also holds if Ω is replaced by the two-dimensional domain D_2^i with Lipschitz boundary. (4.24) and (5.5) thus yield

$$\begin{aligned} &\|(\alpha_i \circ \psi_i) D^\beta (q_1 \circ \psi_i) D^\gamma (q_2 \circ \psi_i)\|_{0, D_2^i} \leqq \|(\alpha_i')^2 D^\beta (q_1 \circ \psi_i) D^\gamma (q_2 \circ \psi_i)\|_{0, D_2^i} \\ &\quad \leqq \tilde{C} \|\alpha_i' D^\beta (q_1 \circ \psi_i)\|_{j, D_2^i} \|\alpha_i' D^\gamma (q_2 \circ \psi_i)\|_{k, D_2^i} \leqq 4\tilde{C} C^2 \|q_1\|_{|\beta| + j, \partial\Omega_-} \|q_2\|_{|\gamma| + k, \partial\Omega_-} \end{aligned}$$

for all β, γ, j, k with $|\beta| + j \leqq 2$, $|\gamma| + k \leqq 2$, and $j + k = 2$. (5.1) follows from this estimate and from (1.17) with the choice $\beta = \gamma = 0$, $j = k = 1$, and (5.2) follows with the choice $j = |\gamma|$, $k = 2 - |\gamma|$.

Similarly, (5.5), (5.6), and (4.24) yield for $|\beta| + |\beta'| + |\gamma| \leqq 2$

$$\begin{aligned} &\left\| (\alpha_i \circ \psi_i) D^\beta \left(\frac{1}{f \circ \psi_i} \right) D^{\beta'} (q_1 \circ \psi_i) D^\gamma (q_2 \circ \psi_i) \right\|_{0, D_2^i} \\ &\leqq \left\| (\alpha_i')^2 D^\beta \left(\frac{1}{f \circ \psi_i} \right) D^{\beta'} (q_1 \circ \psi_i) D^\gamma (q_2 \circ \psi_i) \right\|_{0, D_2^i} \\ &\leqq \begin{cases} \tilde{C} \left\| \alpha_i' D^\beta \left(\frac{1}{f \circ \psi_i} \right) D^{\beta'} (q_1 \circ \psi_i) \right\|_{|\gamma|, D_2^i} \|\alpha_i' D^\gamma (q_2 \circ \psi_i)\|_{2 - |\gamma|, D_2^i} \\ \tilde{C} \left\| \alpha_i' D^\beta \left(\frac{1}{f \circ \psi_i} \right) D^{\beta'} (q_1 \circ \psi_i) \right\|_{2 - |\beta + \beta'|, D_2^i} \|\alpha_i' D^\gamma (q_2 \circ \psi_i)\|_{|\beta + \beta'|, D_2^i} \end{cases} \\ &\leqq \begin{cases} 2CC'\tilde{C} |q_1|_{|\beta + \beta' + \gamma|, \partial\Omega_-} \|q\|_{2, \partial\Omega_-} \\ 2CC'\tilde{C} |q_1|_{2, \partial\Omega_-} \|q_2\|_{|\beta + \beta' + \gamma|, \partial\Omega_-} \end{cases}. \end{aligned}$$

(5.3) follows from these estimates and from (1.18). Exactly in the same way we obtain (5.4) if we use (5.7) instead of (5.6).

Lemma 5.2. *There exist constants $C_7, \ldots, C_9$, only depending on $\partial\Omega_-(f)$, with*

$$|(v_T \cdot \nabla)q|_{0,\partial\Omega_-} \leqq C_7 |q|_{2,\partial\Omega_-} \|v_T\|_{1,\partial\Omega_-}, \tag{5.8}$$

$$|(v_T \cdot \nabla)q|_{m,\partial\Omega_-} \leqq C_8 |q|_{m+1,\partial\Omega_-} \|v_T\|_{2,\partial\Omega_-}, \tag{5.9}$$

$$|||(v_T \cdot \nabla)q|||_{m,\partial\Omega_-} \leqq C_9 |||q|||_{m+1,\partial\Omega_-} \|v_T\|_{2,\partial\Omega_-} \tag{5.10}$$

for $m=0,1$ and for every vector field v_T tangential to $\partial\Omega_-(f)$.

Proof. Let $\Phi_i : D_3 \to U_i \subseteqq \mathbb{R}^3$ and $\psi_i : D_2 \to \partial\Omega$ with $\psi_i(\xi_1, \xi_2) = \Phi_i(\xi_1, \xi_2, 0)$ be the diffeomorphisms introduced in Sect. 1. For $y \in \partial\Omega \cap U_i$ we then have

$$\begin{aligned}(v_T \cdot \nabla)q(y) &= (v_T \cdot \nabla)[(q \circ \psi_i) \circ \psi_i^{-1}](y) = (v_t \cdot \nabla)[(q \circ \psi_i) \circ \Phi_i^{-1}](y)\\ &= \sum_{m=1}^{2} \left[\frac{\partial}{\partial \xi_m}(q \circ \psi_i)\right] \circ \psi_i^{-1}(y) A_m(y) \cdot v_T(y)\end{aligned}$$

with $A_m(x) = \nabla \Phi_{i,m}^{-1}(x) = \left(\frac{\partial}{\partial x_1}\Phi_{i,m}^{-1}, \frac{\partial}{\partial x_2}\Phi_{i,m}^{-1}, \frac{\partial}{\partial x_3}\Phi_{i,m}^{-1}\right)$, where $\Phi_{i,1}^{-1}, \ldots, \Phi_{i,3}^{-1}$ are the components of Φ_i^{-1}. Thus, if $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$ are multi-indices with $|\beta| + |\gamma| \leqq 1$, and if $j + k = 2$, then

$$\begin{aligned}&\left\|(\alpha_i \circ \psi_i) D^\beta\left(\frac{1}{f \circ \psi_i}\right) D^\gamma\{[(v_T \cdot \nabla)q] \circ \psi_i\}\right\|_{0,D_2^i}\\
&= \left\|(\alpha_i \circ \psi_i) D^\beta\left(\frac{1}{f \circ \psi_i}\right) \sum_{m=1}^{2} \sum_{|\beta'|+|\gamma'| \leqq |\gamma|} D^{\beta'}\left[\frac{\partial}{\partial \xi_m}(q \circ \psi_i)\right]\right.\\
&\quad \left.\times D^{\gamma'}[(A_m \cdot v_T) \circ \psi_i]\right\|_{0,D_2^i}\\
&\leqq \sum_{m=1}^{2} \sum_{|\beta'|+|\gamma'| \leqq |\gamma|} \left\|(\alpha_i')^2 D^\beta\left(\frac{1}{f \circ \psi_i}\right) D^{\beta'}\left[\frac{\partial}{\partial \xi_m}(q \circ \psi_i)\right] D^{\gamma'}[(A_m \cdot v_T) \circ \psi_i]\right\|_{0,D_2^i}\\
&\leqq \tilde{C} \sum_{m=1}^{2} \sum_{|\beta'|+|\gamma'| \leqq |\gamma|} \left\|\alpha_i' D^\beta\left(\frac{1}{f \circ \psi_i}\right) D^{\beta'} \frac{\partial}{\partial \xi_m}(q \circ \psi_i)\right\|_{j,D_2^i}\\
&\quad \times \|\alpha_i' D^{\gamma'}[(A_m \cdot v_T) \circ \psi_i]\|_{k,D_2^i}.\end{aligned} \tag{5.11}$$

In the last step we used (4.24). If $k + |\gamma'| \leqq 2$, and if

$$M = \sup\{|D^{\gamma''}(A_m \circ \psi_i)(\xi)| : \xi \in \operatorname{supp} \alpha_i';\ m = 1, 2;\ |\gamma''| \leqq 2\}$$

then we obtain from (5.5) that

$$\begin{aligned}&\|\alpha_i' D^{\gamma'}[(A_m \cdot v_T) \circ \psi_i]\|_{k,D_2^i}\\
&\leqq 4 \sum_{\substack{|v| \leqq k\\ |v+v'+v''| \leqq k+|\gamma'|}} \|D^v \alpha_i' D^{v'}(A_m \circ \psi_i) D^{v''}(v_T \circ \psi_i)\|_{0,D_2^i}\\
&\leqq 4(6M) \sum_{\substack{|v| \leqq k\\ |v+v''| \leqq k+|\gamma'|}} \|D^v \alpha_i' D^{v''}(v_T \circ \psi_i)\|_{0,D_2^i} \leqq 24MC \|v_T\|_{k+|\gamma'|,\partial\Omega_-}.\end{aligned} \tag{5.12}$$

From (5.6) we obtain

$$\left\|\alpha_i' D^\beta\left(\frac{1}{f \circ \psi_i}\right) D^{\beta'} \frac{\partial}{\partial \xi_m}(q \circ \psi_i)\right\|_{j,D_2^i} \leqq C' |q|_{|\beta|+|\beta'|+1+j,\partial\Omega_-}. \tag{5.13}$$

To prove (5.8) we set $\beta=\gamma=0$ and $j=k=1$. Combination of (5.11)–(5.13) yields

$$\left\|(\alpha_i\circ\psi_i)\frac{1}{f\circ\psi_i}[(v_T\cdot\nabla)q]\circ\psi_i\right\|_{0,D_2^i}\leqq 48MCC'\tilde{C}|q|_{2,\partial\Omega_-}\|v_T\|_{1,\partial\Omega_-}.$$

(5.8) follows from this estimate and from (1.18). To prove (5.9) we set $j=|\gamma'|$ and $k=2-|\gamma'|$ in (5.11)–(5.13) and obtain

$$\left\|(\alpha_i\left\|(\alpha_i\circ\psi_i)D^\beta\left(\frac{1}{f\circ\psi_i}\right)D^\gamma\{[(v_T\cdot\nabla)q]\circ\psi_i\}\right\|_{0,D_2^i}$$
$$\leqq 240MCC'\tilde{C}|q|_{|\beta|+|\gamma|+1,\partial\Omega_-}\|v_T\|_{2,\partial\Omega_-}.$$

(5.9) follows from this inequality and from (1.18). The proof of (5.10) is analogous to the proof of (5.9), using (5.7) instead of (5.6).

Lemma 5.3. *There exist constants $K_1,\dots,K_3$, only depending on $\partial\Omega_-(f)$, such that the solution $z=z[g,h,W,v_0,u]$ of (2.3)–(2.6) satisfies*

$$\|z\|_{m,\partial\Omega_-}\leqq K_1[\|h\|_{m,\partial\Omega_-}+\|v_T\|_{2,\partial\Omega_-}|h|_{m,\partial\Omega_-}$$
$$+\|u_T\|_{2,\partial\Omega_-}|n\cdot W|_{m,\partial\Omega_-}+|\nabla_T g|_{m,\partial\Omega_-}],\qquad m=0,1,2 \tag{5.14}$$

$$|z|_{2,\partial\Omega_-}\leqq K_2[|h|_{2,\partial\Omega_-}+\|v_T\|_{2,\partial\Omega_-}|||h|||_{2,\partial\Omega_-}$$
$$+\|u_T\|_{2,\partial\Omega_-}|||n\cdot W|||_{2,\partial\Omega_-}+|||\nabla_T g|||_{2,\partial\Omega_-}], \tag{5.15}$$

$$|||z|||_{2,\partial\Omega_-}\leqq K_3\Bigg[|||h|||_{2,\partial\Omega_-}+\|v_T\|_{2,\partial\Omega_-}\left|\left|\left|\frac{1}{f}h\right|\right|\right|_{2,\partial\Omega_-}$$
$$+\|u_T\|_{2,\partial\Omega_-}\left|\left|\left|\frac{1}{f}n\cdot W\right|\right|\right|_{2,\partial\Omega_-}+\left|\left|\left|\frac{1}{f}\nabla_T g\right|\right|\right|_{2,\partial\Omega_-}\Bigg], \tag{5.16}$$

where $v=v_0+u$, and $n=n(x)$ is the exterior unit normal to $\partial\Omega$.

Proof. From (2.4)–(2.6) we obtain

$$z_{|\partial\Omega_-}=hn+\frac{h}{f}v_T+\frac{1}{f}(n\cdot W)u_T-\frac{1}{f}n\times\nabla_T g.$$

(5.14)–(5.16) are immediate consequences of this equation and of (5.2)–(5.4), since

$$\left\|\frac{1}{f}q\right\|_{m,\partial\Omega_-}\leqq c|q|_{m,\partial\Omega_-},\qquad \left|\frac{1}{f}q\right|_{m,\partial\Omega_-}\leqq c|||q|||_{m,\partial\Omega_-}.$$

In the following we denote by $\partial_n z=(n\cdot\nabla)z$ the normal derivative of z at $\partial\Omega$. For

$$q(x)=(q_1(x),q_2(x),q_3(x))\in\mathbb{R}^3$$

let $D^k q$ denote the vector

$$D^k q=(D^\beta q_i)_{\substack{i=1,2,3\\|\beta|\leqq k}}.$$

Lemma 5.4. *There exist constants* K_4, K_5*, only depending on* $\partial\Omega_-(f)$*, such that the solution* $z=z[g,h,v_0,u]$ *of* (2.3)–(2.6) *satisfies*

$$\|\partial_n z\|_{1,\partial\Omega_-} \leqq K_4\Bigg[\|D^2 v\|_{0,\partial\Omega_-}|z|_{2,\partial\Omega_-} + \|D^2 u\|_{0,\partial\Omega_-} \times \sum_{m=0}^{1} |D^m \operatorname{curl} v_0|_{2-m,\partial\Omega_-}\Bigg], \tag{5.17}$$

$$|\partial_n z|_{1,\partial\Omega_-} \leqq K_5\Bigg[\|D^2 v\|_{0,\partial\Omega_-}|||z|||_{2,\partial\Omega_-} + \|D^2 u\|_{0,\partial\Omega_-} \times \sum_{m=0}^{1} |||D^m \operatorname{curl} v_0|||_{2-m,\partial\Omega_-}\Bigg]. \tag{5.18}$$

Proof. From (1.3) we obtain

$$(v\cdot\nabla)z=[(v_T+(n\cdot v)n)\cdot\nabla]z=(v_T\cdot\nabla)z+f\partial_n z.$$

(2.3) thus implies

$$\partial_n z=-\frac{1}{f}(v_T\cdot\nabla)z+\frac{1}{f}(z\cdot\nabla)v-\frac{1}{f}(u\cdot\nabla)\operatorname{curl} v_0+\frac{1}{f}(\operatorname{curl} v_0\cdot\nabla)u.$$

Therefore (5.9) and (5.3) yield

$$\begin{aligned}\|\partial_n z\|_{1,\partial\Omega_-} &\leqq |(v_T\cdot\nabla)z|_{1,\partial\Omega_-}+|(\nabla v)z|_{1,\partial\Omega_-}\\ &\quad+|(\nabla \operatorname{curl} v_0)u|_{1,\partial\Omega_-}+|(\nabla u)\operatorname{curl} v_0|_{1,\partial\Omega_-}\\ &\leqq C_8\|v_T\|_{2,\partial\Omega_-}|z|_{2,\partial\Omega_-}+C_4\|\nabla v\|_{1,\partial\Omega_-}|z|_{2,\partial\Omega_-}\\ &\quad+C_3\|u\|_{2,\partial\Omega_-}|\nabla \operatorname{curl} v_0|_{1,\partial\Omega_-}+C_4\|\nabla u\|_{1,\partial\Omega_-}|\operatorname{curl} v_0|_{2,\partial\Omega_-}.\end{aligned}$$

(5.17) follows from this inequality. (5.18) is obtained in the same way using (5.10) and (5.4) instead of (5.9) and (5.3).

Lemma 5.5. *There exists a constant* K_6*, only depending on* $\partial\Omega_-(f)$*, such that*

$$\|\partial_n^2 z\|_{0,\partial\Omega_-} \leqq K_6\Bigg[\|D^2 v\|_{0,\partial\Omega_-}(|\partial_n z|_{1,\partial\Omega_-}+|z|_{2,\partial\Omega_-}) + \|D^2 u\|_{0,\partial\Omega_-} \sum_{m=0}^{2} |D^m \operatorname{curl} v_0|_{2-m,\partial\Omega_-}\Bigg]. \tag{5.19}$$

Proof. Observe first that (1.3) and (4.5) imply with $n(x)=(n_1,n_2,n_3)$

$$\begin{aligned}f\partial_n^2 z &= f(n\cdot\nabla)\sum_{i=1}^{3} n_i z_{|i}=\sum_{i=1}^{3} n_i f(n\cdot\nabla)z_{|i}\\ &=\sum_{i=1}^{3} n_i[(v\cdot n)n\cdot\nabla]z_{|i}=\sum_{i=1}^{3} n_i(v\cdot\nabla)z_{|i}-\sum_{i=1}^{3} n_i(v_T\cdot\nabla)z_{|i}\\ &=\sum_{i=1}^{3} n_i[-(v_{|i}\cdot\nabla)z+(z_{|i}\cdot\nabla)v+(z\cdot\nabla)v_{|i}+E_{|i}-(v_T\cdot\nabla)z_{|i}],\end{aligned}$$

hence

$$\partial_n^2 z = -\frac{1}{f}(\partial_n v \cdot \nabla)z + \frac{1}{f}(\partial_n z \cdot \nabla)v + \sum_{i=1}^{3} n_i \frac{1}{f}(z \cdot \nabla)v_{|i} + \frac{1}{f}\partial_n E$$
$$-\frac{1}{f}(v_T \cdot \nabla)\partial_n z + \sum_{i=1}^{3} \frac{1}{f} z_{|i}(v_T \cdot \nabla)n_i .$$

Thus, with (5.1), (5.3), (5.9),

$$\begin{aligned}
\|\partial_n^2 z\|_{0,\partial\Omega_-} \leqq{}& \left\|\frac{1}{f}(\partial_n v \cdot \nabla)z\right\|_{0,\partial\Omega_-} + \left\|\frac{1}{f}(\nabla v)\partial_n z\right\|_{0,\partial\Omega_-} \\
&+ \sum_{i=1}^{3} |(\nabla v_{|i})z|_{0,\partial\Omega_-} + \left\|\frac{1}{f}\partial_n E\right\|_{0,\partial\Omega_-} + |(v_T \cdot \nabla)\partial_n z|_{0,\partial\Omega_-} \\
&+ \sum_{i=1}^{3} |z_{|i}(v_T \cdot \nabla)n_i|_{0,\partial\Omega_-} \\
\leqq{}& \left\|\frac{1}{f}(\partial_n v \cdot \nabla)z\right\|_{0,\partial\Omega_-} + C_1 \|\nabla v\|_{1,\partial\Omega_-} \left\|\frac{1}{f}\partial_n z\right\|_{1,\partial\Omega_-} \\
&+ \sum_{i=1}^{3} C_4 \|\nabla v_{|i}\|_{0,\partial\Omega_-} |z|_{2,\partial\Omega_-} + \left\|\frac{1}{f}\partial_n E\right\|_{0,\partial\Omega_-} \\
&+ C_8 |\partial_n z|_{1,\partial\Omega_-} \|v_T\|_{2,\partial\Omega_-} \\
&+ C_3 \sum_{i=1}^{3} (|z_{|i}|_{0,\partial\Omega_-} \|(v_T \cdot \nabla)n_i\|_{2,\partial\Omega_-}) .
\end{aligned} \tag{5.20}$$

From (5.1) and (5.8) we conclude that

$$\begin{aligned}
\left\|\frac{1}{f}(\partial_n v \cdot \nabla)z\right\|_{0,\partial\Omega_-} ={}& \left\|\frac{1}{f}\{(n \cdot \partial_n v)\partial_n z + [(\partial_n v)_T \cdot \nabla]z\}\right\|_{0,\partial\Omega_-} \\
\leqq{}& C_1 \|n \cdot \partial_n v\|_{1,\partial\Omega_-} \left\|\frac{1}{f}\partial_n z\right\|_{1,\partial\Omega_-} + |[(\partial_n v)_T \cdot \nabla]z|_{0,\partial\Omega_-} \\
\leqq{}& C_1 C \|\partial_n v\|_{1,\partial\Omega_-} |\partial_n z|_{1,\partial\Omega_-} + C_7 \|(\partial_n v)_T\|_{1,\partial\Omega_-} |z|_{2,\partial\Omega_-} \\
\leqq{}& K_7 \|D^2 v\|_{0,\partial\Omega_-} (|\partial_n z|_{1,\partial\Omega_-} + |z|_{2,\partial\Omega_-}) .
\end{aligned} \tag{5.21}$$

Moreover, (4.1) and (5.1), (5.2) imply

$$\begin{aligned}
\left\|\frac{1}{f}\partial_n E\right\|_{0,\partial\Omega_-} \leqq{}& \left\|\frac{1}{f}\partial_n[(\operatorname{curl} v_0 \cdot \nabla)u]\right\|_{0,\partial\Omega_-} + \left\|\frac{1}{f}\partial_n[(u \cdot \nabla)\operatorname{curl} v_0]\right\|_{0,\partial\Omega_-} \\
\leqq{}& C_1 \left\|\frac{1}{f}\partial_n \operatorname{curl} v_0\right\|_{1,\partial\Omega_-} \|\nabla u\|_{1,\partial\Omega_-} \\
&+ C_2 \left\|\frac{1}{f}\operatorname{curl} v_0\right\|_{2,\partial\Omega_-} \|\partial_n \nabla u\|_{0,\partial\Omega_-} \\
&+ C_2 \left\|\frac{1}{f}\partial_n \nabla \operatorname{curl} v_0\right\|_{0,\partial\Omega_-} \|u\|_{2,\partial\Omega_-} \\
&+ C_1 \|\partial_n u\|_{1,\partial\Omega_-} \left\|\frac{1}{f}\nabla \operatorname{curl} v_0\right\|_{1,\partial\Omega_-} \\
\leqq{}& K_8 \|D^2 u\|_{0,\partial\Omega_-} \sum_{m=0}^{2} |D^m \operatorname{curl} v_0|_{2-m,\partial\Omega_-} .
\end{aligned} \tag{5.22}$$

Combination of (5.20)–(5.22) yields (5.19).

Proof of Theorem 2.3. As noted after Theorem 2.3, it suffices to prove (2.12)–(2.14). Note first that (4.1) and (4.24) yield for $j=0,\dots,2$ that

$$\begin{aligned}|E|_{j,\Omega}&\leqq \|E\|_{j,\Omega}\leqq \|(\nabla u)W\|_{j,\Omega}+\|(\nabla W)u\|_{j,\Omega}\\ &\leqq C(\|\nabla u\|_{2,\Omega}\|W\|_{j,\Omega}+\|u\|_{2,\Omega}\|\nabla W\|_{j,\Omega})\\ &\leqq C'\|u\|_{3,\Omega}\|W\|_{3,\Omega}\leqq C'\gamma\|W\|_{3,\Omega},\end{aligned}\tag{5.23}$$

since $u\in V_\gamma$.

(2.12) is an immediate consequence of this estimate, of (5.14), and of Lemma 4.6, if we use in addition that the trace theorem implies

$$\begin{aligned}&\|u_T\|_{2,\partial\Omega_-}\leqq C_1\|u\|_{2,\partial\Omega_-}\leqq C_1\|u\|_{2,\partial\Omega}\leqq C_2\|u\|_{3,\Omega}\leqq C_{2\gamma},\\ &\|v_T\|_{2,\partial\Omega_-}\leqq C_1\|v\|_{2,\partial\Omega}\leqq C_2\|v\|_{3,\Omega},\end{aligned}$$

that

$$\|v\|_{3,\Omega}=\|v_0+u\|_{3,\Omega}\leqq\|v_0\|_{3,\Omega}+\gamma$$

since $u\in V_\gamma$, that Sobolev's inequality yields $\bar v\leqq C\|v\|_{3,\Omega}$, and that $\underline{v}\geqq\underline{v}_0-\hat C\gamma>0$, by Lemma 2.1. We also need (1.20).

To prove (2.13), observe that Lemma 4.4, 4.5, 4.6 and the inequality (5.23) yield

$$\begin{aligned}\|z\|_{2,\Omega}&=(\|z\|_{0,\Omega}^2+|z|_{1,\Omega}^2+|z|_{2,\Omega}^2)^{1/2}\leqq\|z\|_{0,\Omega}+|z|_{1,\Omega}+|z|_{2,\Omega}\\ &\leqq L_\gamma^{1/2}K(L_\gamma,\underline{v}_0,\|v_0\|_{3,\Omega},\bar f,\gamma)\\ &\qquad\times[|z|_{2,\partial\Omega_-}+|z|_{1,4,\partial\Omega_-}+|z|_{1,\partial\Omega_-}+\|z\|_{2,\partial\Omega_-}+\|\operatorname{curl}v_0\|_{3,\Omega}].\end{aligned}\tag{5.24}$$

We use that

$$|z|_{1,\partial\Omega_-}+|z|_{2,\partial\Omega_-}\leqq 2\|D^2z\|_{0,\partial\Omega_-}\leqq C(\|z\|_{2,\partial\Omega_-}+\|\partial_n z\|_{1,\partial\Omega_-}+\|\partial_n^2 z\|_{0,\partial\Omega_-}).\tag{5.25}$$

Moreover, as in the proof of Lemma 4.5 we have

$$\begin{aligned}|z|_{1,4,\partial\Omega_-}&=\Big[\int_{\partial\Omega_-}|z|_1^4dS\Big]^{1/4}\leqq C_1\Big[\sum_{i,j=1}^3\int_{\partial\Omega_-}\Big|\frac{\partial}{\partial x_i}z_j\Big|^4dS\Big]^{1/4}\\ &\leqq C_2\Big[\sum_{i,j=1}^3\Big\|\frac{\partial}{\partial x_i}z_j\Big\|_{1,\partial\Omega_-}^4\Big]^{1/4}\leqq C_3\|\nabla z\|_{1,\partial\Omega_-}\\ &\leqq C_4(\|\partial_n z\|_{1,\partial\Omega_-}+\|z\|_{2,\partial\Omega_-}).\end{aligned}\tag{5.26}$$

Combination of (5.24)–(5.26) and of (5.14)–(5.19) yields (2.13), if we again use the trace theorem, which implies

$$\begin{aligned}&\|D^2u\|_{0,\partial\Omega_-}\leqq\|D^2u\|_{0,\partial\Omega}\leqq C\|u\|_{3,\Omega}\leqq C\gamma,\\ &\|D^2v\|_{0,\partial\Omega_-}\leqq\|D^2v\|_{0,\partial\Omega}\leqq C\|v\|_{3,\Omega}\leqq C(\|v_0\|_{3,\Omega}+\gamma).\end{aligned}$$

We also use (1.20) to estimate $\|q\|_{2,\partial\Omega_-}$ and $|q|_{2,\partial\Omega_-}$ by $|||q|||_{2,\partial\Omega_-}$, and we use that (5.4) yields

$$|||q|||_{2,\partial\Omega_-}=\Big|\Big|\Big|f\Big(\frac{1}{f}q\Big)\Big|\Big|\Big|_{2,\partial\Omega_-}\leqq C_5\|f\|_{2,\partial\Omega_-}\Big|\Big|\Big|\frac{1}{f}q\Big|\Big|\Big|_{2,\partial\Omega_-}.$$

Finally, (2.14) is obtained if we use (2.13) to estimate the term $\|z^{(2)}\|_{2,\Omega}$ in the inequality stated in Lemma 4.8. The proof of Theorem 2.3 and therefore the proof of Theorem 1.1 is complete.

Appendix

Here we prove some results needed in Sect. 4 to integrate with respect to the integral curve coordinates, and we prove Lemma 2.7. Let $\psi_i: D_2 \to \partial\Omega$ be one of the local coordinate systems of $\partial\Omega$ introduced in Sect. 1. For brevity we write

$$\omega(t,\xi)=\omega(t,\psi_i(\xi),u), \qquad \omega(s,\xi)=\omega(s,\psi_i(\xi),u)$$

if $\xi=(\xi_1,\xi_2)\in D_2^i=\psi_i^{-1}(\partial\Omega_-(f))$. Clearly, (t,ξ) and (s,ξ) are local coordinates of Ω. We use these cooridnates for integration in Ω, and therefore need the following result for the Jacobi determinants

$$\tilde{J}(t,\xi)=\det\left(\frac{\partial(\omega_1,\omega_2,\omega_3)}{\partial(t,\xi_1,\xi_2)}\right),$$

$$J(s,\xi)=\det\left(\frac{\partial(\omega_1,\omega_2,\omega_3)}{\partial(s,\xi_1,\xi_2)}\right)=\tilde{J}(t(s),\xi)\frac{\partial t}{\partial s}.$$

Lemma A.1. *For all* (t,ξ) *we have*

$$\frac{\partial}{\partial t}\tilde{J}(t,\xi)=\operatorname{div} v(\omega(t,\xi))\tilde{J}(t,\xi). \tag{A.1}$$

For all (s,ξ) *we have*

$$|J(s,\xi)|=-\frac{f(\xi)}{|v(s,\xi)|}|\partial_{\xi_1}\omega(0,\xi)\times\partial_{\xi_2}\omega(0,\xi)|, \tag{A.2}$$

where f *is the prescribed function in the boundary condition* (1.3), *and where we use the notation*

$$f(\xi)=f(\omega(0,\xi)), \qquad v(s,\xi)=v(\omega(s,\xi)).$$

Proof. The proof of (A.1) is standard, cf. [10, p. 131].

To prove (A.2), note that (A.1) and $\operatorname{div} v=0$ imply $\frac{\partial}{\partial t}\tilde{J}(t,\xi)=0$, hence

$$\begin{aligned}|\tilde{J}(t,\xi)|&=|\tilde{J}(0,\xi)|=|\det(v(\omega(0,\xi),\partial_{\xi_1}\omega(0,\xi),\ \partial_{\xi_2}\omega(0,\xi)))|\\&=|v\cdot(\partial_{\xi_1}\omega\times\partial_{\xi_2}\omega)|=|n\cdot v|\,|\partial_{\xi_1}\omega\times\partial_{\xi_2}\omega|\\&=-f(\xi)\,|\partial_{\xi_1}\omega(0,\xi)\times\partial_{\xi_2}\omega(0,\xi)|.\end{aligned}$$

Here we used (1.3). But

$$|J(s,\xi)|=\frac{1}{\frac{\partial s}{\partial t}}|\tilde{J}(t(s),\ \xi_1,\xi_2)|=-\frac{f(\xi)}{|v(\omega(s,\xi_1,\xi_2))|}|\partial_{\xi_1}\omega\times\partial_{\xi_2}\omega|.$$

This proves (A.2).

Corollary A.2. *If* $q\in L_1(\Omega;\mathbb{R}^m)$ *then*

$$\int_\Omega q(x)dx=\int_{\partial\Omega_-}\int_0^{\ell(y)} q(s,y)\frac{|f(y)|}{|v(s,y)|}dsdS_y.$$

If $q \in L_2(\Omega, \mathbb{R}^m)$, *then*

$$\left\| \int_0^{s(\cdot)} q(\tau, y(\cdot))d\tau \right\|_{0,\Omega} \leqq \left(\frac{\bar{v}}{\underline{v}}\right)^{1/2} L\|q\|_{0,\Omega}, \tag{A.3}$$

where $\underline{v} = \inf_{x\in\Omega} |v(x)|$, $\bar{v} = \sup_{x\in\Omega} |v(x)|$.

If $q \in L_2(\partial\Omega_-(f), \mathbb{R}^m)$, *then*

$$\|q(0, y(\cdot))\|_{0,\Omega} \leqq \left(\frac{\bar{f}}{\underline{v}} L\right)^{1/2} \|q\|_{0,\partial\Omega_-}, \tag{A.4}$$

where $\bar{f} = \sup_{x\in\partial\Omega_-} |f(x)|$.

Proof. The first assertion follows immediately from the integral transform theorem and from (A.2), since

$$dS = |\partial_{\xi_1}\omega(0,\xi) \times \partial_{\xi_2}\omega(0,\xi)|.$$

To prove (A.3), note that the first assertion implies

$$\begin{aligned}
\int_\Omega \left| \int_0^{s(x)} q(\tau, y(x))d\tau \right|^2 dx &= \int_{\partial\Omega_-} \int_0^{\ell(y)} \left| \int_0^s q(\tau, y)d\tau \right|^2 \frac{|f(y)|}{|v(s,y)|} ds dS_y \\
&\leqq \int_{\partial\Omega_-} \int_0^{\ell(y)} s \int_0^s |q(\tau,y)|^2 d\tau \frac{|f(y)|}{|v(s,y)|} ds dS_y \\
&\leqq \int_{\partial\Omega_-} \left[\int_0^{\ell(y)} |q(\tau,y)|^2 d\tau \right] \left[\int_0^{\ell(y)} \frac{s}{|v(s,y)|} ds \right] |f(y)| dS_y \\
&\leqq \frac{L^2}{\underline{v}} \int_{\partial\Omega_-} \int_0^{\ell(y)} |q(\tau,y)|^2 \frac{|f(y)|}{|v(\tau,y)|} \bar{v} d\tau dS_y = \frac{\bar{v}}{\underline{v}} L^2 \|q\|_{0,\Omega}^2.
\end{aligned}$$

Also the inequality (A.4) follows from the first assertion, since

$$\begin{aligned}
\int_\Omega |q(0, y(x))|^2 dx &= \int_{\partial\Omega_-} \int_0^{\ell(y)} |q(0,y)|^2 \frac{|f(y)|}{|v(s,y)|} ds dS_y \\
&\leqq \frac{\bar{f}}{\underline{v}} \int_0^{\ell(y)} ds \int_{\partial\Omega_-} |q(0,y)|^2 dS_y \leqq \frac{\bar{f}}{\underline{v}} L \|q\|_{0,\partial\Omega_-}^2.
\end{aligned}$$

The proof is complete.

Proof of Lemma 2.7. For every $q \in C^1(\bar{\Omega})$ we obtain from (A.3) and (A.4)

$$\begin{aligned}
\|q\|_{0,\Omega} &= \left\| \int_0^{s(\cdot)} \frac{\partial}{\partial\tau} q(\tau, y(\cdot))d\tau + q(0, y(\cdot)) \right\|_{0,\Omega} \\
&\leqq \left(\frac{\bar{v}}{\underline{v}}\right)^{1/2} L \left\| \frac{\partial}{\partial s} q \right\|_{0,\Omega} + \left(\frac{\bar{f}}{\underline{v}} L\right)^{1/2} \|q\|_{0,\partial\Omega_-} \\
&\leqq \left(\frac{\bar{v}}{\underline{v}}\right)^{1/2} L \|\nabla q\|_{0,\Omega} + \left(\frac{\bar{f}}{\underline{v}} L\right)^{1/2} \|q\|_{0,\partial\Omega_-}.
\end{aligned}$$

This estimate is extended to $q \in H_1(\Omega)$ as usual.

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