Math. Ann. 302, 61–79 (1995)

Mathematische
Annalen

On joint moduli spaces

Liaw Huang
Department of Mathematics, Indiana Wesleyan University, 4201 South Washington, Marion, IN 46953, USA (e-mail: lhuang@barnabas.indwes.edu)

Received: 6 December 1993 / Revised version: 16 May 1994

Mathematics Subject Classification (1991): 32G13, 32G08, 14D15

Introduction

The purpose of this paper is to construct the moduli of a holomorphic vector bundle over a compact complex manifold when both the holomorphic structure on the vector bundle and the holomorphic structure on the base manifold are deformed simultaneously. Let E be a holomorphic vector bundle over a compact complex manifold M, and let E_0, M_0 be the underlying C^∞ structure. We call a simultaneous deformation of the holomorphic structures on E and M a *joint deformation* of (E, M), and the set of all such structures the joint moduli space of (E, M). More precisely, consider the set $\mathscr{S} = \{(E, M)$: E is a holomorphic vector bundle over the complex manifold M, E is C^∞ isomorphic to E_0 and M is diffeomorphic to $M_0\}$/equivalence, where (E, M) is equivalent to (E', M') if there is a C^∞ diffeomorphism f: $M \to M'$ and a C^∞ isomorphism of vector bundles $g: E \to f^*E'$, such that with respect to the holomorphic structures on M, M', E, and E', the map f and g are holomorphic. Then set $\mathscr{S}$ gives the joint moduli space of vector bundles E over M with a given C^∞ data E_0 and M_0. We state the main theorem:

Theorem *Suppose M has an ample anticanonical bundle and E is stable with respect to the ample anticanonical class, then joint moduli space $\mathscr{S}$ consisting of all such pairs has naturally a structure of a Hausdorff complex space.*

Specialising to curves of genus greater than one, we have the following corollary

Corollary *If M is a smooth Riemann surface of genus $g > 1$, then there exists a joint moduli of stable vector bundles over the moduli of Riemann surfaces of genus g, such that each fiber consists of the moduli of stable bundles modulo the action of the automorphism group of the base space.*

By assigning to each pair (E, M) the base manifold M, $\mathcal{S}$ maps into the moduli space of the compact complex manifold M such that each fiber consists of the moduli of holomorphic vector bundles over M. However, with the above equivalence relation, if f is a holomorphic automorphism of M, then (E, M) and (f^*E, M) are equivalent. Thus the fiber consists of the moduli of the vector bundle E over M modulo the action of the automorphism group of M.

The proof of our theorem is analytic, and uses the technique in [Ko]: We first construct a differential graded Lie algebra L^*, which is a semi-direct product of the differential graded Lie algebras $A^*(M, TM)$ and $A^*(M, \mathrm{End}(E))$, and show that joint deformations correspond to solutions of the deformation equation in L^*. We then apply Kuranishi's construction (see [Ku2] or [GM2]) to construct a local complete joint deformation space. If the manifold M has ample canonical class and E is stable with respect to the ample canonical class, then following the construction of Narasimhan and Simha [NS], we can glue local deformation spaces together to obtain a joint moduli space.

Finally we study the existence of holomorphic vector bundle structures under small deformations of the base manifold. We show that in certain cases the existence of holomorphic structures is equivalent to the condition that all the Chern classes are (n, n) classes. We have the following theorem:

Theorem *Let M_θ be a small deformation of the complex structure of M. If M is a compact Kähler surface with trivial canonical bundle and E is a simple vector bundle on M, then E admits a holomorphic structure on M_θ if and only if $c_1(E)$ is a $(1, 1)$ class in M_θ.*

Thus there is a relationship between the existence of holomorphic structures and the period map.

The paper is organized as follows. Section 1 studies the local theory of joint deformations. Section 2 gives the construction of the joint moduli. Section 3 studies the existence of holomorphic vector bundle structure under a small deformation of the base manifold.

1 Local theory of joint deformations

(1.0) Introduction

The analytical construction of the joint deformation space uses the concept of differential graded Lie algebra and deformation equation. We first give the basic definitions, referring details to Nijenhuis-Richardson [NR] or Goldman-Millson [GM1].

Definition *A graded Lie Algebra L^* over C is a Z-graded C-vector space*

$$L = \bigoplus_{i \geq 0} L^i$$

with a family of bilinear maps $[,]$: $L^i \times L^j \to L^{i+j}$ *satisfying the following identities*

(1) (*graded skew-commutativity*) $[a, b] = -(-1)^{\deg(a)\deg(b)}[b, a]$

(2) (*Jacobi identity*) $(-1)^{\deg(a)\deg(c)}[a, [b, c]] + (-1)^{\deg(b)\deg(a)}[b, [c, a]] + (-1)^{\deg(c)\deg(b)}[c, [a, b]] = 0.$

A graded Lie Algebra L^* *is called a differential graded Lie algebra if it is provided with a* $\mathbf{C}$*-linear map* d: $L^* \to L^{*+1}$ *such that* d *satisfies*

(1) (*derivation of degree* 1) $d[a, b] = [da, b] + (-1)^{\deg(a)}[a, db]$

(2) (*differential*) $d \circ d = 0$.

Definition *Given a differential graded Lie algebra* (L^*, d), *the deformation equation in* L^* *is the following equation on* L^1

$$da + 1/2[a, a] = 0 .$$

Using these concepts, the analytical construction of a deformation space usually has 3 steps:

(1) Find a differential graded Lie algebra L^* that relates to deformations.

(2) Prove that the deformation equation in L^* corresponds to the integrability condition for deformations.

(3) Obtain a Hodge theory on the differential graded Lie algebra L^*, which is usually infinite dimensional, and apply the Kuranishi construction to obtain an analytic space which is the complete deformation space of the object involved.

From this point of view, joint deformations include the two cases below:

(A) The deformation of complex structures on a given complex manifold. In this case, the differential graded Lie algebra is $A^{0,*}(M, TM)$, and the Lie algebra structure is induced from the Lie bracket of vector fields on M.

(B) The deformation of holomorphic structures on a holomorphic vector bundle E over a complex manifold M. In this case, the differential graded Lie algebra is $A^{0,*}(M, \mathrm{End}(E))$, and the Lie algebra structure is induced from the Lie algebra structure on $\mathrm{End}(E)$.

In order to construct joint deformations, where both the complex structure on M and the holomorphic structure on E vary simultaneously, we need to find a differential graded Lie algebra that contains both (A) and (B). We first construct a semi-direct product L^* of $A^{0,*}(M, TM)$ and $A^{0,*}(M, \mathrm{End}(E))$, and then prove that joint deformations correspond to the deformation equation in L^*.

(1.1) The differential graded Lie algebra L^*

Let M be a compact complex manifold, and E a holomorphic vector bundle on M. Let $A^{0,1}(M, TM)$ be $C^\infty(0, 1)$ forms with value in the tangent bundle of M, and let $A^{0,1}(E, \mathrm{End}(E))$ be $C^\infty(0, 1)$ forms with value in the endomorphism bundle of E. In this section we will define a differential graded Lie algebra L^*, which is a semi-direct product of $A^{0,1}(M, \mathrm{End}(E))$ and $A^{0,1}(M, TM)$.

Let h be a hermitian metric on E, D be the hermitian connection, and let Ω be the curvature form of D. As usual $D = D' + \bar\partial$, and $\Omega \in A^{1,1}(M, \mathrm{End}(E))$.

Let $L^p = \{(a, \theta) | a \in A^{0,p}(M, \mathrm{End}(E)),\ \theta \in A^{0,p}(M, TM)\}$ as a set and let $\bar\partial_R : L^p \to L^{p+1}$ be defined by $\bar\partial_R(a, \theta) = (\bar\partial a - \Omega\theta, \bar\partial\theta)$, where $\Omega\theta$ is induced from the interior product of a (1, 1) form and a vector field. Since $\bar\partial\Omega = 0$, we have $\bar\partial_R \circ \bar\partial_R = 0$. Alternatively, the complex L^* can be defined as follows: let $\{s_i\}$ be a local holomorphic frame in an open trivializing neighborhood U_i, and let g_{ij} be the transition function in $U_i \cap U_j$. Then the collection $\{g_{ij}^{-1} \cdot \partial g_{ij}\}$ defines a 1-cocycle of the sheaf $T^*M \otimes \mathrm{End}(E)$. Let ω_i be the connection form in U_i with respect to the local holomorphic frame s_i, and consider $\{\omega_i\}$ as a 0-cochain with value in the sheaf $T^*M \otimes \mathrm{End}(E)$. Then on the overlap $U_i \cap U_j$, the connection forms satisfy $\omega_i = g_{ij}^{-1} \cdot \partial g_{ij} + g_{ij}^{-1} \cdot \omega_j \cdot g_{ij}$. Thus $R = \bar\partial\omega_i$ defines a global (0, 1) form with value in $T^*M \otimes \mathrm{End}(E)$, and this represent the cohomology of the 1-cocycle $\{g_{ij}^{-1} \cdot \partial g_{ij}\}$ in Dolbeault cohomology. If we skew-symmetrize R, then we obtain the curvature form Ω. So we see from the definition of curvature and skew symmetry that $R\theta = -\Omega\theta$. Now the curvature class $R \in H^1(M, T^*M \otimes \mathrm{End}(E))$ defines an extension $\mathscr{C}$ of the tangent bundle of M by the endomorphism bundle of E

(1) $$0 \to \mathrm{End}(E) \to \mathscr{C} \to TM \to 0,$$

and on $A^*(M, \mathscr{C})$ there is a canonical differential operator $\bar\partial$. It is easily seen that the complex L^* defined earlier is precisely the complex $A^*(M, \mathscr{C})$.

We now define a Lie bracket on L^* by the rule that

(2) $$[(\alpha, \theta), (\alpha', \theta')] = ([\alpha, \alpha'] + D_\theta \alpha' - (-1)^{pq} D_{\theta'}\alpha, [\theta, \theta']),$$

where $\alpha \in A^{0,p}(M, \mathrm{End}(E))$, $\alpha' \in A^{0,q}(M, \mathrm{End}(E))$, $\theta \in A^{0,p}(M, TM)$, and $\theta' \in A^{0,q}(M, TM)$. Before we define $D_\theta\alpha$ for $\alpha \in A^{0,p}(M, \mathrm{End}(E))$, $\theta \in A^{0,q}(M, TM)$, we recall a formula of Cartan: if a is a p-form, and X is a vector field, then

(3) $$L_X a = i_X da + d i_X a,$$

where L_X is the Lie derivative with respect to the vector field X, and i_X is the interior product with the vector field X. Now i_X extends naturally to forms with value in a holomorphic vector bundle by $i_X(\phi \wedge e) = i_X\phi \wedge e$, where ϕ is a differential form and e is a local section of the vector bundle. So we define $D_\theta a$ analogously for θ a vector field and $a \in A^{0,p}(M, \mathrm{End}(E))$ by

(3′) $$D_\theta a = i_\theta Da + D i_\theta a.$$

If θ is a $(0, q)$ form with values in holomorphic vector fields, then by writing $\theta = \phi \wedge X$ locally, we define $D_\theta a$ by

(4) $$D_{\phi \wedge X} a = \phi \wedge D_X a + (-1)^{\deg(\phi)} d\phi \wedge i_X a.$$

Finally, before we prove that L^* with the bracket defined above is a differential graded Lie algebra, we recall that a derivation of degree p of a graded Lie algebra L^* is a homogeneous linear map D: $L^* \to L^*$ of degree p such that

$D([a,b]) = [Da,b] + (-1)^{pr}[a,Db]$, for $a \in L^r$, $b \in L^*$. We denote by $\mathscr{D}(L)$ the space of derivations of L^*. $\mathscr{D}(L)$ becomes a graded algebra if we define $[D,D'] = D \circ D' - (-1)^{pq} D' \circ D$, for D, D' derivation of degree p, q respectively. We now prove the following

Proposition 1.1 *$(L^*, \bar{\partial}_R, [\,,\,])$ is a differential graded Lie algebra.*

Proof. We first define a homomorphism ψ: $A^{0,*}(M,TM) \to \mathscr{D}(A^{0,*}(M,\mathrm{End}(E)))$ by $\psi(\theta)(s) = D_\theta s$, where $\theta \in A^{0,*}(M,TM)$, $s \in A^{0,*}(M,\mathrm{End}(E))$. We need to show that if θ is a q-form with value in TM, then D_θ is a derivation of degree q, and that the map ψ is a homomorphism. If θ is a vector field, the proof that D_θ is a degree 0 derivation is the same as the proof that Lie derivative is a derivation. If θ is a q-form, then the fact that D_θ is a q derivation follows from the formula (4). To show that ψ is a homomorphism, we observe that from the definition of curvature, for $v, w \in A^0(M,TM)$

$$D_v D_w - D_w D_v - D_{[v,w]} = \Omega(v,w)\,, \tag{5}$$

where Ω is the curvature form. Since Ω is chosen to be a (1, 1) form, $\Omega(v,w) = 0$ for $v, w \in A^0(M,TM)$. This extends to $v, w \in A^{0,*}(M,TM)$, and gives $D_v D_w - (-1)^{\deg(w)\deg(v)} D_w D_v = D_{[v,w]}$, which shows that ψ is a homomorphism. Now the fact that $(L^*,[\,,\,])$ is a graded Lie algebra follows from the standard construction of semidirect product of two graded Lie algebras. We state the construction as the following proposition, whose proof can be found in Nijenhuis-Richardson [NR]

Proposition 1.2 *If E and F are graded Lie algebras and $f \to D_f$ is a homomorphism of F into $\mathscr{D}(E)$, then the graded vector space $E \oplus F$ equipped with the product*

$$[(a,f),(b,g)] = ([a,b] + D_f b - (-1)^{pq} D_g a, [f,g])$$

for $a \in E^p$, $b \in E^q$, $f \in F^q$, $g \in F^q$, is a graded Lie algebra. It contains F as a subalgebra, and E as an ideal.

Now it remains to show that the differential $\bar{\partial}_R$ is a derivation of degree 1. For this we compute each term separately: For $a \in A^{0,p}(M,\mathrm{End}(E))$, $b \in A^{0,q}(M,\mathrm{End}(E))$, $f \in A^{0,p}(M,TM)$, $g \in A^{0,q}(M,TM)$,

$$\begin{aligned}(6)\ \bar{\partial}_R[(a,f),(b,g)] &= \bar{\partial}_R([a,b] + D_f b - (-1)^{pq} D_g a, [f,g]) \\ &= \bar{\partial}[a,b] + \bar{\partial}(D_f b) - (-1)^{pq}\bar{\partial}(D_q a) + R\cdot[f,g], \bar{\partial}[f,g])\,. \\ [\bar{\partial}_R(a,f),(b,g)] &= [(\bar{\partial}a + R\cdot f, \bar{\partial}f),(b,g)] \\ &= ([\bar{\partial}a + R\cdot f, b] + D_{\bar{\partial}f} b \\ &\quad - (-1)^{(p+1)q} D_g(\bar{\partial}a + R\cdot f), [\bar{\partial}f, g])\,. \\ [(a,f),\bar{\partial}_R(b,g)] &= [(a,f),(\bar{\partial}b + R\cdot g, \bar{\partial}g)] \\ &= ([a, \bar{\partial}b + R\cdot g] + D_f(\bar{\partial}b + R\cdot g) \\ &\quad - (-1)^{p(q+1)} D_{\bar{\partial}g}\, a, [f, \bar{\partial}g])\,.\end{aligned}$$

Now we assume the following identities:

$$\bar\partial(D_f b) = D_{\bar\partial f} b + (-1)^p D_f(\bar\partial b) + [Rf, b] \tag{7}$$

$$(-1)^p(D_f(Rg)) - (-1)^{(p+1)q} D_g(Rf) = R\cdot[f, g]\,. \tag{8}$$

Substituting (7) and (8) into (6), we obtained the desired formula: $\bar\partial_R[(a, f), (b, g)] = [\bar\partial_R(a, f), (b, g)] + (-1)^p[(a, f), \bar\partial_R(b, g)]$, showing that L^* is a differential graded Lie Algebra.

To prove formula (7) we observe that it is enough to prove it locally. Let $\{s_i\}$ be a local holomorphic section of E, and let $\{\omega_i\}$ be the connection form. Let b be a $\mathrm{End}(E)$ valued $(0, q)$ form, and in the local frame $\{s_i\}$ it is expressed as a matrix of $(0, q)$ forms. Then we have

$$Db = db + \omega \wedge b - (-1)^q b \wedge \omega\,, \tag{9}$$

and for $f \in A^{0,p}(M, TM)$,

$$D_f b = L_f b + \omega(f) \wedge b - (-1)^{pq} b \wedge \omega(f)\,. \tag{9'}$$

Thus

$$\begin{aligned}\bar\partial(D_f b) &= \bar\partial L_f b + \omega(f) \wedge b - (-1)^{pq} b \wedge \omega(f)\\ &= L_{\bar\partial f} b + (-1)^p L_f\cdot\bar\partial b + \bar\partial\omega(f) \wedge b + (-1)^p\omega(f)\bar\partial b\\ &\quad - (-1)^{pq}\bar\partial b \wedge \omega(f) - (-1)^{pq+q} b \wedge \bar\partial\omega(f)\\ &= (-1)^p D_f\bar\partial b + L_{\bar\partial f} b + \bar\partial\omega(f) \wedge b - (-1)^{pq+q} b \wedge \bar\partial\omega(f)\,.\end{aligned}$$

Now use

$$\bar\partial(\omega(f)) = Rf + \omega(\bar\partial f)\,,$$

we have

$$\begin{aligned}\bar\partial(D_f b) &= (-1)^p D_f\bar\partial b + L_{\bar\partial f} b + (Rf + \omega(\bar\partial f)) \wedge b - (-1)^{pq+q} b \wedge (Rf + \omega(\bar\partial f))\\ &= D_{\bar\partial f} b + (-1)^p D_f\bar\partial b + [Rf, b]\,.\end{aligned}$$

So (7) is proved. Formula (8) is proved similarly. □

(1.2) Deformation equation in L^* and joint deformations

In this section we study joint deformations and show that joint deformations correspond precisely to the solutions of deformation equation in L^*. First of all, we need to understand what happened to the Dolbeault complex when the complex structure on the manifold is deformed. Suppose the deformed complex structure is represented by θ, a $(0, 1)$-form with value in holomorphic vector fields, and let $A_\theta^{0,1}$ be the space of $(0, 1)$-forms with respect to the complex structure θ. Subsequently, we use subscript θ to represent forms, operators, or complexes with respect to the complex structure θ, and if θ is omitted, it is understood that it is with respect to the original complex structure. Each complex structure θ determines a decomposition of $A^1(M, TM)$ into its (1, 0)-part and (0, 1)-part. Let $P_\theta^{0,1}$ be the projection onto the (0, 1)-part. When the complex structure on M is deformed, the space of $(0, p)$-forms and the $\bar\partial$ operator are both deformed. However we can transplant the $\bar\partial_\theta$ operator on $A_\theta^{0,*}(M, TM)$ to an operator on $A^{0,*}(M, TM)$ via the projection $P_\theta^{0,1}$. In other words, we can keep the complex the same, but vary the differential so as to pick up the variation of complex structures on M. In this way the vector bundle on M_θ and M can be compared.

We first review Kodaira-Spencer theory on the deformation of complex structures on M so as to establish some notations and conventions. Let $\theta \in A^{0,1}(M, TM)$. Then θ determines a homomorphism from $T''M$ to $T'M$, where $T''M$ is the antiholomorphic tangent bundle and $T'M$ is the holomorphic tangent bundle. If θ is sufficiently close to 0, then we can assume that the homomorphism $\bar\theta \circ \bar\theta$ does not have eigenvalue 1. Then the observation made by Kodaira-Spencer is that θ determines a complex structure on M if θ satisfies the integrability condition

$$\bar\partial\theta + 1/2[\theta, \theta] = 0 .$$

In fact, $T_\theta''M$ consists of the vectors $\{v + \theta(v): v \in T''M\}$. Thus any C^∞ function f is holomorphic with respect to the complex structure θ if and only if $(\bar\partial + \theta)f = 0$. (Note the convention used here differs from [Ko] or [Ku] by a sign.)

Our goal here is to consider $A_\theta^{0,1}(M, TM)$ as a subcomplex of $A^1(M, TM)$, and try to find an operator d_θ^p making the diagram commutative:

$$\begin{array}{ccc} A^{0,p}(M) & \xrightarrow{d_\theta^p} & A^{0,p+1}(M) \\ \downarrow{\scriptstyle P_\theta} & & \downarrow{\scriptstyle P_\theta} \\ A_\theta^{0,p}(M) & \xrightarrow{\bar\partial_\theta} & A_\theta^{0,p+1}(M) \end{array} \tag{10}$$

where P_θ is the projection, i.e., consider forms in $A^{0,p}(M)$ as in $A^p(M)$ and then take the $(0, p)$-part with respect to θ. We remark that P_θ is a C^∞ bundle isomorphism between T'' and T_θ''. The Kodaira-Spencer result suggests that for $p = 0$, $d_\theta = \bar\partial + \theta$. More generally, we let θ act on $\phi \in A^{0,p}(M)$ by writing locally $\theta = \psi \otimes X$, with $\psi \in A^{0,1}(M)$, $X \in A^0(M, TM)$, and define

$$\theta\phi = \psi \wedge L_X\phi . \tag{11}$$

Then we have

Proposition 1.3 *If d_θ^p is defined to be $\bar\partial + \theta$, then the diagram* (10) *commutes.*

Proof. Fix a point x in M, and let z^i be local coordinates around x. We first verify for the case $p = 0$. So let f be a function on M, and write θ locally as $\theta = \Sigma\theta_j^i d\bar z^j \otimes \partial/\partial z^i$. Then locally $(\bar\partial + \theta)f = \Sigma(\partial f/\partial\bar z^j + \theta_j^i \cdot \partial f/\partial z^i)d\bar z^j$. Now let ϕ^i be $P_\theta d\bar{z}^i$. Then $P_\theta(\bar{\partial} + \theta)f = \Sigma(\partial f/\partial \bar{z}^j + \theta^i_j \cdot \partial f/\partial z^i)\phi^j$. On the other hand, if we let $v_i = \partial/\partial \bar{z}^i + \theta^j_i \partial/\partial z^j$, then they span the antiholomorphic vectors in $T''_\theta M$. In order to determine ϕ^i, we write $d\bar{z}^i = \Sigma c^i_j v^{j*} + c'^i_j \bar{v}^{j*}$, where $c^i_j = d\bar{z}^i(v_j)$. Now from the definition of v_i, we see that $c^i_j = \delta^i_j$. So $\phi^i = v^{i*}$. Since θ is integrable, we have

$$[v_i, v_j] = 0 \,. \tag{12}$$

So there exists local coordinates in the complex structure θ, denoted by z_θ, such that at x, $d\bar{z}^i_\theta = v^{i*} = \phi^i$. Hence $\bar{\partial}_\theta f = \Sigma \partial f/\partial \bar{z}^i_\theta \cdot d\bar{z}^i_\theta$. So at x, $\bar{\partial}_\theta f = \Sigma v_j \cdot f \otimes v^{j*} = \Sigma(\partial f/\partial \bar{z}^j + \theta^i_j \cdot \partial f/\partial z^i)v^{j*} = \Sigma(\partial f/\partial \bar{z}^j + \theta^i_j \cdot \partial f/\partial z^i)\phi^j = P_\theta(\bar{\partial} + \theta)f$. Since this is true at every x, the proposition for $p = 0$ is proved. If $p > 0$, let $\psi = \Sigma f_I \otimes d\bar{z}^I$, where I is a multi-index of length p. Then $P_\theta(\bar{\partial} + \theta)\psi = P_\theta(\Sigma(\bar{\partial} + \theta)f_I \wedge d\bar{z}^I) = \Sigma(P_\theta(\bar{\partial} + \theta)f_I) \wedge v^{*I}$. Now we claim that $\bar{\partial}_\theta v^{i*} = 0$. Indeed, locally around x, $\{v_i\}$ spans antiholomorphic vectors in T''^*M, and so it is enough to show that $dv^{i*}(v_j, v_k) = 0$. Now we know that

$$dv^{i*}(v_j, v_k) = v_j \cdot v^{i*}(v_k) - v_k \cdot v^{i*}(v_j) - v^{i*}([v_j, v_k]) \,, \tag{13}$$

so $dv^{i*}(v_j, v_k) = 0$ follows from (12) and (13). Thus $\bar{\partial}_\theta(P_\theta \psi) = \bar{\partial}_\theta(\Sigma f_I \otimes v^{*I}) = \Sigma \bar{\partial}_\theta f \wedge v^{*I}$. So the case for $p > 0$ follows from the case $p = 0$. □

Let E be a holomorphic vector bundle over M, and let D be a connection on E. The connection D depends only on the differentiable structure on E, and not on the holomorphic structure of E. We assume that D is integrable, i.e., $D'' = \bar{\partial}$. By type decomposition we write $D = D'_\theta + D''_\theta$. When $\theta \neq 0$, $D''_\theta \circ D''_\theta$ is not necessarily zero. But if $D''_\theta \circ D''_\theta = 0$, then D''_θ gives an holomorphic structure on E with respect to M_θ. Proposition 1.3 extends to $(0, p)$-forms with value in $\mathrm{End}(E)$ if we let θ act on $A^{0,*}(M, \mathrm{End}(E))$ by D_θ as defined in section (1.1). We state it as the following proposition:

Proposition 1.4 *If d_θ is defined by $\bar{\partial} + D_\theta$, then the following diagram commutes*

$$\begin{array}{ccc} A^{0,p}(M, \mathrm{End}(E)) & \xrightarrow{d^p_\theta} & A^{0,p+1}(M, \mathrm{End}(E)) \\ \downarrow{P_\theta} & & \downarrow{P_\theta} \\ A^{0,p}_\theta(M, \mathrm{End}(E)) & \xrightarrow{D''_\theta} & A^{0,p+1}_\theta(M, \mathrm{End}(E)) \,. \end{array} \tag{14}$$

Now every holomorphic structure on E_θ is given by an D''_θ operator satisfying the integrability condition $D''_\theta \circ D''_\theta = 0$. If we fix a connection D on E, then every such operator is given by a $\mathrm{End}(E)$ valued $(0, 1)$ form α_θ such that $(D''_\theta + \alpha_\theta) \circ (D''_\theta + \alpha_\theta) = 0$. Working out the integrability condition on α_θ, we get

$$D''_\theta \circ D''_\theta + D''_\theta \alpha_\theta + 1/2[\alpha_\theta, \alpha_\theta] = 0 \,. \tag{15}$$

We first compute $D''_\theta \circ D''_\theta$.

Lemma 1.5 *Let E be a holomorphic vector bundle over a complex manifold M, and let D be any connection on E. Assume D is hermitian. Let D''_θ be the $(0, 1)$-part of the connection with respect to M_θ, i.e., $D''_\theta = P^{(0,1)}_\theta \circ D$. Let R be the curvature class as defined in Sect. 1. Then $D''_\theta \circ D''_\theta = P_\theta(R\theta)$.*

Proof. Multiply by the curvature form Ω gives a map Ω: $A^*(M, E) \to A^{*+2}(M, E)$, and $\Omega = D \circ D$. On different complex structures θ, $D''_\theta \circ D''_\theta$ corresponds to multiplication by $P_\theta(\Omega)$. Since D is a hermitian connection on a holomorphic vector bundle E, we know that Ω is a $(1, 1)$ form. In local coordinates write $\Omega = \Sigma \Omega^i_{j\alpha,\bar\beta} dz^\alpha \wedge d\bar z^\beta$. We use the same notation as Proposition 1.3. We have $P_\theta(dz^\alpha) = \theta^\alpha_\beta v^{\beta*}$, and $P_\theta(d\bar z^\alpha) = v^{\alpha*}$. Thus $P_\theta(\Omega) = \Sigma\, \Omega^i_{j\alpha,\bar\beta} \theta^{\bar\beta}_\gamma v^{\alpha*} \wedge v^{\gamma*} = P_\theta(R\theta)$. □

Now we are ready to prove the theorem relating the differential graded Lie algebra L^* constructed last section and the integrability condition on α_θ.

Theorem 1.6 *Let E be a holomorphic vector bundle over a complex manifold M, and let D be any hermitian connection on E. Suppose θ is any TM valued $(0, 1)$-form representing a complex structure on M. Let α be an* End(E) *valued $(0, 1)$-form. Then $D''_\theta + P_\theta\alpha$ represents a holomorphic structure on E over M_θ if and only if (θ, α) satisfies the deformation equation in L^*.*

Proof. We first show that Lie bracket on $A^{0,*}(M, \mathrm{End}(E))$ commutes with P_θ.

Lemma 1.7 *If a, $a' \in A^{0,*}(M, \mathrm{End}(E))$, then $P_\theta[a, a'] = [P_\theta a, P_\theta a']$.*

Proof. The bracket operation in $A^{0,*}(M, \mathrm{End}(E))$, being induced by matrix multiplication and exterior multiplication, is algebraic, i.e., does not involve differentiation. Likewise the projection P_θ is also algebraic. Thus it is enough to check the lemma pointwise, which is obvious. □

We return to the proof of the theorem. From formula (15), Lemma 1.5 and Lemma 1.7, $D''_\theta + P_\theta\alpha$ represents a holomorphic structure on E over M_θ if and only if

(16) $$R\theta + (\bar\partial + D_\theta)\alpha + 1/2[\alpha, \alpha] = 0 \quad \text{and}$$

(17) $$\bar\partial\theta + 1/2[\theta, \theta] = 0\,.$$

Now the deformation equation in L^* is

(18) $$\bar\partial_R(\alpha, \theta) + 1/2[(\alpha, \theta), (\alpha, \theta)] = 0\,.$$

Working out the equation using the definition of the differential graded Lie algebra L^*, we see that (18) is equivalent to

(19) $$(\bar\partial\alpha + R\theta, \bar\partial\theta) + 1/2([\alpha, \alpha] + 2D_\theta\alpha, [\theta, \theta]) = 0\,.$$

Separating the two components, we get (16) and (17) respectively. □

Now in order to repeat the Kuranishi construction on L^* to construct a complete joint deformation space, it is necessary to do harmonic theory on L^*, in particular, to prove that $H^1(L^*)$ is finite dimensional. This is provided by the observation in the first section that L^* is $A^{0,*}(M, \mathscr{C})$ as a complex and so we can do the harmonic theory on L^* in the same way as the harmonic theory on $A^{0,*}(M, \mathscr{C})$. Let $\mathscr{M}$ be the Kuranishi space constructed from the complex L^*. We must understand what completeness property the space $\mathscr{M}$ has.

To begin with, we observe that L^0 is a Lie algebra, being the semidirect product of $A^0(M, TM)$ and $A^0(M, \mathrm{End}(E))$. Let G be the Lie group with Lie algebra L^0. In fact, G is the group of C^∞ diffeomorphisms of the total space of E that is fibre preserving and is linear on each fiber. Such diffeomorphisms may not induce identity map on M. The group G acts on the solution of the deformation equation, and thus on the space of joint deformations. A one parameter subgroup of G acts on L^1 by

$$\exp(t\lambda): \alpha \to \exp(t\, ad\, \lambda)(\alpha) + \frac{I - \exp(t\, ad\, \lambda)}{ad\, \lambda}(d\lambda) \tag{20}$$

in terms of power series [see GM1]. Roughly speaking, the Kuranishi space is the set of joint deformations modulo the action of G. More precisely, we have the following theorem, whose proof is exactly the same as the completeness of the Kuranishi space. (See [Ku2] or [GM2])

Theorem 1.8 *Let E be a holomorphic vector bundle over a compact complex manifold M. Then there is a complex space $\mathscr{M}$ parametrizing joint deformations of (E, M), such that if (S, s_0) is another complex space parametrizing joint deformations of (E, M), with $s_0 = (E, M)$, then there is a neighborhood of s_0, denoted by S', and holomorphic maps f: $S' \to \mathscr{M}$, and g: $S' \to G$, such that the family restricted to S' is equal to $(f^*\mathscr{M})^g$, where f^* denotes the pull back family, and $(\ )^g$ denote the action by G.*

Now let $\mathscr{M}_M$ be the Kuranishi space of complex structures on M. Then the forgetful map from L^1 to $A^{0,1}(M, TM)$ given by $(a, \theta) \to \theta$ induces a map π: $\mathscr{M} \to \mathscr{M}_M$. The fiber consists of vector bundles over a fixed complex structure θ on M. However, the fiber is not the moduli of vector bundles over θ, but the moduli of vector bundles over θ modulo the pull back action of the automorphisms of M_θ. This can be seen by looking at the tangent space to the fiber, which is identified with $H^1(M, \mathrm{End}(E))/R \wedge H^0(M, TM)$, where $R \wedge$: $H^0(M, TM) \to H^1(M, \mathrm{End}(E))$ is the coboundary map associated with the exact sequence

$$0 \to \mathrm{End}(E) \to \mathscr{C} \to TM \to 0\,.$$

It is not hard to see that $R \wedge H^0(M, TM)$ gives the infinitesimal action of the pull back action of the holomorphic automorphisms of M on vector bundles over M at the point E. Ideally one would like to have as fibers the moduli of vector bundles over M. However, as we explained in the introduction that automorphisms of M come into play in an essential way, so that it is difficult to have as fibers the moduli of vector bundles over M_θ without the action of the automorphism group.

2 The construction of global joint moduli

(2.0) Introduction

In the preceding section we studied complete local joint deformations, and in this section we patch these local deformation space together to form a joint moduli. In other words we seek conditions on complex manifolds such that the set $\mathscr{S}$ of holomorphic vector bundles E over a compact complex manifold M modulo equivalence has a structure of a complex space. Let M_0 be a real analytic manifold and E_0 be a C^∞ vector bundle on M_0, and let M be a complex manifold diffeomorphic to M_0. Assume M has ample canonical class Φ and let E be a Φ-stable holomorphic vector bundle on M, C^∞ isomorphic to E_0. Let $\mathscr{S} = \{(M, E)$: M has ample canonical class Φ and E is a Φ-stable holomorphic vector bundle on $M\}$ modulo equivalence. We prove in this section that $\mathscr{S}$ has a structure of a Hausdorff complex space. The method we use follows Narasimhan-Simha[NS] in their construction of moduli space of manifolds with ample canonical class. The joint deformation spaces glue together because of the universal properties of the joint deformation space in this case.

We first review some of the results in Narasimhan-Simha. Let M be a compact complex manifold of dimension n, and let K be the canonical bundle of M. Assume K is ample, i.e., K^m gives an embedding of M in some projective space for $m \gg 0$. Narasimhan-Simha proved the following separation theorem on complex structures on M.

Theorem 2.1 [NS] *Suppose S and T are two complex spaces parametrizing complex structures of M. Let s_0, t_0 be two base points of S and T such that M_{s_0} and M_{t_0} have ample canonical bundles. Let $\{s_n\}$ and $\{t_n\}$ be two sequences in S and T converging to s_0 and t_0 respectively. Assume there are holomorphic isomorphisms ϕ_n: $M_{s_n} \to M_{t_n}$. Then ϕ_n converges uniformly to a holomorphic isomorphism ϕ: $M_{s_0} \to M_{t_0}$.*

The proof of the theorem depends on the construction of a generalisation of the Bergman metric on M. Under this metric, holomorphic isomorphisms become isometries. Furthermore they proved that this metric depends continuously on the parameter space of complex structures of M. Hence the sequence of holomorphic isomorphisms ϕ_n are equicontinuous, from which a uniformly convergent subsequence can be extracted. By Montel's theorem the limit isomorphism is holomorphic. The existence of such a metric also shows that the group of automorphisms of M is finite because it is compact and discrete. The discreteness is the result of ample canonical bundle, and compactness follows from the fact that it is a closed subgroup of the group of isometries.

Because Aut(M) is discrete, Narasimhan-Simha proved the following theorem relating the action of Aut(M) on the Kuranishi space of M.

Theorem 2.2 [NS] *Let K be the Kuranishi space of M with $K_{t_0} = M$. By restricting to smaller neighborhood of t_0, we have that for t, $t' \in K$, M_t is isomorphic to $M_{t'}$ if and only if t, t' are in the same orbit of* Aut(M).

Next we discuss some results on stable bundles that we need. Let E be a holomorphic vector bundle over M. We say that E is *stable* with respect to the ample divisor Φ if for every subsheaf F of E with $0 < \text{rank}(F) < \text{rank}(E)$,

$$\frac{c_1(F)\cdot\Phi^{n-1}}{\text{rank}(F)} < \frac{c_1(E)\cdot\Phi^{n-1}}{\text{rank}(E)}$$

The expression $c_1(F)\cdot\Phi^{n-1}/\text{rank}(F)$ is denoted by $\mu(F)$. In what follows stability is always with respect to the anticanonical class. A consequence of the above definition is the following proposition.

Proposition 2.3 *Let E and F be stable vector bundles over M. Let f: $E \to F$ be a sheaf homomorphism. If $\mu(E) = \mu(F)$, then f is an isomorphism unless $f = 0$.*

It is also well known that the moduli of stable bundles is open in the moduli of vector bundles. So if E is stable over M, we may assume that a small joint deformation E_θ is stable over M_θ.

Finally we recall that (E, M) is isomorphic to (E', M') if there is a C^∞ diffeomorphism f: $M \to M'$ and a C^∞ isomorphism of vector bundles $g: E \to f^*E'$, such that with respect to the holomorphic structures on M, M', E, and E', the map f and g are holomorphic. That is, we do not distinguish a vector bundle and its pull back by an automorphism of the base space.

(2.1) A separation theorem for joint deformations

From now on we always assume that M has ample canonical class, and E is stable with respect to the ample canonical divisor. We prove a separation theorem similar to Theorem 2.1.

Theorem 2.4 *Suppose S and T are two complex spaces parametrizing joint deformations of E over M. Let s_0, t_0 be two base points of S and T such that there are two sequences $\{s_n\}$ and $\{t_n\}$ in S and T converging to s_0 and t_0 respectively. Assume that there are holomorphic isomorphisms ϕ_n: $(E_{s_n}, M_{s_n}) \to (E_{t_n}, M_{t_n})$ for all n. Then (E_{s_0}, M_{s_0}) and (E_{t_0}, M_{t_0}) are isomorphic.*

Proof. The proof is a simple combination of Theorem 2.1 and the proof that the moduli of stable bundles is Hausdorff. We recall that if (E_{s_n}, M_{s_n}) and (E_{t_n}, M_{t_n}) are isomorphic, then there are holomorphic isomorphisms f_n: $M_{s_n} \to M_{t_n}$, and g_n: $f_n^* E_{t_n} \to E_{s_n}$. By Theorem 2.1 there is a holomorphic isomorphism f: $M_{s_0} \to M_{t_0}$ of the complex manifolds. So it remains to show that $f^*E_{t_0}$ and E_{s_0} are isomorphic. Following an argument of Okonek, this is a consequence of upper-semicontinuity of cohomology. The isomorphism $g_n: f_n^* E_{t_n} \to E_{s_n}$ gives a nonzero element in $H^0(M, \mathrm{Hom}(f_n^* E_{t_n}, E_{s_n}))$. So we have

$$\dim H^0(M, \mathrm{Hom}(f_n^* E_{t_0}, E_{s_0})) \geqq \limsup \dim H^0(M, \mathrm{Hom}(f_n^* E_{t_n}, E_{s_n})) \geqq 1 .$$

Let g be an nonzero element in $H^0(M, \mathrm{Hom}(f_n^* E_{t_0}, E_{s_0}))$, then g gives an isomorphism between $f^*E_{t_0}$ and E_{s_0} by Proposition 2.3. This completes the proof. □

(2.2) The universal properties of the joint deformation space

Let $\mathscr{M}$ be the joint deformation space for (E, M) constructed in Sect. 1. The local completeness property of the joint deformation space is that any family of joint deformations of (E, M) is locally induced from an analytic map into $\mathscr{M}$. The universal property of $\mathscr{M}$ is that this map is unique. Kuranishi proved the following sufficient condition for universality:

Theorem 2.5 *If $H^0(L^*) = 0$, then $\mathscr{M}$ is universal.*

Under the assumption that M has ample canonical class and E is stable, we have $H^0(L^*) = \mathbb{C}$, because it is part of an exact sequence

$$0 \to H^0(M, \mathrm{End}(E)) \to H^0(L^*) \to H^0(M, TM) .$$

From this it follows that $\mathscr{M}$ is universal.

Now let $\mathrm{Aut}(E, M)$ be the subgroup of $\mathrm{Aut}(M)$ that fixes E. We now need to consider the action of $\mathrm{Aut}(E, M)$ on $\mathscr{M}$. The following proposition is analogous to Theorem 2.2.

Proposition 2.6 *Let $\mathscr{M}$ be the joint deformation space of (E, M) with $\mathscr{M}_{t_0} = (E, M)$. Let t, t' be two points of $\mathscr{M}$. By restricting to a small neighborhood of t_0, (E_t, M_t) is isomorphic to $(E_{t'}, M_{t'})$ if and only if t, t' are in the same orbit of* $\mathrm{Aut}(E, M)$.

Proof. If (E_t, M_t) is isomorphic to $(E_{t'}, M_{t'})$, then M_t must be isomorphic to $M_{t'}$. Thus by applying Theorem 2.2 we see that by restricting to a smaller neighborhood of t_0, we may assume that there is a holomorphic automorphism ϕ in $\mathrm{Aut}(M)$ such that ϕ gives an isomorphism of M_t and $M_{t'}$. Now since $\mathrm{Aut}(M)$ is finite, and the action of $\mathrm{Aut}(M)$ on $\mathscr{M}$ is continuous, by restricting $\mathscr{M}$ to an even smaller neighborhood of t_0, we may assume that only $\mathrm{Aut}(E, M)$ maps $\mathscr{M}$ into itself. Hence ϕ must be in $\mathrm{Aut}(E, M)$. The converse implication is clear. □

(2.3) The construction of joint moduli

Fix an (E, M) in $\mathscr{S}$, and let $\mathscr{M}$ be the joint deformation space of (E, M) such that the conclusion of Proposition 2.6 holds. Let ϕ: $\mathscr{M}/\mathrm{Aut}(E, M) \to \mathscr{S}$ be the natural map, sending E_t over M_t into its isomorphism class in $\mathcal{S}$. The Proposition 2.6 says that this map ϕ is injective. Thus it gives a local coordinate chart on $\mathcal{S}$ because $\mathcal{M}/\mathrm{Aut}(E, M)$ has an analytic space structure [Ca]. Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are two joint deformations around (E_1, M_1) and (E_2, M_2) respectively, and let $U = \phi_1(\mathcal{M}_1) \cap \phi_2(\mathcal{M}_2)$. If U is nonempty, we need to show that $\phi_i^{-1}(U)$ is open for $i = 1, 2$, and that the transition map is holomorphic.

Let $p \in \mathcal{M}_1$ and $q \in \mathcal{M}_2$ such that (E_p, M_p) and (E_q, M_q) are isomorphic. Then by the universal property of $\mathcal{M}_1$ and $\mathcal{M}_2$, there exist neighborhoods W_1 and W_2 of p and q such that W_1 is biholomorphic to W_2. By shrinking W_i if necessary, we may assume that they are $\mathrm{Aut}(E_i, M_i)$-invariant for $i = 1, 2$. Since points in W_i define the same points in $\mathcal{S}$ if and only if they are in the same orbit under $\mathrm{Aut}(E_i, M_i)$, there is a bijective map

$$\phi_2 \circ \phi_1^{-1}\colon W_1/\mathrm{Aut}(E_1, M_1) \to W_2/\mathrm{Aut}(E_2, M_2)\,.$$

$W_i/\mathrm{Aut}(E_i, M_i)$ is open since W_i is open, and $\phi_2 \circ \phi_1^{-1}$ is holomorphic because it is induced from a holomorphic isomorphism of W_1 onto W_2. Moreover $\mathcal{S}$ is Hausdorff by virtue of the separation Theorem 2.4. So we have proved

Theorem 2.7 *The joint moduli space $\mathcal{S}$ has naturally a structure of a Hausdorff complex space.*

If we apply this construction to curves of genus greater than 1, then we obtain the joint moduli of stable bundles over the moduli of curves.

Corollary 2.8 *If M is a smooth Riemann surface of genus $g > 1$, then there exists a joint moduli of stable vector bundles over the moduli of Riemann surfaces of genus g, such that each fiber consists of the moduli of stable bundles modulo the action of the automorphism group of the base space.*

3 The existence of holomorphic structures under small deformation

(3.1) The integrability condition and some consequences

When the complex structure M is varied, the existence of holomorphic structures on E can be expressed as an integrability condition. For convenience, we use a slightly different notation for the integrability condition here. We begin with the proposition proved in [Ko]

Proposition 3.1 *Let E be a C^∞ complex vector bundle over a complex manifold. If D is a connection in E such that $D'' \circ D'' = 0$, then there is a unique holomorphic vector bundle structure in E such that $D'' = \bar{\partial}$. Conversely, if E has a holomorphic structure, then there is a connection D such that $D'' = \bar{\partial}$.*

Now let E be a holomorphic vector bundle over a compact complex manifold M, and let D be a connection on E such that $D'' = \bar{\partial}$. Then any other connection in E can be expressed as $D + \alpha$, for $\alpha \varepsilon A^1(M, \mathrm{End}(E))$. By Proposition 3.1 E has a holomorphic structure with respect to M_θ if and only if

$$(D + \alpha)''_\theta \circ (D + \alpha)''_\theta = 0 \,. \tag{1}$$

Now $(D + \alpha)''_\theta = D''_\theta + \alpha''_\theta$, and $D''_\theta \circ D''_\theta = R_\theta^{0,2}$. This equation can be written as

$$\begin{aligned} 0 &= (D + \alpha)''_\theta \circ (D + \alpha)''_\theta \\ &= D''_\theta \circ D''_\theta + D''_\theta \circ \alpha''_\theta + \alpha''_\theta \circ D''_\theta + \alpha''_\theta \wedge \alpha''_\theta \\ &= R_\theta^{0,2} + D''_\theta \alpha''_\theta + 1/2[\alpha''_\theta, \alpha''_\theta] \,. \end{aligned}$$

We note that for $\alpha, \beta \varepsilon A^1(M, \mathrm{End}(E))$, $[\alpha, \beta] = [\beta, \alpha]$. Thus we have

Proposition 3.2 *Let E be a holomorphic vector bundle over a compact complex manifold M. Let D be a connection such that $D'' = \bar{\partial}$. Then E admits a holomorphic structure over M_θ if and only if*

$$R_\theta^{0,2} + D''_\theta \alpha''_\theta + 1/2[\alpha''_\theta, \alpha''_\theta] = 0 \tag{2}$$

has a solution for $\alpha''_\theta \varepsilon A^{0,1}(M_\theta, \mathrm{End}(E))$.

We now draw several simple consequences of Proposition 3.2:

Suppose E is a rank r projectively flat vector bundle over a compact Kähler manifold M, and let D be a projectively flat connection on E. Let $R = D \circ D$ be its curvature form. Then $R = \omega I_E$ for a complex 2-form ω on M. Take trace in (2), and observe that $[\alpha''_\theta, \alpha''_\theta]$ has zero trace. Then we have $r\omega_\theta^{0,2} = \bar{\partial}_\theta \alpha''_\theta$ for some $\alpha''_\theta \varepsilon A^{0,1}(M_\theta, \mathrm{End}(E))$. Thus if $c_1(E)$ is a $(1, 1)$ class with respect to θ, then $\omega_\theta^{0,2}$ is exact in $A^{0,2}(M_\theta, \mathrm{End}(E))$. Thus (2) is solvable. So we have

Proposition 3.3 *Let E be a projectively flat bundle over a compact Kähler manifold M. Then E has a holomorphic structure with respect to M_θ if and only if $c_1(E)$ is a $(1, 1)$ class in $H^2(M_\theta, \mathbf{C})$.*

Corollary 3.4 *If E is a line bundle over a compact Kähler manifold M, then E has a holomorphic structure with respect to M_θ if and only if $c_1(E)$ is a $(1, 1)$ class in $H^2(M_\theta, \mathbf{C})$.*

Proof. Every line bundle is projectively flat. □

Suppose now that M is a compact Riemann surface, and that E is a stable rank r bundle. Then the integrability condition vanishes for dimensional reasons. Thus we have

Proposition 3.5 *Let E be a stable bundle over a compact Riemann surface. Then E always admits a holomorphic structure with respect to M_θ.*

(3.2) Kähler surfaces with trivial canonical bundle

In this section we solve the integrability condition (2) for Kähler surfaces with trivial canonical bundle. This includes K3 surfaces and Kähler tori. Let M be such a surface, and E a simple holomorphic bundle over M. Let h be a hermitian metric on E, and let D be its hermitian connection. Then on M_θ, we have $D = D'_\theta + D''_\theta$. Let ω_θ be a C^∞ family of metrics on M_θ, and we can define inner products on forms with values in End(E) in the usual manner. We let δ_θ, δ'_θ, δ''_θ be the corresponding adjoint operators, and we have $\delta_\theta = \delta'_\theta + \delta''_\theta$. Define the Laplacian operator $\Delta''_\theta = \delta''_\theta D''_\theta + D''_\theta \delta''_\theta$. We now verify that the operators Δ''_θ are elliptic.

Proposition 3.6 *For each θ, Δ''_θ is a self-adjoint elliptic operator.*

Proof. Since these operators are local, it suffices to check on a local trivializing neighborhood in M_θ. Let U be such a neighborhood in M, and let $\{s_i\}$ be local frames on E. Then $D''_\theta = \bar\partial_\theta + \omega$, where ω is the connection form with respect to the frame $\{s_i\}$. Then the adjoint $\delta''_\theta = \bar\partial^*_\theta + \omega^*$, where ω^* is a 0th order operator consisting of taking interior products. Hence it is clear that the principal part of Δ''_θ is the same as that of $\Box$, the complex Laplace-Beltram operator. Thus it is elliptic. The self-adjointness follows from the same formal computation as the operator $\Box$.

We now summarize results on elliptic operators in the theorem below

Theorem 3.7 *Let Δ''_θ be the second order operator constructed above. Then*

1) *The Kernel of Δ''_θ is finite dimensional.*

2) *Let* $\mathbf{H}_\theta$: $A^{0,p}(M_\theta, \mathrm{End}(E)) \to \mathrm{Ker}\, \Delta''_\theta$ *be the projection. Then there exists a unique linear map* G_θ: $A^{0,p}(M_\theta, \mathrm{End}(E)) \to A^{0,p}(M_\theta, \mathrm{End}(E))$ *of order* 2, *called the Green's operator such that* $\mathrm{Ker}\, G_\theta = \mathrm{Ker}\, \Delta''_\theta$, *and*

a) $G \circ \Delta''_\theta = \Delta''_\theta \circ G$,

b) $u = \mathbf{H}_\theta u + \Delta''_\theta G u$ *for* $u \in A^{0,p}(M_\theta, \mathrm{End}(E))$.

c) $\| Gu \|_{k+2} \leqq c \| u \|_k$, $\| Qu \|_{k+1} \leqq c \| u \|_k$, *where* $Q = \delta''_\theta \circ G$, *for all* $u \in A^{0,p}(M_\theta, \mathrm{End}(E))$.

Remark 3.8 The operator D''_θ does not satisfy $D''_\theta \circ D''_\theta = 0$, unless $\theta = 0$. So there is no cohomological interpretation of $\mathrm{Ker}\, \Delta''_\theta$ as harmonic representatives of some cohomology class. Nevertheless, since Δ''_θ is elliptic, we still have the finite dimensionality of $\mathrm{Ker}\, \Delta''_\theta$, and the construction of Green's operator as above. However G_θ does not necessarily commute with D''_θ or with δ''_θ, and $\mathbf{H}_\theta \bar\partial \xi$ is not necessarily zero.

We now consider Δ''_θ as a C^∞ family of elliptic operators on the C^∞ family of vector bundles $T^{0,p}_\theta(\mathrm{End}(E))$. We then have the following upper-semicontinuity theorem:

Theorem 3.9 Dim $\mathrm{Ker}\, \Delta''_\theta$ *is upper-semicontinuous in θ.*

We now apply these tools to the case that M is a Kähler surface with trivial canonical bundle, and E is a simple holomorphic bundle. First, we have a trace map tr: $A^0(\mathrm{End}(E)) \to A^0(\mathbf{C})$ by taking trace of a matrix pointwise. This then extends to a map tr: $A^{0,p}_\theta(\mathrm{End}(E)) \to A^{0,p}_\theta(\mathbf{C})$. We call an element $\alpha \in A^{0,p}_\theta(\mathrm{End}(E))$ trace-free if $\mathrm{tr}(\alpha) = 0$. We have

Proposition 3.10 *Under the above assumptions on M and E, let α be a trace-free element in $A_\theta^{0,2}(\mathrm{End}(E))$. Then $\alpha = D_\theta''\beta$ for some $\beta \varepsilon A_\theta^{0,1}(\mathrm{End}(E))$.*

Proof. Let $\mathrm{End}^0(E)$ be the bundle of trace-free endomorphisms of E. Then by Serre duality $H^2(M, \mathrm{End}^0(E)) \simeq H^0(M, \mathrm{End}^0(E)) = 0$. By upper-semicontinuity (Theorem 3.9), $\mathrm{Ker}\, \Delta_\theta'' = 0$ for θ small enough. So if α is trace-free,

$$\alpha = \mathbf{H}\alpha + \Delta_\theta'' G_\theta \alpha = 0 + D_\theta''(\delta_\theta'' G_\theta \alpha) + \delta_\theta''(D_\theta'' G_\theta \alpha) = D_\theta''(\delta_\theta'' G_\theta \alpha),$$

since $D_\theta'' G_\theta \alpha = 0$ for dimensional reasons. So we just take β to be $\delta_\theta'' G_\theta \alpha$. □

Remark The above proposition is not necessarily true if M is not a surface, since the term $D_\theta'' G_\theta \alpha$ may not be zero.

We now return to equation (2): $R_\theta^{0,2} + D_\theta''\alpha_\theta'' + 1/2[\alpha_\theta'', \alpha_\theta''] = 0$. Since $[\alpha_\theta'', \alpha_\theta'']$ is trace-free, we have $[\alpha_\theta'', \alpha_\theta''] = D_\theta''(\delta_\theta'' G_\theta[\alpha_\theta'', \alpha_\theta''])$. Thus if (2) has a solution, then $R_\theta^{0,2} = -D_\theta''(\alpha_\theta'' + 1/2\delta_\theta'' G_\theta[\alpha_\theta'', \alpha_\theta''])$. So $R_\theta^{0,2}$ is D_θ''-exact. Conversely, we prove

Theorem 3.1 *Under the assumptions on M and E, if $R_\theta^{0,2} = D_\theta''\phi$ for some $\phi \varepsilon A_\theta^{0,1}(\mathrm{End}(E))$, and if θ is small enough, then equation (2) can be solved for some $\alpha_\theta'' \varepsilon A_\theta^{0,2}(\mathrm{End}(E))$.*

Proof. We form a sequence $a_0, a_1, \ldots$ as follows:

$$a_0 = 0$$

$$a_1 = -\phi$$

$$a_i = -\phi - 1/2 Q_\theta[a_{i-1}, a_{i-1}],$$

where we put $Q_\theta = \delta_\theta'' G_\theta$. We now show that a_i is bounded in $\| \, \|_2$. We prove by induction that $\|a_i - a_{i-1}\|_2 \leqq L\|\phi\|_2/i^2$, if $\|\phi\|_2$ is small enough and $L = \sum_{k=1}^\infty (1/k)^2$. We have

$$\begin{aligned}
\|a_i - a_{i-1}\|_2 &= 1/2\,\|Q_\theta([a_{i-1}, a_{i-1}] - [a_{i-2}, a_{i-2}])\|_2 \\
&= \|Q_\theta([a_{i-1} - a_{i-2}, a_{i-2}] + 1/2[a_{i-1} - a_{i-2}, a_{i-1} - a_{i-2}])\|_2 \\
&\leqq c_\theta \|[a_{i-1} - a_{i-2}, a_{i-2}] \\
&\qquad + 1/2[a_{i-1} - a_{i-2}, a_{i-1} - a_{i-2}]\|_1 \quad \text{by Theorem 3.7(c)} \\
&\leqq c_\theta \|a_{i-1} - a_{i-2}\|_1 \Big(\sum_{k=1,\ldots,i-1} \|a_k - a_{k-1}\|_1 \Big) \\
&\leqq c_\theta \|a_{i-1} - a_{i-2}\|_2 \Big(\sum_{k=1,\ldots,i-1} \|a_k - a_{k-1}\|_2 \Big) \\
&\leqq c_\theta (L \cdot \|\phi\|_2 \cdot 1/(i-1)^2)(L \cdot \|\phi\|_2 \cdot L) \\
&\leqq L \cdot \|\phi\|_k \cdot 1/i^2 \cdot (c_\theta \cdot L^2 \cdot \|\phi\|_2 \cdot i^2/(i-1)^2) \\
&\leqq L \cdot \|\phi\|_k \cdot 1/i^2, \text{ if } c_\theta \cdot L^2 \cdot \|\phi\|_2 \cdot 4 \leqq 1\,.
\end{aligned}$$

It is enough to show that $c_\theta \cdot L^2 \cdot \|\phi\|_2 \cdot 4$ can be made arbitrarily small if θ is small enough. First we observe that c_θ is bounded for θ small, and that ϕ can be taken to be $Q_\theta R_\theta^{0,2}$. Now as $\theta \to 0$, $\|R_\theta^{0,2}\|_2 \to 0$. Thus $c_\theta \cdot L^2 \cdot \|\phi\|_2 \cdot 4 < 1$ if θ is small.

Now $\|a_i\|_2 \leqq \sum_{k=1}^{i} \|a_k - a_{k-1}\|_2 \leqq L^2 \cdot \|\phi\|_2$, which is bounded. So by the principle of uniform boundedness in a Hilbert space, a_k contains a weakly convergence subsequence, which converges to some α. Thus (2) has a weak solution. Now by the standard bootstrap argument since Q_θ is a smoothing operator, α is C^∞. The theorem is proved. □

We now decompose the curvature tensor $R_\theta^{0,2}$ into its trace part and trace-free part. Since the trace-free part is always D_θ''-exact, in order for $R_\theta^{0,2}$ to be D_θ''-exact, it is enough that the trace part, i.e., the first Chern class of E, is D_θ''-exact. Thus it is sufficient that $c_1(E)$ is a (1, 1) class in M_θ. We have the following theorem

Theorem 3.11 *If M is a compact Kähler surface with trivial canonical bundle and E is a simple vector bundle on M, then E admits a holomorphic structure on M_θ if and only if $c_1(E)$ is a (1, 1) class in M_θ.*

Now if we remove the assumption on M and require that $H^2(\mathrm{End}^0(E)) = 0$, then the proof goes through without any change. So we have

Theorem 3.12 *Suppose M is a compact Kähler surface, and E a simple holomorphic vector bundle on M, and suppose $H^2(\mathrm{End}^0(E)) = 0$. Then E admits a holomorphic structure on M_θ if and only if $c_1(E)$ is a (1, 1) class in M_θ.*

References

[Ca] Cartan H.: Quotient d'un espace analytique par un groupe d'automorphismes, Algebraic geometry and topology, pp 90–102, Prince U.P. 1057

[GH] Griffiths P. and Harris: Principles of Algebraic Geometry. John Wiley & Sons, New York, Toronto, 1978

[GM1] Goldman William M. and Millson John J.: The deformation theory of representations of fundamental groups of compact Kähler manifolds, Publ. Math. I.H.E.S., vol. 67 (1988), 43–96

[GM2] Goldman William M. and Millson John J.: The homotopy invariance of the Kuranishi space, Illinois Journal of Math, vol. 34 (1990) 337–367

[Hi] Hirzebruch F.: Topological Methods in Algebraic Geometry, Springer, Berlin-New York-Heidelberg, 1966

[Ki] Kim H.J.: Curvatures and holomorphic vector bundles, Ph.D. thesis, Berkeley, 1985

[Ko] Kobayashi S.: Differential Geometry of Complex Vector Bundles. Princeton University Press, Princeton, 1987

[Kod] Kodaira K.: Complex Manifolds and Deformation of complex Structures. Springer-Verlag, New York, 1986

[KS] Kodaira K. and Spencer D.C.: On deformations of complex analytic structures I, II, III, Ann. of Math. 67 (1958) 328–466, 71 (1960) 43–76

[Ku1] Kuranishi M.: On the locally complete families of complex structures, Ann. Math. 75 (1962) 536–577

[Ku2] Kuranishi M.: deformations of compact complex manifolds. Les Presses de L'universite de Montreal, Montreal, 1971

[NSe] Narasimhan M.S. and Seshadri C.S.: Stable and unitary vector bundles on a compact Riemann surface, Ann. Math. **82** (1965) 540–567

[NS] Narasimhan M.S. and Simha R.R.: Manifolds with ample canonical class, Invent Math. **5** (1968) 120–128

[NR] Nijenhuis, Albert and Richardson R.W. Jr.: Cohomology and deformations in graded Lie algebras, Bull. Amer. Math. Soc., vol. **72** (966), 1–29