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On the Shafarevich and Tate conjectures for hyperkähler varieties

Yves André

Institut de Mathématiques, Case 247, Tour 46-56, 5ème étage, 4, place Jussieu, F-75252 Paris 5, France (e-mail: andre@mathp6.jussieu.fr)

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Contents

1 Problems and results

1.1

In this paper, we are interested in arithmetico-algebraic properties of certain classes of projective varieties, the prototype of which is the class of K3 surfaces (i.e. simply-connected projective smooth surfaces Y such that $H^0(\Omega^2_Y)$ is one-dimensional and generated by a differential form ω which is non-degenerate at every point).

For K3 surfaces, the problems we are about to examine may be stated as follows:

1.1.1) The Shafarevich problem: are there only finitely many polarized K3 surfaces of fixed degree d over a number field K, with good reduction outside a fixed finite set of primes $\{\wp_1, \wp_2, \ldots, \wp_n\}$?

1.1.2) Describe the motive of a K3 surface, and compute the motivic Galois group.

1.1.3) The Tate problem: let Y and Y' be K3 surfaces defined over a number field K. Is any isomorphism of $\mathrm{Gal}(\overline{K}/K)$-modules $H^*_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell) \to H^*_{\mathrm{et}}(Y'_{\overline{K}}, \mathbb{Q}_\ell)$ induced by a $\mathbb{Q}_\ell$-linear combination of algebraic cycles on $Y \times Y'$? Is the image of $\mathrm{Gal}(\overline{K}/K)$ in $GL\,H^*_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)$ as big as possible, i.e. an open Lie subgroup of the motivic Galois group over $\mathbb{Q}_\ell$ (cf. [S94])?

1.2

We shall tackle these problems in the broader context of *hyperkähler varieties* (where problems 1.1.1 and 1.1.3 have been explicitly posed by A. Todorov [T90]). We recall that an even-dimensional, simply-connected, smooth projective variety Y is said to be hyperkähler (or else "irreducible symplectic" [Be83a]) if $H^0(\Omega^2_Y)$ is one-dimensional and generated by a form ω which is non-degenerate at every point.

Let us set out, following A. Beauville (loc. cit. Sect. 6, 7), some simple constructions of hyperkähler varieties Y in any dimension $2r \geqq 2$:

i) for any K3 surface S, take $Y = S^{[r]}$ the punctual Hilbert scheme which parametrizes finite closed subschemes of S of length r; thus for $r = 1$, $Y = S$;
ii) for any abelian surface A, form in the same way $A^{[r+1]}$ and take $Y = K_r$:= the fibre above 0 of the "summation" morphism $A^{[r+1]} \to A$; thus for $r = 1$, Y is the Kummer surface of A;
iii) any projective deformation Y of a hyperkähler variety of type $S^{[r]}$ or K_r.
We shall call these varieties "*of K3 type*". For instance, Beauville and R. Donagi have shown that the variety of lines of the cubic fourfold is of K3 type [BeD85]. Some modular varieties for stable vector bundles on a K3 surface are also of K3 type [Mu84a]. In fact, it seems to be unknown whether there are hyperkähler varieties not of K3 type. We mention at last that varieties of K3 type carry a rich geometry of lagrangian subvarieties [V92].

1.3

By a *polarized* (resp. *very polarized*) variety of degree d over some field K, we mean a variety endowed with a K-rational ample (resp. very ample) numerical equivalence class of line bundles of degree d. We say that a polarized variety has good reduction at some place of K if a smooth *polarized* model exists (cf. 9.1).

Theorem 1.3.1. *Let $\wp_1, \wp_2, \dots, \wp_n$ be primes of a number field K, and let r and d be positive integers. Then there exist only finitely many isomorphy classes of polarized K3 surfaces (resp. of very polarized hyperkähler varieties of dimension $2r$ with second Betti number > 3) of degree d, with good reduction outside $\wp_1, \wp_2, \dots, \wp_n$.*

This gives a positive answer to 1.1.1. By using a result of C. Jordan in reduction theory, one deduces (Sect. 9.7):

Corollary 1.3.2. *For any positive integer m, there are only finitely many orbits for* $PGL\left(4, \mathbb{Z}\left[\frac{1}{2m}\right]\right)$ *among all smooth quartics in the 3-dimensional projective space over* $\mathbb{Z}\left[\frac{1}{2m}\right]$.

In a similar way, using the cohomological interplay between cubic fourfolds and their varieties of lines, we prove (Sect. 9.6, 9.7):

Corollary 1.3.3. *For any positive integer m, there are only finitely many orbits for* $PGL\left(6, \mathbb{Z}\left[\frac{1}{3m}\right]\right)$ *among all smooth cubics in the 5-dimensional projective space over* $\mathbb{Z}\left[\frac{1}{3m}\right]$.

1.4

Let (Y, η) be a polarized variety defined over a subfield K of $\mathbb{C}$, and let k be a positive integer $\leqq \dim Y$. Identifying $\eta_{\mathbb{C}}$ with an element of $H^2(Y_{\mathbb{C}}, \mathbb{Z})(1)/$torsion, we endow $H^{2k}(Y_{\mathbb{C}}, \mathbb{Z})(k)$/torsion with the quadratic form $\langle\, ,\rangle_\eta$ defined by $\langle x, y\rangle_\eta = (-1)^k x \cup y \cup \eta_{\mathbb{C}}^{\dim Y - 2k}$, and we denote by $P^{2k}(Y_{\mathbb{C}}, \mathbb{Z})(k)$ the primitive lattice, i.e. the orthogonal complement of the image of $H^{2k-2}(Y_{\mathbb{C}}, \mathbb{Z})(k-1) \cup \eta_{\mathbb{C}}$ in $H^{2k}(Y_{\mathbb{C}}, \mathbb{Z})(k)$/torsion. This primitive lattice underlies a Hodge structure of weight 0 polarized by $\langle\, ,\rangle_\eta$ [W58, D71a]; we denote by $h_0^{p,q}$ its Hodge numbers.

Motivated by the hyperkähler instance, we introduce the following 'axioms':

A_k: *one has* $h_0^{-1,1} = h_0^{1,-1} = 1, h_0^{0,0} > 0$, *and* $h_0^{p,q} = 0$ *if* $|p-q| > 2$;

B_k: *there exists a smooth connected K-scheme S, a point $s \in S(K)$, and a projective smooth morphism $f\colon \underline{Y} \to S$, such that:*

i) $Y = \underline{Y}_s$,

ii) $\eta_{\mathbb{C}}$ *extends to a section of* $R^2 f^{\mathrm{an}}_{\mathbb{C}*}\mathbb{Z}(1)$/*torsion*,

iii) *the image of the mapping (Universal covering of $S(\mathbb{C})$) $\to$ (Moduli space of Hodge structures on $P^{2k}(Y_{\mathbb{C}}, \mathbb{Z})(k)$ polarized by $\langle\, ,\rangle_\eta$) contains an open subset.*[1]

We shall say that (Y, η) satisfies B_k^+ if moreover

iv) *for each $t \in S(\mathbb{C})$, every element of Hodge type $(0,0)$ in $H^{2k}(\underline{Y}_t, \mathbb{Q})(k)$ is an algebraic class.*

We observe that these axioms do not depend on the given complex embedding of K[2].

Polarized abelian surfaces, surfaces of general type with $p_g = 1$ and $\mathscr{K}^2 = 1$, K3 surfaces and hyperkähler varieties with $b_2 > 3$ satisfy A_1, B_1^+.

[1] One can show that the image of the monodromy homomorphism $\pi_1(S(\mathbb{C}), s) \to O(P^{2k}(Y_{\mathbb{C}}, \mathbb{Z})(k))$ then has finite index (cf. 3.3.3). These axioms are similar to those being considered in [Ra72]

[2] For A_k, we note that the Hodge numbers $h_0^{p,q}$ may be defined algebraically

Cubic fourfolds satisfy A_2, B_2^+ (see 3.3 to 3.6). It turns out that most of our arguments apply to any polarized variety which satisfy A_k, B_k^+ for some k.

1.5

Let us now turn to problem 1.1.2. The notion of motive which we consider here is the 'strong' one defined in [A93]; however our results hold (a fortiori) for motives defined in terms of absolute Hodge cycles (cf. [DM82]).

Let us record one of the equivalent definitions of a motivated cycle in the sense of loc. cit., for a ground field K which is a subfield of $\mathbb{C}$: a *motivated cycle* (in the strong sense) on a smooth projective K-variety X is an element of $H^*(X_{\mathbb{C}}, \mathbb{Q})$ which can be written $pr_*(\alpha \cup (*_X \otimes *_W)\beta)$, where W is an arbitrary (not necessarily connected) smooth projective K-variety, pr is the projection $X \times W \to X$, α and β are algebraic cycles on $X \times W$, and $*$ stands for the Hodge star operator associated with the Kähler metric attached to some polarization defined over K.

All motivated cycles are absolute Hodge. All algebraic cycles are motivated[3]. Remember that one of A. Grothendieck's standard conjectures predicts that $*$ respects algebraic cycles, which would imply that, conversely, all motivated cycles are algebraic. It is proved in [A93] that in a precise sense, the notion of motivated cycle is invariant under $\mathrm{Aut}(\mathbb{C}/K)$, and that the category of *motives* defined in terms of these is tannakian semisimple over $\mathbb{Q}$. In particular, to any motive is attached a motivic Galois group, which is a reductive $\mathbb{Q}$-group (depending by inner twist on the complex embedding of K). We denote by $\mathcal{M}_K(\mathcal{A}b)$ the tannakian subcategory generated by abelian varieties and 0-dimensional varieties.

Theorem 1.5.1. *Let (Y, η) be a polarized variety which satisfies A_k, B_k^+. Then the motive $\not{p}^{2k}(Y)(k)$ attached to $P^{2k}(Y_{\mathbb{C}}, \mathbb{Q})(k)$ is an object of $\mathcal{M}_K(\mathcal{A}b)$.*

In the special case of complex K3 surfaces, this was proved in [A93];[4] let us also mention the work of K. Paranjape [Pa88] concerning K3 surfaces which are desingularizations of the double cover of the plane branched along six lines.

Using Y. Zarhin's description [Za83] of the Hodge group of $P^{2k}(\,\cdot\,, \mathbb{Q})(k)$ for polarized varieties satisfying A_k, we get:

Corollary 1.5.2. *Let (Y, η) be defined over an algebraically closed subfield K of $\mathbb{C}$, and satisfy A_k, B_k^+. Let us denote by $t^{2k}(Y)(k)$ the submotive of $\not{p}^{2k}(Y)(k)$ whose Betti realization is the orthogonal complement $T^{2k}(Y_{\mathbb{C}}, \mathbb{Q})(k)$ of the subspace of $P^{2k}(Y_{\mathbb{C}}, \mathbb{Q})(k)$ generated by algebraic classes. Then $E :\doteq \mathrm{End}\, t^{2k}(Y)(k)$ is a CM field or a totally real, and the restriction of $\langle\,,\rangle_\eta$ to $T^{2k}(Y_{\mathbb{C}}, \mathbb{Q})(k)$ is E-hermitian (resp. E-bilinear). The corresponding*

[3] For instance if X has a K-point P, take $W = \mathrm{Spec}\, K$, $\beta =$ the class of $P \times W$

[4] For this special case, the weaker statement, in terms of absolute Hodge cycles, was claimed in [DM82], where an argument by deformation of the original surface to a Kummer surface is sketched; however, no existence proof is offered of such an algebraic deformation

unitary (*resp. special orthogonal*) *group, viewed as a* $\mathbb{Q}$*-group, coincides with the motivic Galois group of* $\not{h}^{2k}(Y)(k)$.

On the other hand, having proved in [A93] that any Hodge cycle on a complex abelian variety is motivated, we get:

Corollary 1.5.3. *On a product of complex hyperkähler varieties* Y_i (*with* $b_2 > 3$), *cubic fourfolds and abelian varieties* X_j, *any Hodge cycle in* $(\otimes H^2(Y_i)) \otimes (\otimes H^*(X_j))$ *is motivated.*

1.6

Given a polarized variety (Y, η) and a rational prime ℓ, one defines a quadratic form $\langle\,,\rangle_\eta$ on $H^{2k}_{\text{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)$ by $\langle x, y\rangle_\eta = (-1)^k x \cup y \cup \eta^{\dim Y - 2k}$, and one denotes by $P^{2k}_{\text{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)$ the primitive lattice (which is the l-adic realization of $\not{h}^{2k}(Y)(k)$), i.e. the orthogonal complement of the image of $H^{2k-2}_{\text{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k-1) \cup \eta$.

The following result, which generalizes [O80, Ta88, Ta90], gives a partial answer to question 1.1.3:

Theorem 1.6.1. *Let* (Y, η) *be a polarized variety satisfying* A_k, B_k^+ *over a number field K. Then:*

1) *the* $\mathrm{Gal}(\overline{K}/K)$*-module* $P^{2k}_{\text{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)$ *is semisimple,*

2) *every* $\mathrm{Gal}(\overline{K}/K)$*-invariant element in* $P^{2k}_{\text{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)$ *is induced by a* $\mathbb{Q}_\ell$*-linear combination of algebraic cycles,*

3) *the image of* $\mathrm{Gal}(\overline{K}/K)$ *in* $GL(P^{2k}_{\text{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k))$ *is an open Lie subgroup of the l-adic motivic group attached to* $\not{h}^{2k}(Y)(k)$,

4) *let* (Y', η') *be a polarized K-variety satisfying* A_h, B_h^+ (*for some h*); *then any isomorphism of* $\mathrm{Gal}(\overline{K}/K)$*-modules* $P^{2k}_{\text{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k) \equiv P^{2h}_{\text{et}}(Y'_{\overline{K}}, \mathbb{Q}_\ell)(h)$ *is induced by a* $\mathbb{Q}_\ell$*-linear combination of motivated cycles on* $Y \times Y'$.

An interesting example of such an isomorphism is given by the so-called Abel-Jacobi map of a cubic fourfold (3.4).

1.7

Not surprisingly, our proofs rely upon the use of the period mapping and the Kuga-Satake construction (along the lines of [D72] or [PSS73]); in this way, we reduce problems 1.1.1 and 1.1.3 to the analogous problems on abelian varieties, solved by G. Faltings. However, since the Kuga-Satake abelian variety is constructed by analytical means, it is crucial to verify the existence of a model over some *finite* extension of the ground field (Sect. 5.5, 8):[5]

[5] This result is closed in spirit those to of [D72], especially Proposition 6.5, although the question of the existence of a model of the Kuga-Satake variety over a number field is not taken up in [D72], where the occuring abelian K-variety, with property 1.7.1ii), is simply constructed as a *specialization* of the Kuga-Satake variety. Therefore, contrary to a seemingly widespread opinion (see e.g. [080]), 1.7.1 does not follow formally from [D72], even if we fix a complex embedding of K

Main Lemma 1.7.1. *Let (Y,η) a polarized variety defined over a subfield K of $\mathbb{C}$, satisfying A_k, B_k. Then there is an abelian variety ${}_{K'}A$ defined over some finite extension K' of K such that*
i) $({}_{K'}A)_{\mathbb{C}}$ *is the Kuga-Satake variety of* $(Y_{\mathbb{C}},\eta_{\mathbb{C}})$,
ii) *there is a subalgebra* C^+ *of* $\mathrm{End}({}_{K'}A)$ *such that the* $\mathbb{Z}_l[\mathrm{Gal}(\overline{K}/K')]$*-algebra* $\underline{\mathrm{End}}_{C^+}H^1_{\mathrm{et}}(({}_{K'}A)_{\overline{K}},\mathbb{Z}_l)$ *is isomorphic to the even Clifford algebra of the primitive quadratic module* $P^{2K}_{\mathrm{et}}(Y_{\overline{K}},\mathbb{Z}_l)(k)$ *(with its natural Galois action).*

For K3 surfaces, it turns out that ${}_{K'}A$ does not depend on the given complex embedding of K; in other words, the construction of the Kuga-Satake abelian variety of a K3 surface 'does not depend on the topology of $\mathbb{C}$' (8.5).

We shall give two proofs of the main lemma, both of them based on certain rigidity properties of Kuga-Satake families (5.4). The second proof, more delicate, yields an explicit description of the extension K'/K in terms of the Galois action on etale primitive cohomology in degree $2k$ (8.4.3); this is used in the proof of Theorem 1.3.1. At last, we point out that many of the proofs in the text are much simpler in case the $2k^{\text{th}}$ primitive Betti number is odd (e.g. in the case of K3 surfaces).

2 Polarized hyperkähler varieties and cubic fourfolds

2.1

Let us state again our definition: A *hyperkähler variety* over a field K of characteristic 0 is a simply-connected smooth projective K-variety Y of even dimension $2r$, with the property that there exists a section ω of Ω^2_Y, unique up to muliplication by a constant, such that ω^r vanishes nowhere. For $K=\mathbb{C}$, this is equivalent to the existence of a Kähler metric for which the holonomy group is $\mathrm{Sp}(2r)$.

It is known that $H^0(Y,\Omega^p_Y)=0$ if p is odd, and $H^0(Y,\Omega^{2q}_Y)=K\omega^q$ for $0\leqq q\leqq r$ (see [Be83a] Sect. 3). From these basic properties, it follows that:

a) the canonical line bundle is trivial, generated by the section ω^r;

b) the Kodaira dimension is 0, so that no smooth deformation or specialization of Y is ruled, i.e. birationally equivalent to a product $\mathbb{P}^1\times W$ (indeed, a ruled variety has Kodaira dimension $-\infty$, and the Kodaira dimension cannot decrease by specialization – in any characteristic);

c) Y has no infinitesimal automorphism, i.e. $H^0(Y,T_Y)=0$ (by duality, and by a), this means that the Hodge number $h^{2r,1}(Y)$ is 0, and indeed, $h^{2r,1}=h^{1,2r}=h^{1,0}=h^{0,1}=0$);

d) for any prime $\ell, H^1_{\mathrm{et}}(\overline{Y},\mathbb{Z}_\ell)=0$ and $H^2_{\mathrm{et}}(\overline{Y},\mathbb{Z}_\ell)$ is torsionfree (where $\overline{Y}=Y_{\overline{K}}$, $\overline{K}=$ some algebraic closure of K). This follows from the universal coefficient exact sequence $0\to H^1_{\mathrm{et}}(\overline{Y},\mathbb{Z}_\ell)\otimes\mathbb{Z}/\ell\mathbb{Z}\to H^1_{\mathrm{et}}(\overline{Y},\mathbb{Z}/\ell\mathbb{Z})\to \mathrm{Tor}_1(H^2_{\mathrm{et}}(\overline{Y},\mathbb{Z}_\ell),\mathbb{Z}/\ell\mathbb{Z})\to 0$, and the fact that $H^1_{\mathrm{et}}(\overline{Y},\mathbb{Z}/\ell\mathbb{Z})=0$, because $\overline{Y}$ is simply-connected. It follows that over $K=\mathbb{C}$ and for Betti cohomology, $H^1(Y,\mathbb{Z})=0$ and $H^2(Y,\mathbb{Z})$ is torsionfree.

e) the numerical equivalence class group of line bundles on $\overline{Y}$ coincides with the Picard group $\operatorname{Pic}\overline{Y}$. In particular, a polarization of Y is just a $\mathrm{Gal}(\overline{K}/K)$-invariant isomorphism class η of an ample line bundle.

f) for any polarization η, one has $H^i(\overline{Y},\eta) = 0$ if $i > 0$ (this follows from a) and the Kodaira vanishing theorem).

2.2

A family of hyperkähler varieties parametrized by an algebraic or analytic space S is a proper flat morphism $f\colon\ \underline{Y} \to S$, the fibers of which are hyperkähler varieties. A polarization of f is a section $\underline{\eta}$ of $\operatorname{Pic}_S \underline{Y}$ (the relative Picard etale sheaf) such that the fiber $\underline{\eta}_s \in (\operatorname{Pic}\overline{Y}_s)^{\mathrm{Gal}(\overline{K(s)}/K(s))}$ above any point $s \in S$ is a polarization. When $K = \mathbb{C}$, we shall identify $\underline{\eta}$ with its image under the injective morphism $\Gamma(\operatorname{Pic}_S \underline{Y}) \to \Gamma(R^2 f_*^{\mathrm{an}}\mathbb{Z}(1))$.

2.3

A coarse moduli space for polarized hyperkähler varieties (Y,η) with fixed Hilbert polynomial $P(x)$ may be obtained as follows.

2.3.1. Lemma. *Let (Y,η) be defined over K (a field of characteristic 0). Then $\eta^{\otimes P(1)}$ lies in the image of the morphism $\operatorname{Pic} Y \to (\operatorname{Pic}\overline{Y})^{\mathrm{Gal}(\overline{K}/K)}$.*

Proof. In view of the usual exact sequence $\operatorname{Pic} Y \to (\operatorname{Pic}\overline{Y})^{\mathrm{Gal}(\overline{K}/K)} \to \operatorname{Br} K$, it suffices to show that the image of η in the Brauer group of K is annihilated by $\dim H^0(\eta) = P(1)$ (cf. 21f). The effective divisors on $\overline{Y}$ belonging to the class η are in a natural way the $\overline{K}$-valued points of a Severi-Brauer K-variety $|\eta|$ of dimension $P(1) - 1$; as is well-known, $|\eta|$ is the Grassmannian of rank-$P(1)$ right ideals in some simple central K-algebra A of degree $P(1)^2$. Any maximal commutative subfield K' of A is a splitting field for $|\eta|$. Hence the image of η in $\operatorname{Br} K$ lies in the kernel of $\operatorname{Br} K \to \operatorname{Br} K'$; but the exponent of this kernel divides $[K' : K] = P(1)$ (for all this, see [Se68] X Sects. 5, 6).

2.3.2. By a fundamental theorem of T. Matsusaka [Ma72], there is an integer q depending only on $P(x)$ such that for all polarized hyperkähler varieties (Y,η) with Hilbert polynomial $P(x), \eta^{\otimes q}$ is the class of a very ample line bundle. We set $m = q \cdot P(1)$, and $M = P(m) - 1$. Then $\eta^{\otimes m}$ comes from a very ample element of $\operatorname{Pic} Y$.

2.3.3. Let us consider the smooth subvarieties $Z \subseteq \mathbb{P}^M$ such that $(Z, [\mathcal{O}(1)]) \cong (Y, \eta^{\otimes m})$ for some polarized hyperkähler variety (Y,η) with Hilbert polynomial $P(x)$ (over any field of characteristic 0). This is the same as the set of smooth $Z \subseteq \mathbb{P}^M$ such that

i) $\chi(\mathcal{O}_Z(k)) = P(mk)$, all k,

ii) $\mathcal{O}_Z(1)$ is divisible by m in Pic Z,
iii) $h^{0,2}(Z) = 1$, and
iv) the canonical bundle of Z is trivial.

Such subvarieties Z are parametrized by a Zariski *open* subset H_m of the Hilbert scheme $\mathrm{Hilb}^{P(mx)}(\mathbb{P}^M_K)$. We denote by $\underline{Z} \to H_m$ the universal family.

2.3.4. The quotient space $H_m/PGL(M+1)$ exists as an algebraic space, separated and of finite type over $\mathbb{Q}$, or any field of characteristic 0 (see e.g. [MFo82] App. to Ch. 5, for a description of the local charts, using properties b), c) and f) above). It is *a coarse moduli space for polarized hyperkähler varieties with Hilbert polynomial* $P(x)$.

2.3.5. Remark. Using Chow coordinates instead of Matsusaka's theorem, one shows that there exists a *coarse moduli space*, separated and of finite type over K, *for very polarized hyperkähler varieties of degree* d; indeed it is finite disjoint union of spaces $H_1/PGL(P(1))$, for finitely many $P(x)$ (note that by point f) above, "very ample" is an open condition). Similarly, there is a *coarse moduli space*, separated and of finite type over K, *for polarized K3 surfaces of degree* d; indeed, it is a finite disjoint union of spaces $H_3/PGL(P(3))$, for finitely many $P(x)$.

2.4 Cubic fourfolds and their Fano varieties

Let $Y \subseteq \mathbb{P}^5_K$ be a smooth cubic hypersurface, endowed with the polarization $\eta = [\mathcal{O}_Y(1)]$. Let us denote by (F, η') the polarized Fano variety of Y: F is the variety of lines in Y, and η' the class of $\mathcal{O}_F(1)$ in the Plücker embedding [BeD85]. For any point y of Y, there is a pencil of lines passing through y. It follows that the natural morphism $\mathrm{Aut}(Y, \eta) = \mathrm{Aut}\, Y \cap PGL(6) \to \mathrm{Aut}\, F$ is injective; hence $\mathrm{Aut}(Y, \eta)$ is finite (cf. 2.1c)). We denote by Z the incidence variety, and by $p\colon Z \to F$, $q\colon Z \to Y$, the canonical projections.

It turns out that F is a variety of K3 type, more precisely a projective deformation of a fourfold $S^{[2]}$, where S is a K3 surface of degree 14 in $\mathbb{P}^8$.

Let S be the Zariski open subset of $\mathbb{P}^{55}$ which parametrizes smooth cubics in $\mathbb{P}^5$, and let S' be the component of the open Hilbert scheme H_1 which parametrizes (as above) the deformations of the corresponding Plücker-embedded Fano varieties $F \subseteq \mathbb{P}^M_K$.

The natural morphism $S \to S'$ gives rise to a morphism of coarse moduli algebraic spaces $\mu\colon S/PGL(6) \to S'/PGL(M+1)$.

3 The period mapping

3.1

From now onwards and until Sect. 6.5, the ground field is $\mathbb{C}$ (unless otherwise specified: 3.4, 4.4 and 5.5).

Let (Y,η) be a polarized variety and let k be a positive integer as in 1.4; we record the quadratic form $\langle\,,\rangle_\eta$ on $H^{2k}(Y,\mathbb{Z})(k)$/torsion, defined by $\langle x,y\rangle_\eta = (-1)^k x\cup y\cup\eta\cup\ldots\cup\eta \in H^{2\dim Y}(Y,\mathbb{Z})(\dim Y)\cong\mathbb{Z}$. Let $P^{2k}(Y,\mathbb{Z})(k) = P^{2k}(Y,\eta,\mathbb{Z})(k)$ denote the primitive lattice, i.e. the orthogonal complement of the image of $H^{2k-2}(Y,\mathbb{Z})(k-1)\cup\eta$ in $H^{2k-2}(Y,\mathbb{Z})(k)$/torsion. This lattice carries a Hodge structure of weight 0, polarized by $\langle\,,\rangle_\eta$. In the sequel, we assume that the Hodge structure on $P^{2k}(Y,\mathbb{Z})(k)$ is of type $(-1,1)+(0,0)+(1,-1)$ with $h^{1,-1}=1$. Then the rank of $P^{2k}(Y,\mathbb{Z})(k)$ is $N+2$, where $N=h^{0,0}$, and the fact that $\langle\,,\rangle_\eta$ is a polarization means that it induces a non-degenerate quadratic form on $P^{2k}(Y,\mathbb{Z})(k)\otimes\mathbb{R}$, positive on the (0,0)-component, negative on the $(-1,1)+(1,-1)$-component.

Examples. Our examples will be polarized abelian surfaces and hyperkähler varieties (for $k=1$), cubic fourfolds (for $k=2$), and canonically polarized surfaces of general type with $p_g=1$ and $\mathcal{K}^2=1$ (for $k=1$, and denoting by $\mathcal{K}$ the canonical line bundle); the Kodaira dimension is 0, 0, $-\infty$, and 2 (maximal), respectively.

For abelian surfaces, one has $N=3$; for K3 surfaces, $N=19$; for a variety of K3 type and of dimesion >2, one has $N=20$ (resp. $N=4$) if it is a deformation of a $S^{[r]}$ (resp. K_r), cf. Sect. 1 and [Be83a]; for cubic fourfolds, one has $N=20$, cf. [Ra72], while for canonically polarized surfaces with $p_g=1$ and $\mathcal{K}^2=1$, $N=18$ and $P^2(Y,\mathbb{Z})(1)$ is unimodular [C80, T80].

3.2

Let $V_\mathbb{Z}\cong(\mathbb{Z}^{N+2},\langle\,,\rangle)$ be a quadratic lattice of signature $(N+,2-)$ and let us write V for $V_\mathbb{Z}\otimes\mathbb{Q}$, $V_\mathbb{R}$ for $V_\mathbb{Z}\otimes\mathbb{R}$. The Hodge structures of type $(-1,1)+(0,0)+(1,-1)$ on $\mathbb{Z}^{N+2}$ polarized by $\langle\,,\rangle$ are parametrized by $\Omega^\pm := O(N,2)/O(N)\times SO(2)$, which is a disjoint union of two copies of the hermitian symmetric domain attached to Spin $V_\mathbb{R}$; the complex dimension of $\Omega^\pm$ is N.

A *k-marked* polarized variety (Y,η,ε) of type $V_\mathbb{Z}$ is a polarized variety (Y,η) together with an isomorphism of quadratic lattices $\varepsilon\colon(P^{2k}(Y,\mathbb{Z})(k),\langle,\rangle_\eta)\cong V_\mathbb{Z}$. One thus attaches to (Y,η,ε) a point $\mathcal{P}(Y,\eta,\varepsilon)$ in $\Omega^\pm$, called the *period* of (Y,η,ε). Using an auxiliary k-marking, one can attach to any polarized variety (Y,η) ("of type $V_\mathbb{Z}$") a well-defined point in $\Omega^\pm/O(V_\mathbb{Z})$ still called its period, which depends holomorphically on (Y,η) [G71] 9.6.

3.3 *The case of a polarized hyperkähler variety (Y,η)*

Proposition 3.3.1. *Let $f\colon\mathcal{Y}\to\mathcal{S}$ be a local universal projective deformation of (Y,η). By restricting $\mathcal{S}$ if necessary, one can assume that there is a continuous marking on the P^2-lattices of the fibres. Then the induced period mapping $\mathcal{S}\to\Omega^\pm$ is a local isomorphism.*

See [Be83a] Sect. 8, which relies on the smoothness of the Kuranishi family, due to F. Bogomolov [B74].

Corollary 3.3.2 (Todorov). *The "open" Hilbert scheme H_m is smooth.*

Indeed, the local deformation space $\mathcal{S}$ attached by the proposition to $(Y, \eta^{\otimes m})$ is smooth; the local deformation space for the embedded $Y \subset \mathbb{P}^M$ (via $\eta^{\otimes m}$) is open in a $PGL(M+1)$-torsor over $\mathcal{S}$, and locally isomorphic to H_m in the neighborhood of the modulus of $(Y, \eta^{\otimes m})$ [T90]. Hence H_m is smooth (but not necessarily connected).

Corollary 3.3.3. *The period mapping from any connected component S of $H_m(\mathbb{C})$ to $\Omega^{\pm}/O(V_{\mathbb{Z}})$ is a dominant analytic mapping. In particular, for any $s \in S$, the image of $\pi_1(S(\mathbb{C}), s)$ in $O(V_{\mathbb{Z}})$ given by the monodromy of the universal family $\underline{Z}_{|S} \to S$ has finite index in $O(V_{\mathbb{Z}})$; and, if $N > 0$, it is Zariski-dense in $O(V)$ or $SO(V)$.*

The second assertion follows from the first according to the argument of [G71] D1 (cf. also [D72a] 4.4.17); the third assertion follows from the second according to [Bo69] 15.12.

This shows that hyperkähler varieties with $b_2 > 3$ satisfy properties A_1 and B_1 stated in 1.4. (note that η extends to a polarization of the Hilbert family $\underline{Z}_{|S} \to S$ (integrality of the Chern class)). Instead of a component S of H_m, one could take any algebraization of the formal universal projective deformation of (Y, η). We notice that, by Lefschetz' theorem, $B_1 \Leftrightarrow B_1^+$.

Proposition 3.3.4. *Up to isomorphism, there are only finitely many complex polarized hyperkähler varieties with given Hilbert polynomial (resp. very polarized hyperkähler varieties of dimension $2r$ and degree d, resp. polarized K3 surfaces of degree d) with given period in $\Omega^{\pm}/O(V_{\mathbb{Z}})$.*

Proof. Since the local period mapping $\mathcal{S} \to \Omega^{\pm}$ is a local isomorphism (3.3.1), it is enough to show that every fiber of the period mapping $\mathcal{P}\colon S \to \Omega^{\pm}/O(V_{\mathbb{Z}})$ has finitely many connected components. Let Γ be a torsionfree arithmetic subgroup of $SO(V_{\mathbb{Z}})$, and let S_Γ be a scheme, finite etale over S, and $s_\Gamma \in S_\Gamma$ lying above s, such that the image of $\pi_1(S_\Gamma, s_\Gamma)$ in $O(V_{\mathbb{Z}})$ factorizes through Γ (such a scheme exists by the generalized Riemann existence theorem). We have a commutative diagram

$$\begin{array}{ccc} S_\Gamma & \xrightarrow{\mathcal{P}_\Gamma} & \Omega^{\pm}/\Gamma \\ \delta \downarrow & & \downarrow \pi \\ S & \xrightarrow{\mathcal{P}} & \Omega^{\pm}/O(V_{\mathbb{Z}}) . \end{array}$$

According to A. Borel [Bo72], $\mathcal{P}_\Gamma$ is automatically a morphism of schemes. On the other hand, δ is a surjective morphism of algebraic spaces. Therefore, for any $t \in \Omega^{\pm}/O(V_{\mathbb{Z}})$, $\mathcal{P}^{-1}(t)$ is a closed algebraic subspace, being the projection under δ of a finite union of fibres $\mathcal{P}_\Gamma^{-1}(t_\Gamma)$, $t_\Gamma \in \pi^{-1}(t)$; hence $\mathcal{P}^{-1}(t)$ has finitely many components.

3.4 The case of a cubic fourfold

We take up the notations 2.4 again (with $K \subseteq \mathbb{C}$). The algebraic correspondence $p_* q^*$, which is usually called the 'Abel-Jacobi' correspondence, induces an isomorphism in cohomology [BeD85]:

$\alpha\colon H^4(Y_{\mathbb{C}},\mathbb{Z})(2) \to H^2(F_{\mathbb{C}},\mathbb{Z})(1)$. Moreover, one has $\alpha(\eta^2) = \eta'$, and the restriction of $\alpha\colon P^4(Y_{\mathbb{C}},\mathbb{Z})(2) \to P^2(F_{\mathbb{C}},\mathbb{Z})(1)$ is a "quasi-isometry": $\langle\alpha(x), \alpha(y)\rangle_{\eta'} = 6\langle x, y\rangle_\eta$. On the other hand, it follows from [Ra72] that statement 3.3.1 also holds for cubic fourfolds. From this, one derives as in 3.3 that

i) cubic fourfolds satisfy A_2 and B_2 (even B_2^+ in fact, because the Hodge conjecture is known for them, cf. [Z77], or appendix 2 for a very short proof);

ii) the period mapping $S_{\mathbb{C}}/PGL(6) \to \Omega^\pm/O(V_{\mathbb{Z}})$ has finite fibers;

iii) denoting by $\overline{K}$ the algebraic closure of K in $\mathbb{C}$, there is a finite number of $PGL(6,\overline{K})$-orbits among all non-singular cubic fourfolds over $\overline{K}$ with given period in $\Omega^\pm/O(V_{\mathbb{Z}})$.

Since $S_{\mathbb{C}} \to \Omega^\pm/O(V_{\mathbb{Z}})$ factorizes through the period mapping $S'_{\mathbb{C}} \to \Omega^\pm/O(V_{\mathbb{Z}})$,[6] one also deduces that the morphism $\mu\colon S/PGL(6) \to S'/PGL(M+1)$ of 2.4 has finite fibers; this implies:

iv) there is a finite number of $PGL(6,\overline{K})$-orbits among all non-singular cubic fourfolds over $\overline{K}$ whose Plücker-polarized Fano variety is isomorphic to a given polarized hyperkähler variety over $\overline{K}$.

Remark. C. Voisin [V86] has proved a 'Torelli theorem' for cubic fourfolds, but we shall not need this result (from which one may derive that there is at most one $PGL(6,\overline{K})$-orbit as in iv)).

3.5 The case of a canonically polarized surface of general type with $p_g = 1$ and $\mathcal{K}^2 = 1$

The canonical model of such a surface is a smooth complete intersection of two sextics in the weighted projective space $\mathbb{P}(1,2,2,3,3)$ [C80, T80]. We denote here by S the (smooth) Zariski-open subset of $\mathrm{Sym}^2\mathbb{P}^{18}$ which parametrizes such complete intersections (with K ample). There is a coarse moduli space M for such surfaces, which is a smooth rational variety of dimension $18 = N$. The period mapping $S \to \Omega^\pm/O(V_{\mathbb{Z}})$ factorizes through a dominant mapping $\nu\colon M \to \Omega^\pm/O(V_{\mathbb{Z}})$.

From this, one derives that these canonically polarized surfaces satisfy A_1 and B_1^+. However, the analog of 3.3.1 is no longer true: some of the fibers of ν have dimension 2.

3.6 The case of a polarized abelian surface (Y, η)

The identification of $P^2(Y,\mathbb{Q})$ with a direct summand of $\bigwedge^2 H^1(Y,\mathbb{Q})$ gives rise to an exact sequence of $\mathbb{Q}$-algebraic groups $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Sp}(H^1(Y,\mathbb{Q}),\eta) \to SO(V) \to 0$, and to an identification of Ω^+ with the (complex) 3-dimensional

[6] The change of $\langle\,,\rangle_\eta$ into $6\langle\,,\rangle_\eta$ does not affect $\Omega^\pm$ nor $O(V_{\mathbb{Z}})$

Siegel upper half-space. One derives that polarized abelian surfaces satisfy A_1 and B_1^+.

4 The Kuga-Satake construction

4.1

This construction of abelian varieties applies to any polarized Hodge structure of type $(-1,1)+(0,0)+(1,-1)$ on $\mathbb{Z}^{N+2}$ ($N \geqq 0$), polarized by the form $\langle , \rangle$ of signature $(N+,2-)$, see [KS67] and [D72]. We follow the conventions of P. Deligne [D72].

Let G stand for the *even Clifford group*, i.e. the group of invertible elements γ in the even Clifford algebra $C^+(V)$ such that $\gamma V \gamma^{-1} = V$, so that there is an obvious surjective homomorphism of linear algebraic groups over $\mathbb{Q}$: $G \to SO(V)$, with kernel the homothety group; the induced homomorphism $G(\mathbb{Q}) \to SO(V)(\mathbb{Q})$ is still surjective.

The morphism h: $(\prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m) \to SO(V_{\mathbb{R}})$ which describes the Hodge decomposition on $\mathbb{C}^{N+2}$ lifts uniquely to a morphism $\tilde{h}$: $(\prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m) \to G_{\mathbb{R}}$, such that the image of any element λ in the diagonal group $\mathbb{G}_m$ acts as the multiplication by λ. Then the norm $(\prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m) \to \mathbb{G}_m$ coincides with $\mathbb{N} \circ \tilde{h}$, where $\mathbb{N}$ stands for the spinorial norm[7] (the character of G given by $\mathbb{N}$ corresponds to the Tate Hodge structure $\mathbb{Q}(-1)$).

Let $L_{\mathbb{Z}}$ be a free (left) $C^+(V_{\mathbb{Z}})$-module of rank one, and let us write L for $L_{\mathbb{Z}} \otimes \mathbb{Q}$, $L_{\mathbb{R}}$ for $L_{\mathbb{Z}} \otimes \mathbb{R}$. Then $L_{\mathbb{R}}$ is naturally a $G_{\mathbb{R}}$-module (action by left multiplication), and $\tilde{h}$ gives rise to a polarizable Hodge structure of type $(0,1)+(1,0)$ on $L_{\mathbb{Z}}$.

This defines a complex abelian variety $A = A(V_{\mathbb{Z}}, h)$ of dimension 2^N, called the *Kuga-Satake variety* attached to $(V_{\mathbb{Z}}, h)$, by the condition that $H^1(A, \mathbb{Z}) = L_{\mathbb{Z}}$ as a Hodge structure.

This construction applies in particular when h is the image by ε of the Hodge structure on $P^{2k}(Y, \mathbb{Z})(k)$ attached to a k-marked polarized variety (Y, η, ε) satisfying axiom A_k. In fact, up to isomorphism, A does not depend on ε; we write $A = A(Y, \eta)$ or $A(Y)$ to emphasize the geometric origin of $(V_{\mathbb{Z}}, h)$ (omitting k from the notation for short).

Example 4.1.1. The Kuga-Satake abelian variety of a polarized abelian surface is isogeneous to its fourth power [Mo85] 4.5.

Remark 4.1.2. Apart from hyperkähler varieties, surfaces with $p_g = 1$ and $\mathcal{K}^2 = 1$, and cubic fourfolds, there is another specific case of interest where the Kuga-Satake construction takes place, namely in the study of desingularizations of Hilbert modular surfaces [O82].

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[7] Denoting by $*$ the main (anti)involution of the Clifford algebra, $\mathbb{N}a$ is $a^* a$

Variant 4.1.3. Of course, the construction also works if one replaces $(V_{\mathbb{Z}}, h)$ by $(V_{\mathbb{Z}}^{\#}, h^{\#}) := (V_{\mathbb{Z}}, h) \oplus$ the trivial polarized Hodge structure on $\mathbb{Z}$ (with $\langle 1, 1 \rangle = 1$); we denote by $L^{\#}, A^{\#}$, the objects corresponding to L, A. This trivial trick enables us to recover $V_{\mathbb{Z}}$ from $C^+(V_{\mathbb{Z}}^{\#})$, when N is even (cf. Sects. 9, 10).

Since $C(V^{\#}) \cong C(V) \bigotimes^{\mathrm{gr}} C(1)$, and $C(1) \cong \mathbb{Q} \oplus \mathbb{Q}e$, with $e^2 = 1$, one has $C^+(V^{\#}) = C^+(V) \otimes 1 \oplus C^-(V) \otimes e \cong C(V)$ as left $C^+(V)$-modules. If v is any non-isotropic element of $V, C^+(V^{\#})$ is a free left $C^+(V)$-module with basis $1, v$.

Therefore the G-modules L^2 and $L^{\#}$ are isomorphic, and the abelian varieties A^2 and $A^{\#}$ are isogeneous.

Remark 4.1.4. (not used in the sequel). One has a "periodicity isomorphism" $C^+(V) \otimes M_{16}(\mathbb{Q}) \cong C^+(V^{(\#^8)})$, which is an isomorphism both of left $C^+(V)$-modules and of rings, and which comes as a composed isomorphism, as follows ([Ja80] Sect. 4.8, Th. 4.13, Lemma 5): let $v^{\perp}$ be the orthogonal complement of v in V, st the standard quadratic form on $\mathbb{Q}^n$, and put $q = -\langle v, v \rangle$, then $C^+(V) \otimes C(\mathbb{Q}^8, q^8 \cdot \mathrm{st}) \cong C(v^{\perp}, q\langle\, ,\, \rangle) \otimes C(\mathbb{Q}^8, q^8 \cdot \mathrm{st}) \cong C(v^{\perp} \oplus \mathbb{Q}^8, q(\langle\, ,\, \rangle \oplus \mathrm{st})) \cong C^+(V^{(\#^8)})$. Caution: the periodicity isomorphism is an isomorphism of left G-modules, but not of $SO(V)$-modules.

Variant 4.1.5. It will also be useful to apply the Kuga-Satake construction in the case when $(V_{\mathbb{Z}}, h)$ is replaced by $(V_{\mathbb{Z}}^b, h)$, where $V_{\mathbb{Z}}^b$ is the orthogonal complement of some algebraic classes in $P^{2k}(Y, \mathbb{Z})(k)$ (if any), with respect to some positive integral multiple of $\langle\, ,\, \rangle_\eta$ (note that $\operatorname{im} h \subseteq V_{\mathbb{Z}}^b \otimes \mathbb{R}$); we denote by L^b, A^b, the objects corresponding to L, A. As in 4.1.3, one can see that A is isogeneous to a power of A^b.

4.2

Let us denote by C^+ the opposite ring $(\mathrm{End}_{C^+(V_{\mathbb{Z}})} L_{\mathbb{Z}})^{\mathrm{op}}$ of the ring of $C^+(V_{\mathbb{Z}})$-endomorphisms of $L_{\mathbb{Z}}$. There is an isomorphism $C^+ \cong C^+(V_{\mathbb{Z}})$, well-defined up to conjugation. Note that the *right* action of C^+ on $L_{\mathbb{Z}}$ respects the Hodge structure, so that A has complex multiplication by C^+.

Examples. For K3 surfaces, this ring is an order in the matrix ring $M_{2^{10}}(\mathbb{Q})$, see [KS67];[8] for higher dimensional varieties of type K3 or cubic fourfolds, it turns out that the center of C^+ is an order in an imaginary quadratic field, while for canonically polarized surfaces with $p_g = 1$ and $\mathcal{K}^2 = 1$, the center of C^+ is an order in $\mathbb{Q} \oplus \mathbb{Q}$, see also [Sa66] Sect. 2, Remark 3).

On the other hand, one has a canonical ring isomorphism $C^+(V_{\mathbb{Z}}) \cong \underline{\mathrm{End}}_{C^+} L_{\mathbb{Z}}$, which is also an isomorphism of Hodge structures of type $(-1, 1) + (0, 0) + (1, -1)$ if, in the left-hand side, the tensor construction $C^+(\ )$ is understood as a functor on polarized Hodge structures of weight 0. In the

[8] [KS67] uses the opposite of $\langle\ \rangle$ and H_1 instead of H^1, but the result is the same

application to k-marked polarized varieties, we write this isomorphism as: $\psi\colon C^+((P^{2k}(Y,\mathbb{Z})(k),\langle\,,\rangle_\eta))\cong \underline{\mathrm{End}}_{C^+}H^1(A,\mathbb{Z})$. By Artin's comparison theorem, one derives a similar isomorphism $\psi^\wedge$ in etale cohomology (e.g. with coefficients in $\mathbb{Z}^\wedge=\prod_p\mathbb{Z}_p$, or in $\mathbb{A}^f=\mathbb{Z}^\wedge\otimes\mathbb{Q}$).

4.3

A polarization of the Hodge structure on $L_{\mathbb{Z}}$ (or of A) may be obtained as follows [Sa66] Sect. 2, Ex. 3: let us choose a generator of $L_{\mathbb{Z}}$, which amounts to an identification $C^+(V_{\mathbb{Z}})=L_{\mathbb{Z}}=C^+$; and let a be a non zero-divisor in C^+ with $a^*=-a$. Then the skew-symmetric form $\varphi_a\colon \mathrm{Sym}^2L_{\mathbb{Z}}\to\mathbb{Z}(-1)$ given by $\varphi_a(x,y)=\mathrm{tr}(x^*y\,a)$ defines a polarization of $L_{\mathbb{Z}}$ if and only if the symmetric form $\sqrt{-1}\varphi_a(x,\tilde{h}(\sqrt{-1})y)$ is definite positive (this condition on φ_a depends on the component of $\Omega^\pm$ to which h belongs, but not on the location of h in $\Omega^\pm$). One checks that $\pm\varphi_a$ does not change (as well as the equation $a^*=-a$) if one changes the generator of $L_{\mathbb{Z}}$ by multiplying it by an element of $G(\mathbb{Q})$, because the spinorial norm of an element v of $G(\mathbb{Q})\cap C^+(V_{\mathbb{Z}})^*$ is $\mathbb{N}v=\pm1$.

4.4

We pause to introduce some arithmetic groups. We set $G_{\mathbb{Z}}:=G(\mathbb{Q})\cap C^+(V_{\mathbb{Z}})^*$. The homomorphism $G(\mathbb{Q})\to SO(V)(\mathbb{Q})$ maps $G_{\mathbb{Z}}$ to a subgroup of $SO(V_{\mathbb{Z}})$; indeed, for every $\gamma\in G_{\mathbb{Z}}$, conjugation by γ is an automorphism of $C(V_{\mathbb{Z}})$ which respects V, hence also $V_{\mathbb{Z}}$. Thus there is an exact sequence $0\to\mathbb{Z}/2\mathbb{Z}\to G_{\mathbb{Z}}\to SO(V_{\mathbb{Z}})$.

By a well-known lemma of Minkowski-Serre, the principal congruence subgroup Γ_n of level $n>2$ in $G_{\mathbb{Z}}$ (i.e. the subgroup of elements $\equiv 1 \bmod. n$ in $C^+(V_{\mathbb{Z}})$) is torsionfree.[9] Note also that Γ_n lies inside $\mathrm{Spin}\,V=\mathrm{Ker}\,\mathbb{N}$. The image Γ_n^{ad} of Γ_n in $SO(V_{\mathbb{Z}})$ is a subgroup of the principal congruence subgroup of level n in $SO(V_{\mathbb{Z}})$; according to [Bo69] 8.9, it is an arithmetic subgroup, i.e. of finite index in the latter.[10] On the other hand, one reads on the last displayed exact sequence that the homomorphism $\Gamma_n\to\Gamma_n^{\mathrm{ad}}$ is an isomorphism (it will be convenient to identify these two groups). We also introduce $V_{\mathbb{Z}^\wedge}:=V_{\mathbb{Z}}\otimes\mathbb{Z}^\wedge$, $G_{\mathbb{Z}^\wedge}:=G(\mathbb{A}^f)\cap C^+(V_{\mathbb{Z}^\wedge})^*, \mathbb{K}_n:=$ subgroup of $G_{\mathbb{Z}^\wedge}$ consisting in elements $\equiv 1\bmod. n$ in $C^+(V_{\mathbb{Z}^\wedge})$, so that $\Gamma_n=G(\mathbb{Q})\cap\mathbb{K}_n$. We note that the image $\mathbb{K}_n^{\mathrm{ad}}$ of $\mathbb{K}_n$ in $SO(V_{\mathbb{Z}^\wedge})$ is an open subgroup; indeed, for every prime ℓ not dividing $2n$ and such that the quadratic space $V_{\mathbb{Z}}\otimes\mathbb{F}_\ell$ is non-degenerate, the ℓ-component of $\mathbb{K}_n^{\mathrm{ad}}$ is $SO(V_{\mathbb{Z}_\ell})$, because the morphism

[9] In fact, the subgroup of elements of $G_{\mathbb{Z}}(\mathbb{Z}_\ell)$ congruent to $1\bmod. n$ is already torsionfree for any odd prime $\ell|n$, and for $\ell=2$ if n is divisible by 4

[10] As the referee points out, it does not contain congruence subgroups in general, because the congruence subgroups in Spin define a topology for which the completion is an open subgroup of $\mathrm{Spin}(V_{\mathbb{A}^f})$, while the congruence subgroups in SO define a topology for which the completion is the image of this subgroup in $SO(V_{\mathbb{A}^f})$, i.e. a quotient by an infinite abelian group of exponent 2

$G \to SO(V)$ extends to a smooth morphism of group scheme over $\mathbb{Z}_\ell$, surjective on $\mathbb{F}_\ell$-points (cf. [Ja80] 4.14).

The rest of this paragraph will not be used before Sect. 8. *The reader who is interested only in problems* 1.1.2. *or* 1.1.3. *may skip it.*

4.5 Kuga-Satake packages

We axiomatize, in a way suitable for descent arguments, those structures involved in the Kuga-Satake construction which make sense algebraically over a field K of characteristic 0.

Let C^+ be a ring (with unit). Let $V_{\mathbb{Z}^\wedge}$ be a free $\mathbb{Z}^\wedge$-module of rank $N+2 \geqq 2$, endowed with a non-degenerate quadratic form $\langle , \rangle$ and with an isometric action of $\mathrm{Gal}(\overline{K}/K)$. For $n > 2$, let as before $\mathbb{K}_n$ denote the subgroup of the even Clifford group of $V_{\mathbb{A}^f} := V_{\mathbb{Z}^\wedge} \otimes \mathbb{A}^f$ consisting in units in $C^+(V_{\mathbb{Z}^\wedge})$ which are $\equiv 1 \mathrm{mod}.\, n$.

Definition 4.5.1. *A* **Kuga-Satake package** (*or K.-S. package*) *associated to* $(V_{\mathbb{Z}^\wedge}, C^+, n)$ *over* K *is a* 4*-tuple* $(A, \mu, \{\varphi_a\}, \overline{\upsilon})$, *where*

A is an abelian variety over K,

μ is an embedding of C^+ into $\mathrm{End}_K A$,

$\{\varphi_a\}$ is a non-empty collection of polarizations of A, indexed by suitable elements a of C^+,

$\overline{\upsilon}$ is a class in $\mathbb{K}_n \backslash \mathrm{Isom}(C^+(V_{\mathbb{Z}^\wedge}), H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Z}^\wedge))$ (*for the right action of* $\mathbb{K}_n$ *on* $C^+(V_{\mathbb{Z}^\wedge})$), *fixed under* $\mathrm{Gal}(\overline{K}/K)$;

these data being subject to the following conditions:

a) *the left* $C^+(V_{\mathbb{Z}^\wedge})$*-module structure on* $H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Z}^\wedge)$ *induced by any representative of* $\overline{\upsilon}$ *commutes with the right action of* C^+,

b) *the opposite ring of* $\mu(C^+) \otimes \mathbb{Z}^\wedge$ *coincides with* $\underline{\mathrm{End}}_{C^+(V_{\mathbb{Z}^\wedge})} H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Z}^\wedge)$,

c) *an element a of C^+ is called 'suitable' if via some representative υ of $\overline{\upsilon}$ and some identification* $\mathbb{Z}^\wedge \cong \mathbb{Z}^\wedge(1)$, *the formula* $\varphi_a(x, y) = \mathrm{tr}(x^* \, y \, \mu(a))$ $(x, y \in H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Z}^\wedge))$ *defines a polarization of A, which we denote by φ_a.*

Remarks 4.5.2. i) $\overline{\upsilon}$ gives rise to two more geometrically meaningful objects: a Galois isomorphism ϑ: $C^+(V_{\mathbb{Z}^\wedge}) \otimes \mathbb{Z}/n\mathbb{Z} \cong H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Z}/n\mathbb{Z})$, and a Galois isomorphism $\psi^\wedge$: $C^+(V_{\mathbb{Z}^\wedge}) \cong \underline{\mathrm{End}}_{C^+} H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Z}^\wedge)$.

ii) The subring $\mu(C^+)$ of $\mathrm{End}\, A$ is uniquely determined by condition b), as $\mu(C^+) = \mathrm{End}\, A \cap (\underline{\mathrm{End}}_{C^+(V_{\mathbb{Z}^\wedge})} H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Z}^\wedge))^{\mathrm{op}}$.

iii) The polarization φ_a is uniquely determined by condition c); indeed, polarizations defined by the same a but different representatives υ or different identifications $\mathbb{Z}^\wedge \cong \mathbb{Z}^\wedge(1)$ must be equal except for a possible factor in $(\mathbb{N}(\mathbb{K}_n) \cdot \mathbb{Z}^{\wedge *}) \cap \mathbb{Q}^* = \{\pm 1\}$, which is $+1$ by the positivity condition involved in polarizations. Note that $\varphi_a = \varphi_{-a}$ (just change the sign of the identification $\mathbb{Z}^\wedge \cong \mathbb{Z}^\wedge(1)$).

iv) A K.-S. package over K induces one over any extension of K.

4.6

The Kuga-Satake construction described in 4.1, 4.2, 4.3, associates in a transcendental way to any polarized variety (Y, η) satisfying condition A_k a Kuga-Satake package over $K = \mathbb{C}$, associated to $(V_{\mathbb{Z}} \otimes \mathbb{Z}^\wedge, C^+, n)$, depending on the choice of a Γ_n-orbit of generators of the free $C^+(V_{\mathbb{Z}})$-module $L_{\mathbb{Z}}$ (with the notations of 4.1). If $L_{\mathbb{Z}} = C^+(V_{\mathbb{Z}}) = C^+$, with the Γ_n-orbit of the canonical generator, we call the associated K.-S. package the *canonical Kuga-Satake package of* (Y, η) (the level n being understood).

4.7

Definition 4.7.1. *Two K.-S. packages* $(A, \mu, \{\varphi_a\}, \overline{v})$ *and* $(A', \mu', \{\varphi'_{a'}\}, \overline{v}')$ *associated to the same datum* $(V_{\mathbb{Z}^\wedge}, C^+, n)$, *are said to be* **isomorphic** *if there is an isomorphism* $i\colon A \to A'$, *such that* $\mu = i\mu' i^{-1}$, $\overline{v} = i^* \circ \overline{v}'$.

Lemma 4.7.2. i) *if the K.-S. packages* $(A, \mu, \{\varphi_a\}, \overline{v})$ *and* $(A', \mu', \{\varphi'_{a'}\}, \overline{v}')$ *associated to the same datum* $(V_{\mathbb{Z}^\wedge}, C^+, n)$ *are isomorphic, then* $i^*\varphi'_{i\,ai^{-1}} = \varphi_a$;

ii) *K.-S.-packages have no non-trivial automorphism.*

Proof. i) Follows from the argument given in Remark 4.5.2iii). Let i be an automorphism of a K.-S. package. The formula $\overline{v} = i^* \circ \overline{v}$ implies that $i \in \mathbb{K}_n \cap (C^+)^*$ (considered, via some representative of $\overline{v}$, as a subalgebra of $C^+(V_{\mathbb{Z}^\wedge})$ acting on the right on itself). The formula $\mu = i\mu' i^{-1}$ then shows that i is a unit in the center of C^+, congruent to $1 \bmod. n$. By i) it follows that $i^*\varphi_a = \varphi_a$; one concludes that i is a root of unity congruent to $1 \bmod. n$, hence $i = \mathrm{id}$ (one can also remark, more directly, that the center of $\mathbb{K}_n \cap (C^+)^*$ is trivial).

5 The Kuga-Satake construction in a relative setting

5.1

Let S be a smooth connected complex algebraic variety with a distinguished point s, let $f\colon \underline{Y} \to S$ be a projective smooth morphism, and let $\underline{\eta}$ be a section of $R^2 f_*^{\mathrm{an}}\mathbb{Z}(1)/\mathrm{tors}$ such that the fiber $\underline{\eta}_s$ a polarization (we identify the numerical equivalence class group of line bundles on $\underline{Y}_s$ with a sublattice of $H^2(\underline{Y}_s, \mathbb{Z})(1)/\mathrm{tors}$). We denote by $P^{2k} f_*^{\mathrm{an}}\mathbb{Z}(k)$ the local system orthogonal complement of the image of $R^{2k-2} f_*^{\mathrm{an}}\mathbb{Z}(k-1) \cup \underline{\eta}$ in $R^{2k} f_*^{\mathrm{an}}\mathbb{Z}(k)/\mathrm{tors}$ with respect to the quadratic form $\langle x, y\rangle_{\underline{\eta}} = (-1)^k x \cup y \cup \underline{\eta} \cup \ldots \cup \underline{\eta}$. We assume that the variation of Hodge structures carried by $P^{2k} f_*^{\mathrm{an}}\mathbb{Z}(k)$ is of type $(-1,1) + (0,0) + (1,-1)$ with $h^{-1,1} = 1$. We let $N = h^{0,0} \geqq 0$. Let us fix a k-marking ε_s of $(\underline{Y}_s, \underline{\eta}_s)$. Then there is a unique isomorphism $\underline{\varepsilon}$ compatible with ε_s between $\underline{V}_{\mathbf{Z}}$ and the constant quadratic lattice obtained from $(P^{2k} f_*^{\mathrm{an}} \mathbf{Z}(k), \langle\,,\rangle_{\underline{\eta}})$ by pulling it back to the universal covering $\tilde{S}$ of (S^{an}, s).

To fix ideas, we decide that the period mapping $\tilde{S} \to \Omega^{\pm}$ maps to the component Ω^+, not Ω^-. Using the morphism $\tilde{h}$ described in 4.1, one endows the constant local system $L_{\mathbf{Z}}$ (a free left $C^+(V_{\mathbf{Z}})$-module of rank one) on Ω^+ or on $\tilde{S}$ with a variation of polarizable Hodge structure of type $(0,1)+(1,0)$; this defines an analytic family $\tilde{g}$ of abelian varieties parametrized by the analytic space $\tilde{S}$. We shall show that $\tilde{g}$ descends – if not to S itself in general – *at least to a finite unramified covering of S.*

5.2

By the generalized Riemann existence theorem, there exists an algebraic finite connected unramified covering of S, say S_n, and a point s_n of S_n above s, such that the monodromy homomorphism $\pi_1(S_n, s_n) \to O((P^{2k}(\underline{Y}_s, \mathbf{Z})(k), \langle\,,\rangle_{\underline{\eta}_s}))$ $= O(V_{\mathbf{Z}})$ *factorizes through the arithmetic group* Γ_n introduced in 4.4.3 (this is a priori stronger than just requiring that the local system $P^{2k} f_*^{\mathrm{an}} \mathbf{Z}/n\mathbf{Z}$ becomes constant on S_n). There is then a well-defined analytic period mapping $\mathscr{P}\colon S_n^{\mathrm{an}} \to \Omega^+/\Gamma_n$, and the polarizable variation of Hodge structure on Ω^+ attached to $L_{\mathbf{Z}}$, with its right C^+-module structure, is Γ_n-equivariant, hence descends to Ω^+/Γ_n, where it defines an analytic family of abelian varieties (with level n structure, and complex multiplication by C^+). By the theorem of Borel already quoted in 3.4, this is in fact an algebraic family. Pulling back on S_n, this yields an *abelian scheme* $g\colon \underline{A} \to S_n$ (endowed with a level n structure depending only the Γ_n-coset of the marking), and $\tilde{g}$ is nothing but the pull-back of g to $\tilde{S}$.

For any point t of S_n, there is a well-defined Γ_n-conjugacy class of k-markings ε_t of $(\underline{Y}_t, \underline{\eta}_t)$; any such ε_t corresponds to a point $\tilde{t}$ of $S_n \times_{\Omega^+/\Gamma_n} \Omega^+$ above t, to which is associated a canonical isomorphism $(L_{\mathbf{Z}})_{\tilde{t}} \cong H^1(\underline{A}_t, \mathbf{Z})$, respecting the Hodge structure. Hence $\underline{A}_t$ is the Kuga-Satake variety of $(\underline{Y}_t, \underline{\eta}_t, \varepsilon_t)$.[11]

5.3

Note however that the local system on Ω^+/Γ_n induced by $L_{\mathbf{Z}}$ (resp. $C^+(V_{\mathbf{Z}})$) is described by the action of Γ_n by left multiplication (resp. conjugation).[12] Therefore one can reasonably identify $L_{\mathbf{Z}}$ and $C^+(V_{\mathbf{Z}})$ only *up to right multiplication by elements of* Γ_n (on $C^+(V_{\mathbf{Z}})$). Let us fix such an identification. It follows from 4.3 that the collection of polarizations $\{\varphi_a\}$ of the

[11] With slight abuse of language, because we do not assume that $\underline{\eta}_t$ is a polarization, for $t \neq s$

[12] In other words, for any complex point t of S_n, the monodromy of g (resp. of the pull-back on S_n of the morphism f) is given by $\mathscr{P}_*\colon \pi_1(S_n, t) \to \Gamma_n$ (resp. $\mathscr{P}_*$ followed by the isomorphism $\Gamma_n \to \Gamma_n^{\mathrm{ad}}$)

Hodge structures on $L_{\mathbf{Z}}$ (parametrized by Ω^+) gives rise to a collection of polarizations $\{\varphi_{\underline{a}}\}$ of g. On the other hand, using auxiliary markings ε_t of $(\underline{Y}_t, \underline{\eta}_t)$ as above, one obtains for any point t of S_n a well-defined element of $\Gamma_n\backslash\mathrm{Isom}(C^+(P^{2k}(\underline{Y}_t,\mathbf{Z})(k),\langle\,,\rangle_{\underline{\eta}_t}),H^1(\underline{A}_t,\mathbf{Z}))$ (for the right action of Γ_n on $C^+(V_{\mathbf{Z}})$); the corresponding element $\overline{v}_t$ in etale cohomology is the one entering the definition of the canonical K.-S. package of $(\underline{Y}_t,\underline{\eta}_t)$[15].

The isomorphism of 4.2 admits a relative analog: denoting by f_n the pull-back of f on S_n, one has *a canonical isomorphism of sheaves of algebras on* S_n^{an}

$$\underline{\psi}\colon C^+((P^{2k}(f_n^{\mathrm{an}})_*\mathbf{Z}(k),\langle\,,\rangle_{\underline{\eta}})) \cong \underline{\mathrm{End}}_{C^+}R^1g_*^{\mathrm{an}}\mathbf{Z}\,,$$

(which induces a similar isomorphism $\underline{\psi}^{\wedge}$ in etale cohomology).

5.4

Let us now assume that *the monodromy of the morphism f in P^{2k} factorizes through the arithmetic group Γ_n^{ad}* (so that $S_n = S$), *and is Zariski-dense in $SO(V)$*. We note that because Γ_n lies in $\mathrm{Spin}\,V$, the monodromy of f is Zariski-dense in $SO(V)$ if and only if the monodromy of g (in H^1) is Zariski-dense in $\mathrm{Spin}\,V$; this is the case if the monodromy of f is of finite index in Γ_n and $N > 0$, e.g. in the situation occuring with axioms A_k and B_k (following the argument of [G71] D1).

Under this assumption, we point out two rigidity properties of the Kuga-Satake families.

Proposition 5.4.1. *For any commutative flat $\mathbf{Z}$-algebra R without zero-divisor, $\underline{\psi}$ is the unique isomorphism of sheaves of R-algebras*

$$C^+((P^{2k}f_*^{\mathrm{an}}R(k),\langle\,,\rangle_{\underline{\eta}})) \cong \underline{\mathrm{End}}_{C^+}R^1g_*^{\mathrm{an}}R\,.$$

See [D72] 5.7 and 3.5.

Proposition 5.4.2. *For any abelian scheme $g'\colon \underline{A}' \to S$, one has*

$$\mathrm{Hom}_S(\underline{A}',\underline{A}) \cong \mathrm{Hom}_S(R^1g_*^{\mathrm{an}}\mathbf{Z},R^1g'^{\mathrm{an}}_*\mathbf{Z})\,.$$

Proof. According to [D71a] 4.4.12, the conclusion holds if (and only if) both of the following conditions are satisfied: a) $\mathrm{End}_S(\underline{A}) \cong (\mathrm{End}_S(R^1g_*^{\mathrm{an}}\mathbf{Z}))^{\mathrm{op}}$, b) there is no complex embedding ρ of the center Z of $\mathrm{End}_S(\underline{A})$ such that the direct summand $R^1g_*^{\mathrm{an}}\mathbf{Z}\otimes_{Z,\rho}\mathbf{C}$ is of type $(1,0)$.

Remember that for any complex point s of S, $(R^1g_*^{\mathrm{an}}\mathbf{Z})_s$ is identified with $L_{\mathbf{Z}}$ (via a marking ε_s). The monodromy of g is Zariski-dense in $\mathrm{Spin}\,V$; in particular $\Gamma(S^{\mathrm{an}},R^1g_*^{\mathrm{an}}\mathbf{Z}) = 0$, and $(\mathrm{End}_S(R^1g_*^{\mathrm{an}}\mathbf{Z}))^{\mathrm{op}} \cong (\mathrm{End}_{\mathrm{Spin}\,V}V)^{\mathrm{op}} \cap (\mathrm{End}\,L_{\mathbf{Z}})^{\mathrm{op}} = C^+$.

On the other hand, $C^+\otimes\mathbf{Q}$ is a tensor product of quaternion $\mathbf{Q}$-algebras and its center $\mathscr{Z}(C^+\otimes\mathbf{Q})$, which is at most quadratic over $\mathbf{Q}$; keeping this in mind, one may apply loc. cit. 4.4.11, which settles a).

As for b), the case to be ruled out could occur only when $Z \otimes \mathbb{Q} \cong \mathscr{Z}(C^+ \otimes \mathbb{Q})$ is an imaginary quadratic field. Let us write $C^+((P^{2k} f_*^{an}\mathbb{C}(k))_s) \cong \operatorname{End} W^+ \oplus \operatorname{End} W^-$, where W^+ and W^- are the semi-spinorial representations of $G_{\mathbb{C}}$. Then, possibly after changing ρ into its conjugate, the direct summand $R^1 g_*^{an}\mathbb{Q} \otimes_{Z,\rho} \mathbb{C}$ of the $G_{\mathbb{C}}$-representation $R^1 g_*^{an}\mathbb{C} \cong L_{\mathbb{Z}} \otimes \mathbb{C}$ may be identified with a sum of copies of W^+. If it were of type (1,0), then W^- would be of type $(0,1)$, and the bigraded space $C^+((P^{2k} f_*^{an}\mathbb{C}(k))_s)$ would be of type $(0,0)$, which is however not the case.

5.5 *First proof of the main Lemma 1.7.1*

Let (Y, η) be a polarized variety defined over some subfield K of $\mathbb{C}$, which satisfies axioms A_k and B_k. We keep our usual notations $V_{\mathbb{Z}} = (P^{2k}(Y_{\mathbb{C}}, \mathbb{Z})(k), \langle\, , \rangle_\eta), \ldots$. In axiom B_k, we may replace S by a finite etale covering so that the monodromy of $f_{\mathbb{C}}\colon \underline{Y}_{\mathbb{C}} \to S_{\mathbb{C}}$ in P^{2k} is contained in $\Gamma_n = \Gamma_n^{ad}$; it is Zariski-dense in $SO(V)$, and one may consider the Kuga-Satake abelian scheme g: $\underline{A} := \underline{A}(\underline{Y}_{\mathbb{C}}) \to S_{\mathbb{C}}$, together with the identification $\mu\colon C^+ \cong \operatorname{End} \underline{A}$ as above.

The pair (g, μ) descends from $\mathbb{C}$ to the function field $K'(T)$ of some smooth connected algebraic variety T defined over a finite extension K' of K in $\mathbb{C}$ (which comes equipped with a Weil generic point $\tau \in T(\mathbb{C})$): one obtains an abelian scheme $h\colon \underline{B} \to S \times_K T$, and an isomorphism $v\colon C^+ \cong \operatorname{End} \underline{B}$ such that $(h_{(s,\tau)}, v_{(s,\tau)}) = (A(Y_{\mathbb{C}}), \text{inclusion})$. Moreover, one argues as in [D72] 6.5.1 (using 5.4.1 above) that there is a unique isomorphism of local system of rings:

$$(*) \qquad C^+((P^{2k}(f_{T_{\mathbb{C}}}^{an})_* \mathbb{Z}(k), \langle\, , \rangle_{\underline{\eta}})) \cong \underline{\operatorname{End}}_{C^+} R^1 h_*^{an} \mathbb{Z}\,;$$

this gives rise to an isomorphism of sheaves of $\mathbb{Z}^\wedge$-algebras:

$$(**) \qquad C^+((P_{\text{et}}^{2k} f_{T*} \mathbb{Z}^\wedge(k), \langle\, , \rangle_{\underline{\eta}})) \cong \underline{\operatorname{End}}_{C^+} R_{\text{et}}^1 h_* \mathbb{Z}^\wedge$$

(we remark at this point that axiom B_k ii) implies that η extends to a section of $R_{\text{et}}^2 f_{T*} \mathbb{Z}^\wedge(1)/\text{tors}$ over $S \times_K T$, still denoted by $\underline{\eta}$).

Let us now consider the pair (h', v') obtained from (h, v) by the base change $T_{\mathbb{C}} \to S \times_K T$ induced by $s_{\mathbb{C}} \in S(\mathbb{C})$.

Lemma 5.5.1. *(h', v') is isoconstant.*

This amounts to the finiteness of the image Γ of $\pi_1(T(\mathbb{C}), \tau)$ in $\operatorname{Aut}_{C^+} H^1(h'_\tau, \mathbb{Z})$. From the existence of $(*)$, it follows that $\underline{\operatorname{End}}_{C^+} R^1 h'^{an}_* \mathbb{Z}$ is a constant local system on $T(\mathbb{C})$; thus Γ is abelian. According to [D71a] 4.2.9, this implies that Γ is finite.

Replacing K' by a finite extension if necessary, we may assume that T admits a K'-rational point t. By the lemma, $(h_{(s,t)\mathbb{C}}, v_{(s,t)\mathbb{C}}) = (h_{(s,\tau)}, v_{(s,\tau)}) = (A(Y_{\mathbb{C}}), \text{inclusion})$. On the other hand, the fiber of $(**)$ at (s, t) gives a $\operatorname{Gal}(\overline{K}/K')$-invariant isomorphism $\psi^\wedge\colon C^+(P_{\text{et}}^{2k}(Y_{\overline{K}}, \mathbb{Z}^\wedge)(k), \langle\, , \rangle_\eta) \cong \underline{\operatorname{End}}_{C^+} H_{\text{et}}^1(h_{(s,t)_{\overline{K}}}, \mathbb{Z}^\wedge)$ (where $\overline{K}$ stands for the algebraic closure of K in $\mathbb{C}$), which by base change becomes identified with the isomorphism $\psi^\wedge\colon C^+(P^{2k}_{\text{et}}(Y_{\mathbb{C}},\mathbb{Z}^\wedge)(k),\langle\,,\rangle_\eta)\cong \underline{\mathrm{End}}_{C^+}H^1_{\text{et}}(A(Y_{\mathbb{C}}),\mathbb{Z}^\wedge)$ introduced in 4.2. This proves 1.7.1, by putting ${}_{K'}A = h_{(s,t)}$.

6 Proof of Theorem 1.5.1

The reader who is interested only in the Shafarevich problem 1.1.1 may skip sections 6.2.3 to 7.6.

This paragraph is a variation on the following general principle, which already underlay [D72]: *if a local system "of geometric origin" has a unique section (up to multiplication by a rational number), then this section should be motivated at every point.*

6.1

Lemma 6.1.1. *Let S be a smooth connected complex algebraic variety, $s\in S$ a point, and $h\colon \underline{X}\to S$ a projective smooth morphism. Then for any tensor construction of weight zero*[13] $\mathbb{T}H(\underline{X}_s,\mathbb{Q})$ *on* $H(\underline{X}_s,\mathbb{Q})$, *the fixed part* $(\mathbb{T}H(\underline{X}_s,\mathbb{Q}))^{\pi_1(S,s)}$ *is motivated, i.e. is the realization of a submotive of* $\mathbb{T}h(\underline{X}_s)$.

Proof. [A93]: replacing $\underline{X}$ by a suitable disjoint sum of fibered powers $\underline{X}\times_S\cdots\times_S\underline{X}$ and using Poincaré duality and Künneth decomposition, one reduces to the case $\mathbb{T}H(\underline{X}_s,\mathbb{Q})=H^i(\underline{X}_s,\mathbb{Q})$ for some $i\geqq 0$. Let $\overline{X}$ be a smooth compactification of $\underline{X}$, and let j_s denote the inclusion $\underline{X}_s\to\overline{X}$. By [D71a] 4.1.1, $H^i(\underline{X}_s,\mathbb{Q})^{\pi_1(S,s)}=j_s^*H^i(\overline{X},\mathbb{Q})$; hence it is motivated, since the category of motives (in the sense of 1.5) is abelian.

6.2

We consider a projective smooth morphism $f\colon\underline{Y}\to S$, a point s of S, and a section $\underline{\eta}$ of $R^2f^{\text{an}}_*\mathbb{Z}(1)/\text{tors}$ satisfying the assumptions of 5.1 (from which we keep the notation). We assume in addition that the monodromy of f in P^{2k} factorizes through Γ_n and is Zariski-dense in $SO(V)$. This allows to construct the Kuga-Satake abelian scheme $g\colon\underline{A}:=\underline{A}(\underline{Y})\to S$ and the isomorphism $\underline{\psi}\colon C^+(P^{2k}(f^{\text{an}})_*\mathbb{Z}(k))\cong\underline{\mathrm{End}}_{C^+}R^1g^{\text{an}}_*\mathbb{Z}$, with $\underline{\psi}_s=\psi$. The results of 5.4 are available.

Proposition 6.2.1. *ψ is motivated. In particular,* $C^+(p^{2k}(\underline{Y}_s)(k))\cong\underline{\mathcal{E}nd}_{C^+}h^1(\underline{A}_s)$.

Proof. Let us consider the tower of motives $\mathcal{M}=\underline{\mathcal{H}om}(C^+(p^{2k}(\underline{Y}_s)(k)),\underline{\mathcal{E}nd}_{C^+}h^1(\underline{A}_s))\subseteq\underline{\mathcal{H}om}(\bigoplus_{2i\leqq N+2}(p^{2k}(\underline{Y}_s)(k))^{\otimes 2i},\underline{\mathcal{E}nd}\,h^1(\underline{A}_s))\subseteq\underline{\mathcal{H}om}(\bigoplus_{2i\leqq N+2}(h^{2k}(\underline{Y}_s)(k))^{\otimes 2i},\underline{\mathcal{E}nd}\,h^1(\underline{A}_s))$ (via the decomposition $H^{2k}(\underline{Y}_s,\mathbb{Q})(k)=P^{2k}(\underline{Y}_s,\mathbb{Q})(k)\oplus H^{2k-2}(\underline{Y}_s,\mathbb{Q})(k-1)\cup\eta$).

[13] I.e. a finite sum of spaces $H^a(\underline{X}_s,\mathbb{Q})^{\otimes b}\otimes(H^c(\underline{X}_s,\mathbb{Q})^\vee)^{\otimes d}(e)$ with $ab=cd+2e$

Applying 6.1.1 to $\mathbb{T}H(\underline{Y}_s \cup \underline{A}_s, \mathbb{Q}) = \mathrm{Hom}(\bigoplus_{2i \leqq N+2}(H^{2k}(\underline{Y}_s)(k), \mathbb{Q})^{\otimes 2i}, \mathrm{End}\, H^1(\underline{A}_s, \mathbb{Q}))$, one concludes that $H(\mathcal{M})^{\pi_1(S,s)}$ is the realization of a submotive $\mathcal{M}_0$ of $\mathcal{M}$. By 5.4.1, ψ is the unique element of $H(\mathcal{M}_0)$ which is an algebra isomorphism. Because $C^+(\not{h}^{2k}(\underline{Y}_s)(k))$ and $\underline{\mathcal{E}nd}_{C^+}\, h^1(\underline{A}_s)$ are $\mathbb{Q}$-algebras in the tannakian category of motives, and by looking also at the top exterior power, the property of being an algebra isomorphism is preserved under the motivic Galois group. Hence ψ is fixed by the motivic Galois group, i.e. is motivated.

Corollary 6.2.2. *If N is even, $\det \not{h}^{2k}(\underline{Y}_s)(k) := \bigwedge^{N+2} \not{h}^{2k}(\underline{Y}_s)(k)$ is the unit motive.*

Proof. We use the filtration F_j of $C^+(V)$ defined as the image of $V^{\otimes \leqq 2j}$ in $C^+(V)$; this filtration is stable under $O(V)$. We have $F_{N/2+1}C^+(V) = C^+(V)$ and $\mathrm{Gr}_{N/2+1}C^+(V) \cong \mathrm{Det}\, V$, and there is a (non unique) lifting β: $\mathrm{Det}\, V \to C^+(V)$ such that $O(V)$ fixes $\beta(\mathrm{Det}\, V)$. Because the motivic group of $\not{h}^{2k}(\underline{Y}_s)(k)$ is a subgroup of $O(V)$, the filtration corresponds to a filtration by submotives $F_jC^+(\not{h}^{2k}(\underline{Y}_s)(k))$, and $\det \not{h}^{2k}(\underline{Y}_s)(k) \cong \beta \det \not{h}^{2k}(\underline{Y}_s(k)$, which ψ identifies with a submotive $\mathcal{N}$ of $\underline{\mathcal{E}nd}\, h^1(\underline{A}_s)$, on which the motivic group acts through $\{\pm 1\}$. Every element of $H(\mathcal{N})$ is then fixed under the Hodge group, hence is induced by an element of $\mathrm{End}\underline{A}_s \otimes \mathbb{Q}$. Thus $\det \not{h}^{2k}(\underline{Y}_s)(k) \cong \mathcal{N} \cong \mathbb{Q}(0)$, the unit motive.

Corollary 6.2.3. *If N is odd, $\not{h}^{2k}(\underline{Y}_s)(k) \otimes \det \not{h}^{2k}(\underline{Y}_s)(k)$ is (isomorphic to) a submotive of $\underline{\mathcal{E}nd}_{C^+}\, h^1(\underline{A}_s)$.*

Here, $\mathrm{Gr}_{(N+1)/2}C^+(V) \cong \bigwedge^{N+1} V \cong V \otimes \det V$ as $O(V)$-module, and there is a (non unique) lifting β': $V \otimes \det V \to C^+(V)$. Then $\psi \circ \beta'$ identifies $\not{h}^{2k}(\underline{Y}_s)(k) \otimes \det \not{h}^{2k}(\underline{Y}_s)(k)$ with a submotive of $\underline{\mathcal{E}nd}_{C^+}\, h^1(\underline{A}_s)$.

Variant 6.2.4. Let $V^b_{\mathbb{Z}}$ be the orthogonal complement to some algebraic classes in $(P^{2k}(\underline{Y}_s, \mathbb{Z})(k), \langle\, , \rangle_{\underline{\eta}_s})$, let $\mathcal{V}^b$ be the submotive of $\not{h}^{2k}(\underline{Y}_s)(k)$ with realization $V^b = V^b_{\mathbb{Z}} \otimes \mathbb{Q}$, and let us perform, as in 4.1.5, the corresponding Kuga-Satake construction. *Then the isomorphism ψ^b is motivated. In particular, $C^+(\mathcal{V}^b) \cong \underline{\mathcal{E}nd}_{C^{+b}}\, h^1(A^b)$.*

Indeed, ψ^b is by definition a Hodge correspondence, and since the category of polarized Hodge structures of weight zero is semisimple, there exists a Hodge correspondence π inducing a commutative diagram:

$$\begin{array}{ccc} C^+(H(\mathcal{V}^b)) & \rightarrow & C^+(P^{2k}(\underline{Y}_s, \mathbb{Q})(k)) \\ \psi^b \downarrow & & \downarrow \psi \\ \mathrm{End}_{C^{+b}}\, H^1(A^b, \mathbb{Q}) & \underset{\pi}{\leftarrow} & \mathrm{End}_{C^+} H^1(\underline{A}_s, \mathbb{Q}) . \end{array}$$

Note that the top arrow is the realization of the morphism of motives $C^+(\mathcal{V}^b) \to C^+(\not{h}^{2k}(\underline{Y}_s)(k))$ induced by the natural inclusion $\mathcal{V}^b \to \not{h}^{2k}(\underline{Y}_s)(k)$.

According to [A93], any Hodge correspondence (in particular π) on abelian varieties is motivated. It follows from that and 6.2.1 that ψ^b is motivated.

Remarks 6.2.5. i) The corollaries also hold in this ^{b} situation (with the same proof); we refer to them by $6.2.2^b$ and $6.2.3^b$ respectively.

ii) Proposition 6.2.1 and its corollaries apply as well to any fiber $\underline{Y}_t$, even though we do not assume that $\underline{\eta}_t$ (which is an algebraic class after Lefschetz) is a polarization.

iii) In 6.2.1 and 6.2.4, one may replace the quadratic form $\langle x, x\rangle$ by any positive rational multiple which takes integral values.

iv) Because motivated cycles are shown in [A93] to be absolute Hodge-Tate in the sense of A. Ogus [Og82], 6.2.1 gives a new proof of the main result of [Og84].

v) Although the big monodromy assumption forces $\dim V \geqq 3$ in 6.2.1, the instance $\dim V^b = 2$ is allowed in 6.2.4 (and $6.2.2^b$).

6.3

Still, some sign problems prevent us from deducing 1.5.1 from 6.2.1: we must exclude $-\mathrm{id}$ from the motivic Galois group of $\det \mathscr{h}^{2k}(\underline{Y}_s)(k)$ if N is odd, and $(-\mathrm{id}, \mathrm{id})$ from the motivic Galois group of $\mathscr{h}^{2k}(\underline{Y}_s)(k) \oplus \underline{\mathscr{E}nd}\, \mathscr{h}^1(\underline{A}_s)$ if N is even.

In either case, we shall need the deformation lemma of [A93] 0.5 (which is the basis of the proof of the fact that Hodge cycles on complex abelian varieties are motivated):

Lemma 6.3.1. *In the situation of Lemma* 6.1.1, *assume that the horizontal continuation (= parallel transport) of an element* $\xi \in (\mathbb{T}H(\underline{X}_s, \mathbb{Q}))^{\pi_1(S,s)}$ *at some point* $t \in S$ *is motivated. Then* ξ *is motivated.*

Proof. As in 6.1.1, $(\mathbb{T}H(\underline{X}_s, \mathbb{Q}))^{\pi_1(S,s)} = j_s^*\mathbb{T}H(\overline{X}, \mathbb{Q})$, $(\mathbb{T}H(\underline{X}_t, \mathbb{Q}))^{\pi_1(S,s)} = j_t^*\mathbb{T}H(\overline{X}, \mathbb{Q})$. The horizontal continuation of ξ at t generates a copy of the unit motive $\mathbb{Q}(0)$ in $(\mathbb{T}\mathscr{h}(\underline{X}_t))^{\pi_1(S,s)}$. Then, since the category of motives is semisimple, there is a corresponding copy (via j_t^*) of $\mathbb{Q}(0)$ in $\mathbb{T}\mathscr{h}(\overline{X})$; its image by j_s^* is a copy of $\mathbb{Q}(0)$ in $(\mathbb{T}\mathscr{h}(\underline{X}_s))^{\pi_1(S,s)}$, whose realization contains ξ. Hence ξ is motivated.

6.4

Let (Y, η) be a polarized variety over $\mathbb{C}$, satisfying properties A_k, B_k^+ of Sect. 1.4, and let us write A for $A(Y, \eta)$.

Lemma 6.4.1. $\det \mathscr{h}^{2k}(Y)(k)$ *is the unit motive.*

Proof. Axiom B_k brings us back to situation 6.2 (replacing S by a finite etale covering if necessary). If N is even, 6.4.1 follows from 6.2.2. Let us now assume that N is odd. If $P^{2k}(Y, \mathbb{Z})(k)$ happens to contain some non-zero algebraic class v, one may consider the orthogonal complement $V_{\mathbb{Z}}^b$ of v and conclude, via $6.2.4, 6.2.2^b$, that $\det \mathscr{V}^b$ is the unit motive (see also 6.2.5v) for the special case $N = 1$). Hence $\det \mathscr{h}^{2k}(Y)(k) \cong \det \mathscr{V}^b \cong \mathbb{Q}(0)$.

In the general case, one represents $\det P^{2k} f_*^{\mathrm{an}} \mathbb{Q}(k)$ as a local subsystem of $(R^{2k} f_*^{\mathrm{an}} \mathbb{Q}(k))^{\otimes(N+2)}$ (taking into account the $\pi_1(S, s)$-invariance of the primitive decomposition of $H^{2k}(Y, \mathbb{Q})(k)$). Then by 6.3.1 and the previous discussion, it suffices to find some point t of S such that $P^{2k}(\underline{Y}_t, \mathbb{Z})(k)$ contains an algebraic class; or equivalently, by B_k^+ iv), such that $\dim(V \cap H^{k,k}(\underline{Y}_t, \mathbb{C})(k)) > 0$.

By B_k iii), the image of the period mapping contains an open neighborhood of the period $\mathscr{P}(Y, \eta)$ in $\Omega^{\pm}/O(V_{\mathbb{Z}})$. One deduces that there is an open neighborhood $\mathscr{U}$ of s (for the usual topology), and a locally constant k-marking $\underline{\varepsilon}_t$ of $P^{2k}(\underline{Y}_t, \mathbb{Z})(k)$ on $\mathscr{U}$, such that the period mapping induces a surjection from $\mathscr{U}$ to an open neighborhood of the period $\mathscr{P}(\underline{Y}_s, \underline{\eta}_s, \underline{\varepsilon}_s)$ in Ω^+ (or Ω^-), which can be identified with an open subspace of the Grassmannian of (oriented) N-planes in $V_{\mathbb{R}}$. Therefore, there exist exceptional points in S, i.e. points t such that the real N-plane corresponding to $\mathscr{P}(\underline{Y}_t, \underline{\eta}_t, \underline{\varepsilon}_t)$ is defined over $\mathbb{Q}$; in fact the N-planes attached to such exceptional points are dense in the Grassmannian of N-planes in V (see e.g. a discussion of exceptional points in [X85] IX). For such an exceptional point t, one has $\dim(V \cap H^{k,k}(\underline{Y}_t, \mathbb{C})(k)) = N > 0$.

Remark 6.4.2. In the case of a K3 surface $Y = S$, there are at least two alternative arguments avoiding $6.2.2^b$:

i) one can use $S^{[2]}$ and the decomposition $h^2(S^{[2]})(1) \cong h^2(S)(1) \oplus \mathbb{Q}(0)$ (cf. [Be83a] Lemma 2), and conclude by 6.2.2 applied to $S^{[2]}$;

ii) once reduced as above to the case of an exceptional K3 surface S, one can use the description of S given by Shioda-Inose [ShI77] as a quotient of the product of two isogeneous CM elliptic curves, in order to show directly that $\det h^2(S)(1) \cong \mathbb{Q}(0)$.

Proposition 6.4.3. *$h^{2k}(Y)(k)$ is isomorphic to a submotive of $\underline{\mathscr{E}nd}\, h^1(A)$.*

This proposition implies 1.5.1 when $K = \mathbb{C}$.

Proof of 6.4.3. If N is odd, this follows from 6.4.1 and 6.2.3. Let us now assume that N is even. Then, unlike $C^+(V)$, $C^+(V^\#)$ is a faithful representation of $SO(V)$ (viewed as a subgroup of $SO(V^\#)$). Moreover, as $SO(V)$-modules, V is a factor of $V^\#$, itself a factor of $C^+(V^\#)$, which is a factor of $\underline{\mathrm{End}}\, L^\#$; and since the G-modules $L^\#$ and L^2 are isomorphic (4.1.3), $\underline{\mathrm{End}}\, L^\# \cong (\underline{\mathrm{End}}\, L)^4$ as $SO(V)$-modules. But because V is simple, we obtain the existence of an $SO(V)$-embedding $\beta''\colon V \to \underline{\mathrm{End}}\, L = L^\vee \otimes L$ (it may be reassuring to check this in the tables: [OnV88] Table 5, Formulas 8, 9).

On one hand, it follows from 5.2, 5.3 that the image of the monodromy homomorphism $\pi_1(S, s) \to SO(V) \times SO(V) \subseteq GL(P^{2k}(Y, \mathbb{Q})(k)) \times GL(\mathrm{End}\, H^1(A, \mathbb{Q}))$ is contained in the diagonal $\Gamma_n \subseteq SO(V)$. On the other hand, the image of the morphism $h_t\colon \prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to SO(V_{\mathbb{R}}) \times SO(V_{\mathbb{R}})$ which describes the Hodge structure on $P^{2k}(\underline{Y}_t, \mathbb{R})(k) \times \mathrm{End}\, H^1(\underline{A}_t, \mathbb{R})$ is contained in the diagonal $SO(V_{\mathbb{R}})$ for every $t \in S$. Hence β'' gives rise to an embedding of variations of Hodge structures $\beta''\colon P^{2k} f_*^{\mathrm{an}} \mathbb{Q}(k) \to \underline{\mathrm{End}}(R^1 g_*^{\mathrm{an}} \mathbb{Q}(p))$, and then to a morphism of variations of Hodge structures $\underline{\alpha}$: $R^{2k}f_*^{\mathrm{an}}\mathbb{Q}(k) \to \underline{\mathrm{End}}(R^1 g_*^{\mathrm{an}}\mathbb{Q}(p))$.

By 6.3.1, it suffices to show that $\underline{\alpha}$ is motivated at some point $t \in S$, e.g. an exceptional point as in the proof of 6.4.1. This reduces ourselves to showing that if $P^{2k}(\underline{Y}_t, \mathbb{Z})(k)$ contains some non-zero algebraic class v, then the Hodge correspondence $\underline{\beta}_t''$: $P^{2k}(\underline{Y}_t, \mathbb{Q})(k)) \to \operatorname{End} H^1(\underline{A}_t, \mathbb{Q})$ is motivated.

We mimic the argument of 6.2.4: consider the orthogonal complement $V_{\mathbb{Z}}^b$ of v, the associated submotive $\mathscr{V}^b$, and the associated Kuga-Satake variety A^b. There is a Hodge correspondence π inducing a commutative diagram:

$$\begin{array}{ccc} C^+(H(\mathscr{V}^b)) \oplus \mathbb{Q}(0) & \leftarrow & P^{2k}(\underline{Y}_t, \mathbb{Q})(k) \cong H(\mathscr{V}^b) \oplus \mathbb{Q}(0) \\ \psi^b \downarrow & & \downarrow \underline{\beta}_t'' \\ \operatorname{End} H^1(A^b, \mathbb{Q}) \oplus \mathbb{Q}(0) & \underset{\pi}{\to} & \operatorname{End} H^1(\underline{A}_t, \mathbb{Q}). \end{array}$$

The top arrow is a motivated embedding, ψ^b and π are motivated; hence so is $\underline{\beta}_t''$.

Corollary 6.4.4. *$\psi^\#$ is motivated. In particular, $C^+(\mathscr{h}^{2k}(Y)(k) \oplus \mathbb{Q}(0)) \cong \underline{\mathscr{E}nd}_{C^+}\, \mathscr{h}^1(A^\#)$.*

Indeed $C^+(\mathscr{h}^{2k}(Y)(k) \oplus \mathbb{Q}(0))$ is an object of $\mathscr{M}_K(\mathscr{A}b)$ (6.4.3), and one concludes that the Hodge correspondence $\psi^\#$ is motivated by using again the fact that Hodge correspondences on complex abelian varieties are motivated.

Corollary 6.4.5. *The motivic Galois group of $\mathscr{h}^{2k}(Y)(k) \oplus \mathscr{h}^1(A) \oplus \mathscr{h}^1(A^\#)$ is connected.*

Indeed, 6.4.3 shows that it is isomorphic to the motivic group of $\mathscr{h}^1(A \times A^\#)$, which is connected: any representation on which the connected component of 1 acts trivially is generated by Hodge cycles, necessarily motivated by [A93], once again.

6.5

Let K be a subfield of $\mathbb{C}$, let $\overline{K}$ denote the algebraic closure of K in $\mathbb{C}$, and let now (Y, η) be a polarized variety over K, satisfying properties A_k, B_k^+.

Lemma 6.5.1. *There exists a finite extension K' of K such that the Kuga-Satake varieties $A(Y_{\mathbb{C}})$ and $A^\#(Y_{\mathbb{C}})$ admit models ${}_{K'}A$, ${}_{K'}A^\#$ resp. over K', and such that the motivic Galois group of $\mathscr{h}^{2k}(Y_{K'})(k) \oplus \underline{\mathscr{E}nd}\, \mathscr{h}^1({}_{K'}A) \oplus \underline{\mathscr{E}nd}\, \mathscr{h}^1({}_{K'}A^\#)$ is connected. Then the embeddings $C^+ \to \operatorname{End} A(Y_{\mathbb{C}})$, $C^{+\#} \to \operatorname{End} A^\#(Y_{\mathbb{C}})$ descend to K', and ψ and $\psi^\#$ automatically descend to isomorphisms of motives over K'.*

Proof. The existence of ${}_{K'}A$ is guaranteed by 1.7.1, and the existence of a model ${}_{K'}A^\#$ (after replacement of K' by some finite extension) follows from the fact that $A(Y_{\mathbb{C}})^2$ and $A^\#(Y_{\mathbb{C}})$ are isogeneous (4.1.3). The motivic group of

$\not h^{2k}(Y_{\mathbb{C}})(k) \oplus \underline{\mathcal{E}nd}\, h^1(A(Y_{\mathbb{C}})) \oplus \underline{\mathcal{E}nd}\, h^1(A^\#(Y_{\mathbb{C}}))$ equals that of $\not h^{2k}(Y_{\overline{K}})(k) \oplus \underline{\mathcal{E}nd}\, h^1({}_{K'}A_{\overline{K}}) \oplus \underline{\mathcal{E}nd}\, h^1({}_{K'}A^\#_{\overline{K}})$, and is connected by 6.4.5. It follows that the motivic group of $\not h^{2k}(Y_{K'})(k) \oplus \underline{\mathcal{E}nd}\, h^1({}_{K'}A) \oplus \underline{\mathcal{E}nd}\, h^1({}_{K'}A^\#)$ is connected if and only if for some (of for every) prime ℓ, the Zariski-closure of the image of $\mathrm{Gal}(\overline{K}/K')$ in $O(P^{2k}_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)) \times GL(\mathrm{End}\, H^1_{\mathrm{et}}({}_{K'}A_{\overline{K}}, \mathbb{Q}_\ell))$ is connected. This becomes certainly the case after further replacement of K' by a finite extension (for details on such 'standard' properties of motives, we refer to [A93]).

At last, the embeddings $C^+ \to \mathrm{End}\, A(Y_{\mathbb{C}})$, $C^{+\#} \to \mathrm{End}\, A^\#(Y_{\mathbb{C}})$ descend automatically to some finite extension of K', and ψ and $\psi^\#$ being motivated over $\mathbb{C}$ (6.2.1, 6.4.5), they also descend to some finite extension of K'. Since the motivic Galois group of $\not h^{2k}(Y_{K'})(k) \oplus \underline{\mathcal{E}nd}\, h^1({}_{K'}A) \oplus \underline{\mathcal{E}nd}\, h^1({}_{K'}A^\#)$ is connected, they all descend to K' itself.

We now prove a result more precise than 1.5.1:

Theorem 6.5.2. *Let (Y, η) be a polarized variety over a subfield K of $\mathbb{C}$, satisfying properties A_k, B_k^+. Let K' be a finite extension of K and ${}_{K'}A$ be an abelian K'-variety as in* 6.5.1; *let us denote by* $\mathrm{Res}_{K'/K}\, {}_{K'}A$ *the abelian K-variety obtained by Weil's restriction of scalars.*

Then $\not h^{2k}(Y)(k)$ is a factor of $\underline{\mathcal{E}nd}\, h^1(\mathrm{Res}_{K'/K}\, {}_{K'}A)$.

Proof. From 6.4.3, we know that there is an embedding of motives over $\overline{K}$: $\not h^{2k}(Y_{\overline{K}})(k) \to \underline{\mathcal{E}nd}\, h^1({}_{K'}A_{\overline{K}})$; because the motivic group of $\not h^{2k}(Y_{K'})(k) \oplus \underline{\mathcal{E}nd}\, h^1({}_{K'}A)$ is connected, such an embedding automatically descends to an embedding of motives over K': $\not h^{2k}(Y_{K'})(k) \to \underline{\mathcal{E}nd}\, h^1({}_{K'}A)$. On the other hand, remember that restriction of scalars for motives corresponds to induction for the corresponding representations of motivic groups. The motive (over K) $\not h^{2k}(Y)(k)$ is a factor of $\mathrm{Res}_{K'/K}\, \not h^{2k}(Y_{K'})(k)$, $\mathrm{Res}_{K'/K}\, \underline{\mathcal{E}nd}\, h^1({}_{K'}A)$ is a factor of $\underline{\mathcal{E}nd}\, h^1(\mathrm{Res}_{K'/K}\, {}_{K'}A)$, and $\mathrm{Res}_{K'/K}\, h^1({}_{K'}A) = h^1(\mathrm{Res}_{K'/K}\, {}_{K'}A)$; whence the result.

Remarks 6.5.3. i) In this paragraph, one could have only assumed that Hodge classes in $P^{2k}(Y_{\mathbb{C}}, \mathbb{Q})(k)$ are motivated instead of algebraic. This does not really matter here, since algebraicity is obtained without pain in all our examples.

ii) Because Hodge and motivated classes coincide on abelian varieties defined over an algebraically closed subfield $K = \overline{K}$ of $\mathbb{C}$, the Hodge and motivic groups of $\underline{\mathcal{E}nd}\, h^1({}_{K'}A)$ coincide; it follows from 6.5.1 that the Hodge and motivic groups of $\not h^{2k}(Y)(k)$ coincide. Corollary 1.5.2 now follows from Zarhin's description of the Hodge group of $P^{2k}(Y_{\mathbb{C}}, \mathbb{Q})(k)$ [Za83].

7 Proof of Theorem 1.6.1

7.1

Part i) of 1.6.1 follows from 1.5.1 and Faltings' semisimplicity theorem for abelian varieties.

As for the proof of the remaining parts, let us remark beforehand that we may replace the number field K by a finite extension. In particular, we may and shall assume that 6.5.1 is satisfied for (Y,η) and (Y',η') with $K'=K$ (for k and h respectively, and for possibly different respective complex embeddings of K). We thus have at hand K-abelian varieties ${}_KA$, ${}_KA'$, ${}_KA^{\#}$, ${}_KA'^{\#}$ (which we may assume to be polarized), and we may and shall assume in addition that the motivic Galois group (say with respect to the ℓ-adic realization) of $h^1({}_KA)\oplus h^1({}_KA')\oplus h^1({}_KA^{\#})\oplus h^1({}_KA'^{\#})$ is connected.

We denote by V_ℓ the ℓ-adic realization of $p^{2k}(Y)(k)$, endowed with the quadratic form $\langle\,,\rangle_\eta$, and by L_ℓ the ℓ-adic realization of $h^1({}_KA)$. By 6.5.1, L_ℓ is a $\mathbb{Q}_\ell[\mathrm{Gal}(\overline{K}/K)]-C^+$-bimodule, and there is a motivated (hence $\mathrm{Gal}(\overline{K}/K)$-equivariant) isomorphism $C^+(V_\ell)\cong \underline{\mathrm{End}}_{C^+}L_\ell$. Analogous notation for (Y',η') will be understood.

We note that, by the latter connectedness assumption,[14] the motivic Galois group of $h^1({}_KA)\oplus h^1({}_KA')$ is contained in the product of even Clifford groups $G(V_\ell)\times G(V'_\ell)$, which contains a fortiori the image of $\mathrm{Gal}(\overline{K}/K)$ in $GL(L_\ell)\times GL(L'_\ell)$.

7.2

Proof of 1.6.1. ii): By 6.5.2, there exists a motivated embedding $V_\ell\subseteq \underline{\mathrm{End}}\,L_\ell$. Any $\mathrm{Gal}(\overline{K}/K)$-invariant element ξ in V_ℓ gives rise to an element of $\mathrm{End}_{\mathrm{Gal}(\overline{K}/K)}L_\ell$. According to Faltings [FW86], $\mathrm{End}_{\mathrm{Gal}(\overline{K}/K)}L_\ell=(\mathrm{End}\,{}_KA\otimes\mathbb{Q}_\ell)^{\mathrm{op}}$, from which it follows that ξ is a $\mathbb{Q}_\ell$-linear combination of motivated classes – in fact of algebraic classes, because of axiom B_k^+.

7.3

In the sequel, we assume that N is *odd*. The even case is dealt with by the same arguments, after applying throughout the # construction (replacing N by $N+1$). Let us first prove a special case of 1.6.1iv):

Lemma 7.3.1. *Any* $\mathrm{Gal}(\overline{K}/K)$*-equivariant isometry* $i\colon V_\ell\cong V'_\ell$ *is a* $\mathbb{Q}_\ell$*-linear combination of motivated correspondences.*

The isometry i gives rise to a composed $\mathrm{Gal}(\overline{K}/K)$-equivariant algebra isomorphism $j\colon \underline{\mathrm{End}}_{C^+}L_\ell\cong C^+(V_\ell)\cong C^+(V'_\ell)\cong\underline{\mathrm{End}}_{C^{+\prime}}L'_\ell$. Moreover, conjugation by i identifies $C^+\otimes\mathbb{Q}_\ell$ with $C^{+\prime}\otimes\mathbb{Q}_\ell$.

On the other hand, the connectedness assumption embodied in 7.1 implies that the image of $\mathrm{Gal}(\overline{K}/K)$ in $(O(V_\ell)\times O(V'_\ell))\times(GL(\underline{\mathrm{End}}\,L_\ell)\times GL(\underline{\mathrm{End}}\,L'_\ell))$ is contained in the diagonal $SO(V_\ell)\times SO(V'_\ell)$ (we note that $G(V_\ell)\times G(V'_\ell)$ acts on $\underline{\mathrm{End}}\,L_\ell\times\underline{\mathrm{End}}\,L'_\ell$ through $SO(V_\ell)\times SO(V'_\ell)$). If W, resp. W', denotes the spinorial representation of $G(V_\ell\otimes\overline{\mathbb{Q}}_\ell)$, resp. $G(V'_\ell\otimes\overline{\mathbb{Q}}_\ell)$,

[14] Or simply because C^+, $C^{+\prime}$ and polarizations ϕ_a, ϕ'_b are defined over K: C^+ commutes with any element γ of the ℓ-adic motivic group of ${}_KA$, hence γ is a unit in $C^+(V_\ell)$ acting on the left on L_ℓ; it respects ϕ_a, i.e. is a symplectic similitude w.r.t. $(x,y)\to\mathrm{tr}\,x^*ya$, whence $\gamma^*\gamma\in\mathbb{Q}_\ell$, i.e. $\gamma\in G(V_\ell)$

we have SO-equivariant isomorphisms of $\overline{\mathbb{Q}}_\ell$-algebras: $C^+(V_\ell \otimes \overline{\mathbb{Q}}_\ell) \cong \underline{\mathrm{End}}\, W$, $C^+(V'_\ell \otimes \overline{\mathbb{Q}}_\ell) \cong \underline{\mathrm{End}}\, W'$, $\underline{\mathrm{End}}\, L_\ell \otimes \overline{\mathbb{Q}}_\ell \cong M_{2(N+1)/2}(\underline{\mathrm{End}}\, W)$, $\underline{\mathrm{End}}\, L'_\ell \otimes \overline{\mathbb{Q}}_\ell \cong M_{2(N+1)/2}\,(\underline{\mathrm{End}}\, W')$. Therefore j induces a Galois isomorphism of $\overline{\mathbb{Q}}_\ell$-algebras: $J\colon \underline{\mathrm{End}}\, L_\ell \otimes \overline{\mathbb{Q}}_\ell \cong \underline{\mathrm{End}}\, L'_\ell \otimes \overline{\mathbb{Q}}_\ell$. Such an isomorphism comes from an isomorphism of $\overline{\mathbb{Q}}_\ell$-spaces $v\colon L_\ell \otimes \overline{\mathbb{Q}}_\ell \cong L'_\ell \otimes \overline{\mathbb{Q}}_\ell$, unique up to homothety.

On the other hand, using polarizations, J gives rise to a Galois isomorphism $(L_\ell \otimes \overline{\mathbb{Q}}_\ell)^{\otimes 2} \cong (L'_\ell \otimes \overline{\mathbb{Q}}_\ell)^{\otimes 2}$. One deduces that there is a *Galois* isomorphism $v'\colon L'_\ell \otimes \overline{\mathbb{Q}}_\ell \cong L_\ell \otimes \overline{\mathbb{Q}}_\ell$, by applying the following general result to the (connected) Zariski-closure of the image of $\mathrm{Gal}(\overline{K}/K)$ in $GL(L'_\ell \otimes \overline{\mathbb{Q}}_\ell) \times GL(L_\ell \otimes \overline{\mathbb{Q}}_\ell)$:

Sublemma 7.3.2. *Let G be a connected linear algebraic group over a field of characteristic 0. Let W and W' be finite dimensional semi-simple representations of G, such that for some $k > 0$, $W^{\otimes k} \cong W'^{\otimes k}$, Then $W \cong W'$.*

Indeed, taking the quotient of G by its unipotent radical if necessary, we may assume that G is reductive connected. Then $W \cong W'$ amounts to the equality of characters $\mathrm{ch}\, W = \mathrm{ch}\, W'$. Since $\mathrm{ch}\, W^{\otimes k} = (\mathrm{ch}\, W)^k$, the lemma follows from the fact that $\mathrm{ch}\, W$, as a *Laurent polynomial* in the fundamental weights of G with *non-negative integer coefficients*, is completely determined by its k^{th} power.

Now for any γ in the Zariski-closure G_{L_ℓ} of the image of $\mathrm{Gal}(\overline{K}/K)$ in $GL(L_\ell \otimes \overline{\mathbb{Q}}_\ell)$, we find that $\gamma^{-1}(v'v)^{-1}\gamma(v'v)$ lies in the center of $\underline{\mathrm{End}}\, L_\ell \otimes \overline{\mathbb{Q}}_\ell$, i.e. is a homothety of $L_\ell \otimes \overline{\mathbb{Q}}_\ell$. Being a commutator, it is of finite order, and because G_{L_ℓ} is connected, it is id. Hence v belongs to $\mathrm{Hom}_{\mathrm{Gal}(\overline{K}/K)}(L_\ell \otimes \overline{\mathbb{Q}}_\ell, L'_\ell \otimes \overline{\mathbb{Q}}_\ell) = \mathrm{Hom}_{\mathrm{Gal}(\overline{K}/K)}(L_\ell, L'_\ell) \otimes \overline{\mathbb{Q}}_\ell \cong \mathrm{Hom}({}_KA', {}_KA) \otimes \overline{\mathbb{Q}}_\ell$. It follows that J is a $\overline{\mathbb{Q}}_\ell$-linear combination of motivated correspondences, and that j is a $\mathbb{Q}_\ell$-linear combination of motivated correspondences (for brevity: $\mathbb{Q}_\ell$-motivated). Because ψ is motivated, it follows in turn that $C^+(i)$, $\mathrm{Gr}_{(N+1)/2}\, C^+(i)$, and thus i, are $\mathbb{Q}_\ell$-motivated (cf. 6.4.1, 6.2.3).

7.4

Proof of 1.6.1iii). It follows from point ii) (7.2) that the transcendental part $T^{2k}_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)$ of V_ℓ is just the orthogonal complement of $(V_\ell)^{\mathrm{Gal}(\overline{K}/K)}$, and from point i) that the Galois action on it is completely reducible. We denote by $t^{2k}(Y)(k)$ the submotive of $h^{2k}(Y_{\overline{K}})(k)$ with ℓ-adic realization $T^{2k}_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)$, and we set $E = \mathrm{End}(t^{2k}(Y)(k)) \cong \mathrm{End}(t^{2k}(Y_{\overline{K}})(k))$ (by the connectedness assumption 7.1). Then $E \otimes \mathbb{Q}_\ell$ is the commutant of the ℓ-adic motivic group $G_{\mathrm{mot},\ell}$ of $t^{2k}(Y)(k)$, and is contained in $\mathrm{End}_{\mathrm{Gal}(\overline{K}/K)} T^{2k}_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)$.

Lemma 7.4.1. $E \otimes \mathbb{Q}_\ell = \mathrm{End}_{\mathrm{Gal}(\overline{K}/K)} T^{2k}_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)$.

Proof. Let T be a simple $G_{\mathrm{mot},\ell}$-submodule of $T^{2k}_{\mathrm{et}}(\overline{Y}, \mathbb{Q}_\ell)(k)$. We have to show that T is simple as a Galois module. Let T' (resp. T'') be a $G_{\mathrm{mot},\ell}$-submodule of $T^{2k}_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k)$ (resp. of $T^\perp$) supplementary to $T^\perp$ (resp. to

T). Then $T_{\mathrm{et}}^{2k}(\overline{Y}, \mathbb{Q}_\ell)(k) = (T \oplus T')\bigoplus^{\perp} T''$, and either $T' = 0$ or T is totally isotropic. On the other hand, let T_1 be any simple $\mathbb{Q}_\ell[\mathrm{Gal}(\overline{K}/K)]$-submodule of T. Let T_1' (resp. T_1'') be a Galois submodule of $T \oplus T'$ (resp. of $T_1^{\perp}$) supplementary to $T_1^{\perp}$ (resp. T_1). Then $T \oplus T' = (T_1 \oplus T_1')\bigoplus^{\perp} T_1''$, and either $T_1' = 0$ or T_1 is totally isotropic. If $T_1' = T' = 0$, then $(-\mathrm{id}_{T_1}, \mathrm{id}_{T_1^{\perp} \oplus (V_\ell)^{\mathrm{Gal}(\overline{K}/K)}})$ is a Galois isometry of V_ℓ, hence $\mathbb{Q}_\ell$-motivated by 7.3.1. This shows that $T_1 = T$.

If T_1 is totally isotropic, then for any $\lambda \in \mathbb{Q}_\ell^*$, $(\lambda \cdot \mathrm{id}_{T_1}, \lambda^{-1} \cdot \mathrm{id}_{T_1'}, \mathrm{id}_{T_1'' \oplus (V_\ell)^{\mathrm{Gal}(\overline{K}/K)}})$ is a Galois isometry of V_ℓ, hence $\mathbb{Q}_\ell$-motivated by 7.3.1. One concludes again that $T_1 = T$.

Let us now finish the proof of iii). Because $G_{\mathrm{mot},\ell}$ is semisimple (cf. 1.5.2), it suffices to show that it is the Zariski-closure G_{V_ℓ} of the image of Galois in $SO(V_\ell)$ (actually, in $SO(T_{\mathrm{et}}^{2k}(Y_{\overline{K}}, \mathbb{Q}_\ell)(k))$). Remember that there is a Galois embedding $V_\ell \subseteq \underline{\mathrm{End}}\, L_\ell$. According to Tate-Raynaud, L_ℓ is a Hodge-Tate representation of $\mathrm{Gal}(\overline{K}/K)$ (see [Fo82]), hence so is V_ℓ. The ℓ-adic analogs of Zarhin's results, based on ℓ-adic Hodge-Tate decompositions and on a theorem of Kostant, and indicated by Zarhin himself in [Za83] 2.6.c, lead to the same description for G_{V_ℓ} and $G_{\mathrm{mot},\ell}$ as unitary groups.

7.5

Let us now finish the proof of 1.6.1iv). Let i be a Galois isomorphism $V_\ell \cong V_\ell'$, not necessarily isometric. Because N is odd, and because there are SO-isomorphisms $C^+(V_\ell) \cong \bigwedge^{\mathrm{even}} V_\ell$, $C^+(V_\ell') \cong \bigwedge^{\mathrm{even}} V_\ell'$, one draws from i a composed Galois isomorphism (not necessarily an algebra isomorphism) $j\colon \underline{\mathrm{End}}_{C^+} L_\ell \cong C^+(V_\ell) \cong C^+(V_\ell') \cong \underline{\mathrm{End}}_{C^{+\prime}} L_\ell'$. With the notation of 7.3, we have $C^+ \otimes \overline{\mathbb{Q}}_\ell \cong M_{2^{(N+1)/2}}(\overline{\mathbb{Q}}_\ell)$, corresponding to an isotypical decomposition $L_\ell \otimes \overline{\mathbb{Q}}_\ell \cong W^{2^{(N+1)/2}}$ as $G(V_\ell \otimes \overline{\mathbb{Q}}_\ell)$-modules (hence as Galois modules); idem for L_ℓ'. Therefore, using Spin-invariant bilinear forms on the spinorial representations, one draws from j a Galois isomorphism $W^{\otimes 2} \cong W'^{\otimes 2}$. As in 7.3, this implies that $W \cong W'$ as Galois modules, hence the existence of a Galois isomorphism $v'\colon L_\ell' \otimes \overline{\mathbb{Q}}_\ell \cong L_\ell \otimes \overline{\mathbb{Q}}_\ell$, such that $(C^+ \otimes \overline{\mathbb{Q}}_\ell)^{\mathrm{op}}\, v' = v'(C^{+\prime} \otimes \overline{\mathbb{Q}}_\ell)^{\mathrm{op}}$. Moreover, according to Faltings, v' is $\overline{\mathbb{Q}}_\ell$-linear combination of algebraic correspondences.

One deduces from v' a $\overline{\mathbb{Q}}_\ell$-motivated composed isomorphism of algebras: $j'\colon \underline{\mathrm{End}}_{C^{+\prime}} L_\ell' \otimes \overline{\mathbb{Q}}_\ell \cong C^+(V_\ell' \otimes \overline{\mathbb{Q}}_\ell) \cong C^+(V_\ell \otimes \overline{\mathbb{Q}}_\ell) \cong \underline{\mathrm{End}}_{C^+} L_\ell \otimes \overline{\mathbb{Q}}_\ell$. Now $j' \circ j_{\overline{\mathbb{Q}}_\ell}$ yields a Galois automorphism of $C^+(V_\ell \otimes \overline{\mathbb{Q}}_\ell)$, which is necessarily $\overline{\mathbb{Q}}_\ell$-motivated by point iii). In conclusion, j, $\bigwedge^{\mathrm{even}} i$ and thus i are $\mathbb{Q}_\ell$-motivated.

7.6

Corollary 7.6.1. *Let (Y, η) be a polarized K-variety defined over a number field $K \subset \mathbb{C}$, satisfying properties A_k, B_k^+. Let ${}_{K'}A$ be a model of the Kuga-*

Satake abelian variety attached to the P^{2k} *of* $(Y_{\mathbb{C}}, \eta_{\mathbb{C}})$ *over some finite extension* K'/K *(cf.* 1.7.1*). Then the image of* $\mathrm{Gal}(\overline{K}/K')$ *in* $GL(H^1_{\mathrm{et}}(({}_{K'}A)_{\overline{K}}, \mathbb{Q}_\ell))$ *is open in the* ℓ*-adic motivic Galois group of* $\mathfrak{h}^1({}_{K'}A)$.

Proof. We may replace K' by a finite extension, so that the Galois motivic group of $\mathfrak{p}^{2k}(Y_{K'})(k) \oplus \mathfrak{h}^1({}_{K'}A)$ is contained in $G(V_\ell)$ embedded "diagonally" in $SO(V_\ell) \times GL(L_\ell)$. A well-known result of Bogomolov tells that the image of $\mathrm{Gal}(\overline{K}/K')$ (which lies in $G(V_\ell)$ in our case) contains an open subgroup of the homotheties (which constitute the kernel of $G(V_\ell) \to SO(V_\ell)$ in our case). Hence 7.6.1 follows from 1.6.1iii).

8 Descent

This technical paragraph is logically independent from Sect. 5.5 *to* 7.6, *except for* 6.2.2 *which is used in* 8.2.1. *We give a second proof of Lemma* 1.7.1, *but in a much more precise form: we study the effect of conjugation by an arbitrary element* σ *of* $\mathrm{Aut}\,\mathbb{C}$ *on the Kuga-Satake package associated to a polarized variety satisfying axioms* A_k *and* B_k. *The only results of Sects.* 6, 7 *used here are* 6.2.1 *and* 6.2.2.

8.1

We consider a projective smooth morphism $f\colon \underline{Y} \to S$, a point s of S, and a section $\underline{\eta}$ of $R^2 f_*^{\mathrm{an}}\mathbb{Z}(1)/\mathrm{tors}$ satisfying the assumptions of 5.1.

We consider the conjugate morphism $f^\sigma\colon \underline{Y}^\sigma \to S^\sigma$. Because the fiber $\underline{\eta}_s$ a polarization, there is a well-defined conjugate polarization $(\underline{\eta}_s)^\sigma \in H^2(\underline{Y}^\sigma_{s^\sigma}, \mathbb{Z})(1)/\mathrm{tors}$ (viewed as a subgroup of $H^2_{\mathrm{et}}(\underline{Y}^\sigma_{s^\sigma}, \mathbb{Z}^\wedge)(1)/\mathrm{tors}$), which is invariant under $\pi_1^{\mathrm{alg}}(S^\sigma, s^\sigma)$; hence $(\underline{\eta}_s)^\sigma$ extends to a section $\underline{\eta}^\sigma$ of $R^2 f_*^{\sigma\mathrm{an}}\mathbb{Z}(1)/\mathrm{tors}$. On the other hand, the Hodge numbers h^{pq} attached to $P^{2k}(\underline{Y}_s, \underline{\eta}_s, \mathbb{Z})(k)$ and $P^{2k}(\underline{Y}^\sigma_{s^\sigma}, \underline{\eta}^\sigma_{s^\sigma}, \mathbb{Z})(k)$ are the same, because of their alternative algebraic definition; therefore the signature of $V_{\sigma,\mathbb{Z}} := (P^{2k}(\underline{Y}^\sigma_{s^\sigma}, \underline{\eta}^\sigma_{s^\sigma}, \mathbb{Z})(k), \langle\,,\rangle_{\underline{\eta}^\sigma_{s^\sigma}})$ is the signature of $V_{\mathbb{Z}}$, namely $(N+, 2-)$.

Replacing S by a finite etale covering if necessary, we may and shall assume that the monodromy of f (resp. f^σ) in P^{2k} factorizes through Γ_n (resp. through the analogous arithmetic group $\Gamma_{\sigma,n}$ relative to $V_{\sigma,\mathbb{Z}}$. This allows to construct the Kuga-Satake abelian schemes $g\colon \underline{A} := \underline{A}(\underline{Y}) \to S$ and $g_\sigma\colon \underline{A}(\underline{Y}^\sigma) \to S^\sigma$, and we take up the notation of 5.2, 5.3 again. We identify the $C^+(V_{\mathbb{Z}})$-modules $L_{\mathbb{Z}}$ and $C^+(V_{\mathbb{Z}})$, and the rings $C^+ = C^+(V_{\mathbb{Z}}) = \mathrm{End}_S\,\underline{A}$ (same conventions with respect to $V_{\sigma,\mathbb{Z}}\colon L_{\sigma,\mathbb{Z}} = C^+(V_{\sigma,\mathbb{Z}})$, and $C^+(V_{\sigma,\mathbb{Z}}) = C^+_\sigma = \mathrm{End}_{S^\sigma}\,\underline{A}(\underline{Y}^\sigma)$); these identifications are compatible with the canonical ring isomorphisms $\psi\colon C^+(V_{\mathbb{Z}}) = \mathrm{End}_{C^+} L_{\mathbb{Z}}, \psi_\sigma\colon C^+(V_{\sigma,\mathbb{Z}}) = \mathrm{End}_{C^+} L_{\sigma,\mathbb{Z}}$.

We wish to compare g with the σ^{-1}-conjugate of g_σ: $g'\colon \underline{A}' := \underline{A}(\underline{Y}^\sigma)^{\sigma^{-1}} \to S$, under the extra assumption that the *monodromy of f is Zariski-dense* in $SO(V)$.

We write $L'_{\mathbb{Z}} := H^1(\underline{A}'_s, \mathbb{Z})$, $L' = L'_{\mathbb{Z}} \otimes \mathbb{Q}$, $L'_{\mathbb{Z}^\wedge} = L'_{\mathbb{Z}} \otimes \mathbb{Z}^\wedge, \ldots, C^+ = \mathrm{End}_S \underline{A}'$ (identified with a subring of $\mathrm{End}\,\underline{A}'_s$ or of $(\mathrm{End}\,L'_{\mathbb{Z}})^{\mathrm{op}}$).

As an easy consequence of our monodromy assumptions, we record:

Lemma 8.1.1. *$\underline{\mathrm{End}}(R^1 g_*^{\mathrm{an}}\mathbb{Z}/n\mathbb{Z}, R^1 g_*'^{\mathrm{an}}\mathbb{Z}/n\mathbb{Z})$ is a constant local system.*

8.2

Somewhat abusively, we shall simply write σ_V (resp. σ_L) for σ-conjugation on etale cohomology $V_{\mathbb{Z}^\wedge} \to V_{\sigma,\mathbb{Z}^\wedge}$ (which is an isometry) (resp. $L'_{\mathbb{Z}^\wedge} \to L_{\sigma,\mathbb{Z}^\wedge}$).

Lemma 8.2.1. *There is an isometry $\alpha\colon V \cong V_\sigma$ such that $C^+(\alpha^{-1}\sigma_V) \in C^+(SO(V_{\mathbb{A}^f}))$.*

Proof. Because $V_{\mathbb{R}}$ and $V_{\sigma,\mathbb{R}}$ have the same signature, and $V_{\mathbb{A}^f} \cong V_{\sigma,\mathbb{A}^f}$ (via the isometry σ_V), the $\mathbb{Q}$-spaces V and V_σ are isometric (Hasse-Minkowski). If N is odd, $C^+(SO(V_{\mathbb{A}^f})) = C^+(O(V_{\mathbb{A}^f}))$ and the lemma follows. Let us now assume that N is even. By 6.2.2, any generator w of $\det V$ is motivated. Therefore $\det(\sigma_V)\cdot w$ lies in the rational subspace $\det V_\sigma$ of $\det V_{\sigma,\mathbb{A}^f}$; this means that for any (already found) isometry $\alpha\colon V \cong V_\sigma$, the determinant of $\alpha^{-1}\sigma_V$ is an overall sign ± 1, and after changing α by a symmetry if necessary, one obtains that $\alpha^{-1}\sigma_V \in SO(V_{\mathbb{A}^f})$.

8.3

Let γ be an element of $G(\mathbb{A}^f)$ which lifts $C^+(\alpha^{-1}\sigma_V) \in C^+(SO(V_{\mathbb{A}^f}))$, i.e. $\gamma c \gamma^{-1} = C^+(\alpha^{-1}\sigma_V)c$, for all $c \in C^+(V_{\mathbb{A}^f})$. We define $u_{\sigma,\alpha,\gamma}$ to be the composed isomorphism

$$u_{\sigma,\alpha,\gamma}\colon L_{\mathbb{A}^f} \overset{\gamma}{\cong} L_{\mathbb{A}^f} = C^+(V_{\mathbb{A}^f}) \overset{C^+(\alpha)}{\cong} C^+(V_{\sigma\mathbb{A}^f}) = L_{\sigma\mathbb{A}^f} \overset{\sigma_L^{-1}}{\cong} L'_{\mathbb{A}^f}.$$

Lemma 8.3.1. *The isomorphism $u^{\mathrm{ad}}_{\sigma,\alpha,\gamma}\colon \mathrm{End}_{C^+} L_{\mathbb{A}^f} \cong \mathrm{End}_{C^{+\prime}} L'_{\mathbb{A}^f}$ induced by $u_{\sigma,\alpha,\gamma}$ coincides with $(\psi_\sigma^\wedge)^{\sigma^{-1}} \circ \psi^{\wedge -1}$.*

Proof. By definition of γ, $\gamma^{\mathrm{ad}} = \psi^\wedge C^+(\alpha^{-1}\sigma_V)\psi^{\wedge -1} = \psi^\wedge C^+(\alpha)^{-1} C^+(\sigma_V) \circ \psi^{\wedge -1}$ as an automorphism of $\mathrm{End}_{C^+} L_{\mathbb{A}^f}$. One finds $u^{\mathrm{ad}}_{\sigma,\alpha,\gamma} = (\sigma_L^{-1})^{\mathrm{ad}} \circ \psi_\sigma^\wedge \circ C^+(\alpha) \circ \psi^{\wedge -1} \circ \gamma^{\mathrm{ad}} = (\sigma_L^{-1})^{\mathrm{ad}} \circ \psi_\sigma^\wedge \circ C^+(\sigma_V) \circ \psi^{\wedge -1} = (\psi_\sigma^\wedge)^{\sigma^{-1}} \circ \psi^{\wedge -1}$.

Lemma 8.3.2. *Let $t \in T^{k,l}H^1(\underline{A}_s, \mathbb{Q}) = L^{\otimes k} \otimes (L^\vee)^{\otimes l}$ be any tensor invariant under the even Clifford group $G = G(V)$. Then $\mathbb{T}^{k,l}u_{\sigma,\alpha,\gamma}(t)$ lies in the subspace $\mathbb{T}^{k,l}H^1(\underline{A}'_s, \mathbb{Q})$ of $\mathbb{T}^{k,l}H^1_{\mathrm{et}}(\underline{A}'_s, \mathbb{A}^f)$.*

Proof. Because t is G-invariant, $k = l$, and $\mathbb{T}^{k,l}u_{\sigma,\alpha,\gamma}(t) = \sigma_L^{-1}(\mathbb{T}^{k,l}C^+(\alpha))(t)$. Because α is an isometry, $(\mathbb{T}^{k,l}C^+(\alpha))(t)$ is an element of $\mathbb{T}^{k,l}L_\sigma$ invariant under $G(V_\sigma)$; in particular, it is a Hodge cycle $A(\underline{Y}^\sigma_{s^\sigma})$, hence a motivated cycle; therefore $\sigma_L^{-1}(\mathbb{T}^{k,l}C^+(\alpha))(t)$ is a motivated cycle on $\underline{A}'_s$, and lies in rational cohomology.

Lemma 8.3.3. *The isomorphism of $\mathbb{A}^f$-algebras $u^{\mathrm{int}}_{\sigma,\alpha,\gamma}\colon C^+\otimes\mathbb{A}^f\cong C^{+\prime}\otimes\mathbb{A}^f$ induced by conjugation by $u_{\sigma,\alpha,\gamma}$ comes from an isomorphism $C^+\otimes\mathbb{Q}\cong C^{+\prime}\otimes\mathbb{Q}$.*

This is a special case of 8.3.2, with $(k,l)=(1,1)$.

Lemma 8.3.4. *$u_{\sigma,\alpha,\gamma}$ is the fibre at s of an isomorphism $R^1_{\mathrm{et}}g_*\mathbb{A}^f\cong R^1_{\mathrm{et}}g'_*\mathbb{A}^f$.*

Proof. We have to show that $u_{\sigma,\alpha,\gamma}$ is invariant under the action of $\pi_1(S,s)$. By 8.3.1, this is at least the case for $u^{\mathrm{ad}}_{\sigma,\alpha,\gamma}$, since $(\psi^\wedge_\sigma)^{\sigma^{-1}}\circ\psi^{\wedge-1}$ is the fiber at s of an isomorphism $(\underline{\psi}^\wedge_\sigma)^{\sigma^{-1}}\circ\underline{\psi}^{\wedge-1}$ of etale sheaves (5.3). The set of isomorphisms $u\colon L_{\mathbb{A}^f}\cong L'_{\mathbb{A}^f}$ such that $u^{\mathrm{int}}=u^{\mathrm{int}}_{\sigma,\alpha,\gamma}\colon C^+\otimes\mathbb{A}^f\cong C^{+\prime}\otimes\mathbb{A}^f$ and $u^{\mathrm{ad}}=u^{\mathrm{ad}}_{\sigma,\alpha,\gamma}\colon \mathrm{End}_{C^+}L_{\mathbb{A}^f}\cong\mathrm{End}_{C^{+\prime}}L'_{\mathbb{A}^f}$, is stable under $\pi_1(S,s)$; moreover, any such u is deduced from $u_{\sigma,\alpha,\gamma}$ composed with a unit in the center of $C^+\otimes\mathbb{A}^f$. Therefore a suitable rational multiple of $u_{\sigma,\alpha,\gamma}$ generates an *abelian* representation of $\pi_1(S,s)$ in $\mathrm{End}(L_{\mathbb{Z}},L'_{\mathbb{Z}})\otimes\mathbb{Z}^\wedge$. Then, by [D71a] 4.2.9, this representation factorizes through a finite group; and since it is trivial mod. n (8.1.1), it is trivial, i.e. $u_{\sigma,\alpha,\gamma}$ is fixed under $\pi_1(S,s)$.

Proposition 8.3.5. *There exists an S-isogeny $\iota_{\sigma,\alpha}\colon\underline{A}'\to\underline{A}$, and an element $\lambda_{\sigma,\alpha,\gamma}\in(\mathbb{A}^f)^*$, such that $(\iota^*_{\sigma,\alpha})_s=\lambda_{\sigma,\alpha,\gamma}u_{\sigma,\alpha,\gamma}$.*

If moreover $u_{\sigma,\alpha,\gamma}$ comes from an isomorphism $L_{\mathbb{Z}^\wedge}\cong L'_{\mathbb{Z}^\wedge}$, then $\iota_{\sigma,\alpha}$ may be chosen (uniquely up to sign) to be an isomorphism.

Proof. Let us first notice that changing γ modifies $u_{\sigma,\alpha,\gamma}$ only by a factor in $(\mathbb{A}^f)^*$, so that if $\iota_{\sigma,\alpha}$ exists, one can choose it independently of γ. Let us *identify* $C^+\otimes\mathbb{Q}$ with $C^{+\prime}\otimes\mathbb{Q}$ via $u^{\mathrm{int}}_{\sigma,\alpha,\gamma}$, and consider the space U of $C^+\otimes\mathbb{Q}$-equivariant elements of $\mathrm{Hom}_S(\underline{A}',\underline{A})\otimes\mathbb{Q}$. Because the monodromy of f is Zariski-dense in $SO(V)$, the canonical morphism $U\to\mathrm{Hom}_{C^+\otimes\mathbb{Q}}(R^1g^{\mathrm{an}}_*\mathbb{Q},R^1g'^{\mathrm{an}}_*\mathbb{Q})$ is an isomorphism (cf. 5.4.2).

If N is odd, U is one-dimensional, and because $u_{\sigma,\alpha,\gamma}$ is the fibre at s of an element of $U\otimes\mathbb{A}^f$ (Lemma 8.3.1), the proposition follows immediately in this case.

If N is even, U is two-dimensional, and one can at first only deduce that there is an isogeny $\iota\colon\underline{A}'\to\underline{A}$ and a unit z in the center of $C^+\otimes\mathbb{A}^f$, such that $(\iota^*)_s\circ z=u_{\sigma,\alpha,\gamma}$.

Let $t\in\mathbb{T}^{k,l}H^1(\underline{A}_s,\mathbb{Q})=\mathbb{T}^{k,l}L$ be any G-invariant tensor. It follows from Lemma 8.3.2 that $\mathbb{T}^{k,l}(z)\cdot t=((\mathbb{T}^{k,l}(\iota^*)_s)^{-1}\circ\mathbb{T}^{k,l}(u_{\sigma,\alpha,g}))\cdot t\in\mathbb{T}^{k,l}L$; moreover, $\mathbb{T}^{k,l}(z)\cdot t$ is invariant under G, because the actions of z and G on L commute.

Let us denote by Z the two-dimensional torus in $GL(L)$ attached to the center of $C^+\otimes\mathbb{Q}$. Because $G\cap Z=\mathbb{G}_m$ (the homothety group), $Z/\mathbb{G}_m$ acts faithfully on some space of G-invariant tensors $\subset\mathbb{T}^{k,l}L$. On the other hand, the image $\bar{z}$ of z in $Z/\mathbb{G}_m(\mathbb{A}^f)$ belongs to $Z/\mathbb{G}_m(\mathbb{A}^f)\cap GL(\mathbb{T}^{k,l}L)=Z/\mathbb{G}_m(\mathbb{Q})$.

By Hilbert 90, there is an element z' of $Z(\mathbb{Q})$ which lifts $\bar{z}$; we have: $z'z^{-1} \in (\mathbb{A}^f)^*$. Setting $\iota_{\sigma,\alpha} := z'\iota$ and $\lambda_{\sigma,\alpha,\gamma} := z'z^{-1}$, one then has the required equality $(\iota^*_{\sigma,\alpha})_s = \lambda_{\sigma,\alpha,\gamma}\, u_{\sigma,\alpha,\gamma}$.

The second assertion follows: it suffices to replace $\lambda_{\sigma,\alpha,\gamma}$ (and $\iota_{\sigma,\alpha}$ accordingly) by a suitable rational multiple such that $\lambda_{\sigma,\alpha,\gamma} \in (\mathbb{Z}^\wedge)^*$.

8.4

Let K be a field embeddable into $\mathbb{C}$; let $\overline{K}$ be a fixed algebraic closure of K. A geometric object X being given over some subfield of $\overline{K}$, we let $\overline{X}$ stand for the corresponding object over $\overline{K}$ obtained by extension of scalars.

We are now ready to state our results on descent of Kuga-Satake packages (cf. 4.4 from which we adopt the notation), which imply a *stronger version of 1.7.1.*

Lemma 8.4.1. *Let (Y,η) be any polarized variety defined over K, satisfying A_k and B_k, and set $V_{\mathbb{Z}^\wedge} = (P^{2k}_{\mathrm{et}}(\overline{Y},\mathbb{Z}^\wedge)(k), \langle\,,\rangle_\eta)$. Then there is a subgroup of finite index of* $\mathrm{Gal}(\overline{K}/K)$ *which is mapped to the subgroup $\mathbb{K}^{\mathrm{ad}}_n$ of $O(V_{\mathbb{Z}^\wedge})$ under the natural Galois action. More precisely, for any finite extension K' of K, the image of* $\mathrm{Gal}(\overline{K}/K')$ *in $O(V_{\mathbb{Z}^\wedge})$ lies in $\mathbb{K}^{\mathrm{ad}}_n$ if and only if for each prime divisor ℓ of $2n$ and each of the finitely many odd primes ℓ such that $\langle\,,\rangle_\eta$ degenerates* $\mathrm{mod}\,\ell$, *the image of* $\mathrm{Gal}(\overline{K}/K')$ *in $O(V_{\mathbb{Z}^\wedge}\otimes\mathbb{Z}_\ell)$ lies in the subgroup of rotations which are images of elements congruent to* 1 mod. *n of the even Clifford group of $V_{\mathbb{Z}^\wedge}\otimes\mathbb{Z}_\ell$.*

Proof. The first assertion follows the second, and the "only if" part of the second assertion is trivial. Let us concentrate on the "if" part. We contend that the image of $\mathrm{Gal}(\overline{K}/K)$ in $O(V_{\mathbb{Z}^\wedge})$ lies inside $SO(V_{\mathbb{Z}^\wedge})$ if for some ℓ, its ℓ-adic component lies inside $SO(V_{\mathbb{Z}^\wedge}\otimes\mathbb{Z}_\ell)$; since this holds for instance for $\ell = 2$, by assumption, and since the ℓ-adic component of $\mathbb{K}^{\mathrm{ad}}_n$ is $SO(V_{\mathbb{Z}^\wedge}\otimes\mathbb{Z}_\ell)$ if $V_{\mathbb{Z}^\wedge}\otimes\mathbb{F}_\ell$ is a non-degenerate quadratic space and ℓ does not divide $2n$ (see 4.5), this will achieve our goal.

Let ℓ_1, ℓ_2 be rational primes, and let us assume that the image of $\mathrm{Gal}(\overline{K}/K')$ in $O(V_{\mathbb{Z}^\wedge}\otimes\mathbb{Z}_{\ell_1})$ lies inside $SO(V_{\mathbb{Z}^\wedge}\otimes\mathbb{Z}_{\ell_1})$. We may substitute to K' the function field of a finitely generated smooth $\mathbb{Z}$-algebra R (with $\ell_1\ell_2$ invertible in R), and assume that Y extends to a projective smooth scheme over R. We denote abusively by $Y_{\mathfrak{m}}$ its fiber at a maximal ideal $\mathfrak{m}$ of R (with finite residue field $\kappa(\mathfrak{m})$), and we let $\underline{x}$ be a closed point lying above $\mathfrak{m}$ in a connected component $\mathrm{Spec}(R\otimes\overline{\mathbb{Q}})^0$ of $\mathrm{Spec}(R\otimes\overline{\mathbb{Q}})$. For $i = k, k-1$ and $\ell = \ell_1, \ell_2$, the Galois action on $H^{2i}_{\mathrm{et}}(\overline{Y},\mathbb{Z}_\ell)(i)$ factorizes through $\pi_1^{\mathrm{alg}}(\mathrm{Spec}\,R, x)$. Because the determinant of the "geometric part" of the monodromy (action of $\pi_1^{\mathrm{alg}}(\mathrm{Spec}(R\otimes\overline{\mathbb{Q}})^0, x)$) on $\det H^{2i}_{\mathrm{et}}(\overline{Y},\mathbb{Z}_\ell)(i)$ is independent of ℓ, due to its interpretation in integral cohomology, we derive that for $\ell = \ell_1, \ell_2$, the Galois action on $\det H^{2i}_{\mathrm{et}}(\overline{Y},\mathbb{Z}_\ell)(i) \cong \det H^{2i}_{\mathrm{et}}(\overline{Y}_{\kappa(\mathfrak{m})},\mathbb{Z}_\ell)(i)$ factorizes through $\mathrm{Gal}(\overline{\kappa(\mathfrak{m})}/\kappa(\mathfrak{m}))$. It follows from [D74] that the determinant of the Frobenius element at $\mathfrak{m}$ on $H^{2i}_{\mathrm{et}}(\overline{Y}_{\kappa(\mathfrak{m})},\mathbb{Z}_\ell)(i)$ is the same for

$\ell = \ell_1$ and $\ell = \ell_2$. Hence Frobenius acts trivially on $\det P^{2k}_{\text{et}}(\overline{Y}_{\kappa(\mathfrak{m})}, \mathbb{Z}_{\ell_2})(k) \cong \det H^{2k}_{\text{et}}(\overline{Y}_{\kappa(\mathfrak{m})}, \mathbb{Z}_{\ell_2})(k) \otimes (\det H^{2k-2}_{\text{et}}(\overline{Y}_{\kappa(\mathfrak{m})}, \mathbb{Z}_{\ell_2})(k-1))^{\vee}$, because it acts trivially on $\det P^{2k}_{\text{et}}(\overline{Y}_{\kappa(\mathfrak{m})}, \mathbb{Z}_{\ell_1})(k) \cong \det V_{\mathbb{Z}^{\wedge}} \otimes \mathbb{Z}_{\ell_1}$ by assumption.

Remark 8.4.2. If in addition (Y, η) satisfies B^+_k and if one is willing to use 1.5.1 at this stage, one can avoid [D74] thanks to the following remark: $\det \mathfrak{h}^{2k}(Y)(k)$ is a rank one motive in $\mathcal{M}_K(\mathcal{A}\ell)$ of weight 0, hence it is an Artin motive (cf. [DM82]), and one can read whether the image of $\mathrm{Gal}(\overline{K}/K)$ in $O(V_{\mathbb{Z}^{\wedge}})$ lies inside $SO(V_{\mathbb{Z}^{\wedge}})$ on any ℓ-adic component.

Theorem 8.4.3. *Let (Y, η) be a polarized variety defined over K, satisfying A_k and B_k, and set $V_{\mathbb{Z}^{\wedge}} = (P^{2k}_{\text{et}}(\overline{Y}, \mathbb{Z}^{\wedge})(k), \langle\, ,\rangle_{\eta})$. Let us assume that, for $n = 3$ or $n = 4$, the image of* $\mathrm{Gal}(\overline{K}/K)$ *in $O(V_{\mathbb{Z}^{\wedge}})$ lies in $\mathbb{K}^{\text{ad}}_n$. Then:*

i) *for any embedding $\tau\colon K \to \mathbb{C}$, the canonical Kuga-Satake package of $(Y_{\mathbb{C}}, \eta_{\mathbb{C}})$ descends to K, i.e. there exists a Kuga-Satake package $(A_\tau = A_\tau(Y, \eta), \mu_\tau, \{\phi_{\tau,a}\}, \bar{v}_\tau)$ over K associated to the datum $(V_{\mathbb{Z}^{\wedge}}, C^+ = C^+(P^{2k}(Y_{\mathbb{C}}, \mathbb{Z})(k), n)$ whose τ-extension is isomorphic to the canonical Kuga-Satake package of $(Y_{\mathbb{C}}, \eta_{\mathbb{C}})$;*

ii) $\mathrm{Gal}(\overline{K}/K)$ *acts trivially on $H^1_{\text{et}}(\overline{A}_\tau, \mathbb{Z}/n\mathbb{Z})(k))$, i.e. the n-torsion points of $\overline{A}_\tau$ are rational over K;*

iii) *for any two embeddings $\tau, \tau'\colon K \to \mathbb{C}$, there is a K-isogeny $i_{\tau\tau'}\colon A_{\tau'} \to A_\tau$ such that $i_{\tau\tau'}(C^+_{\tau'} \otimes \mathbb{Q})i^{-1}_{\tau\tau'} = C^+_\tau \otimes \mathbb{Q}$ and $i^*_{\tau\tau'} \circ \psi^{\wedge}_\tau = \psi^{\wedge}_{\tau'} \circ i^*_{\tau\tau'}$.*

Example 8.4.4. In applications, it may be natural to take $n = 4$ for quartic surfaces, and $n = 3$ for cubic fourfolds. Let for instance Y be the Fermat quartic surface $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$. For $n = 4$, it is likely that one can take $K = \mathbb{Q}(e^{2i\pi/8})$. Using the fact that the $(-1, 1) + (1, -1)$ component of $P^2(Y_{\mathbb{C}}, \mathbb{Z})(1) \otimes \mathbb{R}$ is defined over $\mathbb{Z}$, and that the restriction of $\langle\, ,\rangle$ to this lattice is equivalent to $-8(x^2 + y^2)$, one can show that A_τ is isogeneous to the product of 2^{19} copies of the elliptic curve with affine equation $u^2 = 1 - v^4$ (a rational image of the Fermat quartic curve); C^+ is an order in $M_{2^{10}}(\mathbb{Q})$, whereas the full endomorphism ring of A_τ is an order in $M_{2^{19}}(\mathbb{Q}(\sqrt{-1}))$.

8.4.5. Proof of the theorem

Step 1. We make the preliminary remark that if $V_{\mathbb{Z}}$ is any quadratic lattice such that $V_{\mathbb{Z}} \otimes \mathbb{Z}^{\wedge} \cong V_{\mathbb{Z}^{\wedge}}$, then $\mathbb{K}^{\text{ad}}_n \cap O(V_{\mathbb{Z}} \otimes \mathbb{Q}) = \Gamma_n$. Indeed, any $\rho \in \mathbb{K}^{\text{ad}}_n \cap O(V_{\mathbb{Z}} \otimes \mathbb{Q}) = \mathbb{K}^{\text{ad}}_n \cap SO(V_{\mathbb{Z}} \otimes \mathbb{Q})$ is the image under $G_{\mathbb{A}^f} \to SO(V_{\mathbb{A}^f})$ of an element of κ of $\mathbb{K}_n$ and of an element γ of $G(\mathbb{Q})$; then $\kappa = q\lambda\gamma$, for some $q \in \mathbb{Q}^*$, $\lambda \in \mathbb{Z}^{\wedge *}$, and it turns out to be the image of $\pm q\gamma \in \mathbb{K}_n \cap G(\mathbb{Q})$, because $n = 3$ or 4.

Step 2. Let us consider a family $f\colon \underline{Y} \to S$ as in axiom B_k, with $\underline{Y}_s = Y$. Replacing S by a finite etale covering if necessary, we may assume that the monodromy homomorphism $\pi_1^{\text{alg}}(S, s) \to O(V_{\mathbb{Z}^{\wedge}})$ factorizes through $\mathbb{K}^{\text{ad}}_n$. Indeed, let us consider the subgroup of $\pi_1^{\text{alg}}(S, s)$ defined as the inverse image of $\mathbb{K}^{\text{ad}}_n$ under the monodromy homomorphism $\pi_1^{\text{alg}}(S, s) \to O(V_{\mathbb{Z}^{\wedge}})$. By the same argument as in 8.4.1, this is an open subgroup, which defines a (connected) finite étale covering of S. Moreover, it is easy to see that s lifts to a K-rational point of this covering. In addition, the image of $\pi_1^{\mathrm{alg}}(S, s) \to SO(V_{\mathbf{Z}^\wedge})$ is then Zariski-dense in SO.

Step 3. Let us fix an embedding $\bar{\tau}$ of $\bar{K}$ into $\mathbf{C}$. Then after extension of the scalars to $\mathbf{C}$, we see that are in the situation 8.1 (using Step 1). We use again the notation $V_{\mathbf{Z}} = P^{2k}(\underline{Y}_{s\mathbf{C}}, \mathbf{Z})(k)$, $V_{\mathbf{Z}^\wedge} = P^{2k}_{\mathrm{et}}(\bar{Y}, \mathbf{Z}^\wedge)(k)$ is identified with $V_{\mathbf{Z}} \otimes \mathbf{Z}^\wedge = P^{2k}_{\mathrm{et}}(\underline{Y}_{s\mathbf{C}}, \mathbf{Z}^\wedge)(k), \ldots$. For any automorphism σ of $\mathbf{C}$ which fixes K, the canonical isomorphism $(Y^\sigma_{\mathbf{C}}, \eta^\sigma_{\mathbf{C}}) \cong (Y_{\mathbf{C}}, \eta_{\mathbf{C}})$ induces an isometry α: $V_{\mathbf{Z}} = V_{\sigma,\mathbf{Z}}$. Because $\alpha^{-1}\sigma$ is nothing but the image of σ^{-1} acting on $P^{2k}_{\mathrm{et}}(\bar{Y}, \mathbf{Z}^\wedge)(k)$, which is by assumption an element of $\mathbb{K}_n^{\mathrm{ad}}$, there exists $\gamma \in \mathbb{K}_n$ with $\gamma^{\mathrm{ad}} = C^+(\alpha^{-1}\sigma)$; any two such elements γ differ by a unit in $\mathbf{Z}^\wedge$ congruent to 1 mod. n. Proposition 8.3.5 applies, and since $n = 3$ or 4, shows existence (and unicity) of an *isomorphism* $\iota_\sigma \colon \underline{A}(\underline{Y}^\sigma_{\mathbf{C}})^{\sigma^{-1}} \to \underline{A}(\underline{Y}_{\mathbf{C}})$ of abelian schemes over $S_{\mathbf{C}}$, such that $(\iota^*_\sigma)_s = \lambda u_{\sigma,\alpha,\gamma_\sigma}$ *for some* $\lambda \in (\mathbf{Z}^\wedge)^*$ *which is congruent to* 1 mod. n.

Step 4. The $(Y_{\mathbf{C}}, \eta_{\mathbf{C}}) \cong (Y^\sigma_{\mathbf{C}}, \eta^\sigma_{\mathbf{C}})$ induces an isomorphism of complex abelian varieties χ: $A(Y_{\mathbf{C}}) \to A(Y^\sigma_{\mathbf{C}})$, given in etale cohomology by the composed isomorphism $\chi^*\colon L_{\sigma\mathbf{Z}^\wedge} = C^+(V_{\sigma\mathbf{Z}^\wedge}) \overset{C^+(\alpha^{-1})}{\cong} C^+(V_{\mathbf{Z}^\wedge}) = L_{\mathbf{Z}^\wedge}$. On the other hand, the isomorphism of complex abelian varieties $((\iota_\sigma)_s)^\sigma\colon A(Y^\sigma_{\mathbf{C}}) \to A(Y_{\mathbf{C}})^\sigma$ is given in etale cohomology by the composed isomorphism

$$((\iota^*_\sigma)_s)^\sigma\colon L^\sigma_{\mathbf{Z}^\wedge} := H^1_{\mathrm{et}}(A(Y_{\mathbf{C}})^\sigma, \mathbf{Z}^\wedge) \overset{\sigma^{-1}}{\cong} L_{\mathbf{Z}^\wedge}$$

$$= C^+(V_{\mathbf{Z}^\wedge}) \overset{\lambda\gamma}{=} C^+(V_{\mathbf{Z}^\wedge}) \overset{C^+(\alpha)}{\cong} C^+(V_{\sigma,\mathbf{Z}^\wedge}) = L_{\sigma,\mathbf{Z}^\wedge}\,.$$

Let us now consider $j_\sigma := ((\iota_\sigma)_s)^\sigma \circ \chi\colon A(Y_{\mathbf{C}}) \to A(Y_{\mathbf{C}})^\sigma$, given on H^1_{et} by

$$j^*_\sigma\colon L^\sigma_{\mathbf{Z}^\wedge} \overset{\sigma^{-1}}{\cong} L_{\mathbf{Z}^\wedge} = C^+(V_{\mathbf{Z}^\wedge}) \overset{\lambda\gamma}{=} C^+(V_{\mathbf{Z}^\wedge}) = L_{\mathbf{Z}^\wedge}\,.$$

Thus, if μ denotes the canonical embedding $C^+ \to \operatorname{End} A(Y_{\mathbf{C}})$, $j^{\mathrm{int}}_\sigma \circ \mu$ coincides with μ^σ as an embedding $C^+ \to \operatorname{End} A(Y_{\mathbf{C}})^\sigma$, since $\lambda\gamma$ commutes with C^+.

Let now κ stand for $\lambda\gamma \in \mathbb{K}_n$, but viewed as an element of $C^+ \otimes \mathbf{Z}^\wedge$, acting on the right on $L_{\mathbf{Z}^\wedge}$. Then by definition of γ, we can write $j^*_\sigma = \upsilon \circ \kappa \circ (\upsilon^\sigma)^{-1}$, where υ stands for the identification $C^+(V_{\mathbf{Z}^\wedge}) = L_{\mathbf{Z}^\wedge}$, and υ^σ denotes the composed isomorphism $C^+(V_{\mathbf{Z}^\wedge}) \overset{C^+(\sigma)}{\cong} C^+(V_{\mathbf{Z}^\wedge}) = L_{\mathbf{Z}^\wedge} \overset{\sigma}{\cong} L^\sigma_{\mathbf{Z}^\wedge}$. In particular, υ and $j^*_\sigma \circ \upsilon^\sigma$ define the same class $\bar{\upsilon}$ in $\mathbb{K}_n \backslash \mathrm{Isom}(C^+(V_{\mathbf{Z}^\wedge}), H^1_{\mathrm{et}}(A(Y_{\mathbf{C}}), \mathbf{Z}^\wedge))$.

One concludes that j_σ establishes an isomorphism between the canonical K.-S. package of $(Y_{\mathbf{C}}, \eta_{\mathbf{C}})$, i.e. $(A(Y_{\mathbf{C}}), \mu, \{\varphi_a\}, \bar{\upsilon})$, and the K.-S. package $(A(Y_{\mathbf{C}})^\sigma, \mu^\sigma, \{\phi^\sigma_a\}, \bar{\upsilon}^\sigma)$ over $\mathbf{C}$ associated to the same datum $(V_{\mathbf{Z}^\wedge}, C^+, n)$ (4.7.1). Because K.-S. packages have no non-trivial automorphism (4.7.2), one can then descend the canonical K.-S. package of $(Y_{\mathbf{C}}, \eta_{\mathbf{C}})$ to K. Moreover, the K.-S. package over K so obtained depends only (up to isomorphism) upon the restriction of τ to K. This proves i).

Step 5. Proof of ii): embodied in the K.-S. package $(A_\tau = A_\tau(Y,\eta), \mu_\tau, \{\phi_{\tau,a}\}, \overline{v}_\tau)$ is a $\mathrm{Gal}(\overline{K}/K)$-equivariant isomorphism $\vartheta\colon C^+(V_{\mathbf{Z}^\wedge}) \otimes \mathbf{Z}/n\mathbf{Z} \cong H^1_{\mathrm{et}}(\overline{A}_\tau, \mathbf{Z}/n\mathbf{Z})$ (4.4.2), and the Galois action on the left-hand side is trivial by assumption.

Step 6. Proof of iii): Let σ be an automorphism of $\mathbf{C}$ such that $\tau' = \sigma \circ \tau$. Proposition 8.3.5 applies and shows that there is an isogeny $i_{\tau\tau'\mathbf{C}}\colon A_{\tau'\mathbf{C}} \to A_{\tau\mathbf{C}}$ satisfying the requirements over $\mathbf{C}$. However, this isogeny is certainly defined over some finite extension of K. In order to conclude that it is defined over K itself, it suffices to show the Zariski-closure of the image of $\mathrm{Gal}(\overline{K}/K)$ in $GL\,H^1_{\mathrm{et}}(\overline{A}_\tau \times \overline{A}_{\tau'}, \mathbf{Z}_\ell)$, for $\ell | n$, is connected. But this follows in a well-known way from the fact that the n-torsion points of $\overline{A}_\tau \times \overline{A}_{\tau'}$ are defined over K.

Corollary 8.4.6. *For any polarized variety (Y,η) over $\mathbf{C}$ satisfying A_k, B_k, and any automorphism σ of $\mathbf{C}$, $A(Y^\sigma,\eta^\sigma)$ is isogeneous to $A(Y,\eta)^\sigma$.*

8.5

Proposition 8.5.1. *If Y is a K3 surface, then for any two embeddings $\tau,\tau'\colon K \to \mathbf{C}$, the isogeny $i_{\tau\tau'}\colon A_{\tau'} \to A_\tau$ of* 8.4.3. iii) *may be chosen to be an isomorphism.*

Proof. It is well-known that the H^2 of K3 surfaces belong to a single isomorphism class of quadratic lattices (this follows from the classification of even unimodular lattices); in particular, $H^2(Y_{\mathbf{C}}, \mathbf{Z})$ and $H^2(Y^\sigma_{\mathbf{C}}, \mathbf{Z})$ are isometric. In fact, there exists an isometry between them which sends $\mathbf{Q}\eta \cap H^2(Y_{\mathbf{C}}, \mathbf{Z})$ to $\mathbf{Q}\eta^\sigma \cap H^2(Y^\sigma_{\mathbf{C}}, \mathbf{Z})$ (see [LP81] for an elementary proof). Therefore $V_{\mathbf{Z}}$ and $V_{\sigma,\mathbf{Z}}$ are isometric, hence there is a ring isomorphism $j\colon C^+ \cong C^{+\prime}$. On the other hand, A and A' are isogeneous by 8.3.5. Using the Skolem-Noether theorem, one sees that there is an isogeny $\iota\colon A' \to A$ such that $j = \iota^{\mathrm{int}}$. By Proposition 5.4.1 (applied to each ℓ-adic component) the isomorphism of sheaves of algebras $\iota^{\mathrm{ad}}\colon \mathrm{End}_{C^+} R^1_{\mathrm{et}} g_* \mathbf{A}^f \cong \mathrm{End}_{C^{+\prime}} R^1_{\mathrm{et}} g'_* \mathbf{A}^f$ is the unique one hence, by 8.3.1, $\iota^{\mathrm{ad}} = (\psi\sigma^\wedge)^{\sigma^{-1}} \circ \psi^{\wedge^{-1}}$ and comes from an isomorphism $\mathrm{End}_{C^+} R^1_{\mathrm{et}} g_* \mathbf{Z}^\wedge \cong \mathrm{End}_{C^{+\prime}} R^1_{\mathrm{et}} g'_* \mathbf{Z}^\wedge$. Because i is an isogeny, one concludes that ι^{ad} maps $\mathrm{End}_{C^+} R^1 g^{\mathrm{an}}_* \mathbf{Z}$ to $\mathrm{End}_{C^{+\prime}} R^1 g'^{\mathrm{an}}_* \mathbf{Z}$; therefore, with the identifications $C^+ \cong C^{+\prime}$ and $\mathrm{End}_{C^+} R^1 g^{\mathrm{an}}_* \mathbf{Z} \cong \mathrm{End}_{C^{+\prime}} R^1 g'^{\mathrm{an}}_* \mathbf{Z}$ (via ι^{int} and ι^{ad} resp.), $\iota_s^* L_{\mathbf{Z}}$ appears as a principal $C^+ - C^+$- bimodule. On the other hand, with the same identifications, $L'_{\mathbf{Z}^\wedge}$ appears as a principal $C^+_{\mathbf{Z}^\wedge} - C^+_{\mathbf{Z}^\wedge}$-bimodule. Since the center of C^+ is $\mathbf{Z}$ here, the fundamental theorem on Picard groups of orders [R75] 37.28 tells that the canonical morphism $\mathrm{Pic}\, C^+ \to \prod_\ell \mathrm{Pic}\, C^+_{\mathbf{Z}_\ell}$ is an isomorphism. Hence $L'_{\mathbf{Z}}$ is a *principal* $C^+ - C^+$-bimodule. It follows that, up to multiplying ι by a factor in $\mathbf{Q}^*$, $\iota_s^* L_{\mathbf{Z}} = L'_{\mathbf{Z}}$, hence ι_s is an isomorphism. It satisfies our requirements over $\mathbf{C}$. Because the n-torsion points of $\overline{A}_\tau \times \overline{A}_{\tau'}$ are defined over K, ι_s must be defined over K.

Remark 8.5.2. If (Y,η) is a polarized K3 surface over K, let us define the extension K'/K by the condition that $\mathrm{Gal}(\overline{K}/K')$ is the inverse image of $\mathbb{K}_4^{\mathrm{ad}}$ under $\mathrm{Gal}(\overline{K}/K') \to O(P^2_{\mathrm{et}}(\overline{Y}, \mathbb{Z}^\wedge)(1))$. Then K'/K is finite (8.4.1), and by 8.4.3–8.5.1, there exists an abelian K'-variety ${}_{K'}A$ such that the main Lemma 1.7.1 holds for *every* complex embedding of K'. We may call ${}_{K'}A$ the *canonical Kuga-Satake variety* of (Y,η).

Corollary 8.5.3. *Let (Y,η) be a polarized K3 surface over some subfield K_0 of $\mathbb{C}$, the field of moduli of the Kuga-Satake variety is K_0 itself, i.e. for any $\sigma \in \mathrm{Aut}(\mathbb{C}/K_0)$, $A(Y_{\mathbb{C}}, \eta_{\mathbb{C}}) \cong A(Y_{\mathbb{C}}, \eta_{\mathbb{C}})^\sigma$.*

9 Proof of Theorem 1.3.1

9.1

We first make precise our definition of good reduction for polarized varieties. Let $R_\wp$ be a discrete valuation ring with fraction field K and maximal ideal $\wp$. We assume $\mathrm{char}\, K = 0$, and fix an algebraic closure $\overline{K}$ of K (resp. $\overline{\kappa}$ of $R_\wp/\wp R_\wp$). According to Matsusaka, a K-rational numerical equivalence class on $\overline{Y} = Y \otimes \overline{K}$ may be identified with a $\mathrm{Gal}(\overline{K}/K)$-invariant element of the Neron-Severi group $NS\,\overline{Y}$ modulo torsion. We say that a polarized K-variety (Y,η) has *good reduction* at $\wp$ if Y extends to a smooth proper scheme $\mathcal{Y}_\wp$ over $R_\wp$, such that under the specialization map associated to $\mathcal{Y}_\wp$: $NS\,\overline{Y} \to NS(\mathcal{Y}_\wp \times \overline{\kappa})$, the image of η remains *ample.*

In the hyperkähler case, it then follows from Theorem 2 of [MaM64], and 2.1b) above, that $(\mathcal{Y}_\wp, \eta_\wp)$ is unique up to isomorphism. We shall slightly generalize 1.3.1:

Theorem 9.1.1. *Let R be a finitely generated commutative flat $\mathbb{Z}$-algebra without zero-divisor, and let K be the fraction field of R. Let r and d be positive integers, and $P(x)$ be a numerical polynomial.*

Then there are only finitely many isomorphy classes of polarized hyperkähler varieties (Y,η) with Hilbert polynomial $P(x)$ and $b_2 > 3$ (resp. of very polarized hyperkähler varieties of dimension $2r$, degree d and $b_2 > 3$, resp. of polarized K3 surfaces of degree d), with good reduction at every prime ideal $\wp$ of R of height one.

9.2

First reduction step. Let us fix $n = 3$ or 4. By localization, we may and shall assume that R is a regular ring, and that n is invertible in R (the most interesting case is of course when R is a *ring of S-integers in some number field*).

We fix an embedding τ of K into $\mathbb{C}$. Let us observe that the P^2 of the complexification of the polarized varieties which occur in the theorem form only finitely many isomorphism classes of quadratic lattices; this follows from the "limited family" argument of 2.3. Therefore, we may impose that $(P^2(Y_{\mathbb{C}}, \mathbb{Z})(1), \langle\,,\rangle_\eta) \cong V_{\mathbb{Z}}$, a fixed quadratic lattice of signature $(N+, 2-)$, with $N > 0$; and we may also fix this isomorphism up to an element of Γ_n. We also impose that the polarized Hodge structure on $V_{\mathbf{Z}}$ corresponds to a point of the component Ω^+/Γ_n of the moduli space.

9.3

Second reduction step. Since Y has good reduction at all primes of height one, it follows from the theorem on the purity of the branch locus that the representation of $\mathrm{Gal}(\overline{K}/K)$ on $P^2_{\mathrm{et}}(\overline{Y}, \mathbf{Z}^\wedge)(1)$ factorizes through $\pi_1^{\mathrm{alg}}(\mathrm{Spec}\, R)$. According to Hermite-Minkowski (and the topological finite generation of the geometric fundamental group of $R/\mathbf{Z}$) there exist only finitely many continuous homomorphisms from $\pi_1^{\mathrm{alg}}(\mathrm{Spec}\, R)$ to the finite group $\prod_{\ell | 2n \text{ or disc}(\,,\,)_\eta}(O(P^2_{\mathrm{et}}(\overline{Y}, \mathbf{Z}_\ell)(1)) \cap (\gamma \mathbb{K}_n^{\mathrm{ad}} \gamma^{-1})_\ell)$, where γ runs over $O(V_{\mathbf{Z}^\wedge})$.

In order to prove Theorem 9.1.1, we may[15] and shall replace R by its finite unramified extension determined by the intersection of the kernels of all these homomorphisms.

In virtue of 8.4.1, we may and shall assume, in addition to the previous constraints upon (Y,η), that $\pi_1^{\mathrm{alg}}(\mathrm{Spec}\, R)$ acts on $P^2_{\mathrm{et}}(\overline{Y}, \mathbf{Z}^\wedge)(1)$ through $\mathbb{K}_n^{\mathrm{ad}}$, so that there is a K-model A_τ of the Kuga-Satake variety of $(Y_{\mathbf{C}}, \eta_{\mathbf{C}})$, such that all the n-torsion points of A_τ are rational over K (8.4.3). Furthermore, if we fix a non-zero-divisor $a \in C^+ := C^+(V_{\mathbf{Z}})$ which satisfies $a^* = -a$ and such that the skew-symmetric form $\varphi_a(x,y) = \mathrm{tr}\,(x^* y a)$ defines a polarization of any weight-one Hodge structure on $C^+(V_{\mathbf{Z}}) \cong L_{\mathbf{Z}}$ parametrized by Ω^+ (see 4.1), the polarization of $A = A_\tau \otimes \mathbf{C}$ given by φ_a descends to a polarization of A_τ. Its degree D depends on the choice of a, but not on A.

Lemma 9.3.1. *If (Y,η) has good reduction at some prime $\wp$ of R of height one, so does A_τ.*

Proof. (cf. [D72]) Let ℓ be an odd prime distinct from the residual characteristic of $\wp$. Using the $\mathrm{Gal}(\overline{K}/K)$-isomorphism $\psi_\tau^\wedge$ embodied in the K.-S. package, and the good reduction hypothesis, we see that the inertia $I_\wp$ at $\wp$ acts trivially on $\mathrm{End}_{C^+} H^1_{\mathrm{et}}(\overline{A}_\tau, \mathbf{Z}_\ell)$. This implies that $I_\wp$ acts on $H^1_{\mathrm{et}}(\overline{A}_\tau, \mathbf{Z}_\ell)$ through the center of $C^+(V_{\mathbf{Z}_\ell})$. On the other hand, since the n-torsion points of A_τ are rational over K, the theory of semistable reduction tells that the inertia is unipotent. It follows that it is trivial, which means, by the Néron-Ogg-Shafarevich criterion, that A_τ has good reduction at $\wp$.

Remark 9.3.2. In this lemma, the good reduction hypothesis appears only in the guise that $I_\wp$ acts trivially on $V_{\mathbf{Z}_\ell}$.

9.4

We see that the K-models of the Kuga-Satake varieties attached to the (very) polarized hyperkähler varieties (Y,η) under consideration form a set of

[15] By Galois descent, because $\mathrm{Aut}(\overline{Y},\overline{\eta})$ is finite

isomorphism classes of *polarized abelian varieties over K of dimension* 2^{b_2-3} *and degree D, with level n-structure, and with good reduction at every prime ideal* $\wp$ of R *of height one.*

By Faltings' theorem, this set is finite. Let us notice that the moduli space of polarized complex abelian varieties over K of dimension 2^{b_2-3} *and degree D, with symplectic level n-structure, is a finite sum of quotients of a Siegel half-space* S^+ *by the principal congruence subgroup* Λ_n *of level n in a symplectic group, and that the natural mapping* $\Omega^+/\Gamma_n \to S^+/\Lambda_n$ *induced by the Kuga-Satake construction (the polarization* φ_a *being understood) is injective; it follows that the isomorphism class of the polarized Kuga-Satake variety with level n-structure determines the period of* $(Y_{\mathbf{C}}, \eta_{\mathbf{C}})$ *in* Ω^+/Γ_n*. Therefore, in virtue of 3.3.2, the complexification of the polarized hyperkähler varieties* (Y, η) *under consideration form a finite set of isomorphism classes; thus, in addition to the constraints that we have imposed before, we may fix the isomorphism class of* $(Y_{\mathbf{C}}, \eta_{\mathbf{C}}$, *marking mod.* $\Gamma_n)$.

9.5

For K3 surfaces, and more generally for varieties of K3 type which are deformations of $S^{[r]}$'s (see 1.2), it happens that the natural homomorphism $\kappa\colon \operatorname{Aut} Y_{\mathbf{C}} \to (\operatorname{Aut} H^2(Y_{\mathbf{C}}, \mathbf{Z})(1))^{\mathrm{op}}$ is injective [Be83b] Sect. 5. It follows that the triple $(Y_{\mathbf{C}}, \eta_{\mathbf{C}}, \text{marking mod.}\, \Gamma_n)$ has no non-trivial automorphism, hence determines the isomorphism class of (Y, η).

However κ is no longer injective in general. Let us for instance consider a variety $Y_{\mathbf{C}}$ of type K_r (see loc. cit.); then the group of $(r+1)$-torsion points of the auxiliary abelian surface A used in the construction of $Y_{\mathbf{C}}$, acting on $A^{[r+1]}$ and thus on $Y_{\mathbf{C}}$ by translation, lies in the kernel of κ.

We shall overcome this difficulty by using good reduction anew.

Lemma 9.5.1. *There are only finitely many isomorphism classes of polarized hyperkähler K-varieties* (Y, η) *with good reduction at every prime* $\wp$ *of R of height one such that, over* $\overline{K}$, $(\overline{Y}, \overline{\eta})$ *lies in a given isomorphism class.*

Proof. Let $\overline{R}_\wp$ denote the integral closure of $R_\wp$ in $\overline{K}$, and let $(\mathcal{Y}_\wp, \eta_\wp)$ denote the smooth proper $R_\wp$-model of (Y, η). We put $\mathcal{G} = \mathbf{Aut}(Y, \eta)$, $\mathcal{G}_\wp = \mathbf{Aut}(\mathcal{Y}_\wp, \eta_\wp)$, for one of the (Y, η) occurring in the lemma. Then the set of isomorphism classes referred to in the lemma can be identified with the subset of $H^1(\overline{K}/K, \mathcal{G})$ of elements lying in the image of $H^1(\overline{R}_\wp/R_\wp, \mathcal{G}_\wp)$ for every $\wp$.

After localizing R once again, one may assume that for every $\wp$, $H^0(\mathcal{Y}_\wp \otimes (\mathrm{R}_\wp/\wp R_\wp), T_{\mathcal{Y}_\wp \otimes (R_\wp/\wp R_\wp)}) = 0$; it follows that the Lie algebra of the fibres of $\mathcal{G}_\wp$ is trivial, and we derive on one hand that $\mathcal{G}_\wp$ is *quasi-finite, unramified,* and *flat* over $R_\wp$ (because the fibers are reduced and non empty, and $R_\wp$ is one-dimensional regular). On the other hand, it is *proper* by a theorem of Matsusaka-Mumford [MaM64] (which can be applied here because the reduction of $\mathcal{Y}_\wp$ mod. $\wp$ is not ruled, cf. 2.1b). Using a well-known result of Grothendieck, one concludes that $\mathcal{G}_\wp$ is *etale finite* over $R_\wp$, for every $\wp$, and so is every $\mathcal{G}_\wp$-torsor.

Now, elements of $H^1(\overline{K}/K, \mathcal{G})$ lying in the image of $H^1(\overline{R}_\wp/R_\wp, \mathcal{G}_\wp)$ may also be interpreted as generic fibres T of $\mathcal{G}_\wp$-torsors $T_\wp$, up to isomorphism. By the Grothendieck-Galois correspondence, the algebra of T is described by an action of $\mathrm{Gal}(\overline{K}/K)$ on $H^0(\mathbf{Aut}(\overline{Y}, \overline{\eta}))$; if T is the generic fibre of a $\mathcal{G}_\wp$-torsor (necessarily etale) for every $\wp$, this action factorizes through $\pi_1^{\mathrm{alg}}(\mathrm{Spec}\, R)$, and the Hermite-Minkowski argument of 9.3 applies to show finiteness.

This proves the lemma, and completes the proof of Theorem 9.1.1.

9.6 Cubic fourfolds

Theorem 9.6.1. *Let R be a finitely generated commutative flat $\mathbb{Z}$-algebra without zero-divisor, and let K be the fraction field of R. Then there are only finitely many orbits for $PGL(6,K)$ among all smooth cubic hypersurfaces in $\mathbb{P}^5_K$ which have good reduction outside every prime ideal $\wp$ of R of height one.*

Here are two ways of proof: One way consists in deducing 9.6.1 from 9.1.1 via the Abel-Jacobi map, making use of Remark 9.3.2. The other way is to mimic the arguments 9.1 to 9.4 in the case of cubic fourfolds. Both ways result in the finiteness of the set of triples $(\overline{Y}, \overline{\eta}, \mathrm{Gal}(\overline{K}/K)$-isometry $\varepsilon_n\colon P^2_{\mathrm{et}}(\overline{Y}, \mathbb{Z}/n\mathbb{Z})(1) \cong V_{\mathbb{Z}^\wedge} \otimes \mathbb{Z}/n\mathbb{Z})$ under consideration.

Let ι be an automorphism of such a triple. Then, because $\mathrm{Aut}(\overline{Y}, \overline{\eta})$ is finite (2.4), the image of ι in $\mathrm{Aut}\, H^4(Y_{\mathbb{C}}, \mathbb{Z})(2)$ is id, being id mod. n.

Writing $\mathrm{Aut}(\overline{Y}, \overline{\eta}) \to (\mathrm{Aut}\, H^4(Y_{\mathbb{C}}, \mathbb{Z})(2))^{\mathrm{op}}$ as a composition of injective morphisms

$$\mathrm{Aut}(\overline{Y}, \overline{\eta}) \to \mathrm{Aut}\, F \to (\mathrm{Aut}\, H^2(F_{\mathbb{C}}, \mathbb{Z})(1))^{\mathrm{op}} \to (\mathrm{Aut}\, H^4(Y_{\mathbb{C}}, \mathbb{Z})(2))^{\mathrm{op}}$$

(see 2.4, [Be83b] Sect. 5, 3.4 resp.), one concludes that $\iota = \mathrm{id}$. Therefore, up to K-isomorphism, there is a unique (Y, η) which induces a given triple $(\overline{Y}, \overline{\eta}, \varepsilon_n)$. This proves 9.6.1.

9.7

Proof of Corollaries 1.3.2, 1.3.3. Let us first recall that the discriminant of a n-ary form ϕ of degree d changes under a linear substitution $\sigma \in GL(n, \mathbb{Q})$ according to the rule $\mathrm{disc}\, \phi\sigma = (\det \sigma)^{d(d-1)^{n-1}}\, \mathrm{disc}\, \phi$. In particular, if both $\mathrm{disc}\, \phi$ and $\mathrm{disc}\, \phi\sigma$ are units in $\mathbb{Z}\left[\frac{1}{dm}\right]$, then so is $\det \sigma$. Let $\det^{(-1)}\left(\mathbb{Z}\left[\frac{1}{dm}\right]^*\right)$ denote the projection in $PGL(n, \mathbb{Q})$ of the preimage of $\mathbb{Z}\left[\frac{1}{dm}\right]^*$ in $GL(n, \mathbb{Q})$ with respect to the determinant map. Then

$\det^{(-1)}\left(\mathbb{Z}\left[\frac{1}{dm}\right]^*\right) / PSL(n,\mathbb{Q}) \cong PGL\left(n,\mathbb{Z}\left[\frac{1}{dm}\right]\right) / PSL\left(n,\mathbb{Z}\left[\frac{1}{dm}\right]\right) \cong$ $\mathbb{Z}\left[\frac{1}{dm}\right]^* / \left(\mathbb{Z}\left[\frac{1}{dm}\right]^*\right)^n$ is a finite group. Hence it is equivalent to say that there are only finitely many $PSL\left(n,\mathbb{Z}\left[\frac{1}{dm}\right]\right)$-orbits among all hypersurfaces of degree d in $\mathbb{P}^{n-1}_{\mathbb{Z}[\frac{1}{dm}]}$ which are smooth over $\mathbb{Z}\left[\frac{1}{dm}\right]$ and belong to a single $PGL(n,\mathbb{Q})$-orbit, *or* that the n-ary forms of degree d with coefficients in $\mathbb{Z}\left[\frac{1}{dm}\right]$, which are $SL(n,\mathbb{Q})$-conjugate to a given one with discriminant in $\mathbb{Z}\left[\frac{1}{dm}\right]^*$, can be divided into a finite number of $SL\left(n,\mathbb{Z}\left[\frac{1}{dm}\right]\right)$-orbits. But for $d \geqq 3, n \geqq 3$, the last statement is a classical result in reduction theory.[16]

On the other hand, Theorem 9.1.1 (resp. 9.6.1) implies that there are only finitely many orbits for $PGL(4,\mathbb{Q})$ (resp. $PGL(6,\mathbb{Q})$) among all smooth quartics (resp. cubics) over $\mathbb{Q}$ which have good reduction outside the prime divisors of $2m$ (resp. $3m$). One concludes that for $n = d = 4$ (resp. $n = 6$, $d = 3$), there are only finitely many orbits for $PGL\left(n,\mathbb{Z}\left[\frac{1}{\mathrm{dm}}\right]\right)$ among all smooth hypersurfaces of degree d in $\mathbb{P}^{n-1}_{\mathbb{Z}[\frac{1}{\mathrm{dm}}]}$.

Remark. It seems to be an interesting problem to determine in general for which pairs (n,d) this property holds.

Appendix 1

Spinorial Shimura varieties

In this logically independent appendix, we give a short discussion of moduli spaces for Kuga-Satake abelian varieties.

The notations being as in 4.1, let $(t_1,\ldots,t_m)$ be a sequence of mixed tensors such that G is the algebraic subgroup of $GL(L)$ which fixes the t_i's.

The Shimura variety attached to $(G,\Omega^\pm)$ is a complex proalgebraic variety $\mathrm{Sh}(G,\Omega^\pm)$ with complex points $\mathrm{Sh}(G,\Omega^\pm)(\mathbb{C}) = G(\mathbb{Q})\backslash\Omega^\pm \times G(\mathbb{A}^f)$ (see [D71b, D78] for a general reference, and [Sa66] for a study of the spinorial case). For $x \in \Omega^\pm$, $g \in G(\mathbb{A}^f)$, we denote by $[x,g]$ the corresponding point in $\mathrm{Sh}(G,\Omega^\pm)(\mathbb{C})$. There is an obvious continuous action of $G(\mathbb{A}^f)$ on the right: $[x,g]g' = [x,gg']$.

It turns out that $\mathrm{Sh}(G,\Omega^\pm)$ is a fine moduli scheme for triples $(A,(\theta_i),\gamma)$ up to "isogeny", where: A is a complex abelian variety, the θ_i's, $i = 0,\ldots,m$, are Hodge cycles, subject to the following condition: there exists an isomorphism δ: $H^1(A,\mathbb{Q}) \cong L$ mapping each θ_i to t_i (in the appropriate tensor

[16] Over $\mathbb{Z}$, this is a well-known result of Jordan [J]; see also [Bo69] 6.5, and [Bo63] 8.10 for the general case

constructions) such that $\delta \circ \widetilde{h} \circ \delta^{-1} \in \Omega^{\pm}$, where $\widetilde{h}$ stands for the morphism $\prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to GL(H^1(A, \mathbb{R}))$ which gives the Hodge structure; and γ is an isomorphism $H^1_{\text{et}}(A, \mathbb{A}^f) \cong L_{\mathbb{A}^f}$ mapping each θ_i to t_i.

To the triple $(A,(\theta_i),\gamma)$, one associates the 'modulus' $[\delta \circ \widetilde{h} \circ \delta^{-1}, \delta_{\mathbb{A}^f} \circ \gamma^{-1}]$. It is understood that $(A,(\theta_i),\gamma)$ and $(A',(\theta_i'),\gamma')$ are isogenous if there exists an isogeny $\iota\colon A \to A'$ such that $\gamma' = \gamma \circ \iota^*$ (this implies that $\iota^*(\theta_i') = (\theta_i)$).

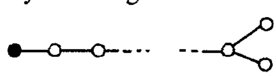
The choice of a $\mathbb{Z}^\wedge$-lattice in $L_{\mathbb{A}^f}$ fixes the universal abelian scheme inside its isogeny class. If the lattice is of the form $L_{\mathbb{Z}} \otimes \mathbb{Z}^\wedge$, then the endomorphism ring of the universal abelian scheme may be identified with C^+.

The quotient $\mathrm{Sh}(G, \Omega^{\pm})/\mathbb{K}_n$ is the fine moduli space for marked Kuga-Satake varieties with level n-structure. The set of connected components of $\mathrm{Sh}_{\mathbb{K}_n}(G, \Omega^{\pm})$ is in a canonical way a principal homogeneous space under the finite class group $\mathbb{Q}_+\backslash(\mathbb{A}^f)^*/\mathbb{N}(\mathbb{K}_n)$; this follows for instance from [D71b] 3.3. Each of these components is the quotient of Ω^+ of Ω^- by an arithmetic group in G.

It is well-known that $\mathrm{Sh}(G, \Omega^{\pm})$ admits a canonical model over the reflex field $E(G, \Omega^{\pm})$, on which $G(\mathbb{A}^f)$ acts continuously.

Lemma. *If rank $V > 4$, then $E(G, \Omega^{\pm}) = \mathbb{Q}$.*

Proof. By definition, $E(G, \Omega^{\pm})$ is the field of definition of the conjugacy class of the morphism $r\colon \mathbb{G}_m \to SO(V \otimes \mathbb{C})$ such that $r(\lambda) \cdot v^{pq} = \lambda^p \cdot v^{pq}$ when v^{pq} has Hodge structure of type (p,q) defines a Hodge structure of type $(-1,1)$ + $(0,0)$ + $(1,-1)$ on $V_{\mathbb{Z}}$, polarized by $\langle , \rangle$. Let T be a maximal torus in $SO(V \otimes \mathbb{C})$, endowed with a system of simple roots. Then according to [D72] 4.6, r is conjugate to the homomorphism $\mathbb{G}_m \to T$ which corresponds to the root labeled by the first left vertex in the Dynkin diagram

(B_n)  or (D_n) 

Except in the case of triality (D_4), this vertex is obviously fixed under $\mathrm{Aut}\,\mathbb{C}$, and one has $E(G, \Omega^{\pm}) = \mathbb{Q}$. In the D_4 case, one should remark in addition that the marked vertex corresponds to the standard orthogonal representation, which is defined over $\mathbb{Q}$ by assumption.

The universal abelian scheme on $\mathrm{Sh}(G, \Omega^{\pm})$ descends to an abelian scheme on the canonical model ${}_{\mathbb{Q}}\mathrm{Sh}(G, \Omega^{\pm})$, and passes to the quotient by $\mathbb{K}_n$.

Appendix 2

A short proof of the Hodge conjecture for cubic fourfolds

The notations are as in 2.4: Y is a complex cubic fourfold, (F, η') its polarized Fano variety, and $\alpha\colon H^4(Y, \mathbb{Q})(2) \to H^2(F, \mathbb{Q})(1)$ the Abel-Jacobi isomorphism. Let x be an element of Hodge type $(0,0)$ in $H^4(Y, \mathbb{Q})(2)$. Then after Lefschetz, $\alpha(x)$ is an algebraic class. On the other hand, the map $L^2\colon H^2(F, \mathbb{Q})(1) \to H^6(F, \mathbb{Q})(3)$ given by the cup-product with η'^2 is an isomorphism (hard Lefschetz' theorem); thus $\beta := {}^t\alpha \circ L^2 \circ \alpha$ is an automorphism of $H^4(Y, \mathbb{Q})(2)$ (induced by an algebraic correspondence). Lefschetz' trace formula gives, for any m, $\mathrm{tr}_{H^4(Y,\mathbb{Q})(2)}\beta^m = \langle \beta^m, \pi_{Y4} \rangle_{Y^2}$, where π_{Y4} stands for the fourth Künneth projector of Y. Since the cohomology of Y is algebraic except in degree 4, π_{Y4} is induced by an algebraic correspondence, and $\langle \beta^m, \pi_{Y4} \rangle_{Y^2} \in \mathbb{Q}$; hence the characteristic polynomial of β has rational coefficients. By Cayley-Hamilton, this implies that β^{-1}, as well as β, is induced by an algebraic correspondence. Therefore $x = \beta^{-1} \circ {}^t\alpha(\alpha(x) \cup \eta'^2)$ is an algebraic class. Q.E.D.

Remark. Because L^2 comes from an isomorphism $H^2(F, \mathbb{Z}[\frac{1}{6}])(1) \to H^6(F, \mathbb{Z}[\frac{1}{6}])(3)$ [BeD85] 6ii, the proof shows more precisely that any element of Hodge type (0,0) in $H^4(Y, \mathbb{Z}[\frac{1}{6}])(2)$ is a $\mathbb{Z}[\frac{1}{6}]$-linear combination of fundamental classes of surfaces on Y.

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