Concerning Triple Systems.

By

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The theory of triple systems of n elements [which we denote by the symbol Δ_n] and of the allied substitution-groups of n letters has been studied by Mr. Netto in his *Substitutionentheorie*, p. 220—235, 1882, (Cole's English translation, pp. 229—239, 1892), and more recently in his paper, *Zur Theorie der Tripelsysteme*, Mathematische Annalen, v. 42, pp. 143—152, 1893.

Two triple systems of the same number of elements are said to be *equivalent* if a 1.1 correspondence between the elements of the two systems can be established in such a way that to the elements of a triple in one system correspond the elements of a triple in the other system. All triple systems equivalent to a particular triple system belong to and constitute a *class* of triple systems.

Mr. Netto finds that n must be of the form $6m+1$ or $6m+3$, say of the form t. He raises two questions:

(a) *Do triple systems Δ_n exist for every positive integer n of the form t?*

(b) *For the same t do all the triple systems Δ_t belong to one class, or are there different classes?*

Towards an answer of the question (a) Mr. Netto in the paper last cited gives four constructions, two conditional (aα) (aβ) and two absolute (aγ) (aδ): viz., the construction

(aα) from a given Δ_t of a Δ_{2t+1};

(aβ) from given Δ_{t_1}, Δ_{t_2} of a $\Delta_{t_1 t_2}$;

(aγ) of a Δ_p where p is a prime of the form $6m+1$;

(aδ) of a Δ_{3q} where q is a prime of the form $6m+5$.

These constructions, with the initial Δ_3 immediately given, enable him to construct a Δ_t for every $t<100$, with the exception of $t=25$, $t=85$. In fact, as he might have stated more generally, these constructions together with a further construction*) $(a\varepsilon)$ of a Δ_t where t is the product of two (different or equal) primes of the form $6m+5$ would suffice for the construction of a Δ_t for every t without exception.

As to the question (b) Mr. Netto states that for each of the cases $t=3, 7, 9$ and apparently 13 there exists but one class**) of triple systems Δ_t.

In this paper by means of a conditional construction (A) of considerable generality (a threefold generalization of $(a\alpha)$) and using the Δ_3, Δ_9, Δ_{13} as given, I show how *to construct at least two distinct sorts of classes of triple systems of t elements for every t of the form* $6m+1$ *or* $6m+3$ *greater than* $t=13$. The word *sort* is used as a generic designation; two distinct classes belong to different sorts; a sort may contain many classes.

The two questions (a) (b) may now be formulated:

(c) *For a given t what is the number* $\varkappa_t$ *of classes of triple systems* Δ_t*?*

We know now

$$\varkappa_3 = 1, \quad \varkappa_7 = 1, \quad \varkappa_9 = 1, \quad \varkappa_{13} = 1 \quad \text{probably***)},$$
$$\varkappa_t \geqq 2 \quad \text{for} \quad t > 13.$$

In a later paper I shall consider certain triple systems whose groups are interesting. The groups are of course at most doubly transitive. Netto's Δ_p and Δ_{3q} have transitive groups which contain cyclic substitutions of all the elements. The Δ_{13} has a simply transitive composite group (Annalen, pp. 148—151). And if there is in fact but one class of Δ_{13}, then we do not for every t have triple systems Δ_t with *doubly* transitive groups. Mr. Netto studied the $\Delta_{3^\varkappa}$ derived from the initial Δ_3 by the $(a\beta)$ process (Substitutionentheorie, pp. 224—234); the groups are doubly transitive and composite. Perhaps the most interesting are the $\Delta_{2^\varkappa-1}$ derived from the initial Δ_3 by the $(a\alpha)$ process; the groups are simple, doubly transitive, with cyclic substitutions of all the $(2^\varkappa - 1)$ elements. [The case $\varkappa = 2$ is an exception; the Δ_3 has the symmetric group of degree 3 which is doubly transitive but composite.]

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*) If such could be given.

**) The Δ_{19} constructed directly by $(a\gamma)$ and the Δ_{19} constructed from a Δ_9 by $(a\alpha)$ are not equivalent.

***) Netto, Annalen, vol. 42, p. 152.

§ 1.

Definitions, notations and introductory theorems.

A triple system of t elements is an arrangement of the t elements into triples*) of elements in such a way that any pair*) of elements enters into one and only one triple of the system. There are in all $\frac{t(t-1)}{6}$ triples. t must have the form $6m+1$ or $6m+3$. We denote hereafter by t a number of the form $6m+1$ or $6m+3$, and by Δ_t a triple system of t elements, and agree for convenience to admit also $t=1$, denoting by Δ_1 a single element. If certain of the triples of a Δ_t determine a triple system of t_1 elements Δ_{t_1} the Δ_t is said to contain the Δ_{t_1}; unless $t=t_1$, $t \geqq 2t_1+1$. Every Δ_t contains t Δ_1 and $\frac{t(t-1)}{6}$ Δ_3.

It is desirable to introduce two new concepts. [First.] *A triple system, repetitions allowed, of s elements*, symbolized $\dot{\Delta}_s$, is an arrangement of the s elements into triples of elements in such a way that every pair of elements enters into one and only one triple, repetitions of elements being allowed in the pairs and triples. s may be any positive integer (see § 2). A Δ_t becomes a $\dot{\Delta}_t$ by adding to the triples of the Δ_t the t triples obtained by taking each element three times. [Second.] *A sub-triple system***) *of t sets of s elements each*, symbolized ${}_t\nabla_s$, is in the first place an arrangement of the t sets into a triple system Δ_t, and then an arrangement of the elements into triples in such a way that any pair of elements not belonging to the same set belong to one and only one triple, the third element of which belongs to the third set of the triple of sets determined in the Δ_t by the two sets to which the first two elements belong. A ${}_t\nabla_s$ contains $\frac{t(t-1)}{6}$ ${}_3\nabla_s$. *If the s elements of each of the t sets of the* ${}_t\nabla_s$ form a Δ_s or a $\dot{\Delta}_s$, then this ${}_t\nabla_s$ is contained in a Δ_{ts} or a $\dot{\Delta}_{ts}$, respectively.

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*) Repetitions not allowed.

**) For the purposes of this paper it is not necessary to study the *more general „sub-triple system"* of s_1 sets of s_2 elements each, in which any two elements of different sets belong to one and only one triple of which the third element belongs to still a different set (but in which the set of the third element is not necessarily determined merely by the sets of the first two elements). It is evident (1) that any triple system Δ_t of t elements decomposes with respect to one of its elements into the $\frac{t-1}{2}$ triples containing that element and a general sub-triple system in the $\frac{t-1}{2}$ pairs of elements associated with it by those triples, and (2) that, conversely, a triple-system of $2s+1$ elements can be made from a general sub-triple system of s pairs of elements.

[If $s_1=3$, this is really identical with the sub-triple system ${}_3\nabla_s$ of the text.]

§ 2.

Construction of a triple-system, repetitions allowed, in s elements Δ_s.

This construction may in general be made in many ways. A Δ_t is immediately derived from every Δ_t. I give here a table of all possible constructions for $s = 1, 2, 3, 4$, and then give a particular construction applicable for every value of s.

$\Delta_{s=1}$.	(1)	aaa.
$\Delta_{s=2}$.	(1)	aaa, abb.
$\Delta_{s=3}$.	(1)	aaa, abc, bbb, ccc.
	(2)	aab, acc, bbc.
$\Delta_{s=4}$.	(1)	aaa, abb, acc, add, bcd.
	(2)	aaa, abb, acd, bcc, bdd.

The case $s = 3$ illustrates the fact that in a Δ_s the number of triples depends not only on s but also on the internal structure of the Δ_s.

Δ_s. *A particular**) *construction* applicable for every value of s. Arrange the s elements cyclically thus, $x_0 x_1 x_2 \dots x_{s-1}$, and let x_i, x_j, x_k be a triple whenever $i + j + k \equiv 0 \pmod{s}$. Thus x_i, x_j (where i and j are either equal or unequal) belong to one and only one triple.

§ 3.

Construction of a sub-triple system ${}_3\nabla_s$.

Take three sets (x) (y) (z) of s elements each and establish arbitrarily among the elements of the different sets a 1 . 1 correspondence by the use of the subscript-notation,

$$x_i \backsim y_i \backsim z_i \quad (i = 0, 1, 2, \dots s-1).$$

Construct in any way (§ 2) in the s elements (x) a Δ_s; let $x_i x_j x_k$ be a triple. Then the system of triples $x_i y_j z_k$ will be a ${}_3\nabla_s$.

§ 4.

Construction of a sub-triple system ${}_3\nabla_{s_1 s_2}$ which shall contain $s_1^2\, {}_3\nabla_{s_2}$.

Take three sets of $s_1 s_2$ elements each. Separate the elements of each set into s_1 sub-sets of s_2 elements each. Form (in any way; § 3)

*) This construction gives for

$s = 3$ $\Delta_{s=3}$ (1), $(a, b, c) = (x_0, x_1, x_2)$,

and for $s = 4$ $\Delta_{s=4}$ (2), $(a, b, c, d) = (x_0, x_2, x_1, x_3)$.

of the three sets of s_1 sub-sets a ${}_3\nabla_{s_1}$; this contains s_1^2 triples of sub-sets. Every such triple of sub-sets may be looked at as three sets of s_2 elements each and so we form (in any way; § 3) to a ${}_3\nabla_{s_2}$. We have then in fact constructed a ${}_3\nabla_{s_1 s_2}$ which contains s_1^2 ${}_3\nabla_{s_2}$.

§ 5.

Construction from a given triple system Δ_t of a sub-triple system ${}_t\nabla_s$.

Take t sets of s elements each. Form a Δ_t in the t sets analogous to the given Δ_t. Construct for each triple of sets of s elements (in any way; § 3) a ${}_3\nabla_s$. We have then formed a sub-triple system ${}_t\nabla_s$.

If we use the *particular* method of §§ 3, 2 in the construction of the ${}_3\nabla_s$ from a triple of sets in the Δ_t, we may distribute the s subscripts $0, 1, 2 \ldots s-1$ over the s elements of a set *arbitrarily for each triple* in which the set enters, or we may allow *one arbitrary* distribution for the elements of each set to suffice. The latter gives the following

Particular construction.

Let the given Δ_t be given in the t elements u_f $(f = 1, 2, \ldots t)$; let $u_\alpha u_\beta u_\gamma$ be any triple.

As elements for our ${}_t\nabla_s$ take

$$x_{fg} \begin{pmatrix} f = 1, 2 & \ldots t \\ g = 0, 1, 2 \ldots & s-1 \end{pmatrix},$$

and at once construct the triples $x_{\alpha i} x_{\beta j} x_{\gamma k}$ where $i+j+k \equiv 0 \pmod{s}$. (α, β, γ are different; i, j, k are not necessarily different). Clearly this ${}_t\nabla_s$ contains a Δ_t in the t elements

$$x_{f0} \ (f = 1, 2 \ldots t).$$

If $s = s_1 s_2$, we may, using § 4 for the various ${}_3\nabla_s$, construct the ${}_t\nabla_{s_1 s_2}$ so as to contain $\frac{t(t-1)}{6} \cdot s_1^2 \; {}_3\nabla_{s_2}$.

§ 6.

Construction of a triple system $\Delta_{t_1 t_2}$ from given triple systems Δ_{t_1}, Δ_{t_2}.

Take t_1 sets of t_2 elements each. Construct in each set of t_2 elements (in each set in any way; one such way is possible by copying the given Δ_{t_2}) a Δ_{t_2}. Construct in the t_1 sets of t_2 elements (in any way; § 5) a ${}_{t_1}\nabla_{t_2}$. Then we have in fact made a $\Delta_{t_1 t_2}$. The $\Delta_{t_1 t_2}$ contains t_1 Δ_{t_2}.

Netto's particular construction.

In each set of t_2 elements construct a copy of the given Δ_{t_2}, and in this way set the elements of the t_1 different sets in 1.1 correspondence.

Containing each such Δ_{t_2} make a Δ_{t_2} (§ 1). Make use of these correspondences and of these Δ_{t_2} in the construction of the ${}_{t_1}\nabla_{t_2}$ (§§ 5, 3). This is the equivalent of Netto's construction $(a\beta)$ of a $\Delta_{t_1 t_2}$ from given Δ_{t_1}, Δ_{t_2} (Substitutionentheorie, § 193; Annalen 42, p. 144). This $\Delta_{t_1 t_2}$ contains t_1 Δ_{t_2} and t_2 Δ_{t_1}.

§ 7.

Construction (A) of a triple system Δ_t, where $t = t_3 + t_1(t_2 - t_3)$, and $t_1 \geqq 3$, $t_2 \geqq 2t_3 + 1$, $t_3 \geqq 1$, from given triple-systems Δ_{t_1}, Δ_{t_2}, Δ_{t_3} of which the Δ_{t_2} contains the Δ_{t_3}.

Take t elements and separate them into t_3 elements (α) and t_1 sets of $t_2 - t_3$ elements each (β_i) $(i = 1, 2 \ldots t_1)$. Construct in the (α) elements a Δ_{t_3} (α) and in the (α) (β_i) elements a Δ_{t_2} (α) (β_i) which contains this Δ_{t_3} (α), $(i = 1, 2 \ldots t_1)$. (These constructions are possible, in accordance with the types of triple systems furnished by the hypothesis.) Construct (in any way, § 5; or, to fix the ideas, by the particular*) method, § 5) in the t_1 sets of $t_2 - t_3$ elements (β) a sub-triple system ${}_{t_1}\nabla_{t_2-t_3}$ (β). We have then in fact constructed the Δ_t required. The pairs of elements determine uniquely a third element, as follows:

$$
\begin{array}{lll}
\alpha' \ \alpha'', & \alpha''' & \text{in the } \Delta_{t_3}\ (\alpha), \\
\alpha \ \beta_i', & \beta_i'' & \text{in the } \Delta_{t_2}\ (\alpha)\ (\beta_i), \\
\beta_i' \ \beta_i''', & \alpha \text{ or } \beta_i^{IV} & \text{in the } \Delta_{t_2}\ (\alpha)\ (\beta_i), \\
\beta_i \ \beta_j, & \beta_k & \text{in the } {}_{t_1}\nabla_{t_2-t_3}\ (\beta).
\end{array}
$$

[If $t_3 = 0$, the preceding construction is exactly that of § 6 for $\Delta_{t_1 t_2}$ from given Δ_{t_1}, Δ_{t_2}.] The explicit hypothesis, that we are given a Δ_{t_2} which contains a Δ_{t_3}, is not needed in the cases $t_3 = 1$, $t_3 = 3$. The Δ_t above constructed contains (at least) one Δ_{t_3} and t_1 Δ_{t_2}, while the *particular* Δ_t contains also one Δ_{t_1}.

For the case $[t_3 = 1,\ t_2 = 3,\ t = 1 + 2t_1]$ this "particular" construction (that is, the construction using the "particular" method of § 5) is Netto's construction $(a\alpha)$ of a $\Delta_{t=1+2t_1}$ from a given Δ_{t_1} (Annalen 42, p. 143).

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*) For illustration of this particular construction (A) and the general construction (A) see §§ 10, 11, 12 and 13 below.

§ 8.

Concerning ${}_3\nabla_2$ and Δ_7.

There is but a single*) class of ${}_3\nabla_2$ of three pairs of elements

$${}_3\nabla_2 \begin{Bmatrix} \beta_1' & \beta_2' & \beta_3' \\ \beta_1'' & \beta_2'' & \beta_3'' \end{Bmatrix}; \quad \beta_1'\beta_2'\beta_3',\ \beta_1'\beta_2''\beta_3'',\ \beta_1''\beta_2'\beta_3'',\ \beta_1''\beta_2''\beta_3'.$$

This is also the only "*general*" sub-triple system (see foot-note, § 1) of three pairs of elements. The four triples are conjugate („gleichberechtigt").

There is but a single class of Δ_7, $(7 = 1 + 3 \,.\, 2)$

$$\Delta_7 \begin{Bmatrix} & \alpha & \\ \beta_1' & \beta_2' & \beta_3' \\ \beta_1'' & \beta_2'' & \beta_3'' \end{Bmatrix}; \quad \begin{matrix} \alpha\beta_1'\beta_1'', \quad \alpha\beta_2'\beta_2'', \quad \alpha\beta_3'\beta_3'', \\ \beta_1'\beta_2'\beta_3',\ \beta_1'\beta_2''\beta_3'',\ \beta_1''\beta_2'\beta_3'',\ \beta_1''\beta_2''\beta_3'. \end{matrix}$$

The seven elements of the Δ_7 are conjugate, the seven triples likewise.

§ 9.

Concerning the ${}_3\nabla_2$ contained in a ${}_3\nabla_s$ constructed by the particular method of §§ 3, 2.

Let (x_f) (y_g) (z_h) be the three sets of s elements each,

$$(f, g, h = 0, 1, 2 \ldots s - 1),$$

and suppose we have in the ${}_3\nabla_s$ a

$${}_3\nabla_2 \quad \begin{Bmatrix} x_{f_1} & y_{g_1} & z_{h_1} \\ x_{f_2} & y_{g_2} & z_{h_2} \end{Bmatrix}.$$

We have then (§§ 3, 2, 8)

$$\begin{aligned} f_1 + g_1 + h_1 &\equiv 0, \\ f_1 + g_2 + h_2 &\equiv 0, \\ f_2 + g_1 + h_2 &\equiv 0, \qquad\qquad (\text{mod. } s), \\ f_2 + g_2 + h_1 &\equiv 0, \\ f_2 + g_2 + h_2 &\not\equiv 0, \end{aligned}$$

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*) The following four ${}_3\nabla_2$ are identical,

$${}_3\nabla_2 \begin{Bmatrix} \beta_1' & \beta_2' & \beta_3' \\ \beta_1'' & \beta_2'' & \beta_3'' \end{Bmatrix}, \quad {}_3\nabla_2 \begin{Bmatrix} \beta_1' & \beta_2'' & \beta_3'' \\ \beta_1'' & \beta_2' & \beta_3' \end{Bmatrix}, \quad {}_3\nabla_2 \begin{Bmatrix} \beta_1'' & \beta_2' & \beta_3'' \\ \beta_1' & \beta_2'' & \beta_3' \end{Bmatrix}, \quad {}_3\nabla_2 \begin{Bmatrix} \beta_1'' & \beta_2'' & \beta_3' \\ \beta_1' & \beta_2' & \beta_3'' \end{Bmatrix};$$

there are likewise four identical with

$${}_3\nabla_2 \begin{Bmatrix} \beta_1'' & \beta_2'' & \beta_3'' \\ \beta_1' & \beta_2' & \beta_3' \end{Bmatrix}.$$

These two ${}_3\nabla_2$ in three pairs of elements $\beta_1'\beta_1''$, $\beta_2'\beta_2''$, $\beta_3'\beta_3''$, are the only ones possible, and they are equivalent, an interchange of the upper strokes (′, ″) changing one ${}_3\nabla_2$ into the other.

and at once deduce from the first four congruences

$$2(f_2+g_2+h_2)\equiv 0 \qquad (\text{mod. } s).$$

These congruences are incompatible, if s is odd; whence

A ${}_3\Delta_{s=2s'+1}$, *with s odd, constructed by the particular method of* §§ 3, 2, *contains no* ${}_3\nabla_2$.

But if s is even, $s=2s'$, we have at once

$$f_2+g_2+h_2\equiv s' \qquad (\text{mod. } 2s'),$$

whence easily

$$f_2\equiv f_1+s',\quad g_2\equiv g_1+s',\quad h_2\equiv h_1+s' \qquad (\text{mod. } 2s').$$

In a ${}_3\nabla_{s=2s'}$, *with s even, constructed by the particular method of* §§ 3, 2, *any triple* $x_{f_1}y_{g_1}z_{h_1}$ *belongs to one* ${}_3\nabla_2$, *of which the other elements are* $x_{f_1+s'}$, $y_{g_1+s'}$ *and* $z_{h_1+s'}$. *There are in all* s'^2 *such* ${}_3\nabla_2$. *There are no other* ${}_3\nabla_2$.

It is to be observed that the elements of each set are paired $[(x_{f_1}, x_{f_1+s'}), (y_{g_1}, y_{g_1+s'})$ and $(z_{h_1}, z_{h_1+s'})]$ independently of the elements of the other sets, and that the ${}_3\nabla_{2s'}$ is in effect first a ${}_3\nabla_{s'}$ on the three sets of s' pairs and then a (particular) ${}_3\nabla_2$ on the elements of the triples of pairs.

A ${}_t\nabla_s$ of t sets of s elements constructed by the particular method of § 5 may contain ${}_3\nabla_2$ of two kinds: *first*, a ${}_3\nabla_2$ of three pairs of elements, the two elements of each pair belonging to the same set of s elements, these three sets forming a triple in the Δ_t of t sets; *second*, a ${}_3\nabla_2$ whose six elements belong to six different sets which form a ${}_3\nabla_2$ of sets in the Δ_t of t sets. The ${}_t\nabla_s$ can contain ${}_3\nabla_2$ of the *first* kind only when s is even $s=2s'$, and then the ${}_t\nabla_{2s'}$ does contain in all $\frac{t(t-1)}{6}\cdot s'^2$ such ${}_3\nabla_2$.

§ 10.

Sorting of the Δ_7 contained in a Δ_t constructed from specified data by the particular method (A).

We recall the four kinds of triples contained in the Δ_t,

$$(\alpha'\,\alpha''\,\alpha''',\ \alpha\,\beta_i'\,\beta_i'',\ \beta_i'\,\beta_i'''\,\beta_i^{IV},\ \beta_i\,\beta_j\,\beta_k),$$

and that a Δ_7 is determined uniquely by any two of its triples which must have a common element. The Δ_7 contained in the Δ_t may be sorted*) as follows into six sorts; it is not to be understood that Δ_7 of these sorts are always found in the Δ_t:

*) This sorting of the Δ_7 holds also for the Δ_t constructed by the *general* method (A) of § 7.

[1⁰] The Δ_7 is contained in the Δ_{t_3} (α).

[2⁰] The Δ_7 has three (α) elements forming a triple and four (β_i) elements and is contained in the Δ_{t_2} (α) (β_i).

[3⁰] The Δ_7 has one (α) element and six (β_i) elements and is contained in the Δ_{t_2} (α) (β_i).

[4⁰] The Δ_7 has seven (β_i) elements and is contained in the Δ_{t_2} (α) (β_i).

[5⁰] The Δ_7 has one (α) element and three pairs of (β) elements belonging to three associated sets (of a triple in the Δ_{t_1} of the t_1 sets (β_i)) and consists of three triples containing the (α) element and the four triples of a ${}_3\nabla_2$ of the ${}_{t_1}\nabla_{t_2-t_3}$ (β) contained in the Δ_t. This ${}_3\nabla_2$ of the ${}_{t_1}\nabla_{t_2-t_3}$ (β) is of the first kind mentioned in § 9. (t_2 and t_3 are both odd, and hence $t_2 - t_3$ is even.)

[6⁰] The Δ_7 has seven (β) elements belonging to seven different sets which form a Δ_7 of sets in the Δ_{t_1} of the t_1 sets. This Δ_7 of seven (β) elements is contained in the ${}_{t_1}\nabla_{t_2-t_3}$ (β).

The particular construction (A) of § 7 for the Δ_t from the specified data was determinate, except that in the particular construction of § 5 for the ${}_{t_1}\nabla_{t_2-t_3}$ (β) the $s = t_2 - t_3$ subscripts were distributed *arbitrarily* over the $t_2 - t_3$ elements of each (β_i) set. It is clear that this arbitrariness may affect the number of Δ_7 of sort [5⁰] contained in the Δ_t, but that it has nothing to do with the numbers of the Δ_7 of the other sorts.

§ 11.

Sorting of the Δ_t constructed from specified data by the particular method (A).

We sort the Δ_t in question according to the number of Δ_7 contained in them, that is, (§ 10), according to the number of Δ_7 of sort [5⁰] contained in them, and denote this latter number by σ. Two Δ_t of the same sort $\sigma_1 = \sigma_2$ are not necessarily equivalent, but two equivalent Δ_t are of the same sort.

In the Δ_t every Δ_7 of sort [5⁰] contains one (α) element and one ${}_3\nabla_2$ of the first kind contained in the ${}_{t_1}\nabla_{t_2-t_3}$ (β), while every such ${}_3\nabla_2$ lies at most in one such Δ_7. There are in all $\frac{t_1(t_1-1)}{6} \cdot \left(\frac{t_2-t_3}{2}\right)^2$ such ${}_3\nabla_2$ (§ 9). This then is the maximum value of σ.

Denote by (x) (y) (z) any three (β_i) sets which form a triple in the Δ_{t_1} of t_1 (β_i) sets, and by x_f, y_g, z_h $(f, g, h = 0, 1, 2 \ldots t_2 - t_3 - 1)$ the elements of those sets. Then one of these ${}_3\nabla_2$ is the

$$
{}_3\nabla_2 \quad \begin{Bmatrix} x_f & y_g & z_h \\ x_{f+\frac{t_2-t_3}{2}} & y_{g+\frac{t_2-t_3}{2}} & z_{h+\frac{t_2-t_3}{2}} \end{Bmatrix}.
$$

$$
f+g+h \equiv 0 \pmod{t_2-t_3}
$$

This ${}_3\nabla_2$ belongs to a Δ_7 of sort $[5^0]$ if and only if the three pairs of (β) elements in the $\Delta_{t_2}(\alpha)(x)$, $\Delta_{t_2}(\alpha)(y)$ and $\Delta_{t_2}(\alpha)(z)$ respectively belong to triples having a common third element, of necessity some (α) element α^0.

In the construction of a Δ_t from given Δ_{t_1}, Δ_{t_2}, Δ_{t_3} of which the Δ_{t_2} contains the Δ_{t_3} (where $t = t_3 + t_1(t_2 - t_3)$ and $t_1 \geqq 3$, $t_2 \geqq 2t_3 + 1$, $t_3 \geqq 1$) the particular method (A) *of* § 7, *by the proper determination of the ${}_{t_1}\nabla_{t_2-t_3}(\beta)$, enables us to construct at least two distinct sorts of Δ_t which contain Δ_7 of sort $[5^0]$, in all cases except the cases*

$$
[t_3 = 1,\ t_2 = 3,\ t = 1 + 2t_1],
$$

in which there is but one sort of Δ_t. This will be seen easily after we consider the following combinations of cases.

Cases $[t_3 = 1,\ t_2 = 3,\ t = 1 + 2t_1]$. There are but two elements in each (β_i) set, which are for every i in a triple with the single (α) element α^0. Thus in these cases there is but one sort of Δ_t, for which σ has its maximum value, $\sigma = \frac{t_1(t_1-1)}{6} \cdot \left(\frac{t_2-t_3}{2}\right)^2 = \frac{t_1(t_1-1)}{6}$. But further it is at once clear that any two Δ_t, $t = 1 + 2t_1$, derived from the same Δ_{t_1} by the particular method (A) of § 7 are equivalent Δ_t; in these cases there is but one *class* of Δ_t.

For the *general cases* also the sort of Δ_t with maximum σ exists, although it is not the only sort existing. To construct such a Δ_t we must choose arbitrarily an (α) element α^0, pair the elements of every (β_i) set (x) by the triples containing α^0, and distribute the subscripts over the $t_2 - t_3$ elements (x) arbitrarily only so as always to make the pairs $x_f\, x_{f+\frac{t_2-t_3}{2}}$ coincide with the pairs determined by α^0. Every Δ_t in question contains a Δ_{t_1}; the one just constructed contains a Δ_{1+2t_1} containing this Δ_{t_1}.

For the cases $[t_3 = 1,\ t_2 > 3,\ t = 1 + t_1(t_2 - 1)]$ the sort of Δ_t with minimum σ, $\sigma = 0$, exists. There is but a single (α) element α^0; it is possible to distribute the $t_2 - t_3 = t_2 - 1 > 2$ subscripts over the elements of the several (β_i) sets (x) so that the pairs $x_f\, x_{f+\frac{t_2-t_3}{2}}$ shall be entirely distinct from the pairs of elements (x) determined by the triples in the $\Delta_{t_2}(\alpha)(x)$ containing the element α^0. For a Δ_t so constructed, $\sigma = 0$.

For the cases $[t_3 \geqq t_1 > 1,\ t = t_3 + t_1(t_2 - t_3)]$ the sort of Δ_t with minimum σ, $\sigma = 0$, exists. There are here, since $t_3 \geqq t_1$, at least as many (α) elements as (β_i) sets. For each (β_i) set select a particular (α) element α^i, different elements corresponding to different sets. Distribute the subscripts over the $t_2 - t_3$ elements x of each (β_i) set so that the pairs $x_f x_{f+\frac{t_2-t_3}{2}}$ shall always coincide with the pairs determined by the triples in the $\Delta_{t_2}(\alpha)(\beta_i)$ containing the element α^i corresponding to the (β_i) set. Then in this Δ_t no ${}_3\nabla_2$ of the ${}_{t_1}\nabla_{t_2-t_3}$ can belong to a Δ_7 of sort $[5^0]$; thus $\sigma = 0$.

More generally, for the cases $[t_3 \geqq 3,\ t = t_3 + t_1(t_2 - t_3)]$ sorts of Δ_t exist with σ less than the maximum. For in the Δ_{t_1} of the t_1 (β_i) sets select any triple of sets $(x)(y)(z)$, and let them correspond with three (α) elements $\alpha^x\, \alpha^y\, \alpha^z$ $(t_3 \geqq 3)$. Distribute the subscripts in the $t_1 - 3$ other (β_i) sets at random, but in the set (x) let the pairs $x_f\, x_{f+\frac{t_2-t_3}{2}}$ coincide with the pairs determined in the $\Delta_{t_2}(\alpha)(x)$ by the triples containig α^x, and likewise for the sets (y), (z). Then the $\left(\frac{t_2-t_3}{2}\right)^2$ ${}_3\nabla_2$ of the ${}_{t_1}\nabla_{t_2-t_3}$ which lie in the ${}_3\nabla_{t_2-t_3}$ of the three sets $(x)(y)(z)$ do not lead to Δ_7 of the Δ_t. Whence indeed for such Δ_t σ has less than its maximum value.

By a suitable combination of the devices now illustrated for the determination of the ${}_{t_1}\nabla_{t_2-t_3}$, one easily convinces himself of the truth of the italicized statement made above.

§ 12.

Sorting of the Δ_t $(t = 1 + 2t_1)$ constructed from a given Δ_{t_1} by the general method (A) $(t_2 = 3,\ t_3 = 1)$.

It is convenient to give for this case $(t_2 = 3,\ t_3 = 1)$ the general construction of § 7 in a new notation.

Given a Δ_{t_1} in the t_1 elements b_i $(i = 1, 2 \ldots t_1)$. Take a single element α and, corresponding to the t_1 elements b_i, t_1 pairs of elements $\beta_i'\, \beta_i''$. Form in the t_1 pairs (β_i) a Δ_{t_1} corresponding to the Δ_{t_1} in the t_1 elements b_i. In every triple of pairs, for instance, $(\beta_\varkappa'\, \beta_\varkappa'')$, $(\beta_\lambda'\, \beta_\lambda'')$, $(\beta_\mu'\, \beta_\mu'')$, construkt a ${}_3\nabla_2$ in *either* of the two possible ways (see § 8, foot-note),

$$
{}_3\nabla_2 \quad \begin{Bmatrix} \beta_\varkappa' & \beta_\lambda' & \beta_\mu' \\ \beta_\varkappa'' & \beta_\lambda'' & \beta_\mu'' \end{Bmatrix}; \qquad {}_3\nabla_2 \quad \begin{Bmatrix} \beta_\varkappa'' & \beta_\lambda'' & \beta_\mu'' \\ \beta_\varkappa' & \beta_\lambda' & \beta_\mu' \end{Bmatrix}.
$$

The ${}_{t_1}\nabla_2$ so constructed and the t_1 triples $\alpha\beta_i'\,\beta_i''$ $(i = 1, 2 \ldots t_1)$ constitute the Δ_t $(t = 1 + 2t_1)$ required.

In this general construction the only arbitrariness lies in the choice, for every one of the $\frac{t_1(t_1-1)}{6}$ triples of pairs, of one of the two possible ${}_3\nabla_2$. If we construct all these 2^τ Δ_t, where $\tau=\frac{t_1(t_1-1)}{6}$, they are certainly not all essentially distinct; for, if in any Δ_t we interchange the upper indices of all the β symbols we get a Δ_t equivalent to but not identical with the original Δ_t; that is to say, the 2^τ Δ_t form $2^{\tau-1}$ pairs of equivalent Δ_t. [For $t=7$, $t_1=3$, $\tau=1$; there is but a single Δ_7]. We wish merely to show that

In the construction of a Δ_t from a given Δ_{t_1} $(t=1+2t_1\geqq 15)$ the general method of § 7 *enables us to construct at least two distinct sorts of Δ_t which contain Δ_7 of sort* $[5^0]$.

We notice that any Δ_{u_1} contained in the Δ_{t_1} (b) leads to a Δ_{1+2u_1} contained in the $\Delta_{t=1+2t_1}$, however it be constructed; this Δ_{1+2u_1} contains the element α. Conversely, if any particular Δ_t contains a Δ_u which contains the element α, then the Δ_{t_1} (b) contains a $\Delta_{\frac{u-1}{2}}$, and every Δ_t contains a corresponding Δ_u containing α. In particular, every Δ_t contains t_1 Δ_3 and $\frac{t_1(t_1-1)}{6}$ Δ_7 which contain α.

Indicate by Δ_u^* a Δ_u contained in the Δ_t and not containing the element α. A Δ_u^* contains u elements (β), one of each of u pairs. The presence of a Δ_u^* shows presence of a Δ_u in the Δ_{t_1} (b), but not conversely. We may then sort the Δ_t, given us by the *general* construction (A) of § 7, according to the number of Δ_u^* which they contain. Two equivalent Δ_t belong to the same sort.

We study the sorting of the Δ_t with respect to the number $\bar{\sigma}$ of $\Delta_{t_1}^*$ contained. Such a $\Delta_{t_1}^*$ contains one element of each of the t_1 pairs $\beta_i'\beta_i''$.

Netto's $\Delta_{t=1+2t_1}$ contains a $\Delta_{t_1}^*$. It is constructed by the *particular* method (A), say by choosing the ${}_3\nabla_2\begin{Bmatrix}\beta_\varkappa' & \beta_\lambda' & \beta_\mu'\\ \beta_\varkappa'' & \beta_\lambda'' & \beta_\mu''\end{Bmatrix}$ for every triple of pairs; then in the elements β_i' $(i=1,2,\ldots t_1)$ there is a $\Delta_{t_1}^*$. Every $\Delta_{t=1+2t_1}$ containing one $\Delta_{t_1}^*$ is equivalent to Netto's $\Delta_{t=1+2t_1}$. Thus of our $\Delta_{t=1+2t_1}$ *there is but a single sort with* $\bar{\sigma}>0$, and this sort contains but a single class of triple systems, Netto's. [If the given Δ_{t_1} (b)contains $\bar{\sigma}_1$ $\Delta_{\frac{t_1-1}{2}}$, then our $\bar{\sigma}=\bar{\sigma}_1+1$].

The sort with $\bar{\sigma}=0$ *also exists.* I prove this by constructing a triple system $\overline{\Delta}_t$ of this sort. For the ${}_3\nabla_2$ of this $\overline{\Delta}_{t_1}$ I choose the ${}_3\nabla_2\begin{Bmatrix}\beta_\varkappa' & \beta_\lambda' & \beta_\mu'\\ \beta_\varkappa'' & \beta_\lambda'' & \beta_\mu''\end{Bmatrix}$ for every triple of pairs except one, say (1 2 3), for which I choose the other ${}_3\nabla_2 \begin{Bmatrix} \beta_1'' & \beta_2'' & \beta_3'' \\ \beta_1' & \beta_2' & \beta_3' \end{Bmatrix}$. Then the triples of $\overline{\Delta}_t$ differ from those of Netto's Δ_t only in the triples of elements with the subscripts $1, 2, 3$. It is convenient to replace β_i $(i=1, 2, 3)$ by x_i and β_i $(i=4, 5 \ldots t_1)$ by y_i. The triples of $\overline{\Delta}_t$ are then of the forms $x_1''x_2''x_3''$, $x_{i_1}''x_{i_2}'x_{i_3}'$ $(i_1, i_2, i_3 = 1, 2, 3$ in any order) $x_i'y_j'y_k'$, $x_i'y_j''y_k''$, $x_i''y_j'y_k''$, $y_i'y_j'y_l'$, $y_i'y_j''y_l''$.

This $\overline{\Delta}_t$ contains no $\Delta_{t_1}^*$; $\bar{\sigma}=0$. For if it were to contain a $\Delta_{t_1}^*$, this $\Delta_{t_1}^*$ would contain one element of each of the t_1 pairs (x_i) (y_j) $(i=1, 2, 3;\ j=4, 5 \ldots t_1)$, and by proper adjustment of the subscript notation such a $\Delta_{t_1}^*$ would have one or other of the forms,

$\Delta_{t_1}^*$(I) in the elements $x_1''x_2''x_3''$; $y_4'y_5' \ldots y_{\varrho+3}'$; $y_{\varrho+4}''y_{\varrho+5}'' \ldots y_{t_1}''$;

$\Delta_{t_1}^*$(II) in the elements $x_1''x_2'x_3'$; $y_4'y_5' \ldots y_{\varrho+3}'$; $y_{\varrho+4}''y_{\varrho+5}'' \ldots y_{t_1}''$;

where in each case ϱ is any integer from 1 to $\overline{t_1-3}$.

$\Delta_{t_1}^*$(I), $3(x_i'') + \varrho(y_j') + \sigma(y_k'') = t_1$ elements; $\varrho + \sigma = t_1 - 3$. x_1'' is connected with the $\varrho(y_j')$ by ϱ triples whose third elements must be $\varrho(y_k'')$, and vice versa; thus $\varrho = \sigma$; y_{t_1}'' is connected with the other $\sigma - 1 = \varrho - 1$ (y_j'') and the $3(x_i'')$ by $\varrho + 2$ triples whose third elements must be $\varrho + 2$ (y_j'); but there are only $\varrho(y_j')$ elements. *Thus $\overline{\Delta}_t$ contains no $\Delta_{t_1}^*$ (I).*

$\Delta_{t_1}^*$(II); $1(x_1'') + 2(x_2', x_3') + \varrho(y_j') + \sigma(y_k'') = t_1$ elements; $\varrho + \sigma = t_1 - 3$.

x_1'' is connected with the $\varrho(y_j')$ by ϱ triples whose third elements must be $\varrho(y_k'')$, and vice versa; thus $\varrho = \sigma$; y_{t_1}'' is connected with the $\varrho(y_j')$ and the $2(x_2', x_3')$ by $\varrho + 2$ triples whose third elements must be $\varrho + 2$ elements with double accents ($''$); but there are only ϱ such elements available, viz., the other $\varrho - 1(y_k'')$ and the $1(x_1'')$. *Thus $\overline{\Delta}_t$ contains no $\Delta_{t_1}^*$ (II).*

§ 13.

Conditional construction B.

If we are given a Δ_{t_1} and a Δ_{t_2} which contains a Δ_{t_3},

$$(t_1 \geqq 3,\ t_3 \geqq 1,\ t_2 \geqq 2t_3 + 1),$$

we can construct at least two distinct sorts of Δ_t $\left(t = t_3 + t_1(t_2 - t_3) > 13\right)$ which contain Δ_7 of sort [5⁰].

This we have seen for the cases $[t_3=1,\ t_2=3,\ t=1+2t_1>13]$

by use of the general construction (A) in § 12, and for all other cases by use of the particular construction (A) in § 11.

Since we have Δ_1, Δ_3 directly given and Δ_7 (§ 8) and Δ_9 (§ 6) known, and since every Δ_t $(t \geqq 3)$ contains Δ_1, Δ_3, we have the following corollaries:

If we are given a $\Delta_{t'}$ $(t' = 6m' + 1$ or $6m' + 3)$, we can construct at least two distinct sorts of Δ_t $(t > 13)$ which contain Δ_7 of sort $[5^0]$, where

Residues (mod. 72)

$$(\mathrm{B}_1)\quad t = 1 + t'.2 \quad = \begin{cases} 12m' + 3 \equiv 3,\ 15,\ 27,\ 39,\ 51,\ 63; \\ 12m' + 7 \equiv 7,\ 19,\ 31,\ 43,\ 55,\ 67; \end{cases}$$

$$(\mathrm{B}_2)\quad t = 1 + 3(t'-1) = \begin{cases} 18m' + 1 \equiv 1,\ 19,\ 37,\ 55; \\ 18m' + 7 \equiv 7,\ 25,\ 43,\ 61; \end{cases}$$

$$(\mathrm{B}_3)\quad t = 3 + 3(t'-3) = \begin{cases} 18m' - 3 \equiv 15,\ 33,\ 51,\ 69; \\ 18m' + 3 \equiv 3,\ 21,\ 39,\ 57; \end{cases}$$

$$(\mathrm{B}_4)\quad t = 3 + t'.4 \quad = \begin{cases} 24m' + 7 \equiv 7,\ 31,\ 55; \\ 24m' + 15 \equiv 15,\ 39,\ 63; \end{cases}$$

$$(\mathrm{B}_5)\quad t = 3 + t'.6 \quad = \begin{cases} 36m' + 9 \equiv 9,\ 45; \\ 36m' + 21 \equiv 21,\ 57. \end{cases}$$

If we are given a $\Delta_{t'}$ $(t' = 6m' + 1$ or $6m' + 3)$ which contains a Δ_7, we can construct at least two distinct sorts of Δ_t which contain Δ_7 of sort $[5^0]$, where

$$(\mathrm{B}_6)\quad t = 7 + 3(t'-7) = \begin{cases} 18m' - 11 \equiv 7,\ 25,\ 43,\ 61; \\ 18m' - 5 \equiv 13,\ 31,\ 49,\ 67. \end{cases}$$

These corollaries require the further hypotheses that

$$\text{in } (\mathrm{B}_1)\,(\mathrm{B}_2)\,(\mathrm{B}_4)\,(\mathrm{B}_5) \qquad t' \geqq 3,\ m' \geqq \begin{cases}1\\0\end{cases},$$

$$\text{in } (\mathrm{B}_3) \qquad t' \geqq 7,\ m' \geqq \begin{cases}1\\1\end{cases},$$

and

$$\text{in } (\mathrm{B}_6) \qquad t' \geqq 15,\ m' \geqq \begin{cases}3\\2\end{cases}.$$

§ 14.

Absolute construction of triple systems.

We admit the existence of and make use of Δ_1, Δ_3, Δ_7, Δ_9 *and* Δ_{13}, *and can construct at least two distinct sorts of triple systems* Δ_t $(t > 13)$ *which contain* Δ_7, *where* $t = 6m + 1$ *or* $6m + 3$ *and* m *has any positive integral value.*

Δ_{13} is constructed by Netto's construction $(a\gamma)$ (Annalen v. 42, p. 145).

The first five corollaries $B_1 \ldots B_5$ apply for $t' = 3 \ldots 13$ to show that we can construct as desired,

$$[B_1] \begin{Bmatrix} \Delta_{15}, \Delta_{27} \\ \Delta_{19} \end{Bmatrix}; \quad [B_2] \begin{Bmatrix} \Delta_{19}, \Delta_{37} \\ \Delta_{25} \end{Bmatrix}; \quad [B_3] \begin{Bmatrix} \Delta_{15}, \Delta_{33} \\ \Delta_{21} \end{Bmatrix};$$

$$[B_4] \begin{Bmatrix} \Delta_{31}, \Delta_{55} \\ \Delta_{15}, \Delta_{39} \end{Bmatrix}; \quad [B_5] \begin{Bmatrix} \Delta_{45}, \Delta_{81} \\ \Delta_{21}, \Delta_{57} \end{Bmatrix};$$

that is, we can construct as desired the Δ_t for $t = 15 \ldots 39$.

We may now apply all six corollaries for $t' = 3 \ldots 39$ (since the Δ_t $(t > 13)$ were constructed so as to contain a Δ_7) and thus construct as desired the Δ_t for $t = 43 \ldots 97$.

This process continued indefinitely reaches every number t. In fact the column headed "Residues mod. 72" to the right of the table of corollaries shows that the forms given contain all integers t of the form $6m + 1$ or $6m + 3$.

Chicago, 29. April 1893.