Math. Ann. 284, 681–699 (1989)

Mathematische Annalen

A local index theorem for non Kähler manifolds*

Jean-Michel Bismut

Département de Mathématique, Université Paris-Sud, F-91405 Orsay, France

Let M be a compact connected oriented spin Riemannian manifold of even dimension. Then any Euclidean connection ∇ on TM lifts to the corresponding Hermitian vector bundle of spinors F. The local index theorem of Patodi [P1], Gilkey [Gi1, 2], Atiyah-Bott-Patodi [ABP] asserts that if ∇^L is the Levi-Civita connection of TM, and if D^L is a Dirac operator acting on twisted spinors associated with the connection ∇^L, then as $t \downarrow 0$, the supertrace $\mathrm{Tr}_s[P_t(x,x)]$ of the heat kernel of $\exp(-tD^2)$ converges to the Atiyah-Singer characteristic polynomial [AS] naturally associated with the connection ∇^L and with the considered connection on the twisting bundle.

The first purpose of this paper is to find sufficient (and almost necessary) conditions for a local index theorem to hold when D is the Dirac operator associated with a connection ∇ which does not necessarily coincide with ∇^L. In fact let T be the torsion of ∇, and let B be the three form which is the antisymmetrization of the tensor $X, Y, Z \to \frac{1}{4}\langle T(X, Y), Z\rangle$. In Theorem 1.11, we prove that if the form B is closed, a local index theorem still holds. However, and rather mischeviously, the corresponding Atiyah-Singer polynomial is calculated with a connection which in general differs from ∇, except when $\nabla = \nabla^L$.

If M is a complex manifold equipped with a Hermitian metric whose Kähler form is ω, we know by Atiyah-Bott-Patodi [ABP] that the Riemann-Roch-Hirzebruch theorem can be derived from the Atiyah-Singer index theorem. Also it is known since Patodi [P2] that the local index theorem for an operator of the type $\bar\partial + \bar\partial^*$ holds if ω is closed, i.e. if (M, ω) is Kähler. In this case, the local supertrace converges as $t \downarrow 0$ to the local Riemann-Roch-Hirzebruch polynomial $Td(TM)\,ch(\xi)$ associated with the holomorphic Hermitian connections on TM, and on the twisting bundle ξ.

In Theorem 2.11, we prove that the Kähler condition can be substantially relaxed. In fact we show that if $\bar\partial\partial\omega = 0$, there is still a local Riemann-Roch-Hirzebruch theorem. The local limit only involves forms of type (p, p), but it is no longer given locally by a Riemann-Roch-Hirzebruch polynomial. In fact we

* This work was written while the author was visiting IHES during the academic year 1987–1988

construct on the complexified tangent space $T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ another holomorphic structure than the canonical one, which depends explicitly on the Kähler form ω. The limit index polynomial is evaluated by means of the curvature of the holomorphic Hermitian connection on $T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$. The condition $\bar{\partial}\partial\omega = 0$ is well-known in Hermitian geometry. In particular, a result of Gauduchon [Ga] asserts that any complex surface carries a Kähler form having this property.

The paper is organized as follows. In Sect. 1, we establish a local index theorem for Dirac operators when TM is equipped with a general Euclidean connection. In Sect. 2, we specialize our results to the case where M is a complex manifold.

Applications of our results to Ray-Singer analytic torsion [RS] and Quillen metrics [Q2] will be given in [B3].

The results contained in this paper were announced in [B2].

I. Torsion and the local index theorem

In this section, we prove a local index theorem for Dirac operators on a Riemannian manifold M associated with connections on TM which have non zero torsion.

This section is organized as follows. In a), we prove a Lichnerowicz formula for a wide class of non trivial perturbations of Dirac operators associated with the Levi-Civita connection of TM. In b), we prove an essential symmetry property of the curvature of certain connections on TM with non zero torsion. In c), we prove a local index theorem for a general class of Dirac operators. Finally in d), we apply the results of c) to Dirac operators associated with connections on TM which have non zero torsion.

a) A Lichnerowicz formula for general Dirac operators

Let M be a compact connected Riemannian oriented spin manifold of dimension n. Let F be the Hermitian vector of TM spinors. The Levi-Civita connection ∇^L on TM lifts to a unitary connection on F, which we still note ∇^L.

Let ξ be a complex vector bundle on M, which is equipped with a smooth connection ∇^ξ, whose curvature is noted $(\nabla^\xi)^2$. We here do not assume that ∇^ξ preserves a metric on ξ.

The vector bundle $F \otimes \xi$ is then equipped with the connection $\nabla^L \otimes 1 + 1 \otimes \nabla^\xi$, which we still note ∇^L.

Let $c(TM)$ be the Clifford algebra of TM. If $X \in TM$, let ${}^cX$ be the corresponding element in $c(TM)$. Remember that $c(TM)$ and $\Lambda(T^*M)$ are isomorphic as Z graded vector spaces. In fact, identifying TM and T^*M by the metric, if $e_1, \ldots, e_k$ are orthogonal unit vectors in TM, this isomorphism maps ${}^ce_1 \ldots {}^ce_k \in c(TM)$ into $e_1 \wedge \ldots \wedge e_k \in \Lambda(T^*M)$.

The Clifford algebra $c(TM)$ is Z_2 graded, and the isomorphism of $c(TM)$ with $\Lambda(T^*M)$ preserves the Z_2 grading. The algebra $c(TM) \otimes \operatorname{End} \xi$ is also Z_2 graded. If $B, B' \in c(TM) \otimes \operatorname{End} \xi$, set

$$[B, B'] = BB' - B'B$$
$$[B, B']_s = BB' - (-1)^{\deg B \deg B'} B'B .$$

Let A be a smooth section of $\Lambda^{\text{odd}}(T^*M)\otimes \operatorname{End}\xi$ and let ${}^cA$ be the image of A in $c(TM)\otimes \operatorname{End}\xi$. The vector bundle $\xi = \xi \oplus \{0\}$ is naturally Z_2 graded, the elements of ξ being even. Therefore $\nabla^\xi + A$ is a superconnection on ξ in the sense of Quillen [Q1]. Note that we here use the formalism of Quillen in the case where ξ is trivially graded. Everything which follows extends to the case where ξ is a non trivially Z_2 graded vector bundle.

Then, in the sense of Quillen [Q1], $(\nabla^\xi + A)^2$ is the curvature of the superconnection $\nabla^\xi + A$. $(\nabla^\xi + A)^2$ is a smooth section of $\Lambda^{\text{even}}(T^*M)\otimes \operatorname{End}\xi$. Let ${}^c((\nabla^\xi + A)^2)$ be the image of $(\nabla^\xi + A)^2$ in $c(TM)\otimes \operatorname{End}\xi$.

Let D^L be the Dirac operator acting on the smooth sections of $F\otimes\xi$ naturally associated with the connection ∇^L. If $e_1, \ldots, e_n$ is an orthonormal base of TM, then

$$D^L = \sum_1^n {}^ce_i \nabla^L_{e_i}. \tag{1.1}$$

Let K be the scalar curvature of M. Let $X \to B(X)$ be a smooth one form on TM with value in $c(TM)\otimes \operatorname{End}\xi$. If $(e_1, \ldots, e_n)(x)$ is a locally defined smooth section of the bundle of orthonormal frames in TM, we use the notation

$$\Sigma(\nabla^L_{e_i} + B(e_i))^2 = \sum_1^n (\nabla^L_{e_i} + B(e_i))^2 - B\left(\sum_1^n \nabla^L_{e_i} e_i\right) - \nabla^L_{\sum_1^n \nabla^L_{e_i} e_i}. \tag{1.2}$$

One verifies easily that the operator $\sum(\nabla_{e_i} + B(e_i))^2$ does not depend on the local trivialization $(e_1, \ldots, e_n)$ of the bundle of orthonormal frames, and so is globally defined on M.

If $X \in TM$, the interior multiplication operator i_X as on $\Lambda(T^*M)\otimes \operatorname{End}\xi$.

Theorem 1.1. *The following identities hold*

$$(D^L + {}^cA)^2 = -\sum(\nabla^L_{e_i} + {}^c(i_{e_i}A))^2 + \frac{K}{4} + {}^c((\nabla^\xi + A)^2) + ({}^cA)^2 + \sum({}^c(i_{e_i}A))^2 - {}^c(A^2)$$

$$({}^cA)^2 + \Sigma({}^c(i_{e_i}A))^2 - {}^c(A^2) = \sum_{\substack{i_1 < i_2 < \ldots < i_k \\ k \geq 2}} (-1)^{\frac{k(k+1)}{2}} (1-k)\, {}^c((i_{e_{i_1}} \ldots i_{e_{i_k}} A)^2). \tag{1.3}$$

Proof. In what follows, we omit the summation sign Σ and the sign ^{c} in ${}^ce_i$. Take $(e_1(x), \ldots, e_n(x))$ as before. Then since ${}^cA$ is odd in $c(TM)\otimes \operatorname{End}\xi$

$$(D^L + {}^cA)^2 = (D^L)^2 + [e_i, {}^cA]_s \nabla^L_{e_i} + e_i \nabla^L_{e_i}({}^cA) + ({}^cA)^2. \tag{1.4}$$

By Lichnerowicz formula [L], we know that

$$(D^L)^2 = -(\nabla^L_{e_i})^2 + \frac{K}{4} + {}^c((\nabla^\xi)^2). \tag{1.5}$$

Using (1.4) and (1.5), we find that

$$(D^L + {}^cA)^2 = -\left(\nabla^L_{e_i} - \frac{[e_i, {}^cA]_s}{2}\right)^2 + ({}^cA)^2 + \frac{1}{4}[e_i, {}^cA]_s^2$$

$$+ e_i \nabla^L_{e_i}({}^cA) - \frac{1}{2}[e_i, \nabla^L_{e_i}({}^cA)]_s + \frac{K}{4} + {}^c((\nabla^\xi)^2) \tag{1.6}$$

or equivalently

$$(D^L+A)^2 = -\left(\nabla^L_{e_i} - \frac{[e_i, {}^cA]_s}{2}\right)^2 + ({}^cA)^2 + \frac{1}{4}[e_i, {}^cA]_s^2$$
$$+\frac{1}{2}[e_i, \nabla^L_{e_i}({}^cA)] + \frac{K}{4} + {}^c((\nabla^\xi)^2). \quad (1.7)$$

Let $dx^1, \dots, dx^n$ be the base of T^*M dual to the base $e_1, \dots, e_n$. One verifies easily that since A is odd

$$\begin{aligned}[e_i, {}^cA]_s &= -2{}^c(i_{e_i}A)\\ [e_i, \nabla^L_{e_i}({}^cA)] &= 2{}^c(dx^i \nabla^\xi_{e_i} A)\\ (\nabla^\xi + A)^2 &= (\nabla^\xi)^2 + \sum dx^i \nabla^\xi_{e_i} A + A^2.\end{aligned} \quad (1.8)$$

The first line of (1.3) follows from (1.7), (1.8). Let $I = \{i_1, \dots, i_p\}$ be an ordered subset of $\{1, \dots, n\}$, and assume all the $i_j \in I$ are distinct. Let $|I|$ be the number of elements in I. Set

$${}^ce_I = {}^ce_{i_1}{}^ce_{i_2} \dots {}^ce_{i_p}.$$

Take $k \leqq n$, and let I and J be two ordered subsets of $\{k+1, \dots, n\}$ such that $I \cap J = \emptyset$. Then

$${}^ce_{1\dots k}{}^ce_I{}^ce_{1\dots k}{}^ce_J = (-1)^{k|I|}({}^ce_{1\dots k})^2\,{}^ce_I{}^ce_J.$$

Since $({}^c(e_{1\dots k}))^2 = (-1)^{\frac{k(k+1)}{2}}$, we find that

$${}^ce_{1\dots k}{}^ce_I{}^ce_{1\dots k}{}^ce_J = (-1)^{\left(k|I| + \frac{k(k+1)}{2}\right)}\,{}^ce_I{}^ce_J. \quad (1.9)$$

If $k+|I|$ is odd, then $k|I|$ is even and the sign in (1.9) is $(-1)^{\frac{k(k+1)}{2}}$. If $k+|I|$ is even, $(-1)^{k|I|} = (-1)^k$, and the sign in (1.9) is $(-1)^{\frac{k(k-1)}{2}}$. Since A is odd, using (1.9), we find that

$$({}^cA)^2 = \sum_{i_1 < \dots < i_k} (-1)^{\frac{k(k+1)}{2}}\,{}^c((i_{e_{i_1}} i_{e_{i_2}} \dots i_{e_{i_k}} A)^2). \quad (1.10)$$

Similarly since $i_{e_i}A$ is even, we get

$$({}^ci_{e_i}A)^2 = \sum_{i_1 < i_2 < \dots < i_k} (-1)^{\frac{k(k-1)}{2}}\,{}^c((i_{e_{i_1}} i_{e_{i_2}} \dots i_{e_{i_k}} i_{e_i} A)^2). \quad (1.11)$$

Observe that when k is changed into $k+2$, $(-1)^{\frac{k(k+1)}{2}}$ changes sign. Therefore

$$({}^cA)^2 + \sum({}^ci_{e_i}A)^2 = {}^c(A^2) + \sum_{\substack{k \geqq 1\\ i_1 < \dots < i_k}} (-1)^{\frac{k(k+1)}{2}}(1-k){}^c((i_{e_{i_1}} \dots i_{e_{i_k}} A)^2). \quad (1.12)$$

The second line of (1.3) follows from (1.12). □

Remark 1.2. If A only contains terms of degree 1, the second line in (1.3) is 0. In this case, $D^L + A$ is simply the Dirac operator on $F \otimes \xi$ associated with the connection $\nabla^L \otimes 1 + 1 \otimes (\nabla^\xi + A)$ on the vector bundle $F \otimes \xi$. The first line of (1.3) is then equivalent to the standard Lichnerowicz formula.

Let now B be a smooth section of $\Lambda^3(T^*M)$. Of course, ${}^cB$ acts like ${}^cB\otimes 1$ on $F\otimes\xi$. Let $\|B\|$ be the norm of B in $\Lambda^3(T^*M)$.

Theorem 1.3. *The following identity holds*

$$(D^L+{}^cB)^2=-\sum_1^n(\nabla^L_{e_i}+{}^c(i_{e_i}B))^2+\frac{K}{4}+{}^c((\nabla^\xi)^2)+{}^c(dB)-2\|B\|^2. \tag{1.13}$$

Proof. We use Theorem 1.1 with $A=B$. Observe that since B lies in $\Lambda^3(T^*M)$, $B^2=0$. Therefore

$${}^c((\nabla^\xi+B))^2={}^c((\nabla^\xi))^2+{}^c(dB).$$

Also for $i<j$, since $i_{e_i}i_{e_j}B\in\Lambda^1(T^*M)$, $(i_{e_i}i_{e_j}B)^2=0$. Finally

$$\sum_{i<j<k}(i_{e_i}i_{e_j}i_{e_k}B)^2=\|B\|^2.$$

(1.13) follows. □

b) A symmetry property of the curvature for connections with non zero torsion

Let B be a real smooth section of $\Lambda^3(TM)$.

Definition 1.4. S^B denotes the one form with values in antisymmetric elements of $\mathrm{End}(TM)$ which is such that if $X,Y,Z\in TM$

$$\langle S^B(X)Y,Z\rangle=2B(X,Y,Z). \tag{1.14}$$

∇^B denotes the Euclidean connection on TM

$$\nabla^B=\nabla^L+S^B. \tag{1.15}$$

T^B, R^B denote the torsion and curvature tensors of the connection ∇^B.

Proposition 1.5. *If $X,Y,Z\in TM$, then*

$$\langle T^B(X,Y),Z\rangle=4B(X,Y,Z). \tag{1.16}$$

Conversely, if ∇ is an Euclidean connection on TM whose torsion is T, if the tensor $X,Y,Z\to\langle T(X,Y),Z\rangle$ is antisymmetric, if B is the three form defined by

$$\langle T(X,Y),Z\rangle=4B(X,Y,Z) \tag{1.17}$$

then $\nabla=\nabla^B$.

Proof. By (1.14)

$$\langle T^B(X,Y),Z\rangle=\langle S^B(X)Y,Z\rangle-\langle S^B(Y)X,Z\rangle=4B(X,Y,Z). \tag{1.18}$$

If the connection ∇ is taken as indicated, observe that the connections ∇ and ∇^B are both Euclidean, and have the same torsion $T=T^B$. They necessarily coincide. □

We now prove an essential identity, which extends the well-known symmetry identity on the curvature tensor of the Levi-Civita connection.

Theorem 1.6. *Let B be a real smooth section of $\Lambda^3(T^*M)$. If B is closed, if $X, Y, Z, T \in TM$, then*

$$\langle R^B(X,Y)Z,T\rangle = \langle R^{-B}(T,Z)Y,X\rangle. \tag{1.19}$$

Proof. By (1.14), the tensor $X,Y,Z \to \langle S^B(X)Y,Z\rangle$ is antisymmetric, and the corresponding three form is closed.

Since ∇^L is torsion free, the action of ∇^L on the smooth sections of $\Lambda(T^*M)$ coincides with the exterior differentiation operator d. Let $\underbrace{\quad}$ denote antisymmetrization. Since the form B is closed, if $X, Y, Z, T \in TM$, then

$$\langle(\nabla^L_X S^B)\underbrace{(Y)Z,T}\rangle - \langle(\nabla^L_Y S^B)\underbrace{(Z)T,X}\rangle + \langle(\nabla^L_Z S^B)\underbrace{(T)X,Y}\rangle - \langle(\nabla^L_T S^B)\underbrace{(X)Y,Z}\rangle = 0. \tag{1.20}$$

Also since the tensor $Y,Z,T \to \langle S^B(Y)Z,T\rangle$ is antisymmetric, for any $X \in TM$, the tensor $Y,Z,T \to \langle(\nabla^L_X S^B)(Y)Z,T\rangle$ is still antisymmetric. Therefore the identity (1.20) is valid without antisymmetrization. We then find that

$$\langle(\nabla^L_X S^B)(Y)Z,T\rangle - \langle(\nabla^L_Y S^B)(X)Z,T\rangle = \langle(\nabla^L_T S^B)(Z)X,Y\rangle - \langle(\nabla^L_Z S^B)(T)X,Y\rangle. \tag{1.21}$$

Clearly

$$R^B = (\nabla^L)^2 + \nabla^L S^B + S^B \wedge S^B. \tag{1.22}$$

From (1.21), since $S^{-B} = -S^B$, we get

$$\langle(\nabla^L S^B)(X,Y)Z,T\rangle = \langle \nabla^L S^{-B}(T,Z)Y,X\rangle. \tag{1.23}$$

Similarly, since the tensor $X,Y,Z \to \langle S^B(X)Y,Z\rangle$, is antisymmetric, we get

$$\begin{aligned}
&\langle(S^B(X)S^B(Y) - S^B(Y)S^B(X))Z,T\rangle \\
&\quad = -\langle S^B(T)S^B(Y)Z,X\rangle + \langle S^B(T)S^B(X)Z,Y\rangle \\
&\quad = \langle S^B(Y)Z, S^B(T)X\rangle - \langle S^B(X)Z, S^B(T)Y\rangle \\
&\quad = -\langle S^B(Z)Y, S^B(T)X\rangle + \langle S^B(Z)X, S^B(T)Y\rangle \\
&\quad = \langle(S^B(T)S^B(Z) - S^B(Z)S^B(T))Y,X\rangle.
\end{aligned} \tag{1.24}$$

From (1.24), since $S^{-B} = -S^B$, we find that

$$\langle(S^B \wedge S^B)(X,Y)Z,T\rangle = \langle(S^{-B} \wedge S^{-B})(T,Z)Y,X\rangle. \tag{1.25}$$

Using (1.21)–(1.25) and the fact that (1.19) holds for $B=0$, we obtain (1.19) in full generality. □

c) A local index theorem for a modified Dirac operator

We now assume that M is even dimensional, so that $n=2l$. The vector bundle F splits orthogonally into $F = F_+ \oplus F_-$, where F_+, F_- are the vector bundles of positive and negative spinors.

Let B be a real smooth section of $\Lambda^3(T^*M)$. Then the operator $D^L + {}^cB$ exchanges the sets of smooth sections of $F_+ \otimes \xi$ and $F_- \otimes \xi$. Let $(D^L + {}^cB)_\pm$ be the restriction of $D^L + B$ to smooth sections of $F_\pm \otimes \xi$.

Then the operator $(D^L + {}^cB)_+$ is Fredholm and its index is given by

$$\operatorname{Ind}(D^L + {}^cB)_+ = \dim \operatorname{Ker}(D^L + B)_+ - \dim \operatorname{Ker}(D^L + B)^*_+ . \tag{1.26}$$

Of course $\operatorname{Ind}(D^L + {}^cB)_+$ does not depend on B.

Let dx be the Riemannian volume element on M. For $t > 0$, let $P^B_t(x, y)$ be the C^∞ kernel associated with the operator $\exp(-t(D^L + {}^cB)^2)$. If h is a smooth section of $F \otimes \xi$, then

$$\exp(-t(D^L + {}^cB)^2)h(x) = \int_M P^B_t(x, y)h(y)dy . \tag{1.27}$$

For any $x \in M$, $P^B_t(x, x) \in \operatorname{End}(F \otimes \xi)_x$ is even, i.e. preserves the Z_2 grading of $(F \otimes \xi)_x$. Let $\operatorname{Tr}_s[P^B_t(x, x)]$ be the supertrace of $P^B_t(x, x)$ in the sense of Quillen [Q1]. The McKean-Singer formula [MKS] asserts that for any $t > 0$

$$\operatorname{Ind}(D^L + B)_+ = \int_M \operatorname{Tr}_s[P^B_t(x, x)]dx . \tag{1.28}$$

Note that (1.28) was established in [MKS] when $D^L + B$ is a self-adjoint operator. This is here the case if ∇^ξ is unitary. In full generality, (1.28) was established in [B4, Theorem 1.2] by using the superconnection formalism of Quillen [Q1].

Let $\hat{A}$ be the Hirzebruch polynomial on (n, n) matrices. If C is an antisymmetric real matrix with diagonal blocks $\begin{bmatrix} 0 & x_i \\ -x_i & 0 \end{bmatrix}$, then

$$\hat{A}(C) = \prod_1^l \frac{x_i/2}{sh(x_i/2)} . \tag{1.29}$$

Theorem 1.7. *Assume that $B \in \Lambda^3(T^*M)$ is closed. Then*

$$\lim_{t \to 0} \operatorname{Tr}_s[P^B_t(x, x)]dx = \left\{ \hat{A}\left(\frac{R^{-B}}{2\pi}\right) \operatorname{Tr}\left[\exp\left(-\frac{(\nabla^\xi)^2}{2i\pi}\right)\right] \right\}^{\max} \tag{1.30}$$

and the convergence is uniform in M.

Proof. Since $dB = 0$, by Theorem 1.3, we know that

$$(D^L + {}^cB)^2 = -\sum_1^n (\nabla^L_{e_i} + {}^c(i_{e_i}B))^2 + \frac{K}{4} + {}^c((\nabla^\xi)^2) - 2\|B\|^2 . \tag{1.31}$$

Now using (1.14), we know that

$$(i_{e_i}B) = \tfrac{1}{2} \sum (i_{e_i}B)(e_j, e_k)dx^j dx^k = \tfrac{1}{4} \langle S^B(e_i)e_j, e_k \rangle dx^j dx^k \tag{1.32}$$

and so

$$(D^L + {}^cB)^2 = -\sum_1^n \left(\nabla^L_{e_i} + \frac{1}{4}\langle S^B(e_i)e_j, e_k \rangle {}^ce_j {}^ce_k\right)^2 + \frac{K}{4} - 2\|B\|^2 + {}^c((\nabla^\xi)^2) . \tag{1.33}$$

Observe that the connection on F

$$\nabla^L + \tfrac{1}{4}\langle S^B(\cdot)e_j, e_k \rangle {}^ce_j {}^ce_k \tag{1.34}$$

is exactly the lift to F of the Euclidean connection ∇^B.

We now indicate the principle of the proof of (1.30) along the lines of our previous work [B1, Sect. 4]. Let g^M be the metric of M. Take $x_0 \in M$. For $t>0$, let x_s^t be the Brownian notion in M associated with the metric $\frac{g^M}{t}$. Let E be the corresponding expectation operator.

For $s>0$, let $\tau_0^{s,t}$ be the parallel transport operator from $(F\otimes\xi)_{x_s^t}$ into $(F\otimes\xi)_{x_0}$ with respect to the connection $\nabla^B\otimes 1+1\otimes\nabla^\xi$ on $F\otimes\xi$, which we still note ∇^B. Let U_s^t be the solution of the differential equation

$$\begin{aligned}\frac{dU_s^t}{ds} &= U_s^t\left[-\frac{1}{2}\tau_0^{s,t}c((\nabla^\xi)^2_{x_s^t})\right]\\ U_0^t &= I_{(F\otimes\xi)_{x_0}}.\end{aligned} \tag{1.35}$$

Using formula (1.33) and Ito's formula as in [B1, Theorem 2.5], we know that if h is a smooth section of $F\otimes\xi$, then

$$\exp\left(\frac{-t(D^L+B)^2}{2}\right)h(x_0)=E\left[\exp\left\{-\frac{t}{2}\int_0^1\left(\frac{K}{4}-2\|B\|^2\right)(x_s^t)ds\right\}U_1^t\tau_0^{1,t}h(x_1^t)\right]. \tag{1.36}$$

Let $p_t(x_0,y_0)$ be the heat kernel on M associated with the operator $\exp\left(\frac{t}{2}\Delta\right)$ (where Δ is the Laplace-Beltrami operator for the metric g^M). Let Q_{x_0,x_0}^t be the probability law on $\mathscr{C}([0,1];M)$ of the Brownian bridge starting at x_0 at time 0 and going back to x_0 at time 1 associated with the metric $\frac{g^M}{t}$. Q_{x_0,x_0}^t is simply obtained by disintegration of the probability law of x^t with respect to the map $x^t\to x_1^t$. From (1.36), we find that

$$P_{\frac{t}{2}}^B(x_0,x_0)=p_t(x_0,x_0)E^{Q_{x_0,x_0}^t}\left[\exp\left\{\frac{-t}{2}\int_0^1\left(\frac{K}{4}-2\|B\|^2\right)(x_s^t)ds\right\}U_1^t\tau_0^{1,t}\right]. \tag{1.37}$$

At this stage, we are formally in a situation formally similar to [B1, Sect. 3]. Of course with respect to [B1]

- K is replaced by $K-8\|B\|^2$. However for t small, this has no influence on the asymptotics of $\mathrm{Tr}_s[P_t^B(x_0,x_0)]$.
- The connection ∇^L in [B1] is replaced by ∇^B.

Let $w_s^1 (0\leq s\leq 1)$ be a Brownian bridge in $T_{x_0}M$, with $w_0^1=w_1^1=0$, whose probability law is denoted P_1.

If $X, Y\in TM$, we identify $R^B(X,Y)$ with the two form $Z,T\to\langle Z, R^B(X,Y)T\rangle$. Let $\exp^\wedge$ be the exponential in $\Lambda^{\mathrm{even}}(T^*M)$. From [B1, Theorem 3.10 and 3.18] where we simply replace ∇ by ∇^B, we get

$$\begin{aligned}\lim_{t\downarrow 0}\mathrm{Tr}_s[P_t(x_0,x_0)]dx_0 = &\left\{\int\exp^\wedge\left\{\frac{-i}{4\pi}\int_0^1 R^B(dw^1,w^1)\right\}\right.\\ &\left. dP_1(w^1)\wedge\mathrm{Tr}\left[\exp\left(\frac{-(\nabla^\xi)^2}{2i\pi}\right)\right]\right\}^{\max} \text{ uniformly on } M.\end{aligned} \tag{1.38}$$

By Theorem 1.6, we know that if $Z, T \in TM$

$$\int_0^1 \langle Z, R^B(dw^1, w^1)T\rangle = \int_0^1 \langle R^{-B}(Z, T)w^1, dw^1\rangle .$$

By [B1, Theorem 3.17], we know that

$$\int \exp^{\wedge}\left\{\frac{-i}{4\pi}\int_0^1 \langle R^{-B}w^1, dw^1\rangle\right\} = \hat{A}\left(\frac{R^{-B}}{2\pi}\right). \tag{1.39}$$

(1.30) follows from (1.38) and (1.39). □

Remark 1.8. Of course (1.30) can also be obtained by any other method for the proof of the local index theorem. In particular if one uses the method of Getzler [Ge], there are two possible choices:

• Either one trivializes the bundle of orthonormal frames in TM using the connection ∇^B instead of ∇^L.

• Or one directly rescales the operator (1.31) according to [Ge], but then an exponential transform is needed to overcome a singularity which appears because of S^B.

The proof of (1.30) is also possible by the methods of Berline-Vergne [BeV], with a few obvious modifications.

d) Local index theorem for Dirac operators associated with connections with non zero torsion

We now will use formula (1.30) in a special situation. Namely let ∇ be any arbitrary Euclidean connection on TM. Let S be the tensor defined by the relation

$$\nabla = \nabla^L + S . \tag{1.40}$$

S is a one form with values in antisymmetric tensors on TM. Let T be the torsion of ∇. Classically, if $X, Y, Z \in TM$

$$\begin{gathered} T(X, Y) = S(X)Y - S(Y)X . \\ 2\langle S(X)Y, Z\rangle - \langle T(X, Y), Z\rangle - \langle T(Z, X), Y\rangle + \langle T(Y, Z), X\rangle = 0 . \end{gathered} \tag{1.41}$$

The connection ∇ lifts naturally into a unitary connection on the vector bundle F, which we still note ∇. If $e_1, \dots, e_n$ is an orthonormal base of TM, then

$$\nabla_{\cdot} = \nabla^L_{\cdot} + \sum \tfrac{1}{4}\langle S(\cdot)e_i, e_j\rangle {}^c e_i {}^c e_j . \tag{1.42}$$

Let θ be the one form

$$X \in TM \to \theta(X) = X \in TM .$$

The first line in (1.41) is equivalent to

$$T = S \wedge \theta . \tag{1.43}$$

Let $\langle T \overset{\wedge}{,} \theta\rangle$ be the antisymmetrization of the tensor $X, Y, Z \to \langle T(X, Y), Z\rangle$.

If $dx^1, \dots, dx^n$ is the base of T^*M dual to $e_1, \dots, e_n$, then

$$\langle T \overset{\wedge}{,} \theta\rangle = \tfrac{1}{2}\langle T(e_i, e_j), e_k\rangle dx^i dx^j dx^k . \tag{1.44}$$

Of course

$$\langle T \wedge \theta \rangle = \langle S\theta \wedge \theta \rangle . \tag{1.45}$$

We still denote by ∇ the Hermitian connection on $F \otimes \xi$ $\nabla \otimes 1 + 1 \otimes \nabla^{\xi}$.

Definition 1.9. D_T denotes the Dirac operator acting on the set of smooth sections of $F \otimes \xi$

$$D_T = \sum_1^n {}^c e_i \nabla_{e_i} + \frac{1}{2} \sum_1^n {}^c(S(e_i)e_i) . \tag{1.46}$$

Theorem 1.10. *The operator D_T is self-adjoint. Also the following identities hold*

$$D_T = D^L + \frac{1}{4} {}^c\langle T \wedge \theta \rangle ,$$

$$D_T^2 = -\sum_1^n \left(\nabla_{e_i}^L + \frac{1}{4} {}^c(i_{e_i} \langle T \wedge \theta \rangle) \right)^2 + \frac{K}{4} + {}^c((\nabla^{\xi})^2) + \frac{1}{4} {}^c(d\langle T \wedge \theta \rangle) - \frac{1}{8} \| \langle T \wedge \theta \rangle \|^2 . \tag{1.47}$$

Proof. Let D' be the operator

$$D' = \sum_1^n {}^c e_i \nabla_{e_i} \tag{1.48}$$

if D'^* is the formal adjoint of D', one verifies easily that

$$D'^* = D' + \sum_1^n {}^c(S(e_i)e_i) . \tag{1.49}$$

Therefore

$$D_T = \tfrac{1}{2}(D' + D'^*) \tag{1.50}$$

and so D_T is formally self-adjoint.

Also by (1.42), (1.48), we know that

$$D' = D^L + \tfrac{1}{4} \sum \langle S(e_j)e_j, e_k \rangle {}^c e_i {}^c e_j {}^c e_k . \tag{1.51}$$

Clearly

$$\tfrac{1}{4} \sum \langle S(e_i)e_i, e_j \rangle ({}^c e_i)^2 {}^c e_j + \tfrac{1}{4} \langle S(e_i)e_j, e_i \rangle {}^c e_i {}^c e_j {}^c e_i = -\tfrac{1}{2} \sum {}^c(S(e_i)e_i) . \tag{1.52}$$

From (1.50)–(1.52), we get

$$D_T = D^L + \tfrac{1}{4} {}^c(\sum \langle S(e_i)e_j, e_k \rangle dx^i dx^j dx^k) = D^L + \tfrac{1}{4} {}^c\langle T \wedge \theta \rangle . \tag{1.53}$$

The first line in (1.47) is proved. The second line in (1.47) follows from Theorem 1.3. □

For $t > 0$, let $P_t(x, y)$ be the smooth heat kernel associated with the operator $\exp(-tD_T^2)$. From Theorems 1.7 and 1.10, we deduce.

Theorem 1.11. *Assume that the three form $B = \frac{1}{4}\langle T \wedge \theta \rangle$ is closed. Then*

$$\lim_{t \downarrow 0} \mathrm{Tr}_s[P_t(x_0, x_0)] dx_0 = \left\{ \hat{A}\left(\frac{R^{-B}}{2\pi} \right) \mathrm{Tr}\left[\exp\left(\frac{-(\nabla^{\xi})^2}{2i\pi} \right) \right] \right\}^{\max} \tag{1.54}$$

uniformly on M.

Remark 1.12. Assume that the tensor $X, Y, Z \to \langle T(X, Y), Z\rangle$ is antisymmetric. Then

$$\langle T(X, Y), Z\rangle = \tfrac{1}{3}\langle T \overset{\wedge}{,} \theta\rangle (X, Y, Z) = \tfrac{4}{3}B(X, Y, Z). \tag{1.55}$$

By Proposition 1.5, we find that $\nabla = \nabla^{\frac{B}{3}}$.

From Proposition 1.5, Theorem 1.11 and (1.55), we find that R^{-B} is the curvature of the Euclidean connection on TM with torsion $-3T$.

Remark 1.13. Let G be a compact group equipped with a right and left invariant metric. For $a \in R$, let ${}^a\nabla$ be the connection on TM such that if X, Y are left invariant vector fields

$${}^a\nabla_X Y = a[X, Y]. \tag{1.56}$$

The torsion T of ${}^a\nabla$ is given by

$${}^aT(X, Y) = (2a-1)[X, Y]. \tag{1.57}$$

Also the tensor $X, Y, Z \to \langle [X, Y], Z\rangle$ is antisymmetric and the corresponding three form is closed. Therefore Theorem 1.11 can be used in this case.

II. The local index theorem for non Kähler manifolds

In this section, we specialize the results of Sect. 1 to the case where M is a compact complex manifold equipped with a Hermitian metric, whose Kähler form is denoted ω. More precisely, we study in detail the local index theorem for Dirac operators of the type $\bar\partial + \bar\partial^*$.

In a), we give a formula for $\bar\partial + \bar\partial^*$ and $(\bar\partial + \bar\partial^*)^2$ in terms of Clifford multiplication operators. In b), when $\bar\partial\partial\omega = 0$, we construct an exotic holomorphic structure on the vector bundle $T_R M \otimes_R \mathbb{C}$ which coincides with the natural one if (M, ω) is Kähler. In c), when $\bar\partial\partial\omega = 0$, we obtain a local Riemann-Roch-Hirzebruch Theorem.

a) A formula for $2(\bar\partial + \bar\partial^*)^2$

Let M is a compact connected complex manifold of complex dimension l. $\Lambda T^{*(0,1)}M$ denotes the algebra of forms of type $(0,p)\,(0 \leq p \leq l)$. Let TM be the holomorphic tangent bundle on M, and let $T_R M$ be the corresponding real tangent bundle.

Let g be a Hermitian metric on $T_R M$ and let ω be the corresponding Kähler form. If J is the complex structure of $T_R M$, then for $X, Y \in T_R M$

$$\omega(X, Y) = \langle X, JY\rangle. \tag{2.1}$$

Then $\Lambda T^{*(0,1)}M$ is a $c(T_R M)$ Clifford module. Namely if $X \in T^{(1,0)}M$, $Y \in T^{(0,1)}M$, if $X^* \in T^{*(0,1)}M$ corresponds to X by the metric, set

$$c(X) = \sqrt{2}X^* \wedge; \qquad c(Y) = -\sqrt{2}i_Y. \tag{2.2}$$

Then if $Y, Y' \in T_R M$, one verifies easily that

$$c(Y)c(Y') + c(Y')c(Y) = -2\langle Y, Y'\rangle. \tag{2.3}$$

To make our arguments simpler, we will assume that M is spin, or equivalently [H, Theorem 2.2] that the line bundle $\det(T^{TM}M)$ has a square root λ, which is then a holomorphic line bundle. Note that this assumption is always verified locally, and this is the only fact we need.

Then the metric on $T^{TM}M$ induces a Hermitian metric on λ. Also by [H, Theorem 2.2], if

$$F = \Lambda(T^{*(0,1)}M) \otimes \lambda^* . \tag{2.4}$$

F is the Hermitian bundle of TM spinors on M. Moreover

$$\begin{aligned} F_+ &= \Lambda^{\text{even}}(T^{*(0,1)}M) \otimes \lambda^* \\ F_- &= \Lambda^{\text{odd}}(T^{*(0,1)}M) \otimes \lambda^* \end{aligned} \tag{2.5}$$

and the identification (2.5) also identifies the metrics.

Let ∇^L be the Levi-Civita connection on $T_R M$, and let ∇^{TM} be the holomorphic Hermitian connection on TM. Similarly let ∇^λ, $\nabla^{\lambda*}$ be the holomorphic Hermitian connections on the line bundles λ, λ^*. Of course $\nabla^L = \nabla^{TM}$ if and only if (M, ω) is Kähler, i.e. if the form ω is closed. In the sequel, we do *not* assume that ω is closed.

Therefore F is equipped with two natural connections:

- one is the lift of ∇^L to F and is still noted ∇^L.
- The other is the lift of ∇^{TM} to F and is noted ∇^F. Equivalently ∇^F is also the holomorphic Hermitian connection on F.

Of course ∇^{TM} induces the natural antiholomorphic connection on $\Lambda(T^{*(0,1)}M)$, which we still note ∇^{TM}. Identifying $\Lambda(T^{*(0,1)}M)$ with $\Lambda(T^{(1,0)}M)$ by the metric, ∇^{TM} induces the holomorphic Hermitian connection on $\Lambda(T^{(1,0)}M)$.

Using (2.4) and the uniqueness of holomorphic Hermitian connections, we get

$$\nabla^F = \nabla^{TM} \otimes 1 + 1 \otimes \nabla^{\lambda*} . \tag{2.6}$$

In this section, $d^M = \partial^M + \bar{\partial}^M$ denotes the exterior differentiation operator acting on smooth sections of $\Lambda(T_R^* M)$.

Let T be the torsion of the connection ∇^{TM}. Remember that T maps $T^{(1,0)}M \times T^{(1,0)}M$ (resp. $T^{(0,1)}M \times T^{(0,1)}M$) into $T^{(1,0)}M$ (resp. $T^{(0,1)}M$) and vanishes on $T^{(1,0)}M \times T^{(0,1)}M$. Set

$$S = \nabla^{TM} - \nabla^L . \tag{2.7}$$

Proposition 2.1. *We have the identity of three forms on M*

$$\langle T \wedge \theta \rangle = i(\partial^M - \bar{\partial}^M)\omega . \tag{2.8}$$

Proof. Clearly $\nabla^{TM}\omega = 0$. Also

$$d^M = \nabla^{TM} + i_T . \tag{2.9}$$

Therefore

$$d^M \omega = i_T \omega \tag{2.10}$$

and so

$$d^M \omega = -i \langle (T^{(1,0)} - T^{(0,1)}) \wedge \theta \rangle . \tag{2.11}$$

Using the properties of T listed before, we get

$$\begin{aligned} \partial^M \omega &= -i\langle T^{(1,0)} \underset{,}{\wedge} \theta\rangle \\ \bar{\partial}^M \omega &= i\langle T^{(0,1)} \underset{,}{\wedge} \theta\rangle . \end{aligned} \tag{2.12}$$

(2.8) follows. □

Let ξ be a holomorphic Hermitian vector bundle on M of complex dimension k. Let ∇^ξ be the holomorphic Hermitian connection on ξ, whose curvature is denoted $(\nabla^\xi)^2$.

Let $\bar{\partial}$ be the Dolbeault operator acting on the set Γ of smooth sections of $\Lambda(T^{*(0,1)}M) \otimes \xi$. Γ is naturally equipped with the L^2 Hermitian product

$$\eta, \eta' \in \Gamma \to \int_M \langle \eta \underset{,}{\wedge} * \eta' \rangle . \tag{2.13}$$

Let $\bar{\partial}^*$ be the formal adjoint of $\bar{\partial}$ with respect to the Hermitian product (2.13). Note that

$$\Lambda(T^{*(0,1)}M) \otimes \xi = F \otimes (\lambda \otimes \xi) . \tag{2.14}$$

$\lambda \otimes \xi$ is naturally equipped with the connection $\nabla^\lambda \otimes 1 + 1 \otimes \nabla^\xi$. Therefore, we can define the Levi-Civita Dirac operator D^L acting on Γ as in (1.1).

If we instead equip $T_R M$ with the holomorphic Hermitian connection ∇^{TM} whose torsion is T, we can define the associated Dirac operator D_T as in Definition 1.9.

Theorem 2.2. *We have the following identities*

$$\begin{aligned} D_T &= D^L + \tfrac{1}{4}{}^c\langle T \underset{,}{\wedge} \theta\rangle \\ \sqrt{2}(\bar{\partial} + \bar{\partial}^*) &= D^L - \tfrac{1}{4}{}^c\langle T \underset{,}{\wedge} \theta\rangle . \end{aligned} \tag{2.15}$$

Proof. The first line of (2.15) is a special case of Theorem 1.10.

Let $w_1, \ldots, w_l$ be an orthonormal base of $T^{(1,0)}M$ and let $\bar{w}^1, \ldots, \bar{w}^l$ be the corresponding dual base of $T^{*(0,1)}M$. The operator $\bar{\partial}$ is given by

$$\bar{\partial} = \sum_1^l \bar{w}^i \wedge \nabla^{\xi''}_{\bar{w}_i} . \tag{2.16}$$

We still note by ∇^{TM} the connection $\nabla^{TM} \otimes 1 + 1 \otimes \nabla^\xi$ on $\Lambda(T^{*(0,1)}M) \otimes \xi$. Since ∇^{TM} has torsion T, by (2.9), we find that

$$\sqrt{2}\bar{\partial} = c(w_i)\nabla^{TM}_{\bar{w}_i} + \sqrt{2}\, i_{T^{(0,1)}} . \tag{2.17}$$

Equivalently using (2.2), we get

$$\sqrt{2}\bar{\partial} = c(w_i)\nabla^{TM}_{\bar{w}_i} - \tfrac{1}{4} c(w_i) c(w_j) c(T(\bar{w}_i, \bar{w}_j)) . \tag{2.18}$$

Set $n = 2l$. Let $e_1, \ldots, e_n$ be a real orthonormal base of $T_R M$. Set

$$Y = \sum_1^n S(e_i) e_i . \tag{2.19}$$

By taking adjoints in (2.18), one finds easily that

$$\sqrt{2}\bar{\partial}^* = c(\bar{w}_i)\nabla^{TM}_{w_i} + c(Y^{(0,1)}) + \tfrac{1}{4}c(T(w_i, w_j))c(\bar{w}_j)c(\bar{w}_i). \tag{2.20}$$

From (2.18), (2.20), we get

$$\sqrt{2}(\bar{\partial}+\bar{\partial}^*) = c(w_i)\nabla^{TM}_{\bar{w}_i} + c(\bar{w}_i)\nabla^{TM}_{w_i} + c(Y^{(0,1)}) - \frac{1}{4}c(w_i)c(w_j)c(T(\bar{w}_i, \bar{w}_j))$$
$$- \frac{1}{4}c(T(w_i, w_j))c(\bar{w}_i)c(\bar{w}_j). \tag{2.21}$$

Now

$$\begin{aligned} c(T(w_i, w_j))c(\bar{w}_i)c(\bar{w}_j) &= -2\langle T(w_i, w_j), \bar{w}_i\rangle c(\bar{w}_j) \\ -c(\bar{w}_i)c(T(w_i, w_j))c(\bar{w}_j) &= -2\langle T(w_i, w_j), \bar{w}_i\rangle c(\bar{w}_j) \\ &+2\langle T(w_i, w_j), \bar{w}_j\rangle c(\bar{w}_i) + c(\bar{w}_i)c(\bar{w}_j)c(T(w_i, w_j)). \end{aligned} \tag{2.22}$$

Using the properties of T listed before (2.7), and Eq. (1.41), we get

$$2\langle S(w_i)\bar{w}_i, w_j\rangle = -\langle T(w_i, w_j), \bar{w}_i\rangle. \tag{2.23}$$

Also since $T(w_i, \bar{w}_i) = 0$, $S(w_i)\bar{w}_i = S(\bar{w}_i)w_i$, and so

$$Y = 2S(w_i)\bar{w}_i. \tag{2.24}$$

From (2.22)–(2.24), we find

$$c(T(w_i, w_j))c(\bar{w}_i)c(\bar{w}_j) = c(\bar{w}_i)c(\bar{w}_j)c(T(w_i, w_j)) + 4c(Y^{(0,1)}). \tag{2.25}$$

Using (1.41), we also have

$$\begin{aligned} &\tfrac{1}{4}c(w_i)c(w_j)c(T(\bar{w}_i, \bar{w}_j)) + \tfrac{1}{4}c(\bar{w}_i)c(\bar{w}_j)c(T(w_i, w_j)) \\ &= \tfrac{1}{4}\langle T(e_i, e_j), e_k\rangle c(e_i)c(e_j)c(e_k) \\ &= \tfrac{1}{2}{}^c\langle T \wedge \theta\rangle + \tfrac{1}{2}\sum \langle T(e_i, e_j), e_i\rangle c(e_j) \\ &= \frac{1}{2}{}^c\langle T \wedge \theta\rangle - \frac{c(Y)}{2}. \end{aligned} \tag{2.26}$$

From (2.21)–(2.26), we get

$$\sqrt{2}(\bar{\partial}+\bar{\partial}^*) = c(w_i)\nabla^{TM}_{\bar{w}_i} + c(\bar{w}_i)\nabla^{TM}_{w_i} + \frac{c(Y)}{2} - \frac{1}{2}{}^c\langle T \wedge \theta\rangle \tag{2.27}$$

or equivalently

$$\sqrt{2}(\bar{\partial}+\bar{\partial}^*) = D_T - \frac{1}{2}{}^c\langle T \wedge \theta\rangle. \tag{2.28}$$

From the first line in (2.15) and (2.28), we deduce the second line of (2.15). □

In the sequel, we use the same notations as in Theorems 1.3 and 1.10. Let K be the scalar curvature of M. We now give a form of the Kodaira-Nakano identity for the operator $2(\bar{\partial}+\bar{\partial}^*)^2$.

Theorem 2.3. *The following identity holds*

$$2(\bar\partial+\bar\partial^*)^2 = -\sum_1^n \left(\nabla^L_{e_i} - \frac{\sqrt{-1}}{4}\, {}^c(i_{e_i}(\partial^M-\bar\partial^M)\omega)\right)^2 + \frac{K}{4} + {}^c\!\left((\nabla^\xi)^2 + \frac{1}{2}\operatorname{Tr}[(\nabla^{TM})^2] I_\xi\right) - \frac{\sqrt{-1}}{2}\, {}^c(\bar\partial^M\partial^M\omega) - \frac{1}{8}\,\|(\partial^M-\bar\partial^M)\omega\|^2 . \tag{2.29}$$

Proof. The curvature of the holomorphic Hermitian connection on $\xi\otimes\lambda$ is exactly $(\nabla^\xi)^2+(\nabla^\lambda)^2 I_\xi$. On the other hand

$$(\nabla^\lambda)^2 = \tfrac{1}{2}\operatorname{Tr}[(\nabla^{TM})^2].$$

(2.29) is now a consequence of Proposition 2.1 and of Theorems 2.2 and 1.3. □

Remark 2.4. By Proposition 2.1, the condition

$$d^M\langle T \overset{\wedge}{,} \theta\rangle = 0 \tag{2.30}$$

is equivalent to the condition

$$\bar\partial^M\partial^M\omega = 0. \tag{2.31}$$

Condition (2.31) is well known in the literature. In particular a result by Gauduchon [Ga] asserts that on a complex surface, there exists a Hermitian metric for which (2.31) is verified. In view of Theorem 1.7, condition (2.31) implies the existence of a local index formula for the Euler characteristic of ξ. We will exploit this fact at the end of this section.

b) An exotic holomorphic structure on $T_RM\otimes_R\mathbb{C}$

We now use the notations of Sect. 1. Let B be the real three form $B=\dfrac{\langle T \overset{\wedge}{,} \theta\rangle}{4}$. Equivalently by Proposition 2.1, we have

$$B = \frac{i}{4}(\partial^M-\bar\partial^M)\omega . \tag{2.32}$$

By Theorem 2.2, we know that

$$\sqrt{2}(\bar\partial+\bar\partial^*) = D^{-B}. \tag{2.33}$$

Proposition 2.5. *The following identity holds for $X, Y, Z \in T_RM$*

$$2\langle S^{-B}(X)Y, Z\rangle = -\langle T(X,Y),Z\rangle - \langle T(Y,Z),X\rangle - \langle T(Z,X),Y\rangle . \tag{2.34}$$

In particular

$$\begin{aligned}\langle (S^{-B}-S)(X)Y,Z\rangle &= -\langle T(X,Y),Z\rangle + \langle T(X,Z),Y\rangle \\ \langle (S^{-B}+S)(X)Y,Z\rangle &= -\langle T(Y,Z),X\rangle .\end{aligned} \tag{2.35}$$

The connection ∇^{-B} preserves the complex structure of T_RM.

Proof. By (1.14)–(1.16) we know that

$$2\langle S^{-B}(X)Y,Z\rangle = -\langle T \wedge \theta\rangle (X,Y,Z) \tag{2.36}$$

which is equivalent to (2.34). Comparing with (1.41), we obtain (2.35). Clearly

$$\nabla^{-B} = \nabla^{TM} + S^{-B} - S. \tag{2.37}$$

Using the properties of T and the first line in (2.35), we find that if $Y,Z\in T^{(1,0)}M$, for any $X\in T_R M$

$$\langle (S^{-B}-S)(X)Y,Z\rangle = 0. \tag{2.38}$$

Equivalently $(S^{-B}-S)(X)$ is a complex endomorphism of T_RM. Using (2.37), we find that ∇^{-B} preserves the complex structure of T_RM. □

Definition 2.6. If $X\in T^{(0,1)}M$, $Y\in T^{(1,0)}M$, let $\alpha(X)Y\in T^{*(1,0)}M$ be defined by the fact that if $Z\in T^{(1,0)}M$

$$(\alpha(X)Y)(Z) = i\partial^M\omega(X,Y,Z). \tag{2.39}$$

Let E be the holomorphic vector bundle

$$E = T^{(1,0)}M \oplus T^{*(1,0)}M. \tag{2.40}$$

We will write elements of End E in matrix form with respect to the spliting (2.40) of E. For $X\in T^{(0,1)}M$, set

$$\beta(X) = \begin{pmatrix} 0 & 0 \\ \alpha(X) & 0 \end{pmatrix}. \tag{2.41}$$

Equivalently $\beta(X)$ coincides with $\alpha(X)$ on $T^{(1,0)}M$ and vanishes on $T^{*(1,0)}M$.

Let $\nabla^{E''}$ be the $\bar\partial$ operator which defines the natural holomorphic structure on E.

Theorem 2.7. *If $\bar\partial^M\partial^M\omega = 0$, then $\nabla^{E''}+\beta$ defines a holomorphic structure on E, i.e. $(\nabla^{E''}+\beta)^2 = 0$.*

Proof. Clearly $\beta^2 = 0$. So we must prove that $\nabla^{E''}\beta = 0$.

Let $z^1,\dots,z^l$ be a holomorphic coordinate system on M. It induces a corresponding local holomorphic trivialization of $T^{(1,0)}M$ and of $T^{*(1,0)}M$. Then if Y,Z are holomorphic sections of $T^{(1,0)}M$, by (2.39), $X\in T^{(0,1)}M \to (\beta(X)Y)(Z)$ is the $(0,1)$ form

$$X\in T^{(0,1)}M \to i\partial^M\omega(X,Y,Z).$$

If $X,X'\in T^{(0,1)}M$, it is then clear that

$$(\nabla^{E''}\beta)(X,X')(Y)(Z) = i\bar\partial^M\partial^M\omega(X,X',Y,Z). \tag{2.42}$$

Since $\bar\partial^M\partial^M\omega = 0$, we find that $\nabla^{E''}\beta = 0$. □

From now on we assume that

$$\bar\partial^M\partial^M\omega = 0. \tag{2.43}$$

When E is equipped with the holomorphic structure $\nabla^{E''}+\beta$, we will write $(E, \nabla^{E''}+\beta)$. The metric on $T^{(1,0)}M$ induces a metric on $T^{*(1,0)}M$. We equip E with the Hermitian metric which coincides with the given metrics on $T^{(1,0)}M$ and $T^{*(1,0)}M$ and is such that the spliting (2.40) of E is orthogonal.

Definition 2.8. $\nabla^{E,\beta}$ denotes the unique holomorphic Hermitian connection on the holomorphic vector bundle $(E, \nabla^{E''}+\beta)$.

The vector bundle $T_{\mathbb{C}}M = T_R M \otimes_R \mathbb{C}$ splits into

$$T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M. \tag{2.44}$$

There is a natural conjugations map $T_{\mathbb{C}}M \to T_{\mathbb{C}}M$ which interchanges $T^{(1,0)}M$ and $T^{(0,1)}M$. $T_{\mathbb{C}}M$ is also naturally equipped with a Hermitian metric, such that the splitting (2.44) is orthogonal.

Also the metric identifies the vector bundles $T^{(0,1)}M$ and $T^{*(1,0)}M$ and so defines a holomorphic structure on $T^{(0,1)}M$. We then have the identification

$$T_{\mathbb{C}}M \underset{\omega}{\overset{\sim}{-}} E, \tag{2.45}$$

where $\underset{\omega}{\overset{\sim}{-}}$ indicates that the identification depends on ω.

So $T_{\mathbb{C}}M$ now carries two holomorphic structures, which *both* depend on ω:

- The first is inherited from the natural holomorphic structure $\nabla^{E''}$ of E.
- The second, which we note $\nabla^{T_{\mathbb{C}}M,\beta''}$, corresponds to $\nabla^{E,\beta''}$ via the identification (2.45).

Let $\nabla^{T_{\mathbb{C}}M,\beta}$ be the holomorphic Hermitian connection on $(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M,\beta''})$. Of course $\nabla^{T_{\mathbb{C}}M,\beta}$ correspond to ∇^E via the identification (2.45).

A connection on $T_{\mathbb{C}}M$ is said to be real if it is the complexification of a connection on $T_R M$.

Theorem 2.9. *The connection $\nabla^{T_{\mathbb{C}}M,\beta}$ is real. Moreover the have the identity*

$$\nabla^{T_{\mathbb{C}}M,\beta} = \nabla^B. \tag{2.46}$$

Proof. Remember that

$$\nabla^B = \nabla^L - S^{-B} = \nabla^{TM} - (S + S^{-B}). \tag{2.47}$$

By Proposition 2.5, we find that if $X, Y, Z \in T_{\mathbb{C}}M$

$$-\langle (S+S^{-B})(X)Y, Z\rangle = \langle T(Y,Z), X\rangle. \tag{2.48}$$

If $X \in T^{(0,1)}M$, $Y \in T^{(0,1)}M$, we deduce from (2.48) that

$$(S+S^{-B})(X)Y = 0. \tag{2.49}$$

If $X \in T^{(0,1)}M$, $Y \in T^{(1,0)}M$, by (2.48), we find that $(S+S^{-B})(X)\ Y \in T^{(0,1)}M$. More precisely if $Z \in T^{(1,0)}M$, from Proposition 2.1, we get

$$\langle T(Y,Z), X\rangle = i(\partial^M \omega)(Y, Z, X). \tag{2.50}$$

We deduce from (2.39), (2.48), (2.50) that if $X \in T^{(0,1)}M$, $Y \in T^{(1,0)}M$, when identifying E to $T_{\mathbb{C}}M$

$$-(S+S^{-B})(X)Y=\alpha(X)Y. \tag{2.51}$$

From (2.49), (2.51), we find that if $X \in T^{(0,1)}M$, then

$$\nabla_X^B = \nabla_X^{T_{\mathbb{C}}M, \beta}. \tag{2.52}$$

A similar argument shows that if $X \in T^{(0,1)}M$, (2.52) still holds. The Theorem is proved. □

Remark 2.10. By Proposition 2.5, we know that ∇^{-B} preserves the complex structure of $T_R M$. In particular the curvature R^{-B} is a two form taking values in complex automorphisms of $T_R M$.

Since $\bar{\partial}^M \partial^M \omega = 0$, by Theorem 1.6, we find that R^B is a two form of complex type (1,1). Using [AHS, Theorem 5.1], we can then deduce that there is a holomorphic structure on $T_{\mathbb{C}}M$ such that ∇^B is the associated holomorphic Hermitian connection. Theorem 2.9 has made explicit this new holomorphic structure on $T_{\mathbb{C}}M$.

c) A local Riemann-Roch-Hirzebruch Theorem for non Kähler manifolds

For $t>0$, let $Q_t(x,y)$ be the C^∞ kernel on M associated with the operator $\exp(-t(\bar{\partial}+\bar{\partial}^*)^2)$. Then if $\chi(\xi)$ is the Euler characteristic of ξ, the Mc-Kean Singer formula [MKS] asserts that

$$\chi(\xi) = \int \mathrm{Tr}_s[Q_t(x,x)]\,dx.$$

Remember that ∇^{TM} is the standard holomorphic Hermitian connection on $T^{TM}M$ whose curvature is noted $(\nabla^{TM})^2$. Therefore $\frac{1}{2}\mathrm{Tr}[(\nabla^{TM})^2]$ is the curvature of the holomorphic Hermitian connection on the line bundle λ.

Theorem 2.11. *Assume that $\bar{\partial}^M \partial^M \omega = 0$. Then*

$$\lim_{t\to 0} \mathrm{Tr}_s[Q_t(x,x)]dx = \left\{\hat{A}\left(\frac{R^B}{2\pi}\right)\exp\left(-\frac{1}{2i\pi}\mathrm{Tr}\left[\frac{(\nabla^{TM})^2}{2}\right]\right)\mathrm{Tr}\left[\exp\left(\frac{-(\nabla^{\xi})^2}{2i\pi}\right)\right]\right\}^{\max} \tag{2.53}$$

uniformly on M.

Proof. Theorem 2.11 is an obvious consequence of Theorems 1.7 and 2.2. □

Remark 2.12. Note that the Todd polynomial does not appears as such in (2.53). Of course if ω is closed, – so that $B=0$ – then

$$\hat{A}\left(\frac{R^B}{2\pi}\right)\exp\left(-\frac{1}{2i\pi}\mathrm{Tr}\left[\frac{(\nabla^{TM})^2}{2}\right]\right) = \mathrm{Td}\left(-\frac{(\nabla^{TM})^2}{2i\pi}\right). \tag{2.54}$$

In general, equality (2.54) only holds in cohomology.

References

[ABP] Atiyah, M.F., Bott, R., Patodi, V.K.: On the heat equation and the index theorem. Invent. Math. **19**, 279–330 (1973)

[AHS] Atiyah, M.F., Hitchin, N.J., Singer, I.M.: Self-duality and four dimensional Riemannian geometry. Proc. Roy. Soc. Lond. A **362**, 425–461 (1978)

[AS] Atiyah, M.F., Singer, I.M.: The index of elliptic operators III. Ann. Math. **87**, 546–604 (1968)

[BeV] Berline, N., Vergne, M.: A computation of the equivariant index of the Dirac operator. Bull. Soc. Math. Fr. **113**, 305–345 (1985)

[B1] Bismut, J.M.: The Atiyah-Singer index theorems: a probabilistic approach. I. J. Funct. Anal. **57**, 56–99 (1984)

[B2] Bismut, J.M.: Le théorème d'indice local pour des variétés non kählériennes. C. R. Acad. Sci. Paris Sér. I **308**, 139–142 (1989)

[B3] Bismut, J.M.: Analytic torsion and equivariant Bott-Chern currents (to appear)

[B4] Bismut, J.M.: Formules de Lichnerowicz et théorème de l'indice. Proceedings of the Conference in honor of A. Lichnerowicz. D. Bernard, Y. Choquet-Bruhat (ed.) Vol. Géom. Diff., pp. 11–31. Travaux en cours. Paris: Hermann 1988

[Ga] Gauduchon, P.: Le théorème de l'excentricité nulle. C.R.A.S. Sér. A **285**, 387–390 (1977)

[Ge] Getzler, E.: A short proof of the Atiyah-Singer index theorem. Topology **25**, 111–117 (1986)

[Gi1] Gilkey, P.: Curvature and the eigenvalues of the Laplacian. Adv. Math. **10**, 344–382 (1973)

[Gi2] Gilkey, P.: Invariance theory, the heat equation and the Atiyah-Singer index theorem. Washington: Publish or Perish 1984

[H] Hitchin, N.: Harmonic spinors. Adv. Math. **14**, 1–55 (1974)

[L] Lichnerowicz, A.: Spineurs harmoniques. C. R. Acad. Sci. Paris Sér. A **257**, 7–9 (1963)

[MKS] McKean, H., Singer, I.M.: Curvature and the eigenvalues of the Laplacian. J. Differ. Geom. 1, 43–69 (1967)

[P1] Patodi, V.K.: Curvature and the eigenforms of the Laplacian. J. Differ. Geom. **5**, 233–249 (1971)

[P2] Patodi, V.K.: Analytic proof of the Riemann-Roch-Hirzebruch theorem for Kähler manifolds. J. Differ. Geom. **5**, 25–283 (1971)

[Q1] Quillen, D.: Superconnections and the Chern character. Topology **24**, 89–95 (1985)

[Q2] Quillen, D.: Determinants of Cauchy-Riemann operators over a Riemann surface. Funct. Anal. Appl. **14**, 31–34 (1985)

[RS] Ray, D.B., Singer, I.M.: Analytic torsion for complex manifolds. Ann. Math. **98**, 154–177 (1973)

Received December 3, 1988; in revised form January 16, 1989