Math. Ann. 284, 595–600 (1989)

Mathematische Annalen

Small contractions of four dimensional algebraic manifolds

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1. Introduction

In this paper we study small elementary contractions of four dimensional non-singular projective varieties defined over $\mathbb{C}$.

Let X be a non-singular projective variety. A surjective morphism $f: X \to Y$ onto a normal projective variety Y is said to be an *elementary contraction* if (1) f has connected fibers, (2) the anti-canonical divisor $-K_X$ is f-ample, and (3) all the curves on X which are vertical with respect to f are numerically proportional (i.e., if C_i $(i = 1, 2)$ are curves on X such that $f(C_i)$ are points, there exists a number r such that $(D \cdot C_1) = r(D \cdot C_2)$ for all divisors D on X). (See [KMM] for more general case.) f is called *small*, if it is birational and an isomorphism in codimension one (i.e., there exists an algebraic subset E of X of codimension $\geqq 2$ such that $f: X - E \xrightarrow{\simeq} Y - f(E)$).

There are no small elementary contractions of three dimensional algebraic manifolds. Non-small elementary contractions of algebraic manifolds in dimension four were studied in [A; B1; B2] after [M] in dimension three. The main result of this paper is the following.

(1.1) **Theorem.** *Let X be a non-singular projective variety of dimension four defined over $\mathbb{C}$, and let $f: X \to Y$ be a small elementary contraction. Then the exceptional locus E of f is a disjoint union of its irreducible components E_i $(i = 1, \ldots, n)$ such that $E_i \cong \mathbb{P}^2$ and $N_{E_i/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, where N denotes the normal bundle.*

We note that E may not be irreducible (see (2.6)).

The *flip* of a small elementary contraction $f: X \to Y$ is a birational morphism $f': X' \to Y$ from a normal projective variety X' with only terminal singularities such that the canonical divisor $K_{X'}$ is f'-ample as a $\mathbb{Q}$-divisor (cf. [KMM]).

(1.2) **Corollary.** *Let $f: X \to Y$ be as in* (1.1). *Then there exists a flip $f': X' \to Y$ of f from a non-singular projective variety X'.*

In fact, if $g: Z \to X$ is the blow-up at the center E, then its exceptional locus is a disjoint union of $\mathbb{P}^2 \times \mathbb{P}^1$'s with normal bundles isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)$, and by contracting them to the other direction, we obtain a morphism $g': Z \to X'$ to a compact complex manifold X' with an induced morphism $f': X' \to Y$. Since $K_{X'}$ is f'-ample by construction, X' is actually a projective manifold.

The main part of this work was done while the author was visiting the Max-Planck-Institut für Mathematik at Bonn in winter of 1987.

2. Proof of the main result

Since there are no small elementary contractions of three dimensional non-singular projective varieties, $f(E)$ is a finite set of points in (1.1). Hence (1.1) is a direct consequence of the following (2.1).

(2.1) **Theorem.** *Let (Y, P) be a germ of a normal isolated singularity of dimension four, and let $f: X \to Y$ be a desingularization. Assume that $-K_X$ is f-ample and the exceptional locus $E = f^{-1}(P)$ has dimension at most two. Then $E \cong \mathbb{P}^2$ and $N_{E/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.*

Note that, after localizing at a singularity of Y, we lose the condition on the numerical proportionality of vertical curves. The proof of (2.1) consists of four steps from (2.2) to (2.5).

(2.2) Let $\mu_i: \tilde{E}_i \to E_i$ $(i = 1, \ldots, n)$ be the normalizations of irreducible components E_i of E. We shall prove that $\tilde{E}_i \cong \mathbb{P}^2$ and $\mu_i^* \mathcal{O}_{E_i}(K_X) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$ for all i.

We fix $i = 1$. By [MM, Theorem 5], E_1 is covered by rational curves. Let b be the minimum of the numbers $-(K_X \cdot C)$ for all the rational curves C on E_1 such that $C \not\subset E_i$ for $i \neq 1$, and let C_1 be a curve which gives the minimum b. Let $a: \mathbb{P}^1 \to C_1 \subset X$ be the composition of the normalization and the closed embedding. We consider deformations of a following the argument of [I, (0.4)].

If we take x and x' to be non-singular points of E contained in E_1, we have inequalities (7) and (8) of [loc. cit.]. Thus $2 \operatorname{codim} E_1 \leqq 4 + 1 - b$, hence $\dim E_1 = 2$ and $b = 1$. Let T, T_x and Y be as in [loc. cit.], and let $S = Y \times_T T_x$. Then $\dim T_x = 1$. Let $\tilde{S}$ and $\tilde{D}$ be normalizations of S and T_x, respectively. Then the projection $\pi: \tilde{S} \to \tilde{D}$ is a $\mathbb{P}^1$-bundle.

Now we follow an argument in [Wis]. Let $\beta: \tilde{S} \to \tilde{E}_1$ be the morphism induced by p in [I, (0.4)], $\nu: E'_1 \to \tilde{E}_1$ the minimal resolution, and let $\sigma: S' \to \tilde{S}$ be a birational morphism from a non-singular projective surface S' such that β induces a morphism $\beta': S' \to E'_1$. Let ℓ' be a general fiber of $\pi \circ \sigma$ and let $C' = \beta'(\ell')$. Then

$$-2 = (K_{S'} \cdot \ell') \geqq (\beta'^* K_{E'_1} \cdot \ell') = (K_{E'_1} \cdot C'),$$

since $\ell' \to C'$ is a birational morphism. Suppose that E'_1 is not a minimal surface. Then by [M, (2.1)], we can write $C' \approx B_1 + B_2 + B_3$ for exceptional curves of the first kind B_i $(i = 1, 2)$ (B_1 may be equal to B_2) and a pseudo-effective 1-cycle B_3. But this contradicts the fact that $-K_X$ is ample on E_1 and $(K_X \cdot (\mu_1 \circ \nu)(C')) = -1$. Therefore, E'_1 is isomorphic to either a minimal ruled surface or $\mathbb{P}^2$. In the former case, there is only one curve in the linear system $|C'|$ through x, a contradiction. Hence $E'_1 \cong \mathbb{P}^2$ and $\tilde{E}_1 \cong \mathbb{P}^2$. Since $(K_X \cdot C_1) = -1$, we have $\mu_1^* \mathcal{O}_{E_1}(K_X) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$.

(2.3) We shall show that $\mathcal{O}_X(-K_X)$ is generated by global sections. We use the argument of the base point free theorem (cf. [K1], also [KMM]) like in [C] and [Wil]. We denote by $\mathrm{Bs}\,|-K_X|$ the base locus of the linear system $|-K_X|$, i.e., the support of the cokernel of the natural homomorphism $f^*f_*\mathcal{O}_X(-K_X)\to\mathcal{O}_X(-K_X)$. Supposing that $\mathrm{Bs}\,|-K_X|$ is non-empty, we shall derive a contradiction. The proof occupies from (2.3.1) to (2.3.4).

(2.3.1) We shall prove that a general member D of $|-K_X|$ has at most terminal singularities (cf. [R], also [KMM]).

By resolving the base locus of $|-K_X|$, we obtain a projective birational morphism $\varphi\colon X'\to X$ from a non-singular variety X' and a divisor with only normal crossings $G=\sum_{j=1}^{m} G_j$ which satisfy the following conditions:

(1) $|\varphi^*D|=|D'|+\sum r_jG_j$, where $|D'|$ is base point free for the strict transform D' of D, and the r_j are non-negative integers,

(2) $K_{X'}=\varphi^*K_X+\sum a_jG_j$ for some non-negative integers a_j,

(3) $-\varphi^*K_X-\sum\delta_jG_j$ is $f\circ\varphi$-ample for some $\delta_j\in\mathbb{Q}$ with $0<\delta_j\ll 1$.

Let us fix $\varepsilon\in\mathbb{Q}$ with $0<\varepsilon\ll 1$, and set

$$c=\min_j\,(a_j+1-\varepsilon\delta_j)/r_j.$$

By changing the δ_j if necessary, we may assume that the minimum c is attained only at $j=1$. We set

$$A-B=\sum(-cr_j+a_j-\varepsilon\delta_j)G_j$$

with $B=G_1$.

Now suppose that $a_j+1<2r_j$ for some j. Then a $\mathbb{Q}$-divisor

$$\begin{aligned} C &=_{\mathrm{def}} -\varphi^*K_X-K_{X'}+A-B\\ &\approx cD'-(2-c)\varphi^*K_X-\sum\varepsilon\delta_jG_j \end{aligned}$$

is $f\circ\varphi$-ample for a suitably chosen ε such that $2-c\geqq\varepsilon$. By [KMM, 1.2.3], $H^1(X',-\varphi^*K_X+\ulcorner A\urcorner-B)=0$. Hence

(2.3.1.1) $H^0(X',-\varphi^*K_X+\ulcorner A\urcorner)\to H^0(B,-\varphi^*K_X+\ulcorner A\urcorner)$ is surjective. Since $\mathcal{O}_{\mathbb{P}^2}(1)$ is generated by global sections, and since $\ulcorner A\urcorner\geqq 0$, the right hand side of (2.3.1.1) does not vanish. But the left hand side is naturally isomorphic to $H^0(X,-K_X)$. Since $\varphi(B)\subset\mathrm{Bs}\,|-K_X|$, we get a contradiction. Thus $a_j+1\geqq 2r_j$ for all j. By the adjunction formula, we have

$$K_{D'}=\varphi^*K_D+\sum(a_j-r_j)(G_j\cap D').$$

Since $a_j-r_j\geqq r_j-1>0$ if $r_j\geqq 2$, D has only terminal singularities.

(2.3.2) Since $-K_X$ is f-ample, we have $H^1(X,\mathcal{O}_X)=0$ by [KMM, 1.2.3]. Let $L=-K_X|_D$. Then we obtain $\mathrm{Bs}\,|-K_X|=\mathrm{Bs}\,|L|$ from the exact sequence

$$0\to\mathcal{O}_X\to\mathcal{O}_X(-K_X)\to\mathcal{O}_D(L)\to 0.$$

Let $f_D\colon D\to f(D)$ be the restriction of f, and let $E_D^{(1)}$ (resp. $E_D^{(2)}$) be the union of all the one (resp. two) dimensional irreducible components of the exceptional locus $E_D = E \cap D$ of f_D. We shall show that $\mathrm{Bs}\,|L| = E_D^{(2)}$.

Let us take an arbitrary point x in $E_D - E_D^{(2)}$. Since $K_D = 0$ by the adjunction formula, we have $R^i f_{D*}\mathcal{O}_D = 0$ for $i > 0$. Hence irreducible components of $E_D^{(1)}$ are non-singular rational curves by the formal function theorem. Then we can take an effective Cartier divisor M on D such that $M \cap E_D = \{x\}$ and such that $L - M$ is f_D-nef. By [KMM, 1.2.3], $H^1(D, L-M) = 0$, and the homomorphism $H^0(D, L) \to H^0(M, L)$ is surjective. Hence $x \notin \mathrm{Bs}\,|L|$.

(2.3.3) Let H be a general member of an f_D-very ample linear system. When we shrink Y if necessary, H splits into a disjoint union $H = H_1 + H_2$ such that $H_i \cap E_D^{(i)} = \emptyset$ for $i = 1, 2$. Then the linear system $|H_2|$ gives a projective birational morphism $f_D^{(2)}: D \to V$ which contracts $E_D^{(2)}$ to isolated singular points of a variety V.

Supposing that $E_D^{(2)} \neq \emptyset$, we let S be a connected component of $E_D^{(2)}$, let V_1 be a small neighborhood of $v = f_D^{(2)}(S)$ in V, and let $D_1 = f_D^{(2)-1}(V_1)$. Let $L_1 = L|_{D_1}$, and let M_1 be a general member of $|L_1|$. Then by (2.3.2), M_1 can be extended to a member of $|L|$. Hence $\mathrm{Bs}\,|L_1| = S$.

(2.3.4) We shall show that $\mathrm{Bs}\,|L_1|$ does not contain the whole S, and derive a contradiction.

Let M_0 be a member of $|L_1|$ with high multiplicities along S. For example, we take M_0 as the sum of a general member of $|L_1|$ plus a high multiple of the pull back of a hyperplane section of V_1 through v. As in (2.3.1), we take a projective birational morphism $\varphi_1: D_1' \to D_1$ from a non-singular variety D_1' and a divisor with only normal crossings $G = \sum G_j$. Instead of (1) there, we assume

(1′) $\varphi_1^* M_0 = \sum r_j G_j$, where some of the G_j are strict transforms of the irreducible components of M_0.

Then the similarly defined number c is small enough to give $1 - c \geqq \varepsilon$ for a suitable ε, and hence we have the relative ampleness of a $\mathbb{Q}$-divisor C. Since $\varphi_1(B)$ is not contained in $\mathrm{Bs}\,|L_1|$, we are done.

(2.3.5) Since $|-K_X|$ is base point free, we have $h^0(E_i, -K_X) \geqq 3$ for all i. Hence $E_i \cong \mathbb{P}^2$.

(2.4) Let us fix an arbitrary irreducible component E_1 of E. We shall prove that there are at most a finite number of jumping lines of the normal bundle $N = N_{E_1/X}$. If this is proved, then by [V, p. 248] and [OSS, 2.1.4. on p. 205], we have $N \cong \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$ with $a + b = -2$, since $(K_X \cdot \ell) = -1$ for a line ℓ on E_1. If $(a, b) \neq (-1, -1)$, then we have $H^0(E_1, N) \neq 0$ and $H^1(E_1, N) = 0$. Hence E_1 deforms inside X, a contradiction. Thus $(a, b) = (-1, -1)$.

Let ℓ_0 be an arbitrary line on $E_1 \cong \mathbb{P}^2$ which is not contained in any other irreducible component of E. We shall prove that ℓ_0 is not a jumping line of N. This observation was inspired by [F, (2.3)].

Since $|-K_X|$ is base point free, the homomorphism $H^0(X, -K_X) \to H^0(E_1, -K_X)$ is surjective. Hence we can find a non-singular member D_0 of $|-K_X|$ such that $D_0 \cap E_1 = \ell_0$. We take D_0 general enough so that $D_0 \cap E$ is one dimensional.

By taking a general holomorphic curve in $|-K_X|$ through $D_0 \in |-K_X|$, we construct a flat family $\pi: \mathscr{D} \to \Delta$ of members of $|-K_X|$ parametrized by a small disc $\Delta = \{t \in \mathbb{C}; |t| < \varepsilon\}$; $\mathscr{D}$ is a divisor on $X \times \Delta$ and $D_t = \pi^{-1}(t) \in |-K_X|$ for $t \in \Delta$. The morphism f induces a projective birational morphism $f': \mathscr{D} \to \mathscr{V}'$ onto a flat family $\sigma': \mathscr{V}' \to \Delta$ of members of $|-K_Y|$ over Δ.

Let ℓ' be the sum of the irreducible components of $D_0 \cap E$ other than ℓ_0, and let H be a general member of an f'-very ample linear system on $\mathscr{D}$. If we take Δ small enough, H splits into a disjoint union $H = H_1 + H_2$ such that $H_1 \cap \ell_0 = \varnothing$ and $H_2 \cap \ell' = \varnothing$. The linear system $|H_1|$ gives a projective birational morphism $\tilde{f}: \mathscr{D} \to \mathscr{V}$ onto a normal variety $\mathscr{V}$ with a projection $\sigma: \mathscr{V} \to \Delta$. Then $\mathscr{V}$ has only terminal singularities, and hence is Cohen-Macauley. Therefore, all the fibers $V_t = \sigma^{-1}(t)$ are normal. For each t, the morphism $f_t = \tilde{f}|_{D_t}$ is the contraction of a curve $\ell_t = D_t \cap E_1$ to the unique isolated singular point of V_t. Since $R^1 f_{t*}\mathcal{O}_{D_t} = 0$, there are three possibilities for the splitting type $(a(\ell_t), b(\ell_t))$ of N at ℓ_t: $(1, -3)$, $(0, -2)$ or $(-1, -1)$.

Let $\mathscr{V}^0$ be a small neighborhood of the singular locus of $\mathscr{V}$, and let $\mathscr{D}^0 = \tilde{f}^{-1}(\mathscr{V}^0)$. We set $V_t^0 = \mathscr{V}^0 \cap V_t$ and $D_t^0 = \mathscr{D}^0 \cap D_t$. Then by [R, (1.1) and (1.14)], we may think of $\mathscr{V}^0$ as the total space of a two parameter family of rational double points of surfaces $\lambda: \mathscr{V}^0 \to S = \{(t, s); |t| < \varepsilon, |s| < \varepsilon\}$ such that $\sigma = \mathrm{pr}_1 \cdot \lambda$, and $\tilde{f}$ gives a family of partial resolutions of fibers of λ.

Let L be a divisor on $\mathscr{D}^0$ such that $(L \cdot \ell_0) < 0$. We shall show that $\dim R^1 f_{t*}\mathcal{O}_{D_t^0}(L)$ is locally constant. By [P, Theorem 3], there exists another family $\tilde{f}^+: \mathscr{D}^+ \to \mathscr{V}^0$ of partial resolutions of fibers of λ such that the strict transform L^+ of L on $\mathscr{D}^+$ is $\tilde{f}^+$-ample. Then $R^1 f_{t*}^+ \mathcal{O}_{D_t^+}(L^+) = 0$ for all t, where $D_t^+ = (\tilde{f}^+)^{-1}(V_t^0)$ and $f_t^+ = \tilde{f}^+|_{D_t^+}$. Since $f_{0*}^+ \mathcal{O}_{D_0^+}(L^+) = \mathcal{O}_{V_0^0}(L^0)$ for the strict transform L^0 of L on $\mathscr{V}^0$, all the sections of $\mathcal{O}_{V_0^0}(L^0)$ are locally liftable to those of $\mathcal{O}_{\mathscr{V}^0}(L^0)$. But we also have $f_{0*}\mathcal{O}_{D_0^0}(L) = \mathcal{O}_{V_0^0}(L^0)$, and hence $\dim R^1 f_{t*}\mathcal{O}_{D_t^0}(L)$ is locally constant.

In the case where $(L, \ell_0) = -1$, we have $R^1 f_{t*}\mathcal{O}_{D_t^0}(L) \neq 0$ if and only if $(a(\ell_t), b(\ell_t)) = (1, -3)$. In case $(L, \ell_0) = -2$, $\dim R^1 f_{t*}\mathcal{O}_{D_t^0}(L) = 1$ (resp. $\geqq 2$) if $(a(\ell_t), b(\ell_t)) = (-1, -1)$ (resp. $= (0, -2)$). This shows that $(a(\ell_t), b(\ell_t))$ is locally constant on t, and hence ℓ_0 is not a jumping line of N.

(2.5) We shall show that E is irreducible. This completes the proof of (2.1). First, suppose that there are two irreducible components E_1 and E_2 of E such that $\dim(E_1 \cap E_2) = 1$. Since $N_{E_1/X}$ is negative, there exists a proper bimeromorphic morphism $X \to X_0$ to a complex space X_0 which contracts E_1. But $E_1 \cap E_2$ is not contractible on E_2, a contradiction. Thus $E_i \cap E_j$ consists of finite number of points for distinct irreducible components E_i and E_j of E. Let D be a general member of $|-K_X|$. Since Y is Cohen-Macauley, $f(D) \in |-K_Y|$ is Gorenstein, and hence is normal (cf. [K2, 8.7]). Thus $D \cap E$ is connected, and E is irreducible.

(2.6) *Example.* Let V be a non-singular projective variety of dimension four such that K_V is ample, and let C (resp. S) be a one (resp. two) dimensional non-singular subvariety of V. Assume that C and S intersect transversally at points P_i $(i = 1, \ldots, n)$. Let $\alpha: V' \to V$ be the blow-up with center C, and let $\beta: X \to V'$ be the blow-up with center S', the strict transform of S by α. Then the strict transforms E_i $(i = 1, \ldots, n)$ of $\alpha^{-1}(P_i)$ by β are isomorphic to $\mathbb{P}^2$ with $E_{E_i/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, and the contraction of them is a small elementary contraction with reducible exceptional locus.

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Received November 24, 1988