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Even Unimodular 8-Dimensional Quadratic Forms Over $\mathbb{Q}(\sqrt{2})$

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Dedicated to Professor O. T. O'Meara on his 60th birthday

Introduction

The problem of classifying integral positive definite quadratic forms (or lattices) has been studied by many people. One frequent approach is the method of Kneser [6], which involves studying lattices adjacent to a given one. An example of this is Niemeier's complete classification of 24-dimensional even positive definite unimodular lattices over $\mathbb{Z}$ [9]. Unfortunately, this method in general requires very extensive calculations and it is difficult to verify that the enumeration is complete. By adopting approaches based on the theory of modular forms, algebraic coding, and Siegel's analytic theory of quadratic forms, it is possible to simplify the classification and indeed to give proofs of the completeness of the enumeration, see e.g. [13, 2, 3]. In this paper we consider even positive definite unimodular lattices over the ring of integers R in $\mathbb{Q}(\sqrt{2})$. By even we mean that $B(x, x) \in 2R$. There is a unique genus of such lattices in each dimension that is a multiple of 4. It is known that the 4 dimensional genus has only one class [12] which we denote by Δ_4'. Computations of the Minkowski-Siegel mass [5] suggest that further algebraic classification is feasible only for dimension 8. Several classes have been found in the 8-dimensional genus, including one with an empty root system [12, 5]. Recall that the root system of a lattice is the set of vectors of norm 2. We shall complete the enumeration. Our approach is based on a combination of Kneser's method and Siegel's theory of quadratic forms. Siegel's mass formula and his theorem for degree one Hilbert-Eisenstein series are used to verify that our enumeration is complete. A neighbor graph for the genus will be given in the last section. We note that the graph provides an alternate check of the completeness of the genus. Unless otherwise indicated, all terminology and notations will follow those of [10, 3].

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Enumeration

Let $F=\mathbb{Q}(\sqrt{2})$, R the ring of integers of F. Then $R=\mathbb{Z}[\varepsilon]$, where $\varepsilon=1+\sqrt{2}$ is the fundamental unit of F. Aside from the classical root systems of ADE-types, there are two new irreducible root systems over R [8], namely

$$\Delta_n(n\geqq 2)=\{z\in I_n|B(z,e_1+\ldots+e_n)\equiv 0\ (\mathrm{mod}\sqrt{2})\}$$
$$=\langle\sqrt{2}e_1,e_1+e_2,\ldots,e_1+e_n\rangle$$

and

$$\Delta_4'=\Delta_4+\langle(e_1+\ldots+e_4)/\sqrt{2}\rangle=\langle\sqrt{2}e_1,(e_1+\ldots+e_4)/\sqrt{2},e_1+e_3,e_1+e_4\rangle$$

where $\{e_i\}$ is an orthonormal basis and $I_n=\langle e_1,\ldots,e_n\rangle$. We have

$$\det\Delta_4'=1$$

so Δ_4' generates an even unimodular lattice, which yields the only class in the quaternary genus. In the genus of 8-dimensional even unimodular lattices, it is known that there are at least 5 classes [12, 5], given here by their root system configurations:

$$E_8,\ 2\Delta_4',\ \Delta_8,\ 2D_4\quad\text{and}\quad\emptyset\ .$$

An even unimodular lattice with the root system Γ will be denoted by L_Γ or just Γ when there is no confusion.

The classes E_8, $2\Delta_4'$ and L_{2D_4} can be obtained by the Kneser method as neighbors of L_{Δ_8} with respect to the prime $\sqrt{2}$. On the other hand, the existence of an empty root lattice $L_\emptyset$ was shown using a technique which is analogous to the Construction A of [7] in coding theory. Now $L_\emptyset$ is not adjacent to the classes E_8, $2\Delta_4'$, L_{Δ_8} and L_{2D_4}. To see this, let K be a neighbor of $L_\emptyset$. Then

$$K=Rx+(L_\emptyset)_x$$
$$=Rx+\{y\in L_\emptyset|B(y,x)\in R\}$$

for some $x\notin L_\emptyset$, $\sqrt{2}x\in L_\emptyset$, $Q(x)\in 2R$. If K is equivalent to E_8, $2\Delta_4'$, L_{Δ_8} or L_{2D_4}, then K contains a binary sublattice

$$B\cong\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

adapted to the base $\{u,v\}$. It is clear that $u,v\notin L_\emptyset$, hence there exist $c_1,c_2\in R-\sqrt{2}R$ such that

$$u=c_1x+y_1\quad\text{and}\quad v=c_2x+y_2$$

for some $y_1,y_2\in(L_\emptyset)_x$. Since $|R/\sqrt{2}R|=2$, we may assume that $c_1=c_2=1$. It follows that $u-v\in(L_\emptyset)_x\subset L_\emptyset$, which is impossible because $B(u-v,u-v)=2$. This suggests that there is a "missing link" between $L_\emptyset$ and the remaining classes. This will be filled by a lattice with the root system $4\Delta_2$. Specifically, consider 4 copies of Δ_2 in the 8-dimensional space:

$$4\Delta_2=\langle\sqrt{2}e_1,e_1+e_2\rangle\perp\langle\sqrt{2}e_3,e_3+e_4\rangle\perp\langle\sqrt{2}e_5,e_5+e_6\rangle\perp\langle\sqrt{2}e_7,e_7+e_8\rangle\ .$$

Then

$$L_{4\Delta_2}=4\Delta_2+\langle(\varepsilon e_1+e_2+\varepsilon e_3+e_4)/\sqrt{2}\,,\ (\varepsilon e_5+e_6+\varepsilon e_7+e_8)/\sqrt{2}\ , \\ e_1+e_3+e_5+e_7\,,\ (\varepsilon e_1+e_2+\varepsilon e_5+e_6)/\sqrt{2}\rangle\ . \qquad (1)$$

Alternately, $L_{4\Delta_2}$ can be obtained by the neighborhood method using the base lattice

$$2\Delta_4'=\langle\sqrt{2}e_1,(e_1+\ldots+e_4)/\sqrt{2},e_1+e_3,e_1+e_4\rangle\perp$$

$$\langle\sqrt{2}e_5,(e_5+\ldots+e_8)/\sqrt{2},e_5+e_7,e_5+e_8\rangle$$

and the vector $(\varepsilon e_1+e_2+\varepsilon e_3+e_4+\varepsilon e_5+e_6+\varepsilon e_7+e_8)/2$. Using (1) for $L_{4\Delta_2}$ and taking its neighbor which contains $(\varepsilon e_1+e_2+\varepsilon e_3+e_4+\varepsilon e_5+e_6+\varepsilon e_7+e_8)/2$, one obtains a lattice that has an empty root system. We have

Proposition 1. *There exists an 8-dimensional even unimodular lattice $L_{4\Delta_2}$, over $\mathbb{Q}(\sqrt{2})$ which has the root system $4\Delta_2$.*

Remark. It is not clear that each of the known root systems gives rise to a unique class of lattices. In particular, there may be more than one class of lattice which has an empty root system. We will show that there are no further classes in our genus via Siegel's mass formula. Alternate checks will be given using Siegel's theorem for degree one Hilbert-Eisenstein series and the neighbor graph of the genus.

Algebraic Descent

Let L be an R-lattice of $\operatorname{rank}_R(L)=m$. The algebraic descent L_0 of L is the $\mathbb{Z}$-lattice $L_0=L$ of $\operatorname{rank}_{\mathbb{Z}}(L_0)=2m$ together with the quadratic form Q_0 defined by

$$Q_0(x):=Tr_{F/\mathbb{Q}}(Q(x)/2\sqrt{2}\varepsilon)\ .$$

If (L,Q) is even unimodular over R, then (L_0,Q_0) is even unimodular over $\mathbb{Z}$. For $Q(x)=a+b\varepsilon$, $a,b\in\mathbb{Z}$ we have

$$Q_0(x)=Tr_{F/\mathbb{Q}}((a+b\varepsilon)/2\sqrt{2}\varepsilon)=a$$

so that

$$Q_0(x)=2 \text{ iff } Q(x)\in\{2,2\sqrt{2}\varepsilon,2(1+2\varepsilon)\}\ .$$

If u,v are any two roots in L, then $B(u,v)=0,\ \pm1,\ \pm\sqrt{2}$, hence $B_0(u,v)=0,\ \pm1,\ \mp1$ respectively. Let Γ be an irreducible root system over $\mathbb{Q}(\sqrt{2})$. In [5], it was shown that if Γ is "old" (i.e. the classical ADE root system) then the algebraic descent Γ_0 is 2 copies of Γ, whereas if $\Gamma=\Delta_n$ $(n\geqq2)$, Δ_4' then Γ_0 is D_{2n}, E_8 respectively. It follows that the 8-dimensional even unimodular lattices E_8 and $2\Delta_8'$ both descend to $2E_8$, while L_{Δ_8} descends to $L_{D_{16}}$. By considering also the vectors of Q-length $2\sqrt{2}\varepsilon$, it was determined that L_{2D_4} descends to $2E_8$. We shall determine the algebraic descent of

$L_{4\Delta_2}$. In this case there are 384 vectors of Q-length $2\sqrt{2}\varepsilon$ in $L_{4\Delta_2}$ given by

$$M_{ij}=\{(*\varepsilon e_i * e_{i+1} * \varepsilon e_j * e_{j+1})/\sqrt{2}, (*\varepsilon e_i * e_{i+1} * e_j * \varepsilon e_{j+1})/\sqrt{2}\ ,$$
$$(* e_i * \varepsilon e_{i+1} * \varepsilon e_j * e_{j+1})/\sqrt{2}, (* e_i * \varepsilon e_{i+1} * e_j * \varepsilon e_{j+1})/\sqrt{2}\}$$

where i, j are *odd* integers satisfying $1 \leqq i < j \leqq 7$ and $*$ denotes an arbitrary sign. Each pair of i, j gives a family of 64 vectors and there are 6 such families. The vectors in each family belong to the same irreducible component upon algebraic descent. Moreover for any two families M_{ij}, $M_{i'j'}$, there exist w, w' from M_{ij}, $M_{i'j'}$, respectively such that $B_0(w, w') \neq 0$. This shows that the descent of $L_{4\Delta_2}$ must be $L_{D_{16}}$. Indeed, the root system $4\Delta_2$ descends to $4D_4$ which accounts for the remaining 96 roots in D_{16}. Thus

Proposition 2. *The algebraic descent of $L_{4\Delta_2}$ is the 16-dimensional even unimodular lattice over $\mathbb{Z}$ with the root system D_{16}.*

Remark. By a similar argument, one can show that the algebraic descent of $L_\emptyset$ is also $L_{D_{16}}$.

Mass Formula

Let L be a positive definite integral R-lattice of rank m. The Minkowski-Siegel mass of the genus of L is given by

$$M(L)=\sum_{i=1}^{h} \frac{1}{e(L_i)}$$

where $\{L_1, \ldots, L_h\}$ is a set of distinct representatives of the isometry classes in the genus of L, and $e(L_i)$ is the order of the orthogonal group $O(L_i)$ of L_i. If L is even unimodular, we have the following formula from [5] (see also [11])

$$M(L)=\frac{4^{1-m} L_F(m/2, \chi_m) \prod_{i=1}^{m/2-1} \zeta_F(2i)}{(\sqrt{8})^{\frac{-m(m-1)}{2}} \prod_{i=1}^{m} \pi^i \Gamma^{-2}\left(\frac{i}{2}\right)}$$

where $\chi_m(p)=\left(\frac{-1}{p}\right)^{m/2}$, $L_F(s, \chi_m)=\prod_p (1-\chi_m(p) N_p^{-s})^{-1}$ and $\zeta_F(\cdot)$ is the Dedekind zeta function.

Let M_m denote the mass for the genus of rank m. Then

$$M_4=\frac{4^{-3}(\zeta_F(2))^2}{(\sqrt{8})^{-6} \prod_{i=1}^{4} \pi^i \Gamma^{-2}\left(\frac{i}{2}\right)}=\frac{1}{2^8 \cdot 3^2}$$

and

$$M_8=\frac{4^{-7}\zeta_F^2(4)\zeta_F(2)\zeta_F(6)}{(\sqrt{8})^{-28} \prod_{i=1}^{8} \pi^8 \Gamma^{-2}\left(\frac{i}{2}\right)}=\frac{11^2 \cdot 19^2}{2^{18} \cdot 3^5 \cdot 5^2 \cdot 7}\ .$$

In order to verify that our earlier enumeration for the 8-dimensional genus is complete, it is essential to compute the order of the automorphism group of each lattice. For those lattices with nonempty root system, we follow a method in [2]. First we decompose the root lattice in L into irreducible components

$$L_1 \perp \ldots \perp L_t \,.$$

Then we let $G_2(L)$ be the factor group of $O(L)$ by the normal subgroup $S(L)$ consisting of those elements which leave invariant all the L_i. Moreover, let $G_0(L)$ be the normal subgroup of $S(L)$ consisting of those elements which, for all i, act trivially on $L_i^{\#}/L_i$. Here $L_i^{\#}$ is the dual lattice of L_i. Finally, we let $G_1(L)$ be the factor group $S(L)/G_0(L)$. If we denote $g_k(L)=|G_k(L)|$ for $0 \leqq k \leqq 2$, then $e(L)=g_0 g_1 g_2$. With the help of a computer we have the following table

L_i	$g_0(L_i)$	$g_1(L_i)$	$g_2(L_i)$	$e(L_i)$
E_8	$2^{14}3^5 5^2 7$	1	1	$2^{14}3^5 5^2 7$
$2\Delta_4'$	$(2^8\cdot 3^2)^2$	1	2	$2^{17}\cdot 3^4$
Δ_8	$2^{15}\cdot 3^2\cdot 5\cdot 7$	1	1	$2^{15}\cdot 3^2\cdot 5\cdot 7$
$2\Delta_4$	$(2^3\cdot 4!)^2$	6	2	$2^{14}\cdot 3^3$
$4\Delta_2$	$(2^3)^4$	2^3	$4!$	$2^{18}\cdot 3$
$\emptyset$	—	—	—	$2^{14}\cdot 3^2\cdot 5\cdot 7$

The lattice $L_\emptyset$ has a base consisting of vectors of norm 4. The automorphism group of $L_\emptyset$ is computed by considering the permutations of its 3360 norm 4 vectors. Upon summing the reciprocals of the $e(L_i)$ we obtain

$$\sum_1^6 \frac{1}{e(L)} = \frac{11^2\cdot 19^2}{2^{18}\cdot 3^2\cdot 5^2\cdot 7}$$

which is exactly the mass predicted by the mass formula. Thus we have:

Theorem. *There are precisely 6 distinct classes of even unimodular lattices of rank 8 over $\mathbb{Q}(\sqrt{2})$ which are distinguished by their root systems $E_8, 2\Delta_4', \Delta_8, 2D_4, 4\Delta_2$ and $\emptyset$.*

Remark. In the quaternary case, the order of the automorphism group of Δ_4' is computed to be $2^8\cdot 3^2$. Hence the mass formula verifies that Δ_4' is the unique class in that genus.

Theta Series

Let H be the upper half plane. A Hilbert modular form of weight k for the Hilbert modular group $SL_2(\mathbb{Z}[\varepsilon])$ is a holomorphic function f on H^2 satisfying the condition

$$f\left(\frac{az_1+b}{cz_1+d}, \frac{\bar{a}z_2+\bar{b}}{\bar{c}z_2+\bar{d}}\right)=(cz_1+d)^k(\bar{c}z_2+\bar{d})^k f(z_1, z_2)$$

for any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}[\varepsilon])$. Here $\bar{a}$ is the conjugation of a. Every Hilbert modular form f has a Fourier expansion of the form

$$f(z)=\sum_{\nu\gg 0} c_f(\nu)e^{2\pi i\sigma(\nu z/2\sqrt{2}\,\varepsilon)}=\sum c_f(a+b\varepsilon)\,[a,b]\ ,$$

where $[a,b]=\exp\left[2\pi i\left(\left(\frac{a+b\varepsilon}{2\varepsilon\sqrt{2}}\right)z_1+\left(\overline{\frac{a+b\varepsilon}{2\varepsilon\sqrt{2}}}\right)z_2\right)\right]$. Let L be an even unimodular lattice over $\mathbb{Q}(\sqrt{2})$ of rank m. Then the theta series of L

$$\Theta_L(z)=\sum_{x\in L} e^{2\pi i\sigma\left(\frac{Q(x)z}{2\varepsilon\sqrt{2}}\right)}$$

$$=\sum_{\nu\gg 0} c_L(\nu)e^{2\pi i\sigma\left(\frac{\nu z}{2\varepsilon\sqrt{2}}\right)}$$

is a Hilbert modular form of weight $\frac{m}{2}$. Here $c_L(\nu)=\#\{x\in L\,|\,Q(x)=2\nu\}$. If $L_1=L$, $L_2,\dots,L_h$ is a complete representative system of the distinct classes in the genus $\operatorname{gen}(L)$ of L, then Siegel's theorem [11] on the average number of representations of a number by $\operatorname{gen}(L)$ is given by

$$\frac{1}{M(L)}\sum_{i=1}^{h}\frac{\theta_{L_i}(z)}{e(L_i)}=G_{\frac{m}{2}}(z)\ , \tag{2}$$

where $G_{\frac{m}{2}}(z)=1+\sum c_{\frac{m}{2}}(\nu)e^{2\pi i\sigma\left(\frac{\nu z}{2\varepsilon\sqrt{2}}\right)}$ is the Eisentein series of weight $\frac{m}{2}$. From [4] we have

$$c_{\frac{m}{2}}(\nu)=b_{\frac{m}{2}}\sum_{(\beta)|(\nu)}(\operatorname{sign}N\beta)^{\frac{m}{2}}|N\beta|^{\frac{m}{2}-1}$$

and

$$b_{\frac{m}{2}}=\frac{(2\pi)^m\sqrt{8}}{\left(\Gamma\left(\frac{m}{2}\right)\right)^2 8^{\frac{m}{2}}\zeta_{\mathbb{Q}(\sqrt{2})}\left(\frac{m}{2}\right)}\ .$$

For $m=4$ and 8, we compute

$$b_2=\frac{(2\pi)^4\sqrt{8}}{8^2\zeta_{\mathbb{Q}(\sqrt{2})}(2)}=48$$

$$b_4=\frac{(2\pi)^8\sqrt{8}}{(3!)^2 8^4\zeta_{\mathbb{Q}(\sqrt{2})}(4)}=\frac{480}{11}\ .$$

Applying (2) to the genus of 8-dimensional even unimodular lattices, we have

$$\frac{1}{M_8}\times\sum_{i=1}^{h}\frac{c_{L_i}(1)}{e(L_i)}=b_4=\frac{480}{11}\ .$$

Using the 6 lattices in the genus and $M(L)=M_8$, we have

$$\frac{1}{M_8}\times\sum_{i=1}^{6}\frac{c_{L_i}(1)}{e(L_i)}=\frac{2^{18}\cdot 3^2\cdot 5^2\cdot 7}{11^2\cdot 19^2}$$
$$\cdot\left(\frac{2^4\cdot 3\cdot 5}{2^{14}\cdot 3^5\cdot 5^2\cdot 7}+\frac{2^5\cdot 3}{2^{17}\cdot 3^4}+\frac{2^7}{2^{15}\cdot 3^2\cdot 5\cdot 7}+\frac{2^4\cdot 3}{2^{14}\cdot 3^3}+\frac{2^5}{2^{18}\cdot 3}\right)=\frac{480}{11}\ .$$

This calculation shows that the only lattices which admit vectors of quadratic norm 2 are exactly the five classes E_8, $2\Delta_4'$, L_{Δ_8}, L_{2D_4} and $L_{4\Delta_2}$. Since the uniqueness of $L_\emptyset$ has been established previously by the mass formula, it follows that there are precisely 6 distinct classes in the 8-dimensional even unimodular genus over $\mathbb{Q}(\sqrt{2})$.

Remark. For the quarternary genus of even unimodular lattices over $\mathbb{Q}(\sqrt{2})$, (2) gives

$$\frac{1}{M_4}\times\sum_{i=1}^{h}\frac{c_{L_i}(1)}{e(L_i)}=b_2=48\ .$$

Since $c_{\Delta_4'}(1)=48$ this shows that Δ_4' is the only class in the genus.

Neighbor Graph

We close this paper by constructing the neighbor graph of the genus of 8-dimensional even unimodular lattices over $\mathbb{Q}(\sqrt{2})$ at the prime $p=(\sqrt{2})$. For a lattice L in the genus, the vertices of the graph $R(L,p)$ are those lattices $M\in\operatorname{gen}L$ such that $M_q=L_q$ for all primes $q\neq p$. Two vertices M and M' are joined by an edge in $R(L,p)$ if $[M:M\cap M']=[M':M\cap M']=Np$. In this case, we say that M and M' are neighbors (or adjacent) in $R(L,p)$. $R(L,p)$ is a connected graph and it contains a representative of every class in the genus of L. For each lattice in the graph, the number of neighbors is the same as the number of isotropic lines in L/pL. A result in [1, p. 21] shows that if $\dim FL=2m$, where m is the Witt index of FL at p, then this number is given by

$$\frac{[(Np)^m-1][(Np)^{m-1}+1]}{Np-1}\ .$$

It follows that each lattice in $R(L,p)$ has exactly 135 neighbors. We will now present the graph. Since the number of neighbors is quite large, we will not give the details and just remark that they can be obtained by the neighbor method. Moreover, we will identify isometric lattices by their root systems.

Let $N(L,K,p)$ denote the number of neighbors of L that are isometric to K. We have the following table of $N(L,K,p)$ for the lattices in the 8-dimensional genus over $\mathbb{Q}(\sqrt{2})$:

$L \backslash K$	E_8	$2\Delta_4'$	Δ_8	$2D_4$	$4\Delta_2$	$\emptyset$
E_8	0	0	135	0	0	0
$2\Delta_4'$	0	18	36	0	81	0
Δ_8	2	35	28	70	0	0
$2D_4$	0	0	3	96	36	0
$4\Delta_2$	0	6	0	64	49	16
$\emptyset$	0	0	0	0	105	30

If L and K are neighbors, then it is shown in [1, p. 48] that

$$\frac{N(L,K,p)}{N(K,L,p)}=\frac{e(L)}{e(K)}\,. \tag{3}$$

If follows from (3) that the graph obtained above agrees with our earlier computations of the orders of automorphism groups, thus giving an additional check of the completeness of our list.

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Note added in proof. A theoretical proof of the uniqueness of the empty root lattice is also possible based on a method similar to that used in [3] for the $\mathbb{Q}(\sqrt{5})$ case.