Appl Math Optim 15:251–277 (1987)

Applied Mathematics and Optimization
© 1987 Springer-Verlag New York Inc.

Existence, Uniqueness, and Regularity Results for the Two-Body Contact Problem

Paolo Boieri,[1] Fabio Gastaldi,[2] and David Kinderlehrer[3]

[1] Dipartimento di Matematica del Politecnico, Torino, Italy

[2] Istituto di Analisi Numerica del C.N.R., Pavia, Italy

[3] School of Mathematics, University of Minnesota, Minneapolis, USA

Communicated by D. Kinderlehrer

Abstract. The problem of contact between two elastic bodies is studied under the assumption of nonzero initial gap in the potential contact region. The related variational inequality is stated and existence, uniqueness, and local regularity results are proved for its solution.

0. Introduction

The problem of contact between two deformable bodies was first considered by Hertz in 1882 [11]. Consideration of a case with special geometry allowed him to decide the shape of the contact region and the stress distribution on it, in good agreement with experimental results (see [21], p. 193). Generalizations of this procedure have been followed by a number of authors who studied this sort of problem using the method of Muskhelishvili ([24] and [7], see also [9] for a recent review), based on singular integral equations. While they provide good estimates of the stress distribution, these techniques require knowing the contact region in advance.

In 1959 Signorini [26] introduced the so called "ambiguous condition" to describe contact phenomena where the contact region is itself one of the unknowns. The application of variational methods to this formulation proved to be effective: existence and uniqueness results were obtained by Fichera (see [5] and [6]). Successive work has been done in the framework of the theory of variational inequalities introduced by Lions and Stampacchia [20]. In particular, in [4] an extensive treatment of contact problems in elasticity is given, including mechanical formulation, mathematical tools, and basic results (see also [25]).

Contact between two elastic bodies is studied in this paper, under reasonably general assumptions on the initial geometry. In particular, a nonzero initial gap between the two bodies is permitted. These new features distinguish this work from previous ones, where the zero initial gap assumption is made. We recall [2] (elastic body on elastic half plane) and [12]. References [14] and [10] study the problem with nonzero initial gap, mainly from the point of view of numerical approximation.

The approach we use here follows that of [15] and [16], where several properties for the problem of an elastic body supported by a rigid punch are obtained: in particular, regularity results and estimates on the contact region.

In Section 1 we introduce the mechanical problem and present a description of the contact condition, using Kalker's approach [13]. Section 2 is devoted to a precise mathematical formulation of the problem in the form of a variational inequality; the related boundary-value problem is presented. The existence theorem is stated under the assumption that the external forces satisfy a compatibility condition. This requirement is necessary since the variational inequality is not coercive. This also affects uniqueness, which is true up to a class of rigid-body motions. Such a class is completely characterized in the case of a flat potential contact region. In Section 3 we prove the local $\mathbf{H}^2$ regularity for the solution using the difference-quotients technique. Further regularity results, for example Hölder continuity for the gradient of the solution in two space dimensions, are given in Section 4, based on penalization and hole-filling procedures.

It is also possible to obtain these results via the application of Gehring's reverse Hölder inequality, first applied in differential equations by Meyers and Elcrat [22] (cf. [17] and Giaquinta's book [8]), but we do not present this calculation here.

1. The Mechanical Problem and Nonpenetration Condition

We consider two elastic bodies in the space $\mathbf{R}^n$. Their initial position is described by the reference coordinates $x \equiv (x_1, \dots, x_n) \equiv (\hat{x}, x_n)$. We denote by Ω^a and Ω^b the reference configuration of the two bodies. They are subjected to external forces, hence undergo a deformation, after which a particle x occupies the position $y(x)$ and

$$u(x) \equiv y(x) - x \tag{1.1}$$

is its displacement. A portion $\Gamma^a \subset \partial\Omega^a$ may come into frictionless contact with points that in the reference configuration belong to $\Gamma^b \subset \partial\Omega^b$. Γ^a and Γ^b are referred to as "potential contact areas."

Our first aim is to identify a suitable class of admissible displacements based on the notion, inspired by conservation of mass, that after a motion the two bodies should not intersect each other. Although expressing such a kinematic condition poses no difficulty, it must be suitably linearized in order that our problem take on the features of a variational inequality. Different linearizations provide different solutions and it is difficult to decide which is the most appropriate. For instance, it is not clear that an arbitrary linearization leads to a well-posed problem. Here we choose one which gives each body equal status, regardless of their geometries. It has a natural mechanical interpretation in terms of the surface tractions.

We assume that there is an explicit representation

$$\Gamma^{a,b}: \quad x_n = \varphi^{a,b}(\hat{x}), \qquad \hat{x} \in \Xi^{(1)} \tag{1.2}$$

for an open $\Xi \subset \mathbf{R}^{n-1}$, independent of Ω^a and Ω^b and that $\varphi^a(\hat{x}) - \varphi^b(\hat{x}) \geq 0$ for $\hat{x} \in \Xi$. Let us denote by $x^{a,b}$, $y^{a,b}$, and $u^{a,b}$ the initial position of a material point in $\Omega^{a,b}$, its final position, and its displacement, namely, $u^{a,b}(x^{a,b}) = y^{a,b}(x^{a,b}) - x^{a,b}$. Let $\mu^a = (\nabla \varphi^a(\hat{x}), -1)$ and $\mu^b = (\nabla \varphi^b(\hat{x}), 1)$ be the normal directions on Γ^a and Γ^b, respectively.

Suppose that the final contact region may be represented implicitly as

$$h(y) = 0 \tag{1.3}$$

with the nonpenetration condition taking the form

$$\begin{cases} h(y^a) \geq 0 & \text{for} \quad y^a = y^a(x^a), \quad x^a \in \Gamma^a, \\ h(y^b) \leq 0 & \text{for} \quad y^b = y^b(x^b), \quad x^b \in \Gamma^b. \end{cases} \tag{1.4}$$

Now, if we assume $\partial h/\partial y_n > 0$, then there exists a function ψ: $\mathbf{R}^{n-1} \to \mathbf{R}$ such that (1.3) holds if and only if

$$y_n = \psi(\hat{y}). \tag{1.5}$$

So, (1.4) becomes

$$\begin{cases} x_n^a + u_n^a - \psi(\hat{x}^a + \hat{u}^a) \geq 0, \\ -x_n^b - u_n^b + \psi(\hat{x}^b + \hat{u}^b) \geq 0. \end{cases} \tag{1.6}$$

It is now convenient to restrict the general formula (1.6) to the case of displacements $u^{a,b}$ which are small with respect to the linear dimensions of the bodies. Retaining only linear terms in the expansion of ψ, we have

$$\begin{cases} x_n^a + u_n^a - \psi(\hat{x}^a) - \nabla\psi(\hat{x}^a) \cdot \hat{u}^a \geq 0, \\ -x_n^b - u_n^b + \psi(\hat{x}^b) + \nabla\psi(\hat{x}^b) \cdot \hat{u}^b \geq 0. \end{cases} \tag{1.7}$$

This is true for any particle $x^{a,b} \in \overline{\Omega^{a,b}}$. If, in addition, $x^{a,b} \in \Gamma^{a,b}$ and $\hat{x}^a = \hat{x}^b \equiv \hat{x}$, then adding the two inequalities of (1.7) gives

$$\begin{aligned} 0 \leq \varphi^a(\hat{x}) - \varphi^b(\hat{x}) + u_n^a(\hat{x}, \varphi^a(\hat{x})) - u_n^b(\hat{x}, \varphi^b(\hat{x})) \\ - \nabla\psi(\hat{x}) \cdot \hat{u}^a(\hat{x}, \varphi^a(\hat{x})) + \nabla\psi(\hat{x}) \cdot \hat{u}^b(\hat{x}, \varphi^b(\hat{x})). \end{aligned} \tag{1.8}$$

(1) Here and in the following, the superscript "a, b" is used for properties that hold for both a and b.

Now, recalling (1.1),

$$\begin{aligned}\nabla\psi(\hat{y})\cdot\hat{u}^{a,b}(\hat{x},\varphi^{a,b}(\hat{x}))\\ &=\psi_{,\lambda}(\hat{y})\hat{u}^{a,b}_{\lambda}(\hat{x},\varphi^{a,b}(\hat{x}))^{(2)}\\ &=\psi_{,\lambda}(\hat{x})\hat{u}^{a,b}_{\lambda}(\hat{x},\varphi^{a,b}(\hat{x}))+\psi_{,\lambda\mu}\hat{u}^{a,b}_{\lambda}(\hat{x},\varphi^{a,b}(\hat{x}))\hat{u}^{a,b}_{\mu}(\hat{x},\varphi^{a,b}(\hat{x}))\\ &=\psi_{,\lambda}(\hat{x})\hat{u}^{a,b}_{\lambda}(\hat{x},\varphi^{a,b}(\hat{x})),\end{aligned}\tag{1.9}$$

where higher-order terms have been dropped.

We shall now make the assumption that the change of the normal vectors $\mu^{a,b}$ is small, which is usual within the frame of the linear approximation (see [13]). This means that

$$\nabla\varphi^a(\hat{x})\cong\nabla\psi(\hat{y})\cong\nabla\varphi^b(\hat{x}).\tag{1.10}$$

Inserting this and (1.9) into (1.8) we have that

$$\begin{aligned}0\le\varphi^a(\hat{x})-\varphi^b(\hat{x})+u^a_n(\hat{x},\varphi^a(\hat{x}))-u^b_n(\hat{x},\varphi^b(\hat{x}))\\ -\nabla\varphi^a(\hat{x})\cdot\hat{u}^a(\hat{x},\varphi^a(\hat{x}))+\nabla\varphi^b(\hat{x})\cdot\hat{u}^b(\hat{x},\varphi^b(\hat{x})),\end{aligned}$$

or

$$[\mu^a\cdot u^a](\hat{x},\varphi^a(\hat{x}))+[\mu^b\cdot u^b](\hat{x},\varphi^b(\hat{x}))\le\varphi^a(\hat{x})-\varphi^b(\hat{x}).\tag{1.11}$$

Owing to (1.10), in (1.11) we could identify μ^a with $-\mu^b$, obtaining a simpler version of the linearized nonpenetration condition. However, in the next section we will study a boundary condition of type (1.11), where μ^a and $-\mu^b$ are possibly different.

The condition (1.11) (gap condition) defines the class of admissible displacements. Among them, the actual displacement $u^{a,b}$ is the minimizer of the total energy

$$\begin{aligned}\mathbf{E}(v^a,v^b)\equiv\tfrac{1}{2}\int_{\Omega^a}a^a_{ijkl}(x)v^a_{i,j}(x)v^a_{k,l}(x)\,dx+\tfrac{1}{2}\int_{\Omega^b}a^b_{ijkl}(x)v^b_{i,j}(x)v^b_{k,l}(x)\,dx\\ -\langle T^a,v^a\rangle-\langle T^b,v^b\rangle,\end{aligned}\tag{1.12}$$

where $a^{a,b}_{ijkl}(x)$ is the elastic tensor and $\langle T^{a,b},v^{a,b}\rangle$ denotes the work of the external forces. These can be either volume forces or surface stresses given on $\Gamma^{a,b}_s\subset\partial\Omega^{a,b}$. A further restriction for the admissible displacements is that they can be prescribed on a portion $\Gamma^{a,b}_d\subset\partial\Omega^{a,b}$.

In the following sections, we will give a more precise formulation of the mechanical problem described so far.

---

(2) Here and in the following these conventions are assumed:

1. $v_{,j}(x)\equiv(\partial/\partial x_j)v(x)$;
2. Greek subscripts range from 1 to $n-1$; Roman from 1 to n;
3. Sum over repeated indices is understood.

2. Mathematical Formulation and Existence Theorem

Let $\Omega^{a,b} \subset \mathbf{R}^n$ be a bounded domain with smooth boundary

$$\partial\Omega^{a,b} = \bar{\Gamma}_d^{a,b} \cup \bar{\Gamma}_s^{a,b} \cup \bar{\Gamma}^{a,b}, \tag{2.1}$$

where $\Gamma_d^{a,b}$, $\Gamma_s^{a,b}$, $\Gamma^{a,b}$ are mutually disjoint open smooth $(n-1)$ manifolds, $\Gamma_d^{a,b}$ and $\Gamma_s^{a,b}$ possibly empty. $\Gamma^{a,b}$ is nonempty and admits an explicit representation as

$$x_n = \varphi^{a,b}(\hat{x}), \tag{2.2}$$

for $\hat{x} \in \Xi$, $\Xi \subset \mathbf{R}^{n-1}$ independent of Ω^a and Ω^b, with $\varphi^a(\hat{x}) - \varphi^b(\hat{x}) \geq 0$, $\forall \hat{x} \in \Xi$. The outer normal vector to $\partial\Omega^{a,b}$ is denoted by $\mu^{a,b}$; set $\nu^{a,b} \equiv \mu^{a,b}/|\mu^{a,b}|$.

Let, for integer nonnegative m, $[\mathbf{H}^m(\Omega^{a,b})]^n$ be the space of the vector functions belonging to $[\mathbf{L}^2(\Omega^{a,b})]^n$ along with all derivatives up to the order m. The space $[\mathbf{H}^1(\Omega^{a,b})]^n$ is endowed with the norm

$$\|v\|^2_{[\mathbf{H}^1(\Omega^{a,b})]^n} \equiv \|v\|^2_{[\mathbf{L}^2(\Omega^{a,b})]^n} + \sum_{i,j=1}^{n} \|v_{i,j}\|^2_{[\mathbf{L}^2(\Omega^{a,b})]^n}. \tag{2.3}$$

In $\Omega^{a,b}$ a tensor field $a_{ijkl}^{a,b}(x) \in \mathbf{C}^\infty(\bar{\Omega}^{a,b})$ is defined, satisfying

$$a_{ijkl}^{a,b}(x) = a_{jikl}^{a,b}(x) = a_{ijlk}^{a,b}(x), \qquad \forall x \in \Omega^{a,b} \tag{2.4}$$

and

$$\begin{cases} \exists \alpha^{a,b} > 0 \quad \text{s.t.} \quad a_{ijkl}^{a,b}(x) m_{ij} m_{kl} \geq \alpha^{a,b} \|m\|^2, \\ \forall x \in \Omega^{a,b}, \quad \forall m \in [\mathbf{R}^n]^2 \quad \text{with} \quad m_{ij} = m_{ji}, \end{cases} \tag{2.5}$$

where $\|m\|^2 = \sum_{i,j=1}^n m_{ij}^2$.

The linearized strain and stress tensors of $v^{a,b} \in [\mathbf{H}^1(\Omega^{a,b})]^n$ are given by

$$\begin{cases} \varepsilon_{ij}(v^{a,b}) = \frac{1}{2}[v_{i,j}^{a,b} + v_{j,i}^{a,b}], \\ \sigma_{ij}(v^{a,b}) = a_{ijkl}^{a,b}(x) \varepsilon_{kl}(v^{a,b}). \end{cases} \tag{2.6}$$

Define the bilinear form $a^{a,b}(\cdot\,,\cdot)$ on $[[\mathbf{H}^1(\Omega^{a,b})]^n]^2$ as

$$a^{a,b}(u^{a,b}, v^{a,b}) = \int_{\Omega^{a,b}} \sigma_{ij}(u^{a,b}) \varepsilon_{ij}(v^{a,b})\, dx. \tag{2.7}$$

Thanks to (2.4), we have

$$a^{a,b}(u^{a,b}, v^{a,b}) = \int_{\Omega^{a,b}} a_{ijkl}^{a,b}(x) u_{i,j}^{a,b}(x) v_{k,l}^{a,b}(x)\, dx. \tag{2.8}$$

Functions $f^{a,b} \in [\mathbf{L}^2(\Omega^{a,b})]^n$ and $g^{a,b} \in [\mathbf{H}^{-1/2+\varepsilon}(\Gamma_s^{a,b})]^n$, $\varepsilon > 0$, are given (see [19] for the definition and properties of the latter space). We define

$$\langle T^{a,b}, v^{a,b} \rangle \equiv \int_{\Omega^{a,b}} f_i^{a,b}(x) v_i^{a,b}(x)\, dx + {}_{[\mathbf{H}^{-1/2+\varepsilon}(\Gamma_s^{a,b})]^n}\langle g_i^{a,b}, v_i^{a,b} \rangle_{[\mathbf{H}^{1/2-\varepsilon}(\Gamma_s^{a,b})]^n} \tag{2.9}$$

A function $v \equiv (v^a, v^b)$ with $v^{a,b} \in [\mathbf{H}^1(\Omega^{a,b})]^n$ is said to be *admissible* if it satisfies $v^{a,b} = 0$ on $\Gamma_d^{a,b}$ and

$$[\mu^a \cdot v^a](\hat{x}, \varphi^a(\hat{x})) + [\mu^b \cdot v^b](\hat{x}, \varphi^b(\hat{x})) \leq \varphi^a(\hat{x}) - \varphi^b(\hat{x}). \tag{2.10}$$

The set of admissible functions is denoted by

$$\mathbf{K} \equiv \{v \equiv (v^a, v^b) \text{ s.t. } v^{a,b} \in [\mathbf{H}^1(\Omega^{a,b})]^n,$$
$$v^{a,b} = 0 \text{ on } \Gamma_d^{a,b} \text{ and (2.10) is satisfied}\}: \tag{2.11}$$

this is a closed convex nonempty subset of $[\mathbf{H}^1(\Omega^a)]^n \times [\mathbf{H}^1(\Omega^b)]^n$.

The problem we are interested in is to solve the following variational inequality (**VI**):

$$u \equiv (u^a, u^b) \in \mathbf{K} \qquad \text{s.t.} \quad \forall v \equiv (v^a, v^b) \in \mathbf{K},$$

$$a^a(u^a, v^a - u^a) + a^b(u^b, v^b - u^b) \geq \langle T^a, v^a - u^a \rangle + \langle T^b, v^b - u^b \rangle. \tag{2.12}$$

Remark 2.1. Referring to the previous section, we note that **VI** can be interpreted as the principle of least energy applied to the two bodies elastic frictionless contact problem, when the elastic tensor also satisfies $a_{ijkl}^{a,b}(x) = a_{klij}^{a,b}(x)$. Say, (2.12) is equivalent to minimizing the energy (1.12), where the elastic energy is given by (2.8), the work of external forces by (2.9), and the nonpenetration condition by (2.10). For this reason, we will use the elasticity terminology to describe our problem.

The mechanical interpretation of **VI** will be more transparent once shown the equivalence with a boundary value problem.

Theorem 2.2. Let u solve **VI**. If $u^{a,b} \in [\mathbf{H}^{1+\varepsilon}(\Omega^{a,b})]^n$, *then*:

(i) $-\sigma_{ij,j}(u^{a,b}) = f_i^{a,b}$ *in* $\mathbf{L}^2(\Omega^{a,b})$;
(ii) $u^{a,b} = 0$ *in* $[\mathbf{H}^{1/2}(\Gamma_d^{a,b})]^n$;
(iii) $\sigma_{ij}(u^{a,b})\nu_j^{a,b} = g_i^{a,b}$ *in* $\mathbf{H}^{-1/2+\varepsilon}(\Gamma_s^{a,b})$;
(iv) $\sigma_{ij}(u^{a,b})\nu_j^{a,b}\tau_i^{a,b} = 0$ *in* $\mathbf{H}^{-1/2+\varepsilon}(\Gamma^{a,b})$ *for smooth* $\tau^{a,b}$ *with* $\tau^{a,b} \cdot \nu^{a,b} = 0$;
(v) $[u^a \cdot \mu^a](\hat{x}, \varphi^a(\hat{x})) + [u^b \cdot \mu^b](\hat{x}, \varphi^b(\hat{x})) \leq \varphi^a(\hat{x}) - \varphi^b(\hat{x})$, a.e. on Ξ;
(vi) $\sigma_{ij}(u^{a,b})\nu_j^{a,b}\nu_i^{a,b} \leq 0$ *in* $\mathbf{H}^{-1/2+\varepsilon}(\Gamma^{a,b})$;
(vii) *if* P *is an open subset of* Ξ *where* (2.10) *holds strictly, then* $\sigma_{ij}(u^{a,b})\nu_j^{a,b}\nu_i^{a,b} = 0$ in $\mathbf{H}^{-1/2+\varepsilon}(P^{a,b})$, *where* $P^{a,b} \equiv \{(\hat{x}, \varphi^{a,b}(\hat{x})): \hat{x} \in P\}$;
(viii) *if* Q *is an open subset of* Ξ *where* $\varphi^a = \varphi^b \equiv \varphi^0$, *then* $\sigma_{ij}(u^a)\nu_j\nu_i = \sigma_{ij}(u^b)\nu_j\nu_i$ *in* $\mathbf{H}^{-1/2+\varepsilon}(Q^0)$, *where* $Q^0 \equiv \{(\hat{x}, \varphi^0(\hat{x})): \hat{x} \in Q\}$ and $\nu \equiv \nu^a = -\nu^b$.

Proof. An integration by parts in **VI** gives

$$\begin{aligned}
&-{}_{\mathbf{H}^{-1}(\Omega^a)}\langle \sigma_{ij,j}(u^a), v_i^a - u_i^a \rangle_{\mathbf{H}^1(\Omega^a)} - {}_{\mathbf{H}^{-1}(\Omega^b)}\langle \sigma_{ij,j}(u^b), v_i^b - u_i^b \rangle_{\mathbf{H}^1(\Omega^b)} \\
&\quad + {}_{\mathbf{H}^{-1/2}(\partial\Omega^a)}\langle \sigma_{ij}(u^a)\nu_j^a, v_i^a - u_i^a \rangle_{\mathbf{H}^{1/2}(\partial\Omega^a)} \\
&\quad + {}_{\mathbf{H}^{-1/2}(\partial\Omega^b)}\langle \sigma_{ij}(u^b)\nu_j^b, v_i^b - u_i^b \rangle_{\mathbf{H}^{1/2}(\partial\Omega^b)} \\
&\geq \langle T^a, v^a - u^a \rangle + \langle T^b, v^b - u^b \rangle.
\end{aligned} \tag{2.13}$$

We will often make the choice

$$v=(u^a+\eta^a, u^b) \quad \text{or} \quad v=(u^a, u^b+\eta^b), \tag{2.14}$$

with appropriate restrictions on η^a or η^b. In particular, (i) is obtained with $\eta^{a,b}\in[\mathbf{C}_0^\infty(\Omega^{a,b})]^n$ and then (iii) with

$$\eta^{a,b}\in[\mathbf{C}^\infty(\overline{\Omega^{a,b}})]^n, \qquad \eta^{a,b}=0 \qquad \text{in a neighborhood of } \overline{\Gamma^{a,b}}\cup\overline{\Gamma_d^{a,b}}.$$

Under our regularity assumptions, the duality

$${}_{\mathbf{H}^{-1/2}(\partial\Omega^{a,b})}\langle\sigma_{ij}(u^{a,b})\nu_j^{a,b}, v_i^{a,b}-u_i^{a,b}\rangle_{\mathbf{H}^{1/2}(\partial\Omega^{a,b})}$$

breaks into

$${}_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma_d^{a,b})}\langle\cdot\rangle_{\mathbf{H}^{1/2-\varepsilon}(\Gamma_d^{a,b})}+{}_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma_s^{a,b})}\langle\cdot\rangle_{\mathbf{H}^{1/2-\varepsilon}(\Gamma_s^{a,b})}+{}_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma^{a,b})}\langle\cdot\rangle_{\mathbf{H}^{1/2-\varepsilon}(\Gamma^{a,b})}$$

(note that the natural condition $g\in\mathbf{H}^{-1/2}$ does not allow extending it to 0 outside $\Gamma_s^{a,b}$). Thanks to (ii) (say the definition of $\mathbf{K}$), the first term vanishes; because of (iii), the second cancels with the corresponding one in $\langle T^{a,b}, v^{a,b}-u^{a,b}\rangle$ (see (2.9)). Eventually, (2.13) reduces to

$$\begin{aligned}&{}_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma^a)}\langle\sigma_{ij}(u^a)\nu_j^a, v_i^a-u_i^a\rangle_{\mathbf{H}^{1/2-\varepsilon}(\Gamma^a)}\\&\quad+{}_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma^b)}\langle\sigma_{ij}(u^b)\nu_j^b, v_i^b-u_i^b\rangle_{\mathbf{H}^{1/2-\varepsilon}(\Gamma^b)}\geq 0.\end{aligned} \tag{2.15}$$

Choose now $\eta^{a,b}\in[\mathbf{C}^\infty(\overline{\Omega^{a,b}})]^n$ with $\eta^{a,b}|_{\Gamma^{a,b}}\in[\mathbf{C}_0^\infty(\Gamma^{a,b})]^n$ in (2.14). Adopting the notation, for $z\in\mathbf{R}^n$,

$$z_\nu^{a,b}\equiv z\cdot\nu^{a,b}, \qquad z_\tau^{a,b}\equiv z-z_\nu^{a,b}\nu^{a,b}, \tag{2.16}$$

(2.15) becomes

$$\begin{aligned}&{}_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma^{a,b})}\langle\sigma_{ij}(u^{a,b})\nu_j^{a,b}, (\eta_\tau^{a,b})_i\rangle_{\mathbf{H}^{1/2-\varepsilon}(\Gamma^{ab})}\\&\quad+{}_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma^{ab})}\langle\sigma_{ij}(u^{a,b})\nu_j^{a,b}, (\eta^{a,b}\cdot\nu^{a,b})\nu_i^{a,b}\rangle_{\mathbf{H}^{1/2-\varepsilon}(\Gamma^{a,b})}\geq 0.\end{aligned} \tag{2.17}$$

If $\eta^{a,b}\cdot\nu^{a,b}=0$ and $\eta_\tau^{a,b}$ is arbitrary, the second term in (2.17) vanishes; (2.17) becomes an equality and we get (iv).

Going back to the general form of (2.17), we find that

$${}_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma^{a,b})}\langle\sigma_{ij}(u^{a,b})\nu_j^{a,b}\nu_i^{a,b}, (\eta^{a,b}\cdot\nu^{a,b})\rangle_{\mathbf{H}^{1/2-\varepsilon}(\Gamma^{a,b})}\geq 0. \tag{2.18}$$

If we choose $\eta^{a,b}$ with $\eta^{a,b}\cdot\nu^{a,b}\leq 0$ (note that the corresponding $v\equiv(v^a, v^b)$ belongs to $\mathbf{K}$), we derive (vi).

To prove (vii), it is enough to note that, for any smooth function $\eta^{a,b}$ such that $\eta^{a,b}|_{\Gamma^{a,b}}$ has support included in $P^{a,b}$, there exists a $\lambda_\eta^{a,b}\in\mathbf{R}^+$ such that $\forall\lambda\in\mathbf{R}$, if $|\lambda|\leq\lambda_\eta^{a,b}$ then $v=(u^a+\lambda\eta^a, u^b)$ and $v=(u^a, u^b+\lambda\eta^b)$ belong to $\mathbf{K}$. Hence, (2.18) becomes

$${}_{\mathbf{H}^{-1/2+\varepsilon}(P^{a,b})}\langle\sigma_{ij}(u^{a,b})\nu_j^{a,b}\nu_i^{a,b}, \eta^{a,b}\cdot\nu^{a,b}\rangle_{\mathbf{H}^{1/2-\varepsilon}(P^{a,b})}=0,$$

and this implies (vii). To show (viii), it is enough to take $v^{a,b}=u^{a,b}+\eta^{a,b}$ with $\eta^{a,b}\in[\mathbf{C}^\infty(\overline{\Omega^{a,b}})]^n$, $\eta^{a,b}|_{\Gamma^{a,b}}\in[\mathbf{C}_0^\infty(Q^0)]^n$, $\eta_\tau^{a,b}|_{Q^0}=0$, $\eta_\nu^{a,b}|_{Q^0}=\hat{\eta}\nu$, $\hat{\eta}\in\mathbf{C}_0^\infty(Q^0)$ (remember that on Q^0, $\nu\equiv\nu^a=-\nu^b$ and the notation (2.16)). The function $v\equiv(v^a, v^b)$ obviously belongs to $\mathbf{K}$ and (2.15) gives

$${}_{\mathbf{H}^{-1/2+\varepsilon}(Q^0)}\langle[\sigma_{ij}(u^a)-\sigma_{ij}(u^b)]\nu_j\nu_i, \hat{\eta}\rangle_{\mathbf{H}^{1/2-\varepsilon}(Q^0)}\geq 0.$$

The sign of $\hat{\eta}$ is not fixed, hence this must be an equality: $\hat{\eta}$ being arbitrary, the assertion follows. □

Remark 2.3. System (i) of Theorem (2.2) is elliptic: in fact, due to the symmetry properties (2.4), the inequality (2.5) implies that there exists an $\alpha_0 \in \mathbf{R}^+$ such that, for any ξ, $\eta \in \mathbf{R}^n$,

$$a_{ijkl}^{a,b}(x)\xi_i\xi_k\eta_j\eta_i \geq \alpha_0 \|\xi\|^2 \|\eta\|^2. \tag{2.19}$$

Our main tools in the proof of the existence theorem for **VI** are Korn's inequality and the structure of the nullspace of $a^{a,b}(\cdot,\cdot)$, the rigid-body motions. As for the former, we recall that, for some constant $C \geq 0$ (see [4])

$$\int_{\Omega^{a,b}} \varepsilon_{ij}(v^{a,b})\varepsilon_{ij}(v^{a,b})\,dx + \|v^{a,b}\|^2_{[\mathbf{L}^2(\Omega^{a,b})]^n} \geq C\|v^{a,b}\|^2_{[\mathbf{H}^1(\Omega^{a,b})]^n}:$$

thanks to (2.4) and (2.5), this implies

$$\int_{\Omega^{a,b}} a_{ijkl}^{a,b}(x)v_{i,j}^{a,b}(x)v_{k,l}^{a,b}(x)\,dx + \|v^{a,b}\|^2_{[\mathbf{L}^2(\Omega^{a,b})]^n} \geq C\|v^{a,b}\|^2_{[\mathbf{H}^1(\Omega^{a,b})]^n}. \tag{2.20}$$

The set M of infinitesimal rigid motions is defined as $M \equiv M^a \times M^b$, where

$$\begin{aligned} M^{a,b} &\equiv \{\zeta^{a,b} \in [\mathbf{H}^1(\Omega^{a,b})]^n\colon a^{a,b}(\zeta^{a,b},\zeta^{a,b}) = 0\} \\ &= \{\zeta^{a,b}\colon \zeta_i^{a,b} = c_i^{a,b} + D_{ij}^{a,b}x_j,\ c^{a,b} \in \mathbf{R}^n,\ D^{a,b} \in [\mathbf{R}^n]^2 \text{ with } D_{ij}^{a,b} = -D_{ji}^{a,b}\}. \end{aligned} \tag{2.21}$$

Let $A \equiv (A^a, A^b) \in \mathbf{K}$ satisfy $(A^a \cdot \mu^a)(\hat{x}, \varphi^a(\hat{x})) + (A^b \cdot \mu^b)(\hat{x}, \varphi^b(\hat{x})) = \varphi^a(\hat{x}) - \varphi^b(\hat{x})$ and define the cone

$$\mathbf{K}_A \equiv \{v - A, \text{ when } v \in \mathbf{K}\}. \tag{2.22}$$

Theorem 2.4. *The variational inequality* **VI** *has a solution whenever*

$$\langle T^a, \zeta^a\rangle + \langle T^b, \zeta^b\rangle < 0 \tag{2.23}$$

for all $\zeta = (\zeta^a, \zeta^b) \in M'$, *where*

$$M' \equiv \{\zeta \in M \cap \mathbf{K}_A \text{ s.t. } -\zeta \notin \mathbf{K}_A\}. \tag{2.24}$$

The solution is unique up to an element of

$$M^0 \equiv \{\zeta \in M \text{ s.t. } \langle T^a, \zeta^a\rangle + \langle T^b, \zeta^b\rangle = 0\}. \tag{2.25}$$

Proof. Existence is a well-known result: see, for instance, [6] and [20] or [18]; see also [1]. Note that, in order to get the existence theorem, the hypothesis $g^{a,b} \in [\mathbf{H}^{-1/2+\varepsilon}(\Gamma_s^{a,b})]^n$ is a little stronger than necessary: the natural framework would require the use of the space $[\mathbf{H}^{-1/2}(\Gamma_s^{a,b})]^n$; so, $T^{a,b}$ turns out to be continuous on $[\mathbf{H}^1(\Omega^{a,b})]^n$.

About uniqueness, we observe that, if u is a solution and $\zeta \in M^0$, then the function $w \equiv u + \zeta$ is another solution, as soon as it belongs to $\mathbf{K}$. Indeed, $a^{a,b}(w^{a,b}, v^{a,b} - w^{a,b}) = a^{a,b}(u^{a,b}, v^{a,b} - u^{a,b})$ and $\langle T^{a,b}, w^{a,b}\rangle = \langle T^{a,b}, u^{a,b}\rangle$. Conversely, if u and w are two solutions, then it is elementary to see that $\zeta \equiv w - u \in M$. Inserting $v = w$ (resp. u) in the **VI** solved by u (resp. w), we get that $\langle T^a, \zeta^a\rangle + \langle T^b, \zeta^b\rangle = 0$. □

Remark 2.5. Condition (2.23) is a "compatibility condition" on the data in order to have a solution for **VI**. This is due to the lack of coerciveness of $a^{a,b}(\cdot, \cdot)$ in the general case. However, when Γ_d^a and Γ_d^b have positive measure,[3] M' turns out to be empty, hence no restriction is imposed on $T^{a,b}$; note that a rigid-body motion can vanish on a set of nonzero $n-1$ measure if and only if it vanishes identically.

Remark 2.6. The wide generality of the geometry of $\Omega^{a,b}$ does not allow a more detailed interpretation of M' and M^0. However, this is possible in special cases, as we shall see in a moment.

In the remaining part of this section we will investigate a particular problem. Precisely, we will make the following:

Assumption. *Γ_d^a and Γ_d^b are empty and $\varphi^a = \varphi^b \equiv 0$.* (2.26)

In this case, $\Gamma^{a,b} = \Xi \subset \{(\hat{x}, x_n) \text{ with } x_n = 0\}$, and $\nu^a = -\nu^b = (0, 0, \ldots, 0, -1)$. Since we can now choose $A \equiv 0$, $\mathbf{K} \equiv \mathbf{K}_A$ is a cone; condition (2.10) becomes

$$v_n^a - v_n^b \geq 0 \quad \text{on } \Xi.$$

It is immediate that M' is given by the infinitesimal rigid-body motions ζ with $\zeta_n^a - \zeta_n^b > 0$ on Ξ.

Identifying the rigid motions that affect uniqueness requires some remarks. First, we derive some easy consequences of the compatibility condition (2.23), which will be assumed to hold true henceforth.

Lemma 2.7. *For all $\zeta \in \mathbf{K} \cap M$ with $-\zeta \in \mathbf{K}$ we have that*

$$\langle T^a, \zeta^a \rangle + \langle T^b, \zeta^b \rangle = 0.$$

Proof. It follows from [16], using the compatibility condition and the fact that $\mathbf{K}$ is a cone. □

Let us introduce the notations:

$$F_i^{a,b} \equiv \langle T_i^{a,b}, 1 \rangle, \qquad i = 1, \ldots, n, \tag{2.27}$$

$$M_{ij}^{a,b} \equiv \langle T_j^{a,b}, x_i \rangle - \langle T_i^{a,b}, x_j \rangle, \qquad i, j = 1, \ldots, n. \tag{2.28}$$

Lemma 2.8. *We have:*

(i) $F_\lambda^{a,b} = 0$, $\lambda = 1, \ldots, n-1$;
(ii) $F_n^a = -F_n^b < 0$;
(iii) $M_{\lambda\mu}^{a,b} = 0$, λ, $\mu = 1, \ldots, n-1$;
(iv) $M_{\lambda n}^a = -M_{\lambda n}^b$, $\lambda = 1, \ldots, n-1$.

---

[3] In this case, the problem becomes coercive and the general theory of [20] can be applied: this gives existence and uniqueness for the solution, again without any compatibility condition on the external forces.

Proof. (i) For fixed $\lambda < n$, the function $\eta = (\eta^a, \eta^b)$ with $\eta_i^a = \delta_{i\lambda}$, $i = 1, \ldots, n$, $\eta^b \equiv 0$ belongs to $\mathbf{K}$ along with its opposite: Lemma 2.7 yields the result for F_λ^a. Analogous procedure works for F_λ^b.

(iii) Similarly, we can prove that $M_{\lambda\mu}^{a,b} = 0$, taking η with $\eta_i^{a,b} = x_\lambda \delta_{i\mu} - x_\mu \delta_{i\lambda}$, $\eta^{b,a} \equiv 0$ and applying Lemma 2.7.

(ii) Let ε, k be real numbers, with $\varepsilon > 0$: the function $\eta = (\eta^a, \eta^b)$ with $\eta_\mu^{a,b} = 0$ for $\mu = 1, \ldots, n-1$, $\eta_n^a = k$, $\eta_n^b = k - \varepsilon$ belongs to $\mathbf{K}$, while $-\eta \notin \mathbf{K}$. By the compatibility condition we must have

$$0 > \langle T^a, \eta^a \rangle + \langle T^b, \eta^b \rangle = k(F_n^a + F_n^b) - \varepsilon F_n^b.$$

This cannot be true for all $\varepsilon > 0$ and for all k, unless $F_n^b \geq 0$, in which case for $\varepsilon \to 0$ we get $k(F_n^a + F_n^b) \leq 0$. Since no restriction of sign is made on k, it must be $F_n^a + F_n^b = 0$.

It remains to show that $F_n^b > 0$. Assume $F_n^b = 0$: plugging into the compatibility condition the function ζ with $\zeta_\mu^{a,b} = 0$, $\forall \mu < n$, $\zeta_n^a = 0$, $\zeta_n^b = 1$, we derive an immediate contradiction, so (ii) is proved.

(iv) The proof is similar to that of (ii), hence we omit it. □

A further property which shows an intrinsic interest is the following, cf. [5].

Theorem 2.9. *There exists a point $x_0 \in (\operatorname{conv} \Xi)^\circ$ such that for all $\zeta \in M$*

$$\langle T^a, \zeta^a \rangle + \langle T^b, \zeta^b \rangle = F_n^a \zeta_n^a(x_0) + F_n^b \zeta_n^b(x_0). \tag{2.29}$$

Proof. Using the representation (2.21) for $\zeta \in M$, Lemma 2.8 gives the following, where the superscript a, b has been dropped when unnecessary:

$$\begin{aligned}\langle T^{a,b}, \zeta^{a,b} \rangle &= c_i F_i + \tfrac{1}{2} D_{ij} M_{ji} = c_n F_n + \tfrac{1}{2} D_{\lambda n} M_{n\lambda} + \tfrac{1}{2} D_{n\lambda} M_{\lambda n} \\ &= c_n F_n + D_{n\lambda} M_{\lambda n} = F_n^{a,b} \zeta_n^{a,b}(x_0^{a,b}),\end{aligned}$$

$$\text{where} \quad x_0^{a,b} \equiv \frac{1}{F_n^{a,b}} (M_{1n}^{a,b}, \ldots, M_{(n-1)n}^{a,b}, 0). \tag{2.30}$$

Furthermore, (ii) and (iv) of Lemma 2.8 imply that $x_0^a = x_0^b$. Denoting by x_0 their common value, we claim that $x_0 \in (\operatorname{conv} \Xi)^\circ$. For, if this were not true, a function $\zeta \in M$ would exist such that $\zeta^b \equiv 0$, $\zeta_n^a > 0$ on Ξ°, $\zeta_n^a(x_0) \leq 0$. For this function, (2.23) implies that

$$0 > \langle T^a, \zeta^a \rangle + \langle T^b, \zeta^b \rangle = \langle T^a, \zeta^a \rangle = F_n^a \zeta_n^a(x_0),$$

which contradicts (ii) of Lemma 2.8. □

As an obvious consequence of this theorem, we have:

Corollary 2.10. *If $\zeta \in M^0$, then*

$$[\zeta_n^a - \zeta_n^b](x_0) = 0. \tag{2.31}$$

Going back to the question of uniqueness, let u and $w = u + \zeta$ be two solutions of **VI** with $g^{a,b} \in [\mathbf{H}^{1/2}(\Gamma_s^{a,b})]^n$: we already know that $\zeta \in M^0$ (see Theorem 2.4) and that (2.31) holds. Our aim is to show that the function

$$\chi \equiv \zeta_n^a - \zeta_n^b \tag{2.32}$$

vanishes identically on Ξ. The proof of this assertion is based on the following:

Lemma 2.11. *Let u be a solution of* **VI**. *Then*

$$\sigma_{nn}(u^{a,b}) \in \mathbf{L}^1(\Xi) \quad \textit{and} \quad \int_\Xi \sigma_{nn}(u^{a,b})\, ds = -F_n^a.$$

Proof. The procedure of [15] works also in our case, as soon as we prove that u is locally $\mathbf{H}^2$: indeed, u turns out to be $\mathbf{H}^2$ regular outside any neighorhood of $\partial\Xi \cup \partial\Gamma_s^{a,b}$ (see Theorem 3.3). □

Using Lemma 2.11 and, once more, the $\mathbf{H}^2$ local regularity, it is easy to see that property (viii) of Theorem 2.2 is actually true, say:

Lemma 2.12. $\sigma_{nn}(u^a) = \sigma_{nn}(u^b)$ *a.e. on* Ξ.

We are now able to prove our main theorem:

Theorem 2.13. *Under assumption* (2.26), *the function χ defined in* (2.32) *is identically zero. Hence, the solution of* **VI** *is unique up to an element of*

$$M^{00} \equiv \{\zeta \in M^0 \colon \zeta \in \mathbf{K} \text{ and } -\zeta \in \mathbf{K}\}. \tag{2.33}$$

Proof. By contradiction, assume that χ does not vanish identically. Anyhow, it does vanish at x_0 (see Corollary 2.10) which belongs to $(\operatorname{conv}\Xi)^\circ$. Then, the set $\{(\hat{x}, 0) \colon \chi(\hat{x}, 0) = 0\}$ is a nontrivial hyperplane that splits Ξ into two parts with nonempty interior. Denote with Ξ^+ (resp. Ξ^-) the part where $\chi > 0$ (resp. <0). We have

$$\begin{cases} w_n^a - w_n^b = u_n^a + \zeta_n^a - u_n^b - \zeta_n^b \geq \chi > 0 & \text{on } \Xi^+, \\ u_n^a - u_n^b = w_n^a - \zeta_n^a - w_n^b + \zeta_n^b \geq -\chi > 0 & \text{on } \Xi^-. \end{cases} \tag{2.34}$$

Theorem 2.2 and Lemma 2.11 imply that

$$\sigma_{nn}(w^{a,b}) = 0 \quad \text{a.e. on } \Xi^+, \qquad \sigma_{nn}(u^{a,b}) = 0 \quad \text{a.e. on } \Xi^-. \tag{2.35}$$

Since $\sigma_{nn}(w^{a,b}) = \sigma_{nn}(u^{a,b} + \zeta^{a,b}) = \sigma_{nn}(u^{a,b}) + \sigma_{nn}(\zeta^{a,b})$ and $\sigma_{nn}(\zeta^{a,b}) = 0$, we derive, for instance, that $\sigma_{nn}(w^{a,b}) = 0$ a.e. on Ξ. But this is impossible, since Lemma 2.11 and Lemma 2.8(ii) require that $\int_\Xi \sigma_{nn}(w^{a,b})\, ds > 0$. This completes the proof. □

The result we have just obtained is physically meaningful: at the equilibrium, only rigid motions that do not separate the two bodies are allowed without increasing energy.

3. Local Regularity

In this section we prove the local $\mathbf{H}^2$ regularity of a solution u of **VI**. Since this result is well known in the interior of $\Omega^{a,b}$ and on $\Gamma_d^{a,b} \cup \Gamma_s^{a,b}$, we need only consider the potential contact area. Let $\hat{x}_0 \in \Xi$ and define

$$\begin{aligned} \mathbf{K}_{\hat{x}_0} \equiv \{v \equiv (v^a, v^b) \text{ s.t. } v^{a,b} \in [\mathbf{H}^1(\Lambda^{a,b} \cap \Omega^{a,b})]^n,\ v^{a,b} = u^{a,b} \\ \text{on } \partial\Lambda^{a,b} \cap \Omega^{a,b} \text{ and (2.10) is satisfied for } \hat{x} \in \Lambda\}, \end{aligned} \tag{3.1}$$

where $\Lambda^{a,b}$ is a smooth neighborhood of $(\hat{x}_0, \varphi^{a,b}(\hat{x}_0))$. Assume that the projections of $\Lambda^{a,b} \cap \Gamma^{a,b}$ on $\{x_n = 0\}$ coincide, and denote them by Λ.

It is easy to see that the restriction to $\Lambda^{a,b} \cap \Omega^{a,b}$ of u solves a variational inequality of the same type as **VI**, with convex set $\mathbf{K}_{\hat{x}_0}$, volume integrals restricted to $\Lambda^{a,b} \cap \Omega^{a,b}$ and no boundary terms in $T^{a,b}$. Our first step is to symmetrize the geometry of the local problem, by means of the change of variables

$$x \to z \qquad \text{with} \quad \hat{z} = \hat{x}, \quad z_n = x_n - \tfrac{1}{2}[\varphi^a(\hat{x}) + \varphi^b(\hat{x})]. \tag{3.2}$$

The local variational inequality becomes:

$$\begin{cases} \text{to find } (\tilde{u}^a, \tilde{u}^b) \in \tilde{\mathbf{K}}_{\hat{x}_0} \text{ such that for all } (\tilde{v}^a, \tilde{v}^b) \in \tilde{\mathbf{K}}_{\hat{x}_0} \\ \tilde{a}^a(\tilde{u}^a, \tilde{v}^a - \tilde{u}^a) + \tilde{a}^b(\tilde{u}^b, \tilde{v}^b - \tilde{u}^b) \geq \langle \tilde{T}^a, \tilde{v}^a - \tilde{u}^a \rangle + \langle \tilde{T}^b, \tilde{v}^b - \tilde{u}^b \rangle, \end{cases} \tag{3.3}$$

where quantities with a tilde denote the transformed of the corresponding ones through (3.2). In particular,

$$\tilde{a}^{a,b}(\tilde{u}^{a,b}, \tilde{v}^{a,b}) \equiv \int_{\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b}} \tilde{a}_{ijkl}^{a,b}(z) \tilde{u}_{i,j}^{a,b}(z) \tilde{v}_{k,l}^{a,b}(z)\, dz, \tag{3.4}$$

$$\begin{aligned} \tilde{\mathbf{K}}_{\hat{x}_0} = \{\tilde{v} \equiv (\tilde{v}^a, \tilde{v}^b) \text{ s.t. } \tilde{v}^{a,b} \in [\mathbf{H}^1(\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b})]^n,\ \tilde{v}^{a,b} = \tilde{u}^{a,b} \text{ on } \partial\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b} \\ \text{and } [\tilde{v}^a \cdot \tilde{\mu}^a](\hat{z}, \tilde{\varphi}^a(\hat{z})) + [\tilde{v}^b \cdot \tilde{\mu}^b](\hat{z}, \tilde{\varphi}^b(\hat{z})) \leq \tilde{\varphi}^a(\hat{z}) - \tilde{\varphi}^b(\hat{z}) \\ \text{for } \hat{z} \in \Lambda\}. \end{aligned} \tag{3.5}$$

Note that $\tilde{\mu}^{a,b}$ may not be the normal vector to $\tilde{\Gamma}^{a,b}$, in general. For fixed Λ^a, a suitable choice of Λ^b makes $\tilde{\Lambda}^a$ and $\tilde{\Lambda}^b$ symmetric with respect to $\{z_n = 0\}$. A reflection of $\tilde{\Lambda}^b$ around the hyperplane $\{z_n = 0\}$ allows us to state the local variational inequality in $\tilde{\Lambda}^a \cap \tilde{\Omega}^a$. Define the operator

$$R^b: \mathbf{R}^n \to \mathbf{R}^n, \qquad R^b: (\hat{z}, z_n) \to (\hat{z}, -z_n). \tag{3.6}$$

Our local problem can be stated in $\tilde{\Lambda}^a \cap \tilde{\Omega}^a$ by means of the functions $\tilde{v}^a(z)$ and $\tilde{v}^b(R^b z)$, where $z \in \tilde{\Lambda}^a \cap \tilde{\Omega}^a$. The convex (3.5) becomes

$$\begin{aligned} \tilde{\mathbf{K}}_{\hat{x}_0} = \{\tilde{v} \equiv (\tilde{v}^a, \tilde{v}^b) \text{ s.t. } \tilde{v}^a, \tilde{v}^b \circ R^b \in [\mathbf{H}^1(\tilde{\Lambda}^a \cap \tilde{\Omega}^a)]^n, \\ \tilde{v}^a = \tilde{u}^a \text{ and } \tilde{v}^b \circ R^b = \tilde{u}^b \circ R^b \text{ on } \partial\tilde{\Lambda}^a \cap \tilde{\Omega}^a \\ \text{and } ([\tilde{v}^a \cdot \tilde{\mu}^a] + [(\tilde{v}^b \circ R^b) \cdot (\tilde{\mu}^b \circ R^b)])(\hat{z}, \tilde{\varphi}^a(\hat{z})) \leq 2\tilde{\varphi}^a(\hat{z}) \text{ for } \hat{z} \in \Lambda\}. \end{aligned} \tag{3.7}$$

The unilateral boundary condition in (3.7) involves two vectors $\tilde{\mu}^a$ and $\tilde{\mu}^b \circ R^b$ which are in general different from each other. It is convenient to introduce the orthogonal matrices $E^{a,b}(\hat{z})$ defined as follows:

$$\tilde{\nu}_i^a = E_{ij}^a N_j, \qquad \tilde{\nu}_k^b \circ R^b = E_{kj}^b N_j, \tag{3.8}$$

where N is the unit normal vector to $\tilde{\Gamma}^{\alpha}$.[4] According to this, the gap condition becomes

$$\{[\tilde{u}_i^a|\tilde{\mu}^a|E_{ij}^a+(\tilde{u}_k^b\circ R^b)|\tilde{\mu}^b\circ R^b|E_{kj}^b]N_j\}(\hat{z},\tilde{\varphi}^a(z))\le 2\tilde{\varphi}^a(\hat{z}) \qquad \text{for} \quad \hat{z}\in\Lambda. \tag{3.9}$$

Hence, it is natural to express the variational inequality in terms of the functions

$$U_j^{a,b}\equiv|\tilde{\mu}^{a,b}\circ R^{a,b}|(\tilde{u}_k^{a,b}\circ R^{a,b})E_{kj}^{a,b}, \tag{3.10}$$

where R^a denotes the identity, to shorten notations. This leads to the following form for the local variational inequality:

$$\begin{cases} \text{to find } (U^a, U^b)\in K_{\hat{x}_0} \text{ such that for all } (V^a, V^b)\in K_{\hat{x}_0} \\ A^a(U^a, V^a-U^a)+A^b(U^b, V^b-U^b) \\ \qquad \ge \displaystyle\int_{\tilde{\Lambda}^a\cap\tilde{\Omega}^a,} \{F_i^a(z)[V^a-U^a]_i(z)+F_i^b(z)[V^b-U^b]_i(z)\,dz, \end{cases} \tag{3.11}$$

where ($V^{a,b}$ defined according to (3.10))

$$\begin{aligned} K_{\hat{x}_0}=\{V\equiv(V^a, V^b)\text{ s.t. } &V^{a,b}\in[\mathrm{H}^1(\tilde{\Lambda}^a\cap\tilde{\Omega}^a)]^n,\ V^{a,b}=U^{a,b}\text{ on }\partial\tilde{\Lambda}^a\cap\tilde{\Omega}^a \\ &\text{and } [(V^a+V^b)(\hat{z},\tilde{\varphi}^a(\hat{z}))-2\tilde{\varphi}^a(\hat{z})N(\hat{z},\tilde{\varphi}^a(\hat{z}))] \\ &\times N(\hat{z},\tilde{\varphi}^a(\hat{z}))\le 0 \text{ for } \hat{z}\in\Lambda\}; \end{aligned} \tag{3.12}$$

$$\begin{aligned} A^{a,b}(U^{a,b}, V^{a,b})\equiv &\int_{\tilde{\Lambda}^a\cap\tilde{\Omega}^a} A_{ijkl}^{a,b}U_{i,j}^{a,b}V_{k,l}^{a,b}\,dz+\int_{\tilde{\Lambda}^a\cap\tilde{\Omega}^a} B_{ijk}^{a,b}U_i^{a,b}V_{j,k}^{a,b}\,dz \\ &+\int_{\tilde{\Lambda}^a\cap\tilde{\Omega}^a} C_{ijk}^{a,b}U_{i,j}^{a,b}V_k^{a,b}\,dz+\int_{\tilde{\Lambda}^a\cap\tilde{\Omega}^a} D_{ij}^{a,b}U_i^{a,b}V_j^{a,b}\,dz \\ &=\tilde{a}^{a,b}(\tilde{u}^{a,b},\tilde{v}^{a,b}); \end{aligned} \tag{3.13}$$

$$F_i^{a,b}\equiv\frac{1}{|\tilde{\mu}^{a,b}\circ R^{a,b}|}(\tilde{f}_k^{a,b}\circ R^{a,b})E_{ki}^{a,b}. \tag{3.14}$$

In (3.13) the coefficients

$$A_{ijkl}^{a,b}\equiv\frac{1}{|\tilde{\mu}^{a,b}\circ R^{a,b}|}(\tilde{a}_{rhtm}^{a,b}\circ R^{a,b})\gamma_{jh}\gamma_{lm}E_{ri}^aE_{tk}^a, \tag{3.15}$$

$B_{ijk}^{a,b}$, $C_{ijk}^{a,b}$, $D_{ij}^{a,b}$, are smooth combinations of $\tilde{a}_{ijkl}^{a,b}$ with coefficients depending on $\tilde{\mu}^{a,b}$, $E^{a,b}$ and on the diagonal matrix

$$\gamma_{ij}=\begin{cases} 1, & i=j<n, \\ -1, & i=j=n, \\ 0, & i\ne j. \end{cases} \tag{3.16}$$

---

[4] $E^{a,b}$ is a rotation which transforms N into $\tilde{\nu}^a$ and $\tilde{\nu}^b\circ R^b$ respectively. Of course, if Γ^a and Γ^b were symmetric with respect to $\{x_n=0\}$, then we have $E_{ij}^a=\delta_{ij}$ and $E_{kj}^b=\gamma_{kj}$ (see (3.16)).

After a rigid-body motion, we can assume $z_0=0$ and $N(z_0)=(0,0,\ldots,0,-1)$. Note that this can affect the ellipticity condition in the form (2.5), but not in the form (2.19). For this reason, we will study the local problem under the property (2.19), assuming that (2.20) remains true in $\tilde{\Lambda}^a \cap \tilde{\Omega}^a$.

The next step is to straighten $\partial\tilde{\Lambda}^a \cap \tilde{\Omega}^a$. Define

$$y(z)=(\hat{z}, z_n-\tilde{\varphi}^a(\hat{z})) \tag{3.17}$$

for $\hat{z}\in\Lambda$. Under this transformation, let

$$G_\rho \equiv \{y\in\mathbf{R}^n: |y|<\rho, y_n>0\}$$

be the image of $\tilde{\Lambda}^a \cap \tilde{\Omega}^a$, for ρ small. In $\overline{G_\rho}$ we define a system of smooth orthonormal vectors $\tau_1(y),\ldots,\tau_n(y)$ such that $\tau_\mu(\hat{y},0)$ is tangent to $\partial\tilde{\Lambda}^a \cap \tilde{\Omega}^a$ and

$$\tau_n(\hat{y},0)=-N(\hat{z},\tilde{\varphi}^a(\hat{z})), \qquad \hat{z}\in\Lambda. \tag{3.18}$$

Any function $W\in[\mathbf{L}^2(\tilde{\Lambda}^a \cap \tilde{\Omega}^a)]^n$ admits a representation on the system $\tau_1(y),\ldots,\tau_n(y)$ as follows:

$$W(z)=\tilde{W}_l(y)\tau_l(y), \tag{3.19}$$

where $\tilde{W}_l(y)$ are suitable real numbers.

The unilateral condition in (3.12) reads

$$[\tilde{V}_n^a+\tilde{V}_n^b+2\tilde{\Phi}_n](\hat{y},0)\geq 0, \tag{3.20}$$

where $\tilde{\Phi}_n$ is implicitly defined by $\tilde{\varphi}^a(\hat{z})N(\hat{z},\tilde{\varphi}^a(\hat{z}))=-\tilde{\Phi}_l(\hat{y})\tau_l(\hat{y})$. For,

$$W(z)\cdot N(z)=-W(z)\cdot\tau_n(y)=-\tilde{W}_l(y)\tau_l(y)\cdot\tau_n(y)=-\tilde{W}_n(y), \tag{3.21}$$

where $z=(\hat{z},\tilde{\varphi}^a(\hat{z}))$, $y=(\hat{y},0)$ and (3.18) and (3.19) have been used. So, the convex $K_{\hat{x}_0}$ becomes

$$\begin{aligned}\tilde{\mathbf{K}}=\{(\tilde{V}^a,\tilde{V}^b)^{(5)}\in[\mathbf{H}^1(G_\rho)]^n\times[\mathbf{H}^1(G_\rho)]^n: \tilde{V}^{a,b}=\tilde{U}^{a,b}\\ \text{on } \partial G_\rho\cap\{y_n>0\} \text{ and (3.20) is satisfied}\}.\end{aligned} \tag{3.22}$$

We have

$$\begin{aligned}A^{a,b}(U^{a,b},V^{a,b})&=\tilde{A}^{a,b}(\tilde{U}^{a,b},\tilde{V}^{a,b})\\ &\equiv\int_{G_\rho}\{[\tilde{A}^{a,b}_{ijkl}\tilde{U}^{a,b}_{i,j}+\tilde{A}^{a,b}_{ikl}\tilde{U}^{a,b}_i](\tilde{V}^{a,b})_{k,l}\\ &\qquad+[\tilde{B}^{a,b}_{ikl}\tilde{U}^{a,b}_{k,l}+\tilde{A}^{a,b}_{ik}\tilde{U}^{a,b}_k](\tilde{V}^{a,b})_i\}\,dy\end{aligned} \tag{3.23}$$

(5) $\tilde{V}^{a,b}$ denotes the vector of components $\tilde{V}^{a,b}_i$.

for suitable smooth coefficients $\tilde{A}^{a,b}_{ijkl}$, $\tilde{A}^{a,b}_{ikl}$, $\tilde{B}^{a,b}_{ikl}$, $\tilde{A}^{a,b}_{ik}$. The variational inequality (3.11) becomes:

$$\begin{cases} \text{to find } (\tilde{U}^a, \tilde{U}^b) \in \tilde{\mathbf{K}} \text{ such that for all } (\tilde{V}^a, \tilde{V}^b) \in \tilde{\mathbf{K}}, \\ \tilde{A}^a(\tilde{U}^a, \tilde{V}^a - \tilde{U}^a) + \tilde{A}^b(\tilde{U}^b, \tilde{V}^b - \tilde{U}^b) \\ \quad \geq \int_{G_\rho} \{\tilde{F}^a_i(\tilde{V}^a - \tilde{U}^a)_i + \tilde{F}^b_i(\tilde{V}^b - \tilde{U}^b)_i\}\, dy. \end{cases} \tag{3.24}$$

We now use the difference quotient technique to prove the local $\mathbf{H}^2$ regularity for $\tilde{U}^{a,b}$. Let $\eta \in C^\infty(\overline{G_\rho})$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ in $G_{\rho/4}$, $\eta = 0$ outside $G_{\rho/2}$. Let $\tilde{\Psi}_l$ be a smooth extension to G_ρ of $\tilde{\Phi}_l$. We plug into (3.24) the test function

$$\tilde{V}^{a,b}(y) = \tilde{U}^{a,b}(y) + \eta^2(y)[\tilde{U}^{a,b}(y+h) - \tilde{U}^{a,b}(y) + \tilde{\Psi}(y+h) - \tilde{\Psi}(y)],$$

where $h \in \mathbf{R}$ is small and $y+h$ is shorthand for $(y_1, \dots, y_\mu + h, \dots, y_n)$, with $1 \leq \mu \leq n-1$. It is easy to check that $(\tilde{V}^a, \tilde{V}^b) \in \tilde{\mathbf{K}}$.

With the notation $\Delta W(y) \equiv W(y+h) - W(y)$, the principal terms in (3.24) (see (3.23)) are

$$\int_{G_\rho} \tilde{A}^{a,b}_{ijkl}(y)\, \tilde{U}^{a,b}_{i,j}(y) \{\eta^2(y)[\Delta \tilde{U}^{a,b}(y) + \Delta\tilde{\Psi}(y)]\}_{k,l}\, dy. \tag{3.25}$$

Writing (3.24) in the variable $(y+h)$, then choosing

$$\tilde{V}^{a,b}(y+h) = \tilde{U}^{a,b}(y+h) - \eta^2(y+h)[\Delta \tilde{U}^{a,b}(y) + \Delta\tilde{\Psi}(y)],$$

the principal terms are

$$\int_{G_\rho} \tilde{A}^{a,b}_{ijkl}(y+h)\, \tilde{U}^{a,b}_{i,j}(y+h) \{-\eta^2(y+h)[\Delta \tilde{U}^{a,b}(y) + \Delta\tilde{\Psi}(y)]\}_{k,l}\, dy. \tag{3.26}$$

Adding together (3.25) and (3.26), we have

$$\begin{aligned} &-\int_{G_\rho} [\Delta \tilde{U}^{a,b}(y) + \Delta\tilde{\Psi}(y)]_{k,l} \Delta[\eta^2 \tilde{A}^{a,b}_{ijkl} \tilde{U}^{a,b}_{i,j}](y)\, dy \\ &\qquad -\int_{G_\rho} [\Delta \tilde{U}^{a,b}(y) + \Delta\tilde{\Psi}(y)]_k \Delta[\eta^2_{,l} \tilde{A}^{a,b}_{ijkl} \tilde{U}^{a,b}_{i,j}](y)\, dy. \end{aligned} \tag{3.27}$$

Consider the first integral in (3.27). Adding and subtracting the quantity $\eta^2(y)\tilde{A}^{a,b}_{ijkl}(y)\,\tilde{U}^{a,b}_{i,j}(y+h)$, a suitable grouping yields

$$\begin{aligned} &-\int_{G_\rho} \Delta \tilde{U}^{a,b}_{i,j}(y) \Delta \tilde{U}^{a,b}_{k,l}(y) \eta^2(y) \tilde{A}^{a,b}_{ijkl}(y)\, dy \\ &\qquad -\int_{G_\rho} \tilde{U}^{a,b}_{i,j}(y+h) \Delta \tilde{U}^{a,b}_{k,l}(y) \Delta[\eta^2(y) \tilde{A}^{a,b}_{ijkl}(y)]\, dy \\ &\qquad -\int_{G_\rho} \Delta \tilde{U}^{a,b}_{i,j}(y) \Delta\tilde{\Psi}_{k,l}(y) \eta^2(y) \tilde{A}^{a,b}_{ijkl}(y)\, dy \\ &\qquad -\int_{G_\rho} \tilde{U}^{a,b}_{i,j}(y+h) \Delta\tilde{\Psi}_{k,l}(y) \Delta[\eta^2(y) \tilde{A}^{a,b}_{ijkl}(y)]\, dy \\ &\quad = -[I^{a,b}_1 + I^{a,b}_2 + I^{a,b}_3 + I^{a,b}_4]. \end{aligned} \tag{3.28}$$

Inequality (2.20) implies (see the Appendix)

$$\tilde{C}\|\eta\Delta\tilde{U}^{a,b}\|^2_{[\mathbf{H}^1(G_\rho)]^n}\le\tilde{A}^{a,b}(\eta\Delta\tilde{U}^{a,b},\ \eta\Delta\tilde{U}^{a,b})+\|\eta\Delta\tilde{U}^{a,b}\|^2_{[\mathbf{L}^2(G_\rho)]^n},$$

hence, as shown in [15],

$$\tilde{C}\|\eta\Delta\tilde{U}^{a,b}\|^2_{[\mathbf{H}^1(G_\rho)]^n}\le I_1^{a,b}+\mathrm{LOT},\tag{3.29}$$

where LOT denotes in general a sum of "lower order terms." These are integrals containing the solution only in the forms:

$$\begin{aligned}&\text{(i) } \Delta\tilde{U}^{a,b}_{i,j}(y)\tilde{U}^{a,b}_{k,l}(y);\\&\text{(ii) } \Delta\tilde{U}^{a,b}_{i,j}(y)\tilde{U}^{a,b}_{k}(y);\\&\text{(iii) } \Delta\tilde{U}^{a,b}_{i,j}(y);\\&\text{(iv) } \Delta\tilde{U}^{a,b}_{k}(y)\tilde{U}^{a,b}_{i,j}(y),\end{aligned}\tag{3.30}$$

and so on. Note that the terms in (3.23), (3.27), and (3.28) not taken into account so far are LOT.

We are now able to prove that

$$\left\|\eta\frac{\Delta\tilde{U}^a}{h}\right\|^2_{[\mathbf{H}^1(G_\rho)]^n}+\left\|\eta\frac{\Delta\tilde{U}^b}{h}\right\|^2_{[\mathbf{H}^1(G_\rho)]^n}$$
$$\le C\{\|\tilde{U}^a\|^2_{[\mathbf{H}^1(G_\rho)]^n}+\|\tilde{U}^b\|^2_{[\mathbf{H}^1(G_\rho)]^n}+\|\tilde{F}^a\|^2_{[\mathbf{L}^2(G_\rho)]^n}+\|\tilde{F}^b\|^2_{[\mathbf{L}^2(G_\rho)]^n}\}.\tag{3.31}$$

This is done by inserting (3.28) in (3.29), then using (3.24). About LOT we act as follows. Terms containing (3.30(i)) are treated as $I_2^{a,b}$:

$$|I_2^{a,b}|\le C\sup_{i,j,k,l}\left\{\left[\int_{G_\rho}\eta^2(y)|\tilde{U}^{a,b}_{i,j}(y)|^2\,dy\right]^{1/2}\left[\int_{G_\rho}\eta^2(y)[\Delta\tilde{U}^{a,b}_{k,l}(y)]^2\,dy\right]^{1/2}\right\}$$
$$\le C\left[\frac{1}{\varepsilon}\|\eta\tilde{U}^{a,b}\|^2_{[\mathbf{H}^1(G_\rho)]^n}+\varepsilon\|\eta\Delta\tilde{U}^{a,b}\|^2_{[\mathbf{H}^1(G_\rho)]^n}\right],$$

where $\varepsilon>0$ is arbitrary (Young inequality). An analogous procedure works for (3.30(ii)); an easier one for the remaining LOT. Letting $h\to0$ in (3.31) and recalling that $\eta\equiv1$ in $G_{\rho/4}$, we have $\tilde{U}^{a,b}_{i,\mu k}\in\mathbf{L}^2(G_{\rho/4})$, $\forall i,\ k=1,\ldots,n$, $\forall\mu=1,\ldots,n-1$.

As shown in [15], an estimate for $\|\tilde{U}^{a,b}_{i,nn}\|_{\mathbf{L}^2(G_{\rho/4})}$, $i=1,\ldots,n$, can be obtained in terms of the remaining second derivatives by means of (i) of Theorem 2.2. Eventually we can go back to $u^{a,b}$ by means of smooth linear combinations (see (3.19)), using also the regular operators $R^{a,b}$ and $E^{a,b}$. So, we have proved the following:

Theorem 3.1. *Let* $f^{a,b}\in[\mathbf{L}^2(\Omega^{a,b})]^n$, $g^{a,b}\in[\mathbf{H}^{1/2}(\Gamma_s^{a,b})]^n$. *Any solution of* VI *belongs to* $[\mathbf{H}^2(\Omega^a_\delta)]^n\times[\mathbf{H}^2(\Omega^b_\delta)]^n$, *for any* $\delta>0$, *where*

$$\Omega^{a,b}_\delta\equiv\{x^{a,b}\in\Omega^{a,b}:\ \mathrm{dist}(x^{a,b},\partial\Gamma^{a,b}\cup\partial\Gamma^{a,b}_d\cup\partial\Gamma^{a,b}_s)>\delta\}.\tag{3.32}$$

Thanks to the Sobolev embedding theorem, Theorem 3.1 immediately yields the following continuity result.

Corollary 3.2. *Under the same assumptions as in Theorem* 3.1, *any solution of* **VI** *belongs to*:

1. $[\mathbf{C}^{0,\alpha}(\Omega_\delta^a)]^2 \times [\mathbf{C}^{0,\alpha}(\Omega_\delta^b)]^2$ for all $\alpha \in [0, 1[$, if $n=2$;
2. $[\mathbf{C}^{0,1/2}(\Omega_\delta^a)]^3 \times [\mathbf{C}^{0,1/2}(\Omega_\delta^b)]^3$, if $n=3$.

If $n=2$, a stronger regularity result will be proved in Section 4, namely local Hölder continuity for the gradient of $u^{a,b}$. The same method will also provide local Hölder continuity for the solution when $n=4$.

Remark 3.3. If in a neighborhood of $\hat{x}_0 \in \Xi$ it is $\varphi^a(\hat{x}) = \varphi^b(\hat{x})$, then the $\mathbf{H}^2$ regularity near x_0 can be obtained simply by a straightening of the common boundary (see (3.17)), then manipulating the variational inequality (3.24).

4. Further Regularity of the Solution

4.1. The Main Result

To illustrate how additional regularity of the solution is achieved, we shall employ the method of [17] based on penalization and Widman's hole-filling device [27], but limited to the case where the two bodies Ω^a and Ω^b are initially in contact on an open submanifold of the $x_n = 0$ hyperplane. Indeed, since our considerations are local, we set

$$G^a = \{x \in \mathbf{R}^n \colon x_n > 0, |x| < 1\}, \qquad G^b = \{x \in \mathbf{R}^n \colon x_n < 0, |x| < 1\},$$
$$\Gamma^a = \Gamma^b \equiv \Gamma \equiv \{x \in \mathbf{R}^n \colon x_n = 0, |x| < 1\}, \qquad \Gamma_d^{a,b} = \partial G^{a,b} \backslash \bar{\Gamma},$$

and consider the convex

$$\mathbf{K}_G \equiv \{v \equiv (v^a, v^b) \colon v^{a,b} \in [\mathbf{H}^1(G^{a,b})]^n,\ v^a - v^b \geq 0 \text{ on } \Gamma,\ v^{a,b} = \vartheta^{a,b} \text{ on } \Gamma_d^{a,b}\},$$

where $\vartheta^{a,b}$ is a given function in $[\mathbf{H}^1(G^{a,b})]^n$. Suppose that the force distribution is given by

$$\langle T, \zeta \rangle \equiv \int_{G^a} f^a \cdot \zeta^a \, dx + \int_{G^b} f^b \cdot \zeta^b \, dx, \qquad \zeta = (\zeta^a, \zeta^b),$$

where $f^{a,b}$ are suitably smooth, for instance, $f^{a,b} \in [\mathbf{L}^\infty(G^{a,b})]^n$.

Finally, let $u = (u^a, u^b) \in \mathbf{K}_G$ be the solution of

$$a^a(u^a, v^a - u^a) + a^b(u^b, v^b - u^b) \geq \langle T, v - u \rangle, \qquad v \in \mathbf{K}_G. \tag{4.1}$$

Thus problem (4.1) corresponds to the variational inequality **VI** of the preceding sections suitably localized, and somewhat simplified for purposes of exposition. Our object is to prove an integral estimate (see Theorem 4.1 below) which implies Hölder continuity for the solution when $n=4$ and for its derivatives when $n=2$, in, say, a neighborhood of $x=0$. This ought to provide some confidence in the variational approach, not to mention the smoothness assumptions used to derive the constraints appearing in $\mathbf{K}_G$.

To shorten notations, let us set $|\nabla^2 \zeta|^2 = \sum_{i,j,k} (\zeta_{i,jk})^2$ for $\zeta = (\zeta_1, \ldots, \zeta_n)$.

Theorem 4.1. *Let u be the solution of* (4.1). *Then there exist $M>0$ and $\beta>0$ (depending on $f^{a,b}$, $\vartheta^{a,b}$, $a^{a,b}$) such that for $r\le\frac14$ and $|x_0|\le\frac12$*

$$\int_{B_r(x_0)\cap G^a} |\nabla^2 u^a|^2\,dx \le Mr^{2\beta}, \qquad x_0\in\overline{G^a}, \tag{4.2a}$$

and

$$\int_{B_r(x_0)\cap G^b} |\nabla^2 u^b|^2\,dx \le Mr^{2\beta}, \qquad x_0\in\overline{G^b}. \tag{4.2b}$$

Before proving the theorem, let us state its consequences.

Corollary 4.2. *Let $\beta>0$ be the number found in Theorem* 4.1.

(i) *If $n=4$, then $u^{a,b}\in[\mathbf{C}^{0,\lambda}(\overline{G^{a,b}}\cap B_{1/4}(0))]^4$, for all $\lambda<\beta$;*
(ii) *If $n=2$, then $u^{a,b}\in[\mathbf{C}^{1,\beta}(\overline{G^{a,b}}\cap B_{1/4}(0))]^2$.*

Proof of the Corollary. (i) From (4.2a, b) it follows that $u^{a,b}_{i,jk}$ belongs to the Morrey space $\mathbf{L}^{2,2\beta}(G^{a,b})$. Since $u^{a,b}\in[\mathbf{H}^2(G^{a,b})]^4$, we get the Hölder continuity with exponent λ, for all $\lambda<\beta$ (see [3]).

(ii) This is a consequence of (4.2a, b), by application of Morrey's lemma [23]. □

So, we just need to prove Theorem 4.1.

A statement of the complementarity conditions will help to clarify our point of view. With the same method as in Section 2 we can derive from (4.1) the following:

$$\begin{aligned}
&-\sigma_{ij,j}(u^{a,b}) = f_i^{a,b} \quad \text{in } G^{a,b}\\
&\left.\begin{aligned}
u_n^a - u_n^b &\ge 0\\
\sigma_{nn}(u^a) = \sigma_{nn}(u^b) \equiv \sigma_{nn} &\le 0\\
(u_n^a-u_n^b)\sigma_{nn} &\equiv 0\\
\sigma_{n\mu}(u^{a,b}) &= 0
\end{aligned}\right\}\ \text{on } \Gamma,\\
&u^{a,b} = \vartheta^{a,b} \quad \text{on } \Gamma_d^{a,b}.
\end{aligned} \tag{4.3}$$

Set $|||\zeta||| \equiv \zeta^a(\hat{x},0)-\zeta^b(\hat{x},0)$. Due to Theorem 3.1, in (4.3) the equality $|||u_n|||\sigma_{nn}=0$ is intended a.e. on $\Gamma_\delta\equiv\Gamma\cap\{(\hat{x},0)\colon |\hat{x}|<1-\delta\}$, for all $\delta\in\,]0,1[$. Since $|||u_n|||$ actually belongs to $\mathbf{H}^{3/2}(\Gamma_\delta)$, we get that

$$|||u_n|||_{,\mu}\sigma_{nn}=0 \qquad \text{a.e. on } \Gamma_\delta, \quad \delta\in\,]0,1[, \quad \mu=1,\ldots,n-1. \tag{4.4}$$

To obtain (4.2a, b) we shall exploit (4.4) in conjunction with the inequalities, valid for sufficiently small r and constants $c_{n1}, \dots, c_{n(n-1)}$

$$\int_{G_r^a} |\nabla^2 u^a|^2\,dx + \int_{G_r^b} |\nabla^2 u^b|^2\,dx \le \frac{M_0}{r^2}\left\{\int_{H_r^a} (u^a_{n,\mu} - c^a_{n\mu})^2\,dx + \int_{H_r^b} (u^b_{n,\mu} - c^b_{n\mu})^2\,dx\right\} + M_0\left\{\int_{H_r^a} |\nabla^2 u^a|^2\,dx + \int_{H_r^b} |\nabla^2 u^b|^2\,dx\right\} + F(r) \tag{4.5}$$

and

$$\int_{G_r^a} |\nabla^2 u^a|^2\,dx + \int_{G_r^b} |\nabla^2 u^b|^2\,dx \le \frac{M_0}{r^2}\left\{\int_{H_r^a} [\sigma_{nn}(u^a)]^2\,dx + \int_{H_r^b} [\sigma_{nn}(u^b)]^2\,dx\right\} + M_0\left\{\int_{H_r^a} |\nabla^2 u^a|^2\,dx + \int_{H_r^b} |\nabla^2 u^b|^2\,dx\right\} + F(r), \tag{4.6}$$

where M_0 is a suitable positive constant,

$$G_r^{a,b} = G^{a,b} \cap \{x \in \mathbf{R}^n : |x| < r\},$$

$$H_r^{a,b} = G_{2r}^{a,b} \backslash G_r^{a,b},$$

$$0 \le F(r) \le Cr^{2\nu} \qquad \text{for a} \quad C > 0, \quad \text{and for all} \quad \nu \in]0, 1[,$$

and a summation on $\mu = 1, \dots, n-1$ is intended in (4.5). The proof of (4.5) and (4.6) is delayed to Section 4.2.

To understand how (4.2a, b) follow from (4.5) and (4.6) we recall two familiar Poincaré inequalities. For this, let

$$T_r = \{(\hat{x}, 0) : r \le |\hat{x}| \le 2r\}.$$

Lemma 4.3. *Suppose that* $\zeta \in \mathbf{H}^1(H_r^a)$ *satisfies*

$$\text{meas}_{n-1}\{\zeta = 0, x \in T_r\} \ge \tfrac{1}{2}\,\text{meas}_{n-1} T_r.$$

Then

$$\int_{H_r^a} \zeta^2\,dx \le Cr^2 \int_{H_r^a} |\nabla \zeta|^2\,dx,$$

where C *is independent of* r.

Lemma 4.4. *Suppose that* $\zeta \in \mathbf{H}^1(H_r^a)$ *and that* $E \subset T_r$ *satisfies*

$$\text{meas}_{n-1} E \ge \tfrac{1}{2}\,\text{meas}_{n-1} T_r.$$

Then

$$\int_{H_r^a} \zeta^2\,dx \le C\left\{r^2 \int_{H_r^a} |\nabla \zeta|^2\,dx + r^{2-n}\left[\int_E \zeta\,d\hat{x}\right]^2\right\},$$

where C *is independent of* r.

Analogous results hold for H_r^b. Obviously, Lemma 4.3 follows from Lemma 4.4. We regard the proof of Lemma 4.4 to be routine and omit it.

Proof of Theorem 4.1. According to the complementarity condition (4.4), either

$$\operatorname{meas}_{n-1}\{\sigma_{nn}=0,\, x\in T_r\}\geq \tfrac{1}{2}\operatorname{meas}_{n-1} T_r \tag{4.7}$$

or

$$\operatorname{meas}_{n-1}\{|||u_n|||_{,\mu}=0,\, x\in T_r\}\geq \tfrac{1}{2}\operatorname{meas}_{n-1} T_r, \qquad \mu=1,\dots,n-1. \tag{4.8}$$

If (4.7) holds, we employ (4.6) and Lemma 4.3. After simplifying in (4.6) we find that

$$\int_{G_r^a} |\nabla^2 u^a|^2\,dx+\int_{G_r^b}|\nabla^2 u^b|^2\,dx \leq M_1\left\{\int_{H_r^a}|\nabla^2 u^a|^2\,dx+\int_{H_r^b}|\nabla^2 u^b|^2\,dx\right\}+F(r). \tag{4.9}$$

However, if (4.8) holds, there are sets $E_\mu\subset T_r$ such that

$$u^a_{n,\mu}=u^b_{n,\mu}\quad\text{on } E_\mu,\qquad \operatorname{meas}_{n-1}E_\mu\geq\tfrac{1}{2}\operatorname{meas}_{n-1}T_r,\qquad \mu=1,\dots,n-1.$$

Choosing now, in (4.5),

$$c_{n\mu}=\frac{1}{\operatorname{meas}_{n-1}E_\mu}\int_{E_\mu} u^a_{n,\mu}\,d\hat{x}=\frac{1}{\operatorname{meas}_{n-1}E_\mu}\int_{E_\mu} u^b_{n,\mu}\,d\hat{x},$$

we obtain

$$\int_{H_r^{a,b}}(u^{a,b}_{n,\mu}-c_{n\mu})^2\,dx\leq Cr^2\int_{H_r^{a,b}}|\nabla u^{a,b}_{n,\mu}|^2\,dx\leq Cr^2\int_{H_r^{a,b}}|\nabla^2 u^{a,b}|^2\,dx$$

by Lemma 4.4. As will be seen, it is necessary to choose the numbers $c_{n\mu}$ independently of a, b to derive (4.5). In this way we again arrive at (4.9).

Writing

$$\omega(r)=\int_{G_r^a}|\nabla^2u^a|^2\,dx+\int_{G_r^b}|\nabla^2u^b|^2\,dx$$

we have from (4.9),

$$\omega(r)\leq M_1[\omega(2r)-\omega(r)]+F(r)$$

or

$$\omega(r)\leq\lambda\omega(2r)+\frac{1}{1+M_1}F(r),$$

$$\lambda=\frac{M_1}{1+M_1}<1,\qquad |F(r)|\leq Cr^{2\nu}\quad\text{for }\nu\in\,]0,1[.$$

By a well known lemma (see [18]), there exist $M>0$ and $\beta\in\,]0,1[$ such that

$$\omega(r)\leq Mr^{2\beta},\qquad r\text{ small.}$$

This proves Theorem 4.1 in the case $x_0=0$. When $x_0\in\Gamma$, $|x_0|\leq\frac{1}{2}$, and $r\leq\frac{1}{4}$ the procedure is analogous. The remaining cases may be treated in the same manner as [15]. □

Up to verification of (4.5) and (4.6), the proof of Theorem 4.1 is complete. Far from $\partial\Gamma$, when $n=2$ Hölder continuity for the derivatives of $u^{a,b}$ on the remaining part of $\overline{G^{a,b}}$ can be obtained again in the same way as in [15]. Thus we have the following:

Theorem 4.5. *Let* $f^{a,b} \in [\mathbf{L}^\infty(G^{a,b})]^n$. *There exists* $\beta \in]0, 1[$ *such that the solution of* (4.1) *belongs to*

(i) $[\mathbf{C}^{0,\beta}(\overline{G_\delta^a})]^4 \times [\mathbf{C}^{0,\beta}(\overline{G_\delta^b})]^4$, if $n=4$;
(ii) $[\mathbf{C}^{1,\beta}(\overline{G_\delta^a})]^2 \times [\mathbf{C}^{1,\beta}(\overline{G_\delta^b})]^2$, if $n=2$;

for any $\delta \in]0, 1[$, *where*

$$G_\delta^{a,b} \equiv \{x^{a,b} \in G^{a,b} : \operatorname{dist}(x^{ab}, \partial\Gamma) > \delta\}.$$

4.2. Proof of the Estimates (4.5) and (4.6)

In Section 3 we employed a finite difference technique to establish our estimates, so here, to illustrate a different idea, we use penalization. Conditions (4.3) suggest consideration of

$$\begin{aligned}
&-\sigma_{ij,j}(u^{a,b}) = f_i^{a,b} \quad \text{in } G^{a,b},\\
&\left.\begin{aligned} -\sigma_{nn} + \beta_\varepsilon(|||u_n|||) &= 0\\ \sigma_{n\mu}(u^{a,b}) &= 0 \end{aligned}\right\} \quad \text{on } \Gamma,\\
&u^{a,b} = \vartheta^{a,b} \quad \text{on } \Gamma_d^{a,b},
\end{aligned} \tag{4.10}$$

where $\beta_\varepsilon(t) \in \mathbf{C}^\infty(\mathbf{R})$, $\beta_\varepsilon(t)=0$ for $t \geq 0$, $\beta_\varepsilon(t)<0$ for $t<0$ and $\lim_{\varepsilon\to 0}\beta_\varepsilon(t) = -\infty$ for $t<0$. Written in weak form, (4.10) becomes:

$$u^{a,b} \in [\mathbf{H}^1(G^{a,b})]^n, \qquad u^{a,b} = \vartheta^{a,b} \quad \text{on } \Gamma_d^{a,b} \quad \text{and for}$$

$$\zeta^{a,b} \in [\mathbf{H}^1(G^{a,b})]^n, \qquad \zeta^{a,b} = 0 \text{ on } \Gamma_d^{a,b},$$

$$a^a(u^a, \zeta^a) + a^b(u^b, \zeta^b) + \int_\Gamma \beta_\varepsilon(|||u_n|||)|||\zeta_n|||\, d\hat{x} = \langle T, \zeta\rangle. \tag{4.11}$$

The solution $u_\varepsilon^{a,b}$ of (4.11) is smooth in $G_\delta^{a,b}$. This may be shown by applying difference quotients repeatedly. Expression (4.11) is a result of multiplying the equation by $\zeta^{a,b}$ and integrating by parts, using, of course, (4.10). It is easy to check that $u_\varepsilon^{a,b} \to u^{a,b}$ in $[\mathbf{H}^1(G^{a,b})]^n$ and weakly in $[\mathbf{H}^2(G_\delta^{a,b})]^n$.

Now we are able to prove (4.5) and (4.6), beginning with the former. For $\varepsilon>0$, consider the penalized problem (4.11); in the interest of brevity, drop the subscript ε in u_ε and β_ε. We set $\zeta = (\zeta^a, \zeta^b)$ with

$$\zeta_i^{a,b} = -[\eta^2(u_{i,\mu}^{a,b} - c_{i\mu}^{a,b})]_{,\mu}, \tag{4.12}$$

where $c_{i\mu}^{a,b}$ are constants subject to $c_{n\mu}^a = c_{n\mu}^b$ and $\eta \in \mathbf{C}_0^\infty(G^a \cup G^b \cup \Gamma)$ is a cutoff function with

$$\begin{cases} \eta = 1, & |x| \le r, \\ \eta = 0, & |x| \ge 2r, \\ |\nabla \eta| \le \dfrac{2}{r}. \end{cases} \tag{4.13}$$

With this ζ in (4.11), we consider the boundary term first:

$$\begin{aligned} \int_\Gamma \beta(|||u_n|||)|||\zeta_n||| \, d\hat{x} &= -\int_\Gamma \beta(|||u_n|||)\{\eta^2[u^a_{n,\mu} - c^a_{n\mu} - (u^b_{n,\mu} - c^b_{n\mu})]\}_{,\mu} \, d\hat{x} \\ &= -\int_\Gamma \beta(|||u_n|||)\{\eta^2|||u_n|||_{,\mu}\}_{,\mu} \, d\hat{x} \\ &= \int_\Gamma \eta^2 \beta'(|||u_n|||)|||u_n|||^2_{,\mu} \, d\hat{x} \ge 0 \end{aligned}$$

since $c^a_{n\mu} = c^b_{n\mu}$ and $\beta' \ge 0$. Thus

$$a^a(u^a, \zeta^a) + a^b(u^b, \zeta^b) \le \langle T, \zeta \rangle. \tag{4.14}$$

Consider now the term $a^a(u^a, \zeta^a)$:

$$\begin{aligned} a^a(u^a, \zeta^a) &= \int_{G^a} \sigma_{ij}(u^a)\zeta^a_{i,j} \, dx = \int_{G^a} \sigma_{ij,\mu}(u^a)[\eta^2(u^a_{i,\mu} - c^a_{i\mu})]_{,j} \, dx \\ &= \int_{G^a} \sigma_{ij,\mu}(u^a)u^a_{i,j\mu}\eta^2 \, dx + \int_{G^a} \sigma_{ij,\mu}(u^a)(u^a_{i,\mu} - c^a_{i\mu})(\eta^2)_{,j} \, dx \\ &= \int_{G^a} \sigma_{ij}(u^a_{,\mu})u^a_{i,j\mu}\eta^2 \, dx + \int_{G^a} \tau_{ij}(\nabla u^a)u^a_{i,j\mu}\eta^2 \, dx \\ &\quad + \int_{G^a} \sigma_{ij,\mu}(u^a)(u^a_{i,\mu} - c^a_{i\mu})(\eta^2)_j \, dx, \end{aligned}$$

where we have written

$$\sigma_{ij,\mu}(u^a) = \sigma_{ij}(u^a_{,\mu}) + \tau_{ij}(\nabla u^a), \qquad \tau_{ij}(\nabla u^a) = a^a_{ijkl,\mu} u^a_{k,l}.$$

Analogous result holds for $a^b(u^b, \zeta^b)$. After addition, using also (4.14) we obtain that

$$\begin{aligned} &\int_{G^a} \eta^2 \sigma_{ij}(u^a_{,\mu})u^a_{i,j\mu} \, dx + \int_{G^b} \eta^2 \sigma_{ij}(u^b_{,\mu})u^b_{i,j\mu} \, dx \\ &\quad \le \langle T, \zeta \rangle + \left| \int_{G^a} \tau_{ij}(\nabla u^a)u^a_{i,j\mu}\eta^2 \, dx \right| + \left| \int_{G^b} \tau_{ij}(\nabla u^b)u^b_{i,j\mu}\eta^2 \, dx \right| \\ &\quad + \left| \int_{G^a} \sigma_{ij,\mu}(u^a)(u^a_{i,\mu} - c^a_{i\mu})(\eta^2)_{,j} \, dx \right| \\ &\quad + \left| \int_{G_b} \sigma_{ij,\mu}(u^b)(u^b_{i,\mu} - c^b_{i\mu})(\eta^2)_{,j} \, dx \right|. \end{aligned} \tag{4.15}$$

We now apply the technical observation, Lemma 3.2 of [17], to estimate the $\mathbf{L}^2$ norm of the second derivatives of $u^{a,b}$ in terms of the left-hand side of (4.15). Thus, we obtain that

$$\int_{G^a} \eta^2|\nabla^2 u^a|^2\,dx+\int_{G^b} \eta^2|\nabla^2 u^b|^2\,dx\le J^a+J^b+[\text{r.h.s. of (4.15)}], \tag{4.16}$$

where

$$\begin{aligned} J^{a,b}=2\int_{G^{a,b}} & a^{a,b}_{ijkl}\eta\eta_{,j}(u^{a,b}_{i,\mu}-c^{a,b}_{i\mu})u^{a,b}_{k,l\mu}\,dx \\ &+\int_{G^{a,b}} a^{a,b}_{ijkl}\eta_{,l}\eta_{,j}(u^{a,b}_{i,\mu}-c^{a,b}_{i\mu})(u^{a,b}_{k,\mu}-c^{a,b}_{k\mu})\,dx \\ &+\int_{G^{a,b}} \eta^2(u^{a,b}_{i,\mu}-c^{a,b}_{i\mu})^2\,dx+\int_{G^{a,b}}[\eta_{,j}(u^{a,b}_{i,\mu}-c^{a,b}_{i\mu})]^2\,dx \\ &+\int_{G^{a,b}} \eta\eta_{,j}u^{a,b}_{i,j\mu}(u^{a,b}_{i,\mu}-c^{a,b}_{i\mu})\,dx. \end{aligned}$$

Owing to the weak convergence of $u^{a,b}_\varepsilon$ to $u^{a,b}$ in $[\mathbf{H}^2(G^{a,b}_\delta)]^n$, using also the weak l.s.c. of the norm, we may pass to the limit when $\varepsilon\to 0$, obtaining (4.16) for the limit function $u^{a,b}$.(6)

In the right-hand side of (4.16) we apply Young's inequality to the various terms, including those in $\langle T,\zeta\rangle$, and obtain expressions of the following types:

(i) $\gamma\int_{G^{a,b}}\eta^2|\nabla^2 u^{a,b}|^2\,dx$;

(ii) $\dfrac{1}{\gamma}\int_{G^{a,b}}\eta^2|\nabla u^{a,b}|^2\,dx$;

(iii) $\int_{G^{a,b}_r}|u^{a,b}_{i,\mu}-c^{a,b}_{i\mu}|^2\,dx$;

(iv) $\dfrac{1}{\gamma r^2}\int_{H^{a,b}_r}|u^{a,b}_{i,\mu}-c^{a,b}_{i\mu}|^2\,dx$;

(v) $\int_{G^{a,b}_r}|f|^2\,dx$,

where γ is a small parameter. Let us examine the various cases:

(i) This term is moved to the left-hand side.

(ii) Since any derivative of $u^{a,b}$ belongs to $\mathbf{H}^1(G^{a,b}_\delta)$, from Sobolev embeddings we get that it also belongs to $\mathbf{L}^q(G^{a,b}_\delta)$, where $q=2n/(n-2)$ if $n>2$ and q is arbitrary if $n=2$. Then,

$$\int_{G^{a,b}}\eta^2|\nabla u^{a,b}|^2\,dx\le\left[\int_{G^{a,b}}|\nabla u^{a,b}|^q\right]^{1/q}[\operatorname{meas}(G^{a,b}_r)]^{2\nu/n}\le Cr^{2\nu}$$

for all $\nu\in\,]0,1[$ (actually, $\nu\in\,]0,1]$, if $n>2$).

(6) Remember that in (4.16) $u^{a,b}$ stands for $u^{a,b}_\varepsilon$.

(iii) Applying Poincaré inequality (Lemma 4.4) and taking as $c_{i\mu}^{a,b}$ the average of $u_{i,\mu}^{a,b}$ on $G_r^{a,b}$, we are able to estimate this term by means of the quantity $Cr^2 \int_{G^{a,b}} |\nabla^2 u^{a,b}|^2 \, dx$.

(iv) For $i = \lambda < n$, let $c_{\lambda\mu}^{a,b} =$ average of $u_{\lambda,\mu}^{a,b}$ on $H_r^{a,b}$ and apply Poincaré inequality (Lemma 4.4). Thus

$$\frac{1}{r^2} \int_{H_r^{a,b}} |u_{\lambda,\mu}^{a,b} - c_{\lambda\mu}^{a,b}|^2 \, dx \le C \int_{H_r^{a,b}} |\nabla^2 u^{a,b}|^2 \, dx, \qquad \lambda = 1, \dots, n-1.$$

Since $u_{i,nn}^{a,b}$ may be expressed in terms of $u_{i,h\lambda}^{a,b}$, $\lambda < n$, the corresponding term may be added to the left by increasing the constants on the right (cf. Section 3). Defining $F(r)$ as the sum of the terms of type (ii), (iii) and (v), we get (4.5). Note that it was necessary to choose $c_{n\mu}^a = c_{n\mu}^b$ to obtain cancellation in the boundary terms.

Proof of 4.6. Again, let $\varepsilon > 0$ and $u = u_\varepsilon$, $\beta = \beta_\varepsilon$. Set, for fixed $\mu = 1, \dots, n-1$, $\zeta = (\zeta^a, \zeta^b)$,

$$\zeta_i^{a,b} = -\eta^2 u_{i,\mu\mu}^{a,b} \tag{4.17}$$

and let $c_{ij}^{a,b}$ be two matrices of constants with $c_{in}^{a,b} = 0$. We place ζ in (4.11) and consider, again, the boundary term first:

$$\begin{aligned}\int_\Gamma \beta(|||u_n|||)|||\zeta_n||| \, d\hat{x} &= -\int_\Gamma \beta(|||u_n|||)\eta^2|||u_{n,\mu\mu}||| \, d\hat{x} \\ &= \int_\Gamma \beta'(|||u_n|||)\eta^2|||u_n|||_{,\mu}^2 \, d\hat{x} + \int_\Gamma \beta(|||u_n|||)(\eta^2)_{,\mu}|||u_n|||_{,\mu} \, d\hat{x} \\ &\ge \int_\Gamma \beta(|||u_n|||)(\eta^2)_{,\mu}|||u_n|||_{,\mu} \, d\hat{x} \\ &= \int_\Gamma \sigma_{nn}(u^{a,b})(\eta^2)_{,\mu}|||u_n|||_{,\mu} \, d\hat{x}\end{aligned}$$

from (4.10). As $\varepsilon \to 0$, according to the boundary condition (4.4) and the weak convergence of $u_\varepsilon^{a,b}$ to $u^{a,b}$ in $[\mathrm{H}^2(G_\delta^{a,b})]^n$,

$$\lim_{\varepsilon \to 0} \int_\Gamma \beta(|||u_n|||)|||\zeta_n||| \, d\hat{x} \ge 0.$$

Turning now to the volume integrals, since $c_{in}^{a,b} = 0$,

$$a^a(u^a, \zeta^a) = \int_{G^a} \sigma_{ij}(u^a)\zeta_{i,j}^a \, dx = \int_{G^a} [\sigma_{ij}(u^a) - c_{ij}^a]\zeta_{i,j}^a \, dx,$$

so

$$a^a(u^a, \zeta^a) = \int_{G^a} \sigma_{ij,\mu}(u^a)u_{i,j\mu}^a\eta^2 \, dx + \int_{G^a} [\sigma_{ij}(u^a) - c_{ij}^a]u_{i,j\mu}^a(\eta^2)_{,\mu} \, dx.$$

Rearranging this in a manner similar to the preceding derivation, applying the technical Lemma 3.2 of [17], letting $\varepsilon \to 0$, and using Young's inequality, we find that the limit function $u^{a,b}$ satisfies

$$\int_{G^a} \eta^2 |\nabla^2 u^a|^2\,dx + \int_{G^b} \eta^2 |\nabla^2 u^b|^2\,dx$$
$$\le \frac{C}{\gamma\tau^2}\left[\int_{H_r^a} |\sigma_{ij}(u^a) - c_{ij}^a|^2\,dx + \int_{H_r^b} |\sigma_{ij}(u^b) - c_{ij}^b|^2\,dx\right] + L^a + L^b. \tag{4.18}$$

In $L^{a,b}$ there are terms of the same kind as (i), (ii), (iii), and (v) of the first part of this section. The first term of the right-hand side of (4.18) shows three types of summands.

For $i < n$ and $j < n$, choose $c_{ij}^{a,b}$ = average of $\sigma_{ij}(u^{a,b})$ on $H_r^{a,b}$ and apply the Poincaré lemma. For $i = \mu < n$ and $j = n$, we have $\sigma_{n\mu}(u^{a,b}) = 0$ on Γ (see (4.3)), and Lemma 4.3 may be used. Finally, $\sigma_{nn}(u^{a,b})$ appears in the final form of (4.6).

Again, $u_{i,nn}^{a,b}$ may be found by the equation. This permits absorption of the second derivatives integral over $G^{a,b}$, and the proof of (4.6) is complete. □

Acknowledgment

One of the authors (D.K.) was partially supported by NSF Grant MCS 83-01345.

Appendix

In order to get (3.29) a coerciveness-type inequality has been used: it is not trivial to see how it can be derived from (2.20). Let us prove the following:

Lemma A.1. *Assume* (2.20) *holds. Then there exists a constant* $\tilde{C} > 0$ *such that for all* $\tilde{V} \equiv (\tilde{V}^a, \tilde{V}^b) \in [\mathbf{H}^1(G_\rho)]^n \times [\mathbf{H}^1(G_\rho)]^n$, *with* $\tilde{V}^{a,b} = 0$ *on* $\partial G_\rho \cap \{y_n > 0\}$,

$$\tilde{A}^{a,b}(\tilde{V}^{a,b}, \tilde{V}^{a,b}) + \|\tilde{V}^{a,b}\|^2_{[\mathbf{L}^2(G_\rho)]^n} \ge \tilde{C}\|\tilde{V}^{a,b}\|^2_{[\mathbf{H}^1(G_\rho)]^n}. \tag{A.1}$$

Proof. From (2.20) it follows immediately that

$$\int_{\Lambda^{a,b}\cap\Omega^{a,b}} a_{ijkl}^{a,b} v_{i,j}^{a,b} v_{k,l}^{a,b}\,dx + \|v^{a,b}\|^2_{[\mathbf{L}^2(\Lambda^{a,b}\cap\Omega^{a,b})]^n} \ge C\|v^{a,b}\|^2_{[\mathbf{H}^1(\Lambda^{a,b}\cap\Omega^{a,b})]^n}, \tag{A.2}$$

for all $v^{a,b} \in [\mathbf{H}^1(\Lambda^{a,b} \cap \Omega^{a,b})]^n$ with $v^{a,b} = 0$ on $\partial\Lambda^{a,b} \cap \Omega^{a,b}$, where C is the same as in (2.20). After the change of variables (3.2), it is easy to see that there exists $C_1 > 0$ such that

$$\tilde{a}^{a,b}(\tilde{v}^{a,b}, \tilde{v}^{a,b}) + \|\tilde{v}^{a,b}\|^2_{[\mathbf{L}^2(\tilde{\Lambda}^{a,b}\cap\tilde{\Omega}^{a,b})]^n} \ge C_1\|\tilde{v}^{a,b}\|^2_{[\mathbf{H}^1(\tilde{\Lambda}^{a,b}\cap\tilde{\Omega}^{a,b})]^n}, \tag{A.3}$$

for all $\tilde{v}^{a,b} \in [\mathbf{H}^1(\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b})]^n$ with $\tilde{v}^{a,b} = 0$ on $\partial\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b}$.

After the change of variables (3.6), using the definitions (3.10) and (3.13), we have that there exists $C_2 > 0$ such that

$$A^{a,b}(V^{a,b}, V^{a,b}) + \| V^{a,b} \|^2_{[\mathbf{L}^2(\tilde{\Lambda}^a \cap \tilde{\Omega}^a)]^n} \geq C_2 \| V^{a,b} \|^2_{[\mathbf{H}^1(\tilde{\Lambda}^a \cap \tilde{\Omega}^a)]^n}, \tag{A.4}$$

for all $V^{a,b} \in [\mathbf{H}^1(\tilde{\Lambda}^a \cap \tilde{\Omega}^a)]^n$ with $V^{a,b} = 0$ on $\partial\tilde{\Lambda}^a \cap \tilde{\Omega}^a$. Let us detail this step. From (3.13) we have

$$A^{a,b}(V^{a,b}, V^{a,b}) = \tilde{a}^{a,b}(\tilde{v}^{a,b}, \tilde{v}^{a,b}). \tag{A.5}$$

Furthermore,

$$\begin{aligned}\int_{\tilde{\Lambda}^a \cap \tilde{\Omega}^a} V_i^{a,b} V_i^{a,b}\, dz &= \int_{\tilde{\Lambda}^a \cap \tilde{\Omega}^a} \{[|\tilde{\mu}^{a,b} \circ R^{a,b}|(\tilde{v}_k^{a,b} \circ R^{a,b}) E_{ki}^{a,b}] \\ &\qquad \times [|\tilde{\mu}^{a,b} \circ R^{a,b}|(\tilde{v}_j^{a,b} \circ R^{a,b}) E_{ji}^{a,b}]\, dz\} \\ &= \int_{\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b}} |\tilde{\mu}^{a,b}|^2 \tilde{v}_k^{a,b} \tilde{v}_k^{a,b}\, dz,\end{aligned}$$

which is equivalent to $\| \tilde{v}^{a,b} \|^2_{[\mathbf{L}^2(\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b})]^n}$. As for the derivatives, we have

$$\int_{\tilde{\Lambda}^a \cap \tilde{\Omega}^a} V_{i,j}^{a,b} V_{i,j}^{a,b}\, dz \leq C_3 \| \tilde{v}^{a,b} \|^2_{[\mathbf{H}^1(\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b})]^n}.$$

Eventually, recalling (A.3) and (A.5),

$$\begin{aligned}A^{a,b}(V^{a,b}, V^{a,b}) + \| V^{a,b} \|^2_{[\mathbf{L}^2(\tilde{\Lambda}^a \cap \tilde{\Omega}^a)]^n} &= \tilde{a}^{a,b}(\tilde{v}^{a,b}, \tilde{v}^{a,b}) + \| \tilde{v}^{a,b} \|^2_{[\mathbf{L}^2(\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b})]^n} \\ &\geq C_1 \| \tilde{v}^{a,b} \|^2_{[\mathbf{H}^1(\tilde{\Lambda}^{a,b} \cap \tilde{\Omega}^{a,b})]^n} \\ &\geq C_2 \| V^{a,b} \|^2_{[\mathbf{H}^1(\tilde{\Lambda}^a \cap \tilde{\Omega}^a)]^n},\end{aligned}$$

which is (A.4). In an analogous way we can derive (A.1) from (A.4). Note that the vanishing of $\tilde{V}^{a,b}$ on $\partial G_\rho \cap \{y_n > 0\}$ entails an analogous behavior for the corresponding functions at each intermediate step. This is obvious for $V^{a,b}$ (see (3.19)), while for $\tilde{v}^{a,b}$ we can read (3.10) as a linear system with nonsingular matrix. □

References

1. Baiocchi C, Gastaldi F, Tomarelli F (1984) Inéquations variationnelles non coercives. CR Acad Sci Paris 299:647–650
2. Boucher M (1975) Signorini's problem in viscoelasticity. In: de Pater-Kalker (ed), Proceedings of the Symposium on the Mechanics of Contact Between Deformable Bodies. Delft University Press, Delft, pp 41–53
3. Campanato S (1963) Proprietà di inclusione per spazi di Morrey. Ricerche Mat 12:68–85
4. Duvaut G, Lions JL (1976) Inequalities in Mechanics and Physics. Springer-Verlag, New York
5. Fichera G (1964) Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. Mem Accad Naz Lincei (8) 7:91–140
6. Fichera G (1972) Existence theorems in elasticity. In: Flugge (ed) Handbuch der Physik, vol VIa/2. Springer-Verlag, Berlin, pp. 347–389; Boundary value problems of elasticity with unilateral constraints. Ibid, pp 391–424
7. Galin LA (1961) Contact problems in the theory of elasticity. North Carolina State College, Raleigh

8. Giaquinta M (1983) Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Princeton University Press, Princeton
9. Gladwell GML (1980) Contact Problems in the Classical Theory of Elasticity. Sijthoff and Noordhoff, Alphen aan den Rijn
10. Haslinger J, Hlavacek I (1982) Contact between elastic perfectly plastic bodies. Appl Math 27:27–45
11. Hertz H (1882) Uber die Beruhrung fester elasticher Korper. J Reine Angew Math 92:156–171
12. Jarusek J (1983) Contact problems with bounded friction. Coercive case. Czechoslovak Math J 33:254–278
13. Kalker JJ (1977) Variational principles of contact elastostatics. J Inst Math Applic 20:199–219
14. Kikuchi N, Oden JT (1979) Contact problems in elasticity. TICOM Report 79-8, University of Texas at Austin
15. Kinderlehrer D (1981) Remarks about Signorini's problem in linear elasticity. Ann Scuola Norm Sup Pisa (4) 8:605–645
16. Kinderlehrer D (1982) Estimates for the solution and its stability in Signorini's problem. Appl Math Optim 8:159–188
17. Kinderlehrer D (1982) Remarks about Signorini's problem. Nonlinear PDE and Their Applications, Collège de France Seminar III. Pitman Research Notes, 70, Pitman, London, pp 234–251
18. Kinderlehrer D, Stampacchia G (1980) An Introduction to Variational Inequalities and Their Applications. Academic Press, New York
19. Lions JL, Magenes E (1972) Nonhomogeneous Boundary Value Problems and Applications, vol 1. Springer-Verlag, Berlin
20. Lions JL, Stampacchia G (1967) Variational inequalities. Comm Pure Appl Math 20:493–519
21. Love AEH (1927) A Treatise on the Mathematical Theory of Elasticity. Cambridge University Press, London
22. Meyers NG, Elcrat A (1975) Some results on regularity for solutions of nonlinear elliptic systems and quasiregular functions. Duke Math J 42:121–136
23. Morrey ChB Jr (1966) Multiple Integrals in the Calculus of Variations. Springer-Verlag, Berlin
24. Muskhelishvili NI (1953) Some Basic Problems of the Mathematical Theory of Elasticity. Noordhoff, Groningen
25. Necas J, Hlavacek I (1981) Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction. Elsevier, Amsterdam
26. Signorini A (1959) Questioni di elasticità non linearizzata e semi-linearizzata. Rend Mat 18:95–139
27. Widman KO (1971) Hölder continuity of elliptic systems. Manuscripta Math 5:299–308

Accepted 11 August 1986