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Dynamical Evolution of Elasto-Perfectly Plastic Bodies*

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Abstract. We prove the existence of a displacement field and of a stress field that satisfy the dynamical equation for continuous media and the Prandtl-Reuss constitutive law of elasto-perfect plasticity. First we obtain the existence of a displacement rate in a space of functions of bounded deformation, where the constitutive law is satisfied in an integral form, then we show that one can choose a good representative for the stress in such a way that the Prandtl-Reuss law is satisfied almost everywhere with respect to the deformation measure.

1. Introduction and Statement of the Main Results

The aim of this paper is to prove the existence of a displacement field $u(x, t)$ and of a stress field $\sigma(x, t)$ that satisfy the dynamical equation for continuous media with suitable initial and boundary conditions, and that also satisfy the Prandtl-Reuss constitutive law for elasto-perfect plasticity in a strong sense.

In this section we shall give a self-contained exposition of our main results and we shall leave the proofs for the following sections. We shall study Problem 1.1 in the form of Problem 1.2; Theorem 1.3 states the existence of a weak solution, Theorem 1.4 states that our solution satisfies the consititutive law in a strong integral form, Theorems 1.5 and 1.6 make precise in what pointwise sense the

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constitutive law is satisfied and, finally, Theorem 1.8 describes the pointwise behavior of the solution at the Dirichlet boundary.

Let us fix notations and be more precise. We shall consider an open bounded connected set $\Omega \subset \mathbb{R}^3$ with a class C^2 boundary; Γ_D and Γ_N will be two disjoint open subsets of $\partial\Omega$ with $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$ and $\partial\Gamma_D = \partial\Gamma_N$ a Lipschitz submanifold of $\partial\Omega$. We take a number $\Gamma > 0$ and we set

$$Q = \Omega \times (0, \Gamma), \qquad \Sigma_D = \Gamma_D \times (0, \Gamma), \qquad \Sigma_N = \Gamma_N \times (0, \Gamma).$$

We shall set

$$M_s = \{\alpha \in (\mathbb{R}^3)^3 \mid \alpha_{ij} = \alpha_{ji}\}$$

and $\alpha^D = \alpha - \frac{1}{3}(\operatorname{tr} \alpha) I$ will be the deviator of α, where I is the identity matrix. We shall denote by K some fixed bounded closed convex set in the space $M_s^0 = \{\alpha \in M_s \mid \operatorname{tr} \alpha = 0\}$. Recall that $\alpha^D \in M_s^0$ for all $\alpha \in M_s$.

We shall denote by $A = \{A_{ijhk}\}_{i,j,h,k=1,2,3}$ the elasticity coefficients matrix; we shall assume that A is constant in space-time, $A_{ijhk} = A_{hkij} = A_{jihk}$ and

$$A_{ijhk}\xi_{ij}\xi_{hk} \geq c_0|\xi|^2 \qquad \text{for all} \quad \xi \in M_s.$$

The summation convention over repeated indices is used.

We shall be concerned with the following problem:

Problem 1.1. Find a displacement field $u(x, t)$: $\bar{\Omega} \times [0, T] \to \mathbb{R}^3$ and a stress field $\sigma(x, t)$: $\bar{\Omega} \times [0, T] \to M_s$, such that

$$\frac{\partial^2}{\partial t^2} u(x, t) - \operatorname{div} \sigma(x, t) = f(x, t) \quad \text{in } Q,$$

$$\sigma(x, t) \cdot n(x, t) = F(x, t) \quad \text{on } \Sigma_N,$$

$$u(x, t) = g(x, t) \quad \text{on } \Sigma_D,$$

$$u(x, 0) = u_0(x),$$

$$\frac{\partial u}{\partial t}(x, 0) = u_1(x) \quad \text{in } \Omega$$

$$\sigma(x, 0) = \sigma_0(x),$$

$$\left.\begin{aligned} &\sigma^D(x, t) \in K, \\ &\varepsilon(\dot{u})(x, t) = A\dot{\sigma}(x, t) + \lambda(x, t), \\ &\lambda(x, t) \cdot (\sigma(x, t) - \tau) \geq 0 \qquad \text{for all} \quad \tau \in M_s \quad \text{such that } \tau^D \in K. \end{aligned}\right\} \tag{1.1}$$

We have set

$$\varepsilon(v) = \left\{\frac{\partial}{\partial x_i} v_j + \frac{\partial}{\partial x_j} v_i\right\}_{i,j=1,2,3},$$

$$\operatorname{div} \sigma = \left\{\frac{\partial}{\partial x_j} \sigma_{ij}\right\}_{i=1,2,3} \quad \text{and} \quad \sigma \cdot n = \{\sigma_{ij} n_j\}_{i=1,2,3},$$

where n is the outward unit normal to $\partial\Omega$. By $\dot{u}(x,t)$ and $\dot{\sigma}(x,t)$ we have denoted $(\partial/\partial t)u(x,t)$ and $(\partial/\partial t)\sigma(x,t)$, respectively.

Condition (1.1) is the Prandtl–Reuss law. Clearly one must have $\lambda=\lambda^D$. The choice of the convex set K depends on the yield criterion that one uses; in the case of the von Mises criterion one has $K=\{\alpha\in M_s^0 \| \alpha| \le \sqrt{2}\, y_0\}$, where y_0 is the yield constant.

Introducing the displacement rate $v(x,t)=(\partial/\partial t)u(x,t)$ and the plastic rate of deformation $\dot{\varepsilon}_p=\varepsilon(v)-A\dot{\sigma}$, one easily sees that Problem 1.1 is equivalent to:

Problem 1.2. Find $v(x,t)$ and $\sigma(x,t)$ such that

$$
\begin{aligned}
&\frac{\partial v}{\partial t}-\operatorname{div}\sigma=f \quad \text{in } Q,\\
&\sigma\cdot n=F \quad \text{on } \Sigma_N,\\
&v=\gamma \quad \text{on } \Sigma_D,\\
&\left.\begin{aligned} v(\cdot,0)&=v_0\\ \sigma(\cdot,0)&=\sigma_0\end{aligned}\right\} \quad \text{in } \Omega,\\
&\sigma^D(x,t)\in K,\\
&\dot{\varepsilon}_p(x,t)\cdot(\sigma(x,t)-\tau)\ge 0 \qquad \text{for all } \quad \tau\in M_s,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{such that } \quad \tau^D\in K,
\end{aligned}
\tag{1.2}
$$

where $v_0=u_1$ and $\gamma=(\partial/\partial t)g$.

Now we give our results for Problem 1.2, from which one can easily recover corresponding results for Problem 1.1, if one wishes. First we have an existence theorem. The prescribed initial and boundary values for v and σ are given as traces of functions v^* and σ^* defined in Q. From now on we denote $\partial/\partial t$ by ∂_t.

Theorem 1.3. *Suppose there exist functions $v^*(x,t)$ and $\sigma^*(x,t)$ satisfying the following assumptions*:

$$
\begin{aligned}
&v^*,\partial_t v^*\in L^\infty(0,T;L^2(\Omega,\mathbb{R}^3)),\\
&\partial_t\varepsilon(v^*)\in L^2(0,T;L^2(\Omega,M_s)),\\
&\sigma^*,\partial_t\sigma^*\in L^\infty(0,T;L^\infty(\Omega,M_s)),\\
&\partial_t^2\sigma^*\in L^1(0,T;L^\infty(\Omega,M_s)),\\
&\operatorname{div}\sigma^*\in L^\infty(0,T;L^2(\Omega,\mathbb{R}^3)),
\end{aligned}
\tag{1.3}
$$

and such that the boundary and initial conditions are given as

$$
\begin{aligned}
&v^0(x)=v^*(x,0)=0 \quad \text{in } \Omega,\\
&\sigma^0(x)=\sigma^*(x,0) \quad \text{in } \Omega,\\
&\gamma(x,t)=v^*(x,t) \quad \text{on } \Sigma_D,\\
&F(x,t)=\sigma^*(x,t)\cdot n(x) \quad \text{on } \Sigma_N.
\end{aligned}
$$

Assume also that

$$\begin{aligned} &f \in L^\infty(0, T; L^2(\Omega, \mathbb{R}^3)), \\ &\partial_t(f + \operatorname{div} \sigma^* - \partial_t v^*) \in L^1(0, T; L^2(\Omega)), \\ &f(\cdot, 0) + \operatorname{div} \sigma^*(\cdot, 0) - \partial_t v^*(\cdot, 0) \in L^2(\Omega). \end{aligned} \tag{1.4}$$

Finally, assume that one can find σ^* also satisfying the following safe load condition:

$$\sigma^{*D}(x, t) \in K \quad \textit{and} \quad \operatorname{dist}(\sigma^{*D}(x, t), \partial K) \geq c_1 > 0 \qquad \forall x, t. \tag{1.5}$$

Then there exist, and they are unique, two functions

$$\begin{aligned} &v \in L^\infty(0, T; L^2(\Omega, \mathbb{R}^3)), \\ &\sigma \in L^\infty(0, T; L^2(\Omega, M_s)) \end{aligned} \tag{1.6}$$

such that

$$\partial_t v \in L^\infty(0, T; L^2(\Omega, \mathbb{R}^3)), \tag{1.7}$$

$$\operatorname{div} \sigma \in L^\infty(0, T; L^2(\Omega, M_s)),$$

$$\partial_t v - \operatorname{div} \sigma = f \quad \textit{in } Q, \tag{1.8}$$

$$\begin{aligned} &v(\cdot, 0) = 0 \quad \textit{in } \Omega, \\ &v \cdot n = v^* \cdot n \quad \textit{on } \Sigma_D, \\ &\sigma(\cdot, 0) = \sigma^*(\cdot, 0) \quad \textit{in } \Omega, \\ &\sigma \cdot n = \sigma^* \cdot n \quad \textit{on } \Sigma_N, \end{aligned} \tag{1.9}$$

and such that the constitutive law is satisfied as follows: for almost all $t \in [0, T]$ one has

$$\sigma^D(x, t) \in K, \qquad \mathcal{L}^3 \textit{ a.e. in } \Omega, \tag{1.10}$$

$$\int_\Omega A\dot{\sigma} \cdot (\sigma - \tau) - \int_\Omega (v^* - v) \cdot \operatorname{div}(\sigma - \tau) - \int_\Omega \varepsilon(v^*) \cdot (\sigma - \tau) \leq 0 \tag{1.11}$$

for all $\tau(x) \in L^2(\Omega, M_s)$ such that $\tau^D \in K$, $\mathcal{L}^3$ a.e. in Ω, $\operatorname{div} \tau \in L^2(\Omega, \mathbb{R}^3)$, $\tau \cdot n = \sigma^*(\cdot, t) \cdot n$ on Γ_N. Moreover, for almost all $t \in [0, T]$ one has $v(\cdot, t) \in BD(\Omega)$, and

$$\sup_{t \in [0, T]} \int_\Omega |\varepsilon(v(\cdot, t))| < +\infty, \tag{1.12}$$

where, for every fixed t, we denote by $\int_\Omega |\varepsilon(v(\cdot, t))|$ the total variation in Ω of the measure $\varepsilon(v(\cdot, t))$.

We remark that (1.3) and (1.4) are just fairly natural assumptions on the data. The safe load condition is a condition on the boundary force F and it is needed to obtain estimate (1.12). While the requirement $\sigma^* \in K$ is natural, the whole condition (1.5) is a little less natural; on the other hand, it is not clear how to do without it, and all the existence results obtained so far in BD spaces in elasto-perfect plasticity had to assume some condition of this type, or some other quite strong smallness condition on the load [8], [11], [12], [4], [1]. Compare also with [6] and [7].

The proof of Theorem 1.3 is given in Sections 2 and 3; Problem 1.2 is approximated by problems for elasto-plastic materials with viscosity. For every value $\mu > 0$ of the viscosity coefficient, we can solve the problem and find functions v^μ, σ^μ, then we get estimates independent of μ and we obtain v, σ as the limit of v^μ, σ^μ for $\mu \to 0$. For earlier work on this problem, with a similar type of approach, we refer to [5], [8], [11], and [12], where an extensive bibliography can also be found, but we point out that our approximating problems are different from those used by the authors mentioned, and we use instead a constitutive law of elasto-plasticity with viscosity that has been proposed in [10].

We notice that Theorem 1.3 does not say anything about the tangential component of v taking the prescribed value v^* on the Dirichlet boundary Σ_D. We notice also that, so far, the constitutive law is satisfied only in the weak form (1.11), where the strain rate $\varepsilon(v)$ does not appear, and the related derivatives appear on the test function τ. Both these facts are now to be considered. In fact, as we know that $\varepsilon(v)$ is a measure, we would like to prove that our solution satisfies the constitutive law in some strong sense, and, to do that, we would like to perform an integration by parts in (1.11). It is not immediately clear that this is possible, because we would get, for instance, a term of the type $\int_\Omega \varepsilon(v) \cdot (\sigma - \tau)$, where $\varepsilon(v)$ is in general just a measure and $(\sigma - \tau)$ is not continuous. However, in recent papers [9], [4], [1]-[3], a meaning has been given to the scalar product $(\sigma - \tau, \varepsilon(v))$ as a measure in Ω, and to the tangential component $[(\sigma - \tau) \cdot n]_{\tan}(x)$ of the normal trace of $\sigma - \tau$ on $\partial\Omega$ as an $L^\infty(\partial\Omega, \mathbb{R}^3)$ function. We shall use these results, that are briefly recalled in Section 4, to get the following:

Theorem 1.4. *For almost all $t \in [0, T]$ one has*

$$\int_\Omega (\alpha - \tau, \dot{\varepsilon}_p) + \int_{\Gamma_D} (v^*(x) - v(x)) \cdot [(\sigma - \tau) \cdot n]_{\tan}(x)\, dH^2 \geq 0 \tag{1.13}$$

for all $\tau \in L^2(\Omega, M_s)$ such that $\tau^D(x) \in K$, $\mathscr{L}^3$ a.e. in Ω, $\operatorname{div} \tau \in L^2(\Omega, \mathbb{R}^3)$, *$\tau \cdot n = \sigma^*(\cdot, t) \cdot n$ on Γ_N.*

Notice that the measure $(v^* - v)\, dH^2|_{\Gamma_D}$ can be interpreted as a plastic strain rate at the boundary. Notice also that $\dot{\varepsilon}_p = \dot{\varepsilon}_p^D$ and that we may write σ^D, τ^D instead of σ, τ in (1.13).

Now, in order to get as close as possible to (1.2), starting from Theorem 1.4, we can prove that our solution satisfies the constitutive law in a suitable pointwise sense.

To the convex set K we associate the map $\mathscr{F}: M_s^0 \to S^K$ defined as

$$\mathscr{F}(\alpha) = \{\beta \in K \mid P_K(\beta + \alpha) = \beta\}, \tag{1.14}$$

where $P_K: M_s^0 \to K$ is the projection on K. We remark that if $\alpha \neq 0$ one has $\beta \in \mathscr{F}(\alpha)$ if and only if α is in the normal cone of ∂K at the point β. It follows that one can write the second formula in (1.2) as

$$\sigma^D(x, t) \in \mathscr{F}(\dot{\varepsilon}_p(x, t)) \qquad \text{for all } x, t. \tag{1.15}$$

We cannot prove that our solution satisfies exactly (1.15), but if we set $⨍_{B_\rho(x)} \sigma^D = (1/\mathscr{L}^3(B_\rho(x))) \int_{B_\rho(x)} \sigma^D(x)\, dy$, and denote by $(\dot{\varepsilon}_p/|\dot{\varepsilon}_p|)(x)$ the density function of the measure $\dot{\varepsilon}_p$ with respect to its total variation $|\dot{\varepsilon}_p|$, we have the following result.

Theorem 1.5. *If* (1.13) *holds for* v *and* α *as in Theorem* 1.1, *then one has*

$$\int_A \left\{\operatorname{dist}\left(⨍_{B_\rho(x)} \sigma^D, \mathscr{F}\left(\frac{\dot{\varepsilon}_p}{|\dot{\varepsilon}_p|}(x)\right)\right)\right\}|\dot{\varepsilon}_p| \to 0 \qquad \text{for} \quad \rho \downarrow 0 \tag{1.16}$$

for all $A \Subset \Omega$.

Let us draw some consequences. If we write $\dot{\varepsilon}_p = \dot{\varepsilon}_p^{\mathrm{a}} + \dot{\varepsilon}_p^{\mathrm{s}}$, where $\dot{\varepsilon}_p^{\mathrm{a}}$ and $\dot{\varepsilon}_p^{\mathrm{s}}$ are the absolutely continuous and the singular part of $\dot{\varepsilon}_p$ with respect to Lebesgue measure in Ω, from Theorem 1.5 we get in particular, for all $A \Subset \Omega$,

$$\int_A \operatorname{dist}\left(⨍_{B_\rho(x)} \sigma^D, \mathscr{F}\left(\frac{\dot{\varepsilon}_p}{|\dot{\varepsilon}_p|}(x)\right)\right)|\dot{\varepsilon}_p|^{\mathrm{a}} \to 0.$$

Moreover, by the Lebesgue point theorem we have

$$\int_A \left|⨍_{B_\rho(x)} \sigma^D - \sigma(x)\right| |\dot{\varepsilon}_p|^{\mathrm{a}} \to 0$$

and it follows that

$$\sigma^D(x) \in \mathscr{F}\left(\frac{\dot{\varepsilon}_p}{|\dot{\varepsilon}_p|}(x)\right), \qquad |\dot{\varepsilon}_p|^{\mathrm{a}} \text{ a.e. in } \Omega.$$

Now we go back to our solution $v(x, t)$, $\sigma(x, t)$ and, for almost all $t \in [0, T]$, we consider a Borel set E_t such that $\mathscr{L}^3(E_t) = |\dot{\varepsilon}_p|^{\mathrm{s}}(\Omega \setminus E_t) = 0$. Then, if we define a function $S^D(x, t)$: $\Omega \times [0, T] \to M_s^0$ as

$$S^D(x, t) = \begin{cases} \sigma^D(x, t) & \text{if } x \in \Omega \setminus E_t, \\ \alpha(x, t) & \text{if } x \in E_t, \end{cases}$$

where $\alpha(x, t)$: $E_t \to M_s^0$ is *any* $|\dot{\varepsilon}_p(\cdot, t)|$-measurable function with $\alpha(x, t) \in \mathscr{F}((\dot{\varepsilon}_p/|\dot{\varepsilon}_p|)(x, t))$, we obviously have the following result.

Theorem 1.6. *The function* $S(x, t) = \operatorname{tr} \sigma(x, t) I + S^D(x, t)$ *is a representative of* $\sigma(x, t) \in L^1(Q)$ *(hence, it obviously satisfies the equation of motion and the boundary conditions) that satisfies the constitutive law in the following sense: for almost all* t, $S^D(x, t)$ *is defined, for* $(\mathscr{L}^3 + |\dot{\varepsilon}_p|)$ *almost all* $x \in \Omega$

$$S^D(x, t) \in K \qquad \text{for a.a.} \quad t \in [0, T], \quad \text{for} \quad (\mathscr{L}^3 + |\dot{\varepsilon}_p|) \text{ a.a. } x,$$

$$|\dot{\varepsilon}_p(\cdot, t)|(\{x \in \Omega | S^D(x, t) \in K^0\}) = 0 \qquad \text{for a.a. } t,$$

$$S^D(x, t) \in \mathscr{F}\left(\frac{\dot{\varepsilon}_p(\cdot, t)}{|\dot{\varepsilon}_p(\cdot, t)|}(x)\right) \qquad |\dot{\varepsilon}_p(\cdot\, t)| \text{ a.e. in } \Omega,$$

for a.a. t.

Remark 1.7. In general $S^D(x, t)$ is not uniquely defined in the zone where $\dot{\varepsilon}_p$ is singular, but, if the set K is strictly convex, then the map $\mathscr{F}(\alpha)$ is single-valued for $\alpha \neq 0$ and in that case we have

$$S^D(x, t) = \mathscr{F}\left(\frac{\dot{\varepsilon}_p(\cdot, t)}{|\dot{\varepsilon}_p(\cdot, t)|}(x)\right), \qquad |\dot{\varepsilon}_p(\cdot, t)| \text{ a.e. in } \Omega,$$

for a.a. t,

and

$$S^D(x, t) = \lim_{\rho \downarrow 0} \fint_{B_\rho(x)} \sigma^D(y, t)\, dy \qquad \text{in } L^1_{\text{loc}}(\Omega, |\dot{\varepsilon}_p| + \mathscr{L}^3),$$

for a.a. t.

In particular, in the case of the von Mises yield condition, one has

$$S^D(x, t) = \sqrt{2}\, y_0 \frac{\dot{\varepsilon}_p(\cdot, t)}{|\dot{\varepsilon}_p(\cdot, t)|}(x), \qquad |\dot{\varepsilon}_p(\cdot, t)| \text{ a.e.},$$

for a.a. t.

Finally, we have to discuss the behavior of $v(x, t)$ at the Dirichlet boundary. For every vector $n \in \mathbb{R}^3$ with $|n| = 1$, let us consider the bounded closed convex set

$$K_n = \{(\alpha \cdot n)_{\tan} | \alpha \in K\} \subset \mathbb{R}^3, \tag{1.17}$$

where $(\alpha \cdot n)_{\tan} = (\alpha \cdot n) - ((\alpha \cdot n) \cdot n)n$ and the map $\mathscr{F}_n\colon \mathbb{R}^3 \to 2^{(K_n)}$ defined as

$$\mathscr{F}_n(\omega) = \{z \in K_n \,|\, P_{K_n}(z + \omega) = z\}. \tag{1.18}$$

Using again the information contained in (1.13) we can prove:

Theorem 1.8. *For almost all times t, one has either $v(x, t) = v^*(x, t)$ or*

$$[\sigma \cdot n]_{\tan}(x) \in \mathscr{F}_n\left(\frac{v^* - v}{|v^* - v|}(x)\right)$$

for H^2 almost all $x \in \Gamma_D$.

We notice that Theorem 1.8 is a statement about the way the constitutive law is satisfied at the Dirichlet boundary.

Remark 1.9. Again, if K is strictly convex then, for $w \in \mathbb{R}^3$, $w - (\omega \cdot n)n \neq 0$, $\mathscr{F}_n(w)$ is single value and one has $[\sigma \cdot n]_{\tan}(x) = \mathscr{F}_n(((v^* - v)/|v^* - v|)(x))$ for $|v^* - v|\, dH^2$ a.a. $x \in T_D$ (in the case of the von Mises conditions one has $[\sigma \cdot n]_{\tan}(x) = y_0((v^* - v)/|v^* - v|))$. However, we remark that the force density $[\sigma \cdot n]_{\tan}(x)$ is in any case uniquely defined as H^2 a.e. on Γ_D (actually on $\partial\Omega$), independently of K.

A similar thing also happens on the singular support of $\dot{\varepsilon}_p$ in Ω, in this case the singularity is of the type of a discontinuity of the tangential component of the displacement across a rectifiable surface T.

As a final consideration, we should like to observe that Theorems 1.5, 1.6, and 1.8 are based only on Theorem 1.4, which is in turn true every time that (1.10), (1.11), and (1.12) are satisfied. Hence we conclude that, whenever one gets a pair $v(x, t)$, $\sigma(x, t)$ that satisfies the Prandtl–Reuss law in the weak form (1.7), (1.8), and (1.9) (this holds, for instance, for the solutions of the quasi-static problem in [12]), then the constitutive law also holds in the stronger sense.

In our opinion, the meaning that we give to the constitutive law for weak solutions is the strongest possible, without starting a regularity theory. As far as we know, there are no regularity results, for general situations, in the theory of perfect plasticity.

2. Elasto-Plasticity with Viscosity

In this section we study the dynamical problem for a body subject to a constitutive law of elasto-plastic type with viscosity, following the approach proposed in [10]. First, we state the mechanical problem; then, in Theorem 2.2, we collect a few of the results given in [10], in a form which is suitable for our purposes; finally we obtain an estimate, independent of the viscosity coefficient, on our solutions.

Problem 2.1. Find a displacement rate $v(x, t)\colon \bar{\Omega}\times[0, T]\to\mathbb{R}^3$ and a stress field $\tilde{\sigma}(x, t)\colon \bar{\Omega}\times[0, T]\to M_s$ such that

$$\partial_t v - \operatorname{div}\tilde{\sigma} = f \quad \text{in } Q,$$

$$\tilde{\sigma}\cdot n = \sigma^*\cdot n \quad \text{on } \Sigma_N,$$

$$v = v^* \quad \text{on } \Sigma_D,$$

$$\left.\begin{aligned} v(\cdot, 0) &= v_0 \\ \tilde{\sigma}(\cdot, 0) &= \sigma_0 \end{aligned}\right\} \quad \text{in } \Omega,$$

and such that, if we set $\sigma = \tilde{\sigma} - \mu\varepsilon(\dot{u})$, where the constant number $\mu > 0$ is a viscosity coefficient, the following constitutive law is satisfied:

$$\begin{aligned} &\sigma^D(x, t)\in K, \\ &(\varepsilon(v)(x, t) - A\dot{\sigma}(x, t))\cdot(\sigma(x, t) - \tau)\geq 0 \end{aligned} \tag{2.1}$$

for all (x, t) and for all $\tau\in M_s$ such that $\tau^D\in K$.

We remark that the stress $\tilde{\sigma}$ is the sum of two parts: the first part $\mu\in(v)$ is due to the viscosity and is proportional to the strain rate, the remaining part σ is the stress that originates from the elastic reaction to the deformation. The plastic behavior of the body is described then by the requirement that σ^D stays in the convex K and that the nonelastic strain rate $\varepsilon(v) - A\dot{\sigma}$ satisfies the inequality in (2.1).

Let us consider the spaces

$$H = L^2(\Omega, M_s),$$

$$\begin{aligned} V &= \{w\in L^2(\Omega, \mathbb{R}^3)\,|\,\varepsilon(w)\in L^2(\Omega, \mathbb{R}^9),\ w|_{\Gamma_D} = 0\} \\ &= H^{1,2}(\Omega, \mathbb{R}^3)\cap\{w|_{\Gamma_D} = 0\}. \end{aligned}$$

We shall denote by V' the dual space of V, and $\langle\ ,\ \rangle$ will denote the V', V pairing. We shall also consider the convex set

$$\mathscr{K}=\{\tau\in H \mid \tau^D(x)\in K \text{ a.e. in } \Omega\}.$$

One has the following theorem, whose proof is in [10].

Theorem 2.2. *Under the assumptions*

$$\sigma_0\in\mathscr{K},\qquad v_0\in L^2(\Omega),$$

$$v^*\in L^2(0,T;H^{1,2}(\Omega,\mathbb{R}^3)),$$

$$\partial_t v^*\in L^2(0,T;V'),$$

$$v^*(\cdot,0)=v_0,$$

$$\sigma^*\in L^2(0,T;H),$$

$$\operatorname{div}\sigma^*\in L^2(0,T;V'),$$

$$f\in L^2(0,T;V'),$$

for any fixed number $\mu>0$, there exists a unique pair of functions $v(x,t)$, $\sigma(x,t)$ such that

$$v-v^*\in L^2(0,T;V),$$

$$\partial_t v\in L^2(0,T;V'),$$

$$\sigma\in L^2(0,T;\mathscr{K}),$$

$$\partial_t\sigma\in L^2(0,T;H),$$

$$v(\cdot,0)=v_0,$$

$$\sigma(\cdot,0)=\sigma_0,$$

and such that, if $\tilde\sigma=\sigma+\mu\varepsilon(v)$, one also has

$$\int_0^T\langle\partial_t(v-v^*),w\rangle\,dt+\int_0^T\int_\Omega(\tilde\sigma-\sigma^*)\cdot\varepsilon(w)\,dx\,dt$$
$$=\int_0^T\langle\operatorname{div}\sigma^*+f-\partial_t v^*,v\rangle\,dt\qquad\textit{for all}\quad w\in L^2(0,T;v),\tag{2.2}$$

$$\int_0^T\int_\Omega(\varepsilon(v)-A\dot\sigma)\cdot(\sigma-\tau)\,dx\,dt\ge 0\qquad\textit{for all}\quad\tau(x,t)\in L^2(0,T;\mathscr{K}).\tag{2.3}$$

Remark 2.3. From (2.2) one gets $\partial_t v-\operatorname{div}\tilde\sigma=f$ (hence $\operatorname{div}\tilde\sigma\in L^2(0,T;V')$) and $\tilde\sigma\cdot n=\sigma^*\cdot n$ on Σ_N in the usual sense of traces in $H^{-1/2}(\Sigma_N)$.

Remark 2.4. By standard techniques, one obtains from (2.3) that for almost all $t \in (0, T)$ one has

$$\int_\Omega (\varepsilon(v)(x,t) - A\dot{\sigma}(x,t)) \cdot (\sigma(x,t) - \tau(x))\,dx \geq 0 \qquad \text{for all} \quad \tau(x) \in \mathcal{K}. \tag{2.4}$$

And, if (2.4) holds for some t, then one has, for the same t,

$$\int_\Omega A\dot{\sigma}(x,t) \cdot (\sigma(x,t) - \tau(x))\,dx + \langle \operatorname{div}(\tilde{\sigma}(\cdot,t) - \tau), v(\cdot,t) - v^*(\cdot,t)\rangle$$
$$+ \mu \int_\Omega |\varepsilon(v)(x,t)|^2\,dx \leq \int_\Omega \varepsilon(v^*)(x,t) \cdot (\tilde{\sigma}(x,t) - \tau(x))\,dx$$
$$\text{for all} \quad \tau(x) \in \mathcal{K} \quad \text{such that } \operatorname{div} \tau \in V', \quad \tau \cdot n = \sigma^*(\cdot,t) \cdot n \quad \text{on } \Gamma_N. \tag{2.5}$$

Now we prove a basic estimate, including an L^1 estimate for the strain rate, that is independent of the viscosity coefficient μ. The key assumption will be the safety condition (1.5) (see [8], [11], [12], and [1]).

Theorem 2.5. *Under the assumptions of Theorem 2.2, if we also assume that*

$$v^* \in L^\infty(0, T; L^2(\Omega)),$$
$$\partial_t \varepsilon(v^*) \in L^2(0, T; L^2(\Omega)),$$
$$\partial_t \sigma^* \in L^\infty(0, T; L^\infty(\Omega)),$$
$$\partial_t^2 \sigma^* \in L^1(0, T; L^\infty(\Omega)),$$
$$\sigma_0 = \sigma^*(\cdot 0) - \mu\,\varepsilon(v_0),$$
$$\partial_t(f + \operatorname{div} \sigma^* - \partial_t v^*) \in L^1(0, T; L^2(\Omega)),$$
$$f(\cdot, 0) + \operatorname{div} \sigma^*(\cdot, 0) - \partial_t v^*(\cdot, 0) \in L^2(\Omega),$$

and if the safety condition (1.5) *holds, then, for the functions $v(x,t)$, $\sigma(x,t)$ given in Theorem* 2.2, *we also have*

$$\sup_{t \in [0,T]} \left\{ \int_\Omega |\partial_t \sigma(x,t)|^2\,dx + \int_\Omega |\partial_t v(x,t)|^2\,dx + \int_\Omega |\varepsilon^D(v)(x,t)|\,dx \right\} + \mu \int_0^T \int_\Omega |\partial_t \varepsilon(v)|^2\,dx\,dt \leq c_2, \tag{2.6}$$

where c_2 depends on the data, but not on μ.

Proof. In the following proof we should use the difference quotients

$$\partial_t^h v(x,t) = \frac{1}{h}(v(x,t+h) - v(x,t)),$$

$$\partial_t^h \sigma(x,t) = \frac{1}{h}(\sigma(x,t+h) - \sigma(x,t)).$$

With the intent of making the proof a little less cumbersome, instead of these difference quotients we shall write derivatives with respect to time. The results that we get are the same that we would get by using difference quotients.

We take the equation of motion (2.2), we differentiate it with respect to time, we take $\partial_t(v-v^*)$ as a test function and we perform an integration by parts in the term containing $\operatorname{div}(\tilde{\sigma}-\sigma^*)$ to get

$$\int_0^t \int_\Omega \partial_t^2(v-v^*)^2 \cdot \partial_t(v-v^*)\, dx\, ds + \int_0^t \int_\Omega \partial_t(\tilde{\sigma}-\sigma^*)\partial_t\varepsilon(v-v^*)\, dx\, ds$$

$$= \int_0^t \int_\Omega \partial_t(f+\operatorname{div}\sigma^* - \partial_t v^*)\partial_t(v-v^*)\, dx\, ds.$$

Recalling that $\tilde{\sigma}=\sigma+\mu \in (v-v^*)+\mu \in (v^*)$, and integrating with respect to time where it is needed, we obtain

$$\begin{aligned}
&\tfrac{1}{2}\int_\Omega |\partial_t(v-v^*)(x,t)|^2\, dx - \frac{1}{2}\int_\Omega |\partial_t(v-v^*)(x,0)|^2\, dx \\
&\quad + \mu \int_0^t \int_\Omega |\partial_t\varepsilon(v-v^*)|^2\, dx\, ds = -\mu \int_0^t \int_\Omega \partial_t\varepsilon(v^*)\cdot \partial_t\varepsilon(v-v^*)\, dx\, ds \\
&\quad - \int_0^t \int_\Omega \partial_t\sigma \cdot \partial_t\varepsilon(v-v^*)\, dx\, ds + \int_0^t \int_\Omega \partial_t(f+\operatorname{div}\sigma^* - \partial_t v^*)\partial_t(v-v^*) \\
&\quad + \int_\Omega \partial_t\sigma^*(x,t)\varepsilon(v-v^*)(x,t)\, dx - \int_\Omega \partial_t\sigma^*(x,0)\varepsilon(v-v^*)(x,0)\, dx \\
&\quad - \int_0^t \int_\Omega \partial_t^2\sigma^* \cdot \varepsilon(v-v^*)\, dx\, ds.
\end{aligned} \tag{2.7}$$

By inequality (2.3) we have

$$\int_0^t \int_\Omega (\partial_t\varepsilon(v) - \partial_t A\dot{\sigma})\cdot \dot{\sigma}\, dx\, ds \geq 0. \tag{2.8}$$

From (2.8) we then get

$$\int_0^t \int_\Omega (A\partial_t\dot{\sigma}) \leq \int_0^t \int_\Omega \partial_t\varepsilon(v)\cdot \partial_t\sigma \tag{2.9}$$

and it follows that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \dot{\sigma}(x,t)\cdot A\dot{\sigma}(x,t)\, dx - \frac{1}{2}\int_\Omega \dot{\sigma}(x,0)\cdot A\dot{\sigma}(x,0)\, dx \\
&\quad \leq \int_0^t \int_\Omega \partial_t\varepsilon(v)\cdot \partial_t\sigma\, dx\, ds.
\end{aligned} \tag{2.10}$$

Summing (2.7) and (2.10), and taking into account the positive definiteness of the matrix A, we get

$$c_3\left\{\int_{\Omega_t} |\dot{\sigma}|^2\,dx+\int_{\Omega_t} |\partial_t(v-v^*)|^2\,dx+\mu\int_0^t\int_\Omega |\partial_t\varepsilon(v-v^*)|^2\,dx\,ds\right\}$$

$$\leq \int_{\Omega_0} |\dot{\sigma}|^2\,dx+\int_{\Omega_0} |\partial_t(v-v^*)|^2\,dx+\mu\int_0^t\int_\Omega |\partial_t\varepsilon(v^*)|^2\,dx\,ds$$

$$+\sup_{s\in[0,t]}\left[\int_{\Omega_s}|\sigma|^2\,dx\right]^{1/2}\|\partial_t\varepsilon(v^*)\|_{L^1(0,t;L^2(\Omega))}$$

$$+\sup_{s\in[0,t]}\left[\int_{\Omega_s}|\partial_t(v-v^*)|^2\,dx\right]^{1/2}\|\partial_t(f+\operatorname{div}\sigma^*-\partial_t v^*)\|_{L^1(0,t;L^2(\Omega))}$$

$$+\sup_{s\in[0,t]}\int_{\Omega_s}|\varepsilon(v-v^*)|(x,s)\,dx\{\|\partial_t^2\sigma^*\|_{L^1(0,t;L^\infty(\Omega))}$$

$$+\|\partial_t\sigma^*\|_{L^\infty(0,t,L^\infty(\Omega))}\},$$

where we write Ω_t to signify that the integral is taken for t fixed, and $c_3=c_3(c_0)$. Taking the supremum on both sides, we get

$$\sup_{s\in[0,t]}\int_{\Omega_s}|\dot{\sigma}|^2\,dx+\sup_{s\in[0,t]}\int_{\Omega_s}|\partial_t(v-v^*)|^2\,dx$$

$$+\mu\int_0^t\int_\Omega|\varepsilon(v-v^*)|^2\,dx\,ds\leq c_4+c_5\sup_{s\in[0,t]}\int_{\Omega_s}|\varepsilon(v-v^*)|\,dx, \tag{2.11}$$

where c_4 and c_5 depend on various norms of the data.

Now we have to estimate $\int_{\Omega_s}|\varepsilon(v-v^*)|\,dx$. By the safe load condition (1.5), for almost all $(x,t)\in Q$, we have that

$$(\varepsilon(v)-A\dot{\sigma})\cdot(\sigma-\sigma^*)\geq c_1|\varepsilon(v)-A\dot{\sigma}|. \tag{2.12}$$

By (2.12) we get, for almost all t,

$$\int_{\Omega_t}|\varepsilon(v)|\,dx\leq\int_{\Omega_t}|A\dot{\sigma}|\,dx+c_6\int_{\Omega_t}(\varepsilon(v)-A\dot{\sigma})\cdot(\sigma-\sigma^*)\,dx, \tag{2.13}$$

where $c_6=c_1^{-1}$. Now we use the equation of motion (2.2) with the test function $(v-v^*)$ to estimate $\int_{\Omega_t}\varepsilon(v)\cdot(\sigma-\sigma^*)\,dx$. We have

$$\int_{\Omega_t}\varepsilon(v)(\sigma-\sigma^*)\,dx\leq\|f+\operatorname{div}\sigma^*-\partial_t v^*\|_{L^2(\Omega_t)}\|v-v^*\|_{L^2(\Omega_t)}$$

$$+\|\partial_t(v-v^*)\|_{L^2(\Omega_t)}\|v-v^*\|_{L^2(\Omega_t)}+\|\sigma-\sigma^*\|_{L^2(\Omega_t)}\|\varepsilon(v^*)\|_{L^2(\Omega_t)}$$

$$+\tfrac{3}{2}\mu\int_{\Omega_t}|\varepsilon(v)|^2+\frac{1}{2}\mu\int_{\Omega_t}|\varepsilon(v^*)|^2. \tag{2.14}$$

By (2.12), (2.13), and (2.14) we then get

$$\sup_{s\in[0,t]}\int_{\Omega_s}|\sigma|^2\,dx+\sup_{s\in[0,t]}\int_{\Omega_s}|\partial_t(v-v^*)|^2\,dx+\mu\int_0^t\int_\Omega|\varepsilon(v-v^*)|^2\,dx\,ds\leq$$

$$\leq c_7+c_8\left\{\sup_{s\in[0,t]}\int_{\Omega_s}|\sigma-\sigma^*|^2\,dx+\sup_{s\in[0,t]}\int_{\Omega_s}|v-v^*|^2\,dx\right.$$

$$\left.+\mu\sup_{s\in[0,t]}\int_{\Omega_s}|\varepsilon(v)|^2\,dx\right\}. \tag{2.15}$$

Estimating the right-hand side of (2.15) by an integral of the left-hand side, using the Gronwall Lemma, and recalling again (2.13) and (2.14) we obtain (2.6). □

3. Proof of Theorem 1.3

First Step. Under the hypotheses of Theorem 1.3 there exist functions $v(x, t)$, $\sigma(x, t)$ as in (1.6), that also satisfy (1.7), (1.8), and (1.9).

Proof. For $\mu>0$, let us denote by v^μ, σ^μ the solution obtained in Theorem 2.2 for the problem with viscosity coefficient μ. By estimate (2.6) there exists a sequence v^{μ_j}, σ^{μ_j}, that we shall denote v^j, σ^j, such that

$$v^j\to v\quad\text{in } L^\infty(0,t;L^2(\Omega))\text{ weak*},$$

$$\sigma^j\to\sigma\quad\text{in } L^\infty(0,t;L^2(\Omega))\text{ weak*},$$

for some v and σ. On the other hand, we have

$$\int_{\Omega_t}|\tilde\sigma^j-\sigma^j|^2\,dx=\mu_j^2\int_{\Omega_t}|\varepsilon(v^j)|^2\,dx\leq\mu_j\int_\Omega\int_0^t|\partial_t\varepsilon(v^j)|^2\,dx\,ds\leqslant\mu_jc_2$$

and it follows that one also has

$$\tilde\sigma^j\to\sigma\quad\text{in } L^\infty(0,t;L^2(\Omega))\text{ weak*}.$$

By estimate (2.6), also using the equation of motion to get a bound for $\|\operatorname{div}(\tilde\sigma^j-\sigma^*)\|_{L^\infty(0,T;L^2(\Omega))}$, we have that

$$\left.\begin{array}{l}\partial_t v^j\to\partial_t v\\ \partial_t\sigma^j\to\partial_t\sigma\\ \operatorname{div}\tilde\sigma^j\to\operatorname{div}\sigma\end{array}\right\}\quad\text{in } L^\infty(0,t;L^2(\Omega))\text{ weak*}, \tag{3.1}$$

and we also have that

$$\operatorname{div}v^j\to\operatorname{div}v\quad\text{in } L^\infty(0,t;L^2(\Omega))\text{ weak*}$$

because, as is easily seen from (2.3), one has $\operatorname{div}v^j=\operatorname{tr}(A\partial_t\sigma^j)$. Conditions (1.7), (1.8), and (1.9) now follow.

Second Step. We have

$$\lim_{\mu\to 0}\sup_{t\in[0,T]}\left\{\int_{\Omega_t}|\sigma^\mu-\sigma|^2\,dx+\int_{\Omega_t}|v^\mu-v|^2\,dx+\mu\int_0^t\int_\Omega|\varepsilon(v^\mu)|^2\,dx\,ds\right\}=0. \tag{3.2}$$

Proof. Taking the test function $\tau=\sigma$ in inequality (2.2), and integrating by parts, we get

$$\int_0^t\int_\Omega A\dot\sigma^\mu\cdot(\sigma^\mu-\sigma)\,dx\,ds+\int_0^t\int_\Omega(v^\mu-v^*)\cdot\operatorname{div}(\tilde\sigma^\mu-\sigma)\,dx\,ds$$
$$+\mu\int_0^t\int_\Omega|\varepsilon(v^\mu)|^2\,dx\,ds\le\int_0^t\int_\Omega\varepsilon(v^*)\cdot(\tilde\sigma^\mu-\sigma)\,dx\,ds.$$

Using the equation of motion we obtain

$$\int_{\Omega_t}|\sigma^\mu-\sigma|^2\,dx+\int_{\Omega_t}|v^\mu-v|^2\,dx+\mu\int_0^t\int_\Omega|\varepsilon(v^\mu)|^2\,dx\,ds$$
$$\le c_9\int_0^t\int_\Omega\{\varepsilon(v^*)(\tilde\sigma^\mu-\sigma)-(v-v^*)\operatorname{div}(\tilde\sigma^\mu-\sigma)-(\sigma^\mu-\sigma)\cdot A\partial_t\sigma$$
$$+(v^\mu-v)(f+\operatorname{div}\sigma^*-\partial_t v^*)\}\,dx\,ds+c_9\mu^2\int_{\Omega_0}|\varepsilon(v_0)|^2\,dx \tag{3.3}$$

and (3.2) follows just be noticing that the right-hand side of (3.3) goes to zero uniformly with respect to $t\in[0,T]$, for $\mu\to 0$.

Third Step. If we write inequality (2.5) for v^μ and σ^μ, then, by (3.2), we can take the limit for $\mu\to 0$, for almost all t, and we get (1.11). Finally, (1.12) follows because of the lower semicontinuity of the total variation $\int_{\Omega_t}|\varepsilon^D(v)|$ with respect, for example, to the $L^1(\Omega_t)$-convergence of v.

Finally, if σ_1, v_1 and σ_2, v_2 are two pairs of solutions for our problem, we can write (1.8) and (1.11) for σ_1, v_1 using σ_2, v_2 as test functions, and vice versa, then we can sum and we immediately get that $\sigma_1=\sigma_2$, $v_1=v_2$.

4. Strong Formulation for the Constitutive Law

In this section we prove Theorems 1.4–1.6 and 1.8. We begin by recalling a few known results.

Lemma 4.1. *Assume that $\partial\Omega$ is of class C^2. If $\sigma^D\in L^\infty(\Omega, M_s)$,* $\operatorname{tr}\sigma\in L^2(\Omega)$, $\operatorname{div}\sigma\in L^2(\Omega)$, *then there exists a function $\gamma_\sigma(x)\in L^\infty(\partial\Omega,\mathbb{R}^3)$ that depends linearly on σ, such that*

$$\gamma_\sigma(x)\cdot n(x)=0,\qquad H^2 \text{ a.e. on } \partial\Omega,$$

$$\|\gamma_\sigma\|_{\infty,\partial\Omega}\le\frac{1}{\sqrt{2}}\|\sigma^D\|_{\infty,\Omega},$$

and

$$\int_\Omega \sigma(x)\cdot\varepsilon(u(x))\,dx+\int_\Omega u(x)\cdot \operatorname{div}\sigma(x)\,dx=\int_{\partial\Omega} u(x)\cdot\gamma_\sigma(x)\,dH^2$$

for all $u\in C^1(\bar\Omega)$ *such that* $u\cdot n=0$ *on* $\partial\Omega$. *We shall denote the function* $\gamma_\sigma(x)$ by $[\sigma\cdot n]_{\tan}(x)$.

Proof. This lemma is proved in [4] (Theorem 5.3) in the case where $\operatorname{div}\sigma\in L^3$. That proof extends immediately to our case; compare with the first section of [3]. □

Obviously, the integration by parts formula in Lemma 4.1 also holds for all functions $u\in L^2(\Omega,\mathbb{R}^3)$ such that $\operatorname{div} u\in L^2(\Omega)$, $\varepsilon(u)\in L^1(\Omega)$.

Lemma 4.2. *Assume that* $\partial\Omega$ *is of class* C^2 *and that* $\sigma^D\in L^\infty(\Omega, M_s)$, $\operatorname{tr}\sigma\in L^2(\Omega)$, $\operatorname{div}\sigma\in L^2(\Omega)$. *If* $\delta\in(0,\delta_0)$, *for* δ_0 *sufficiently small, the map* $(x,\delta)\to x-\delta n(x)$ *is a diffeomorphism between* $\partial\Omega\times(0,\delta_0)$ *and* $\{x\in\Omega|\operatorname{dist}(x,\partial\Omega)>\delta_0\}$. *Set*

$$\gamma^\delta(x)=(\sigma(x-\delta n(x))\cdot n(x))$$

and set also $\gamma^\delta_{\tan}=\gamma^\delta-(\gamma^\delta\cdot n)n$. *Then one has*

$$\gamma^\delta_{\tan}\to[\sigma\cdot n]_{\tan}\quad \textit{in } L^\infty(\partial\Omega,\mathbb{R}^3)\ \textit{weak}^*.$$

Proof. It is sufficient to take a function $\omega\in L^1(\partial\Omega,\mathbb{R}^3)$, with $\omega(x)\cdot n(x)=0$, and to extend it to a function $\tilde\omega\in L^2(\Omega)$ with $\varepsilon(\tilde\omega)\in L^1(\Omega)$, $\operatorname{div}\tilde\omega\in L^2(\Omega)$ (a slight modification of Theorem 5.2 of [4]), then one easily sees that

$$\int_{\partial\Omega}\gamma^\delta_{\tan}(x)\cdot\tilde\omega^\delta_{\tan}(x)\to\int_{\partial\Omega}[\sigma\cdot n]_{\tan}(x)\cdot\omega(x),$$

where $\tilde\omega^\delta(x)=\tilde\omega(x-\delta n(x))$. Recalling that $\tilde\omega^\delta\to\omega$ in $L^1(\partial\Omega)$ one has the result □

Lemma 4.3. *Assume that* $\partial\Omega$ *is of class* C^2. If $\sigma^D\in L^\infty(\Omega,M_s)$, $\operatorname{tr}\sigma\in L^2(\Omega)$, $\operatorname{div}\sigma\in L^2(\Omega)$, $\sigma\cdot n=0$ on Υ_N *and* $u\in BD(\Omega)\cap L^2(\Omega)$ *with* $\operatorname{div} u\in L^2(\Omega)$, $u\cdot n=0$ on T_D, *then one has*

$$\int_\Omega(\sigma,\varepsilon(u))+\int_\Omega u(x)\cdot\operatorname{div}\sigma(x)\,dx=\int_{T_D}[\sigma\cdot n]_{\tan}(x)u(x)\,dH^2,\tag{4.1}$$

where $(\sigma,\varepsilon(u))$ *is a bounded real valued measure in* Ω *characterized* [9] *by*

$$\langle(\sigma,\varepsilon(u)),\varphi\rangle=\int_\Omega u(x)\cdot\operatorname{div}(\varphi\sigma)(x)\,dx,\qquad\forall\varphi\in C^1_0(\Omega).$$

Proof. This is basically Theorem 3.2 of [9], but we have $u\in L^2$ and $\operatorname{div}\sigma\in L^2$ instead of $u\in L^{3/2}$ and $\operatorname{div}\sigma\in L^3$. □

We recall that if σ is as in Lemma 4.3 and σ^D is also continuous in Ω, one has $\int_B(\sigma,\varepsilon(u))=\int_B\sigma(x)\cdot\varepsilon(u)$ for all Borel sets $B\subset\Omega$.

Now we can integrate by parts in (1.11).

Proof of Theorem 1.4. If σ and v are as in Theorem 1.3, by Lemma 4.3, for almost all t, for all $\tau(x): \Omega \to M_s$ such that

$$\tau^D \in L^\infty(\Omega), \qquad \operatorname{tr}\tau \in L^2(\Omega), \qquad \operatorname{div}\tau \in L^2(\Omega),$$
$$\tau \cdot n = \sigma^*(\cdot, t) \cdot n \quad \text{on } \Gamma_N \tag{4.2}$$

we have

$$\int_\Omega (\sigma - \tau, \varepsilon(v - v^*)) + \int_\Omega (v - v^*) \cdot \operatorname{div}(\sigma - \tau)\, dx$$
$$- \int_{\Gamma_D} (v - v^*) \cdot [(\sigma - \tau) \cdot n]_{\tan}\, dH^2 = 0$$

and (1.13) follows immediately from (1.11). □

Now we need a simple approximation lemma.

Lemma 4.4. *Let K be a bounded closed convex subset of M_s^0, assume that $\partial\Omega$ is of class C^2 and let $\sigma \in L^2(\Omega, M_s)$ be such that*

$$\left.\begin{array}{l} \sigma^D(x) \in L \\ \operatorname{div}\sigma \in L^2(\Omega) \end{array}\right\} \quad \mathscr{L}^3 \text{ a.e. in } \Omega.$$

Then, for $\rho \in (0, \rho_0)$, for some $\rho_0 > 0$, there exists functions $\sigma_\rho \in L^2(\Omega, M_s) \cap C^0(\Omega)$ such that

$$\sigma_\rho(x) = \fint_{B_\rho(x)} \sigma(y)\, dy \qquad \text{if} \quad \operatorname{dist}(x, \partial\Omega) > 2\rho, \tag{4.3}$$
$$\sup_\rho \|\sigma_\rho^D\|_{\infty,\Omega} \le c_{10}(K),$$
$$\operatorname{div}\sigma_\rho \in L^2(\Omega),$$
$$\sigma_\rho(x) \to \sigma, \quad \mathscr{L}^3 \text{ a.e.},$$
$$\operatorname{tr}\sigma_\rho \to \operatorname{tr}\sigma \quad \text{in } L^2(\Omega),$$
$$\operatorname{div}\sigma_\rho \to \operatorname{div}\sigma \quad \text{in } L^2(\Omega),$$
$$[\sigma_\rho \cdot n] = [\sigma \cdot n] \quad \text{on } \partial\Omega \quad \text{for all } \rho. \tag{4.4}$$

Proof. One can follow [13], i.e., for any fixed ρ, one considers open set $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \ldots$ with $\bigcup \Omega_j = \Omega$, one takes a partition of unity φ_j for covering $A_1 = \Omega_2$, $A_j = (\Omega_{j+1} - \Omega_{j-1})$ if $j \ge 2$, and sets

$$\sigma_\rho(x) = \sum_{j=1}^{\infty} \int_{B_{\tau_j}(x)} (\varphi_j \sigma),$$

where $\tau_j > 0$ have to be chosen depending on Ω_j, φ_j, ρ. If ρ is sufficiently small one can take $\Omega_1 = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \rho\}$, $\varphi_1(x) = 1$ on Ω_1, and $\tau_1 = \rho$ in order to have (4.3). The other properties follows as in [13], except for (4.4), which is easily understood, as one can choose $\delta > 0$ so that $[\sigma \cdot n]$ is arbitrarily close to $\sigma|_{S_\delta} \cdot n$ (where $S_\delta = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) = \delta\}$) and $[\sigma_\rho \cdot n]$ is arbitrarily close to $\sigma_\rho|_{S_\delta} \cdot n$, while $\sigma_\rho|_{S_\delta} \cdot n \to \sigma|_{S_\delta} \cdot n$ for almost every fixed δ, for $\rho \to 0$. □

We remark that we need not have $\sigma_\rho^D(x) \in K$ for all $x \in \Omega$, on the other hand, by (4.3), we do have $\sigma_\rho^D(x) \in K$ if $\operatorname{dist}(x, \partial\Omega) > 2\rho$.

Now we prove the following lemma.

Lemma 4.5. *If $\sigma = \sigma(\cdot, t)$ and $v = v(\cdot, t)$ are such that* (1.13) *holds, σ_ρ are as in Lemma* 4.4 *and we denote by $P_K\sigma_\rho^D(x)$ the projection of $\sigma_\rho^D(x)$ on the convex set K, we have*

$$\int_\Omega (P_K\sigma_\rho^D(x) - \tau_1) \cdot \varepsilon_p - \int_{\Gamma_D} (v^* - v) \cdot ([\sigma \cdot n]_{\tan} - [\tau_2 \cdot n]_{\tan})\, dH^2 \le \omega(\rho), \tag{4.5}$$

where $\lim_{\rho\to 0} \omega(\rho) = 0$, for all pairs of functions $\tau_1: \Omega \to K$, $\tau_2: \Gamma_D \to K$, such that τ_1 is $|\dot\varepsilon_p|$-measurable and τ_2 is $H^2|_{\Gamma_D}$ measurable.

Proof. First we notice that for all τ such that (4.2) holds, using (1.13) we have

$$\begin{aligned}
&-\int_\Omega (\sigma_\rho^D(x) - \tau^D(x), \dot\varepsilon_p) - \int_{T_D} (v^* - v) \cdot ([\sigma \cdot n]_{\tan} - [\tau \cdot n]_{\tan})\, dH^2 \\
&= -\int_\Omega (\sigma^D - \tau^D, \dot\varepsilon_p) - \int_{T_D} (v^* - v)([\sigma \cdot n]_{\tan} - [\tau \cdot n]_{\tan})\, dH^2 \\
&\quad + \int_\Omega A\dot\sigma \cdot (\sigma_\rho - \sigma) - \int_\Omega (v^* - v) \operatorname{div}(\sigma_\rho - \sigma) - \int_\Omega \varepsilon(v^*) \cdot (\sigma_\rho - \sigma) \\
&\le c_{11}\{\|\sigma - \sigma_\rho\|_{L^2(\Omega)} + \|\operatorname{div}(\sigma - \sigma_\rho)\|_{L^2(\Omega)}\} = (\rho),
\end{aligned} \tag{4.6}$$

where $c_{11} = c_{11}(\sigma, v, \sigma^*, v^*)$. Then we remark that for all τ such that

$$\tau \in L^2(\Omega, M_s), \qquad \tau^D(x) \in K, \qquad \mathscr{L}^3 \text{ a.e.}, \qquad \operatorname{div} \tau \in L^2(\Omega), \tag{4.7}$$

one has

$$-\left[\int_\Omega \mathrm{P_K}\sigma_\rho^D \cdot \varepsilon_p - \int_\Omega (\tau, \dot\varepsilon_p)\right] - \int_{T_D} (v^* - v) \cdot ([\sigma \cdot n]_{\tan} - [\tau \cdot n]_{\tan})\, dH^2 \le \omega(\rho). \tag{4.8}$$

In fact, take $\delta > 0$ and take a function $\varphi \in C^1(\partial\Omega)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on Γ_N, $H^2(\{x \in \Gamma_D \mid \varphi(x) > 0\}) < \delta$, then extend it to a function $\varphi \in C^1(\bar\Omega)$ such that again $0 \le \varphi \le 1$ on Ω and $\mathscr{L}^3(\{x \in \Omega \mid \varphi(x) > 0\}) < \delta$. Let τ be such that (4.7) holds and take

$$\alpha = (1 - \varphi)\tau + \varphi\sigma^*,$$

then it is clear that α satisfies (4.2) and by (4.6) we get that the left-hand side of (4.8) is less or equal than

$$\begin{aligned}
&\omega(\rho) + 2c_{10}\left\{\int_{\Omega_{2\rho}} |\dot\varepsilon_p| + \int_{\{x \in \Omega \mid \varphi(x) > 0\}} + \int_{\{x \in \Gamma_D \mid \varphi(x) > 0\}} |v^* - v|\, dH^2\right\} \\
&= \omega(\rho) + \omega_1(\delta),
\end{aligned}$$

where $\Omega_{2\rho}=\{x\in\Omega|\mathrm{dist}(x,\partial\Omega)<2\rho\}$ and where $\omega_1(\delta)\to 0$ for $\delta\to 0$. As δ is arbitrary, (4.8) is proved.

Now consider the Radon measure μ on $\mathbb{R}^3$ defined by

$$\mu(B)=|\dot{\varepsilon}_p|(B\cap\Omega)+H^2(B\cap T_D) \qquad \text{for all Borel sets} \quad B\subset\mathbb{R}^3$$

and the μ-measurable function τ_0: $\mathbb{R}^3\to K$ defined by

$$\tau_0(x)=\begin{cases}\tau_1(x) & \text{in } \Omega,\\ \tau_2(x) & \text{on } T_D.\end{cases}$$

We can approximate τ_0 by a sequence of functions $\alpha_j\in C^1(\mathbb{R}^3, K)$ that converge μ a.e. For all j we have (4.8) and, taking the limit for $j\to\infty$, we get (4.5). □

Proof of Theorems 1.5 and 1.8. We shall use Lemma 4.5 with a particular choice of τ_1 and τ_2, more precisely we can take τ_1 and τ_2 such that the hypotheses of Lemma 4.5 hold and

$$\begin{aligned}&\tau_1(x)\in\mathcal{F}\left(\frac{\dot{\varepsilon}_p}{|\dot{\varepsilon}_p|}(x)\right), \qquad |\dot{\varepsilon}_p| \text{ a.e. in } \Omega,\\ &\tau_2(x)\in\mathcal{F}\left(\frac{v^*-v}{|v^*-v|}(x)\otimes_s n(x)\right), |v^*-v|\, dH^2 \text{ a.e. on } \Gamma_D,\end{aligned} \tag{4.9}$$

where

$$\frac{v^*-v}{|v^*-v|}\otimes_s n=\frac{1}{2}\left[\frac{(v^*-v)_j}{|v^*-v|}n_i+\frac{(v^*-v)_i}{|v^*-v|}n_j\right].$$

In fact, we notice that $\mathcal{F}$ is the subgradient of the polar function I_K^* of the indicator function I_K of K, then we can suitably approximate I_K^* from above by a sequence of smooth functions h_j and consider the sequence of functions

$$\begin{aligned}\alpha^j(x)&=\mathcal{F}^j\left(\frac{\dot{\varepsilon}_p}{|\dot{\varepsilon}_p|}(x)\right),\\ \beta^j(x)&=\mathcal{F}^j\left(\frac{v^*-v}{|v^*-v|}(x)\otimes_s n(x)\right),\end{aligned}$$

where $\mathcal{F}^j$ is the subgradient of h_j. The functions α^j, β^j are $|\dot{\varepsilon}_p|$-measurable and $|v^*-v|\, dH^2|_{\Gamma_D}$-measurable, respectively, and one can extract subsequences that converge weakly so some τ_1, τ_2 that are as in (4.9). We remark that if τ_1 is as in (4.9) then, by the definition of $\mathcal{F}$, we have

$$(\tau_1(x)-P_K\sigma_\rho^D(x))\cdot\frac{\dot{\varepsilon}_p}{|\dot{\varepsilon}_p|}(x)\geq 0, \qquad |\dot{\varepsilon}_p| \text{ a.e.} \tag{4.10}$$

Moreover, if K_n and $\mathcal{F}_n$ are as in (1.17) and (1.18), we have that the functions γ^δ of Lemma 4.2 all belong to the convex set

$$\mathcal{K}_1=\{z\in L^1(\Gamma_D, \mathcal{H}^2)|z(x)\in K_{n(x)}, \mathcal{H}^2 \text{ a.e.}\}$$

and by Lemma 4.2 one also has that $[\sigma\cdot n]_{\tan}\in\mathscr{K}_1$. Finally we notice that if τ_2 is as in (4.9), we have

$$\tau\{\tau_2(x)\cdot n(x)]_{\tan}\in\mathscr{F}_n\left(\frac{v^*-v}{|v^*-v|}(x)\right)$$

and it follows that

$$([\tau_2\cdot n]_{\tan}(x)-[\sigma\cdot n]_{\tan}(x))\cdot\frac{v^*-v}{|v^*-v|}(x)\geq 0,\qquad \mathscr{H}^2 \text{ a.e. on } \Gamma_D. \tag{4.11}$$

Writing (4.5) for τ_1 and τ_2 as in (4.9), and taking into account (4.10) and (4.11), we conclude

$$(\tau_1(x)-P_K\sigma_\rho^D(x))\cdot\frac{\dot\varepsilon_p}{|\dot\varepsilon_p|}(x)\to 0\quad \text{in } L^1(\Omega,|\dot\varepsilon_p|) \tag{4.12}$$

for $\rho\to 0$, and

$$([\tau_2\cdot n]_{\tan}(x)-[\sigma\cdot n]_{\tan}(x))\cdot\frac{v^*-v}{|v^*-v|}(x)=0,$$

$$|v^*-v|\mathscr{H}^2 \text{ a.e. on } \Gamma_D. \tag{4.13}$$

Now, (4.13) proves Theorem 1.8, while Theorem 1.5 follows from (4.12) recalling the following well-known lemma:

Lemma 4.6. *Let $G_1(x,\alpha)$, $G_2(x,\alpha)\colon\Omega\times K\to\mathscr{R}$ be two positive Caratheodory functions such that*

$$G_1(x,\alpha)=0\ \Rightarrow\ G_2(x,\alpha)=0.$$

Then, if $\xi_\rho(x)$: $\Omega\to K$, $\rho\in(0,\rho_0)$, are μ-measurable functions in $x\in\Omega$ for each fixed p such that

$G_1(x,\xi_\rho)\to 0$ *in measure μ, for $\rho\to 0$, then one also has*

$G_2(x,\xi_\rho)\to 0$ *in measure μ, for $\rho\to 0$.*

In fact, if we take

$$G_1(x,\alpha)=(\tau_0(x)-\alpha)\cdot\frac{\dot\varepsilon_p}{|\dot\varepsilon_p|}(x),$$

$$G_2(x,\alpha)=\operatorname{dist}\left(\alpha,\mathscr{F}\left(\frac{\dot\varepsilon_p}{|\dot\varepsilon_p|}(x)\right)\right),$$

$$\xi_\rho(x)=P_K\sigma_p^D(x),$$

by Lemma 4.6 and (4.12) we have

$$\lim_{\rho\to 0}\operatorname{dist}\left(P_K\sigma_\rho^D(x),\mathscr{F}\left(\frac{\dot\varepsilon_p}{|\dot\varepsilon_p|}(x)\right)\right)=0\qquad\text{in } |\dot\varepsilon_p|\text{-measure in } \Omega$$

and by the boundedness of the function $\operatorname{dist}(\alpha,\mathscr{F}((\dot\varepsilon_p/|\dot\varepsilon_p|)(x))$ we get (1.16), which concludes the proof of Theorem 1.5. □

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