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A Numerical Approach to the Infinite Horizon Problem of Deterministic Control Theory*

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Abstract. We are concerned with the Hamilton–Jacobi equation related to the infinite horizon problem of deterministic control theory. Approximate solutions are constructed by means of a discretization in time as well as in the state variable and we prove that their rate of convergence to the viscosity solution is of order 1, provided a semiconcavity assumption is satisfied. A computational algorithm, originally due to R. Gonzales and E. Rofman, is adapted and reformulated for the problem at hand in order to obtain an error estimate for the numerical approximate solutions.

1. Introduction

The main goal of this paper is to provide a constructive method to approach the viscosity solution v of the Hamilton–Jacobi equation related to the infinite horizon deterministic control problem, namely,

$$\text{(HJ)} \qquad \lambda u(x)+\operatorname*{Max}_{a\in A}[-g(x,a)Du(x)-f(x,a)]=0.$$

This is the first step if one is interested in optimal feedback controls (see, e.g., Fleming and Rishel [9]).

We recall that equation (HJ) may have, in general, many generalized (locally Lipschitz) solutions: the relevance of the notion of a viscosity solution is that it

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allows the characterization of a particular solution (see Crandall and Lions [4], Crandall *et al.* [6], and Lions [12]).

Capuzzo Dolcetta [2], in a previous paper, has shown a discretization in time (step h) of the original control problem which leads to the approximate Hamilton–Jacobi equation

$$(\mathrm{HJ}_h) \qquad u_h(x)+\operatorname*{Max}_{a\in A}\,[-(1-\lambda h)u_h(x+hg(x,a))-hf(x,a)]=0.$$

This discretization is significant for the problem at hand since it has been proved (Capuzzo Dolcetta and Ishii [3]) that the rate of convergence of the solutions v_h of (HJ_h) to v, as h tends to zero, is of order 1, provided a semiconcavity assumption on f and g is satisfied.

Now the following question arise: how can we compute v_h? This is done here via a discretization in the state variable (step k) of (HJ_h) by means of finite element techniques. In such a way we obtain the approximate Hamilton–Jacobi equation

$$(\mathrm{HJ}_h^k) \qquad w^k(x_i)=\operatorname*{Min}_{a\in A}[(1-\lambda h)w^k(x_i+hg(x_i,a))+hf(x_i,a)],$$

which must be verified at any vertex x_i of a regular triangulation and which is finally reduced to a finite dimensional problem. The main result of Section 2 shows that the rate of convergence of the solutions v_h^k of (HJ_h^k) toward v_h, as k tends to zero, is of order 1.

As far as a numerical solution of deterministic control problems is concerned, the results presented here have some intersections (see Remark 2) with those of Gonzales and Rofman [11], who considered a direct discretization in the state variable of the Hamilton–Jacobi equation related to optimal stopping problems with impulse and continuous controls, proving the convergence of approximate solutions by means of quite heavy regularization techniques. The new approach, i.e., discretization in time and in the state variable, permits a solution to the infinite horizon problem (whereas in [11] only finite horizon problems are considered), furthermore, it allows a simpler proof of the convergence of approximate solutions to v and provides a better error estimate in the case of the optimal stopping problem with continuous control, to which both results apply.

We refer the reader interested in other aspects of the approximation of solutions of Hamilton–Jacobi equations and in stochastic optimal control problems to Crandall and Lions [5], Lions and Mercier [13], Quadrat [14], and Souganidis [16].

In the last section we discuss the computational aspects of the mixed discretization procedure. The algorithm proposed by Gonzales and Rofman [11] to solve a fixed point problem in finite dimension is reformulated so that it appears as a method to speed up the convergence of a sequence defined recursively by means of a contracting operator. In this way we obtain an error estimate for the approximate solution computed by the algorithm at the nth iteration.

Since it is beyond the aim of this paper to present in detail the numerical implementation of the mixed discretization technique, we refer the reader interested in the practice of the algorithm and in numerical tests to Falcone [7], [8].

2. Discretization Procedure and Convergence of Approximate Solutions

The Hamilton–Jacobi equation (HJ) provides a characterization of the value function of the following infinite horizon problem with discount:

$$\operatorname*{Inf}_{a\in\mathscr{A}} J(x, a(\cdot)) \equiv \operatorname*{Inf}_{a\in\mathscr{A}} \int_0^{+\infty} f(y(s), a(s))\, e^{-\lambda s}\, ds, \tag{2.1}$$

where $\mathscr{A}$, the set of admissible controls, is given by

$$\mathscr{A} = \{a\colon [0, +\infty) \to A,\ a(\cdot) \text{ measurable}\},$$

$$A \text{ is a compact subset of } \mathbb{R}^m, \tag{2.2}$$

and the state y evolves in $\mathbb{R}^n$ according to the following system of differential equations:

$$\left.\begin{aligned} \dot y(s) &= g(y(s), a(s)),\\ y(0) &= x \in \mathbb{R}^n. \end{aligned}\right\} \tag{2.3}$$

In fact, it is well known (see Lions [12] and related references) that the value function

$$v(x) = \operatorname*{Inf}_{a(\cdot)\in\mathscr{A}} J(x, a(\cdot)) \tag{2.4}$$

is the unique viscosity solution of (HJ), provided the following assumptions are satisfied:

$$|g(x, a) - g(x', a)| \le L_g|x - x'|, \qquad |g(x, a)| \le M_g, \qquad \forall x, x' \in \mathbb{R}^n, \quad \forall a \in A \tag{2.5}$$

$$|f(x, a) - f(x', a)| \le L_f|x - x'|, \qquad |f(x, a)| \le M_f, \qquad \forall x, x' \in \mathbb{R}^n, \quad \forall a \in A. \tag{2.6}$$

Moreover, it turns out that v is Lipschitz continuous and this allows us to consider Du in (HJ) as an "almost everywhere" derivative.

Let h be a strictly positive parameter; we shall consider the discretized Hamilton–Jacobi equation (see Capuzzo Dolcetta [2])

$$(\mathrm{HJ}_h) \qquad u_h(x) + \operatorname*{Max}_{a\in A}[-(1-\lambda h)u_h(x + hg(x, a)) - hf(x, a)] = 0$$

and we shall assume that

$$f(x, \cdot) \text{ and } g(x \cdot) \text{ are continuous} \tag{2.7}$$

(this assumption can be suppressed in the case of a finite number of admissible controls, $A \equiv \{1, \ldots, m\}$).

The main motivation to consider (HJ_h) is that its solutions supply a good approximation of v, as is stated by the following theorem.

Theorem 2.1 (Capuzzo Dolcetta and Ishii [3]). *Assume* (2.5), (2.6), *and* (2.7), *then* (HJ_h) *has a unique Lipschitz continuous solution* v_h *and we have*

$$\sup_{x\in\mathbb{R}^n} |v_h(x)| \le \frac{M_f}{\lambda}, \tag{2.8}$$

$$\sup_{x_1\neq x_2} \frac{|v_h(x_1)-v_h(x_2)|}{|x_1-x_2|} \le \frac{L_f}{\lambda - L_g}, \tag{2.9}$$

$$\sup_{x\in\mathbb{R}^n} |v(x)-v_h(x)| \le Ch^{1/2} \tag{2.10}$$

for any $\lambda > L_g$ *and* $h \in [0, 1/\lambda)$.

Under the supplementary semiconcavity assumptions

$$|g(x+z, a) - 2g(x, a) + g(x-z, a)| \le M|z|^2, \tag{2.11}$$

$$f(x+z, a) - 2f(x, a) + f(x+z, a) \le M|z|^2 \quad \text{for any} \quad x, z \in \mathbb{R}^n \quad \text{and} \quad a \in A, \tag{2.12}$$

for any $\lambda > 2L_g$ *and* $h \in [0, 1/\lambda)$ *we have*

$$\sup_{x\in\mathbb{R}^n} |v(x)-v_h(x)| \le Ch \tag{2.13}$$

for some positive constant C.

Remark 1. The constants appearing in (2.10) and (2.13) can be estimated. In particular, once the proof of Theorem 2.1 is known, it is a simple exercise to prove that in (2.10)

$$C \le \max\left\{\frac{L_f}{\lambda}, \frac{2M_g^2}{\lambda}, \frac{L_g}{\lambda}, M_g\right\}\left(\frac{L_f}{\lambda - L_g} + 2\right)^2.$$

Theorem 2.1 states that v_h converges to v locally uniformly but the proof does not suggest an algorithm to compute the sequence of approximate solutions. The main difficulty is that, due to the dynamic programming approach, in (HJ_h) the value of u_h at the point x depends on all possible values of u_h at the points $x + hg(x, a)$, $a \in A$. In order to overcome this difficulty we discretize (HJ_h) in the state variable using finite element techniques.

First we must restrict our problem to a bounded subset of $\mathbb{R}^n$. Let us notice that a solution v_h of (HJ_h) is defined for any $x \in \mathbb{R}^n$ so that the choice of this set will be arbitrary, the only condition being that it contains any discretized trajectory starting in it. In particular, we shall assume that there exists a polyhedron Ω in $\mathbb{R}^n$ such that, for some h,

$$x + hg(x, a) \in \bar{\Omega}, \qquad \forall x \in \bar{\Omega}, \quad \forall a \in A. \tag{2.14}$$

Later we shall discuss how to extend the results to the general case of an open bounded convex set Ω in $\mathbb{R}^n$.

Let $\{S_j\}$ be a finite family of simplices which set up a regular triangulation (see Glowinski *et al.* [10]) of Ω (Ω being a polyhedron $\bigcup_j S_j = \bar{\Omega}$) and verify

$$\max_j\{\text{diam } S_j\} = k. \tag{2.15}$$

Let N be the number of vertices, x_i, of the triangulation. We shall consider the set W^k of piecewise affine functions $w^k: \bar{\Omega} \to \mathbb{R}$ such that w^k is continuous in $\bar{\Omega}$ and the gradient of w^k is constant in the interior of any simplex S_j of the triangulation. We look for a solution in W^k of

$$(\mathrm{HJ}_h^k) \qquad w^k(x_i) = \operatorname*{Min}_{a \in A}[(1-\lambda h) w^k(x_i + hg(x_i, a)) + hf(x_i, a)] \qquad \text{for any vertex } x_i.$$

Clearly, a solution v_h of (HJ_h) verifies (HJ_h^k) and the function that we obtain interpolating the values of v_h on the vertices, that is,

$$\tilde{v}_h(x) = \sum_{j=1}^{N} \lambda_j v_h(x_j) \qquad \text{for} \quad x = \sum_{j=1}^{N} \lambda_j x_j,$$

where $\lambda_j \geq 0$, $\forall j$ and $\sum_{j=1}^{N} \lambda_j = 1$, belongs to W^k.

Theorem 2.2. *Assume* (2.7), *then for any* $h \in [0, 1/\lambda)$ *verifying* (2.14) *there exists a unique solution of* (HJ_h^k) *in* W^k.

Proof. For any affine function u^k, (HJ_h^k) corresponds to

$$u^k(x_i) = \operatorname*{Min}_{a \in A}[(1-\lambda h) \sum_{j=1}^{N} \lambda_j(x_j, a) u^k(x_j) + hf(x_i, a)], \tag{2.16}$$

where

$$\lambda_j \geq 0, \quad \forall j, \qquad \sum_{j=1}^{N} \lambda_j = 1,$$

$$x_i + hg(x_i, a) = \sum_{j=1}^{N} \lambda_j(x_i, a) x_j.$$

So it suffices to prove that there exists a unique vector $U \in \mathbb{R}^N$ such that

$$U = \operatorname*{Min}_{a \in A}[(1-\lambda h)\Lambda(a) U + hF(a)],$$

where $\Lambda(a)$ is an $N \times N$ positive matrix $\Lambda_{ij}(a) = \lambda_j(x_i, a)$ and $F(a) \in \mathbb{R}^N$ is defined by $F_i(a) = f(x_i, a)$, $i = 1, \ldots, N$.

Let us define the operator $T_h: \mathbb{R}^N \to \mathbb{R}^N$

$$(T_h(U))_i \equiv \operatorname*{Min}_{a \in A}[(1-\lambda h)\Lambda(a) U + hF(a)]_i, \qquad i = 1, \ldots, N. \tag{2.17}$$

T_h is a contraction mapping in $\mathbb{R}^N$ which verifies

$$\|T_h(U) - T_h(V)\| \leq (1-\lambda h)\|U - V\|, \qquad \forall U, V \text{ in } \mathbb{R}^N, \tag{2.18}$$

where $\|X\| = \max_{i=1,\ldots,N}|X_i|$.

In fact, denoting the ith row of Λ by Λ_i we have

$$|(T_h(U)-T_h(V))_i|\le(1-\lambda h)\max_{a\in A}|\Lambda_i(a)|\,\|U-V\|,$$

then (2.18) directly follows by the definition of Λ.

By the contraction mapping theorem there will be a unique $V^*\in\mathbb{R}^N$ such that $T_h(V^*)=V^*$. □

Notice that $V^*=(v_h(x_1),\ldots,v_h(x_N))$. We shall denote by v_h^k the solution of (HJ_h^k) obtained by interpolating V^*. We prove that v_h^k converges locally uniformly toward v_h as k tends to zero.

Theorem 2.3. *Assume* (2.5), (2.6), (2.7), (2.14), *and* (2.15), *then for any* $\lambda>L_g$ *and any* $h\in[0,1/\lambda)$

$$\max_{x\in\bar\Omega}|v_h^k(x)-v_h(x)|\le\frac{L_f}{\lambda-L_g}\,k. \tag{2.19}$$

Proof. It is evident that $v_h^k(x)$ coincides with $\tilde v_h(x)$ for any $x\in\bar\Omega$ since $V^*=(v_h(x_1),\ldots,v_h(x_N))$, then the proof simply follows from (2.9). In fact, for any $x\in\bar\Omega$ we have

$$\begin{aligned}|v_h^k(x)-v_h(x)|&\le|v_h^k(x)-\tilde v_h(x)|+|\tilde v_h(x)-v_h(x)|\\&=\left|\sum_{j=1}^N\lambda_j v_h(x_j)-v_h(x)\right|\\&\le\frac{L_f}{\lambda-L_g}\,k.\end{aligned}$$ □

The preceding theorem joined with Theorem 2.1 gives the following estimates of the rate of convergence of v_h^k toward v, the unique viscosity solution of (HJ).

Corollary 2.4. *Assume* (2.5), (2.6), (2.7), (2.14), *and* (2.15), *then for some positive real constant* C *and for any* $\lambda>L_g$ *and* $h\in[0,1/\lambda)$

$$\max_{x\in\bar\Omega}|v_h^k(x)-v(x)|\le Ch^{1/2}+\frac{L_f}{\lambda-L_g}\,k. \tag{2.20}$$

Moreover, if (2.11) *and* (2.12) *hold, for some positive real constant* C *and for any* $\lambda>2L_g$ *and* $h\in[0,1/\lambda)$

$$\max_{x\in\bar\Omega}|v_h^k(x)-v(x)|\le Ch+\frac{L_f}{\lambda-L_g}\,k. \tag{2.21}$$

Remark 2. Gonzales and Rofman [11] showed a constructive approach to the optimal stopping problem with impulse and continuous controls. In their approach the value function v is characterized as the maximum Lipschitz continuous function which satisfies a set of inequalities and approximate solutions w^k are found via a discretization in the state variable. More recently Menaldi and Rofman proved that the sequence w^k converges to the viscosity solution of the related Hamilton-Jacobi equation (see Rofman [15]).

In particular, in the stationary case and for an optimal stopping problem with continuous controls, Gonzales and Rofman obtained the following error estimate:

$$|w^k(x) - v(x)| \leq C(\log k)k^{1/2}, \qquad \forall x \in \Omega,$$

where k always denotes the maximum diameter of the simplices of the triangulation. For that particular problem our estimates (2.20) and (2.21) ameliorate their result since we can write the optimal stopping problem as an infinite horizon problem simply by adding one control, $\hat{a}$, to the set A and defining

$$g(x, \hat{a}) \equiv 0, \qquad \forall x \in \mathbb{R}^n,$$

$$f(x, \hat{a}) \equiv \lambda\psi(x), \qquad \forall x \in \mathbb{R}^n,$$

where ψ represents the stopping cost.

It is worth while observing that if ψ is bounded and Lipschitz continuous, which is the case in [11], then f defined in such a way still satisfies (2.6).

Let us briefly discuss the case of a general open bounded convex set Ω in $\mathbb{R}^n$. Let $\{S_j\}$ be a finite family of simplices which set up a regular triangulation of Ω and verify (2.15). Since all the vertices x_i belong to $\bar{\Omega}$ we have

$$\Omega_k \equiv \bigcup_j S_j \subset \bar{\Omega}. \tag{2.22}$$

Notice that Ω_k is an open polyhedron so that, in order to apply the previous results, we have to check that the following condition is satisfied:

$$\exists h > 0\colon x + hg(x, a) \in \bar{\Omega}_k, \qquad \forall x \in \bar{\Omega}_k, \quad \forall a \in A. \tag{2.23}$$

Clearly, (2.14) is not sufficient since $\Omega_k \subset \bar{\Omega}$.

The next result shows that (2.23) holds provided the mesh of the triangulation is small enough and a classical sufficient condition for the positive invariance of Ω is verified. In the proof we shall make use of the following definitions of the tangent cone, $T_Q(x)$, to a nonempty closed convex subset $Q \subset \mathbb{R}^n$ at $x \in Q$

$$T_Q(x) \equiv \text{cl}\left(\bigcup_{h \geq 0} \frac{1}{h}(Q - x) \right)$$

and of the normal cone $N_Q(x)$ to Q at $x \in Q$

$$N_Q(x) \equiv \{w \in \mathbb{R}^n\colon \langle w, z\rangle \leq 0, \forall z \in T_Q(x)\}.$$

Notice that they are closed convex cones with vertex at the origin and that

$$Q \subset x + T_Q(x).$$

We refer to Aubin and Cellina [1] for a general overview of the properties of tangent and normal cones.

Proposition 2.5. *Let Ω be an open bounded convex set of $\mathbb{R}^n$ and let its boundary be an $(n-1)$-dimensional manifold of class C^1. Let g be continuous and let* (2.5) *be verified. Assume*

$$\langle \eta(x), g(x, a)\rangle < c < 0, \qquad \forall x \in \partial\Omega, \quad \forall a \in A, \tag{2.24}$$

where $\eta(x)$ denotes the outward normal to Ω at the point x, then there exists a regular triangulation of Ω such that (2.23) *is verified.*

Proof. We start by noticing that for any $\varepsilon>0$ it is always possible, eventually refining the mesh, to have

$$\partial\Omega_k\subset\partial\Omega+\varepsilon B, \tag{2.25}$$

where B denotes the unit ball in $\mathbb{R}^n$.

Let F_i be a face of the triangulation belonging to $\partial\Omega_k$. In the relative interior of F_i, $\mathring{F}_i$, the outward normal to Ω_k is constant, let us denote it by η_i. We define the following subset of $\partial\Omega$:

$$F_i'\equiv\{x'\in\partial\Omega | x'=x+\varepsilon\eta, \text{for some } x\in\mathring{F}_i \text{ and } \varepsilon>0\}.$$

Since $\eta(\cdot)$ is continuous over $\partial\Omega$, without loss of generality we can always assume that there exists $y\in F_i'$ such that

$$\eta(y)=\eta_i. \tag{2.26}$$

We have

$$\langle\eta_i, g(x, a)\rangle=\langle\eta(y), g(y, a)\rangle+\langle\eta_i, g(x, a)-g(y, a)\rangle+\langle\eta_i-\eta(y), g(y, a)\rangle$$

then, for F_i and F_i' sufficiently near, namely,

$$\sup_{\substack{x\in F_i\\ x'\in F_i'}}|x-x'|<\varepsilon_1,$$

by the Lipschitz continuity of g we obtain

$$\langle\eta_i, g(x, a)\rangle<c+L_g\varepsilon_1$$

which implies

$$\langle\eta_i, g(x, a)\rangle<c_1<0, \qquad \forall x\in F_i, \quad \forall a\in A. \tag{2.27}$$

Let us now consider $x\in\bigcap_{i\in 1}^m \partial F_i$: it is evident that for such an x, (2.27) is verified for any $i=1,\ldots,m$. Since, for any $i=1,\ldots,m$,

$$\langle\eta_i, y-x\rangle\le 0, \qquad \forall y\in\Omega_k,$$

we will have

$$\langle\eta_i, v\rangle\le 0, \qquad \forall v\in T_{\bar\Omega_k}(x),$$

that is,

$$\eta_i\in N_{\bar\Omega_k}(x), \qquad i=1,\ldots,m.$$

In particular, we notice that $\eta_i\in\partial N_{\bar\Omega_k}(x)$, since for any $z\in F_i$ we have

$$\langle\eta_i, z-x\rangle=0$$

and $z-x$ clearly belongs to $T_{\bar\Omega_k}(x)$.

Moreover, it is easy to prove that, for $\lambda_i>0$,

$$\overline{\text{co}}\{\lambda_1\eta_1,\ldots,\lambda_m\eta_m\}=N_{\bar\Omega_k}(x) \tag{2.28}$$

(otherwise there will be at least one point $w = \lambda_i\eta_i + \lambda_j\eta_j$, $i,j = 1, \ldots, m$, which is not on the boundary of $N_{\bar{\Omega}_k}(x)$). Then, by (2.27), for any $x \in \partial\Omega_k$, and $a \in A$, we have

$$\langle w, g(x, a)\rangle < c_1, \qquad \forall w \in N_{\bar{\Omega}_k}(x),$$

that is, $g(x, a) \in \mathring{T}_{\bar{\Omega}_k(x)}$ (notice that $\mathring{T}_{\bar{\Omega}_k}(x)$ is clearly not empty since $\Omega_k \neq \emptyset$).

Let us now show that for any $x \in \partial\Omega_k$ and $a \in A$ there exists $h_{x,a} > 0$ such that

$$x + h_{x,a}g(x, a) \in \Omega_k. \tag{2.29}$$

In fact, suppose that

$$\forall h, \qquad x + hg(x, a) \notin \Omega_k,$$

we will have

$$g(x, a) \notin \frac{1}{h}(\Omega_k - x),$$

that is a contradiction since

$$\bigcup_{h>0} \frac{1}{h}(\Omega_k - x) = \mathring{T}_{\bar{\Omega}_k}(x).$$

By a compactness argument it can be proved that $h_{x,a}$ can be chosen independently of a.

For any $x \in \Omega_k$ (2.29) is verified since g is bounded. Since Ω_k is open, by the continuity of g we can affirm that for any $x \in \bar{\Omega}_k$ there exists a neighborhood V_x of x such that

$$\forall x' \in V_x \cap \bar{\Omega}_k, \qquad x' + h_x g(x, a) \in \Omega_k, \qquad \forall a \in A.$$

We define

$$\mathcal{O}_h \equiv \{x \in \bar{\Omega}_k : x + hg(x, a) \in \Omega_k, \forall a \in A\}.$$

The sets $\mathcal{O}_h$ are open and constitute a covering of $\bar{\Omega}_k$. So we can take out a finite covering $\{\mathcal{O}_{h_j}\}_{j=1,\ldots,p}$ and affirm that (2.23) is verified for $h = \min_j\{h_j\}$. □

3. The Algorithm

The proof of Theorem 2.2 immediately suggests an algorithm to compute an approximate solution of (HJ). Starting at any point $V_0 \in \mathbb{R}^N$, by applying the operator T_h we obtain the sequence

$$V_n = T_h(V_{n-1}) \tag{3.1}$$

which converges to V^*. Since V^* belongs to $B(0, M_f/\lambda)$, the closed ball with center 0 and radius M_f/λ, the following estimate holds:

$$|V_n - V^*| \leq (1 - \lambda h)^n \delta, \qquad \forall n, \tag{3.2}$$

where

$$\delta = \max_{V \in B(0, M_f/\lambda)} |V_0 - V|.$$

Let us denote by $v_h^{k,n}$ the affine function obtained by interpolating the vector V_n defined in (3.1), then, by Corollary 2.4, we have

$$\max_{x\in\bar{\Omega}}|v_h^{k,n}(x)-v(x)|\leq Ch^{1/2}+(1-\lambda h)^n\delta+\frac{L_f}{\lambda-L_g}k \tag{3.3}$$

for any $\lambda > L_g$ and $h\in[0, 1/\lambda)$ and, provided the semiconcavity assumptions (2.11) and (2.12) are satisfied,

$$\max_{x\in\bar{\Omega}}|v_h^{k,n}(x)-v(x)|\leq Ch+(1-\lambda h)^n\delta+\frac{L_f}{\lambda-L_g}k \tag{3.4}$$

for any $\lambda > 2L_g$ and $h\in[0, 1/\lambda)$.

In practice, for $h\simeq 0$ we expect the convergence of V_n to V^* to be very slow since, in that case, $(1-\lambda h)\simeq 1$. How to overcome this difficulty? Here we can apply a relaxation type algorithm proposed by Gonzales and Rofman [11] to solve a fixed point problem in finite dimension: this algorithm has given good results in their numerical tests. By means of a new formulation we shall prove that it can be reinterpreted as an acceleration method for the sequence generated by the contracting operator T_h and this will provide an error estimate.

Let us briefly recall, for the reader's convenience, some basic properties of the algorithm (to simplify notations in the following, we will write v^n and $\lambda_j(a)$ rather than $v_h^{k,n}$ and $\lambda_j(x_i, a)$):

(1) At the $(nN+i)$th step it modifies the ith component of $\bar{V}_n\equiv(\bar{v}^n(x_1),\ldots,\bar{v}^n(x_N))$ according to the iterative scheme

$$\bar{v}^{n+1}(x_i)\equiv\operatorname*{Min}_{a\in A}\left[(1-\lambda h)\left(\sum_{j=1}^{i-1}\lambda_j(a)\bar{v}^{n+1}(x_j)+\sum_{j=i}^{N}\lambda_j(a)v^n(x_j)\right)+hf(x_i,a)\right] \tag{3.5}$$

(for the original formulation see [11]).

(2) Let $M_h\colon\mathbb{R}^N\to\mathbb{R}^N$ be the operator defined component by component in (3.5), i.e., $M_h(\bar{V}_n)=(\bar{v}^{n+1}(x_1),\ldots,\bar{v}^{n+1}(x_N))$, then the sequence generated by the iterative scheme

$$\bar{V}_n=M_h(\bar{V}_{n-1}) \tag{3.6}$$

will converge to a fixed point for M_h, provided it starts at a point $\bar{V}_0$ such that

$$\bar{V}_0\leq(1-\lambda h)\Lambda(a)\bar{V}_0+hF(a),\qquad\forall a\in A. \tag{3.7}$$

The main characteristic of this algorithm is that once the ith component of $\bar{V}_n$ is modified it is immediately substituted in the right-hand side of (3.5) giving a contribution to the calculus of the $(i+1)$th component.

On the contrary, since by definition

$$V_{n+1}\equiv T_h(V_n)\equiv\operatorname*{Min}_{a\in A}[(1-\lambda h)\Lambda(a)V_n+hF(a)], \tag{3.8}$$

the ith component of V_{n+1} does not appear in the calculus of its $(i+1)$th component.

Let $\mathcal{V}$ denote the following subset of $\mathbb{R}^N$:

$$\mathcal{V} \equiv \{V \in \mathbb{R}^N : V \leq (1-\lambda h)\Lambda(a)V + hF(a), \forall a \in A\},$$

notice that $\mathcal{V}$ is a closed convex set.

The following proposition holds true.

Proposition 3.1. *For any starting point $V_0 \in \mathcal{V}$,*

(1) *$\{V_n\}$ and $\{\bar{V}_n\}$ are monotone nondecreasing;*
(2) *for any n, V_n and $\bar{V}_n$ belong to $\mathcal{V}$.*

Proof. The monotonicity of V_n is trivial, we only prove that $\bar{V}_n$ is monotone.

We have

$$v^0(x_1) \leq (1-\lambda h) \sum_{j=1}^{N} \lambda_j(a) v^0(x_j) + hf(x_1, a), \qquad \forall a \in A,$$

then $v^0(x_1) \leq \bar{v}^1(x_1)$. Since

$$v^0(x_j) \leq \bar{v}^1(x_j), \qquad \forall j < i,$$

implies

$$v^0(x_i) \leq (1-\lambda h)\left[\sum_{j=1}^{i-1} \lambda_j(a)\bar{v}^1(x_j) + \sum_{j=i}^{N} \lambda_j(a) v^0(x_j)\right] + hf(x_i, a), \qquad \forall a \in A,$$

we have

$$v^0(x_i) \leq \bar{v}^1(x_i), \qquad \forall i = 1, \dots, N,$$

that is, $V_0 \leq \bar{V}_1$.

Since $\bar{V}_{n-1} \leq \bar{V}_n$ implies

$$\bar{v}^n(x_1) \leq (1-\lambda h) \sum_{j=1}^{N} \lambda_j(a)\bar{v}^{n-1}(x_j) + hf(x_1, a)$$

$$\leq (1-\lambda h) \sum_{j=1}^{N} \lambda_j(a)\bar{v}^{n}(x_j) + hf(x_1, a), \qquad \forall a \in A,$$

we have $\bar{v}^n(x_1) \leq \bar{v}^{n+1}(x_1)$. Then, repeating the same argument that we have used before, we prove that

$$\bar{v}^n(x_i) \leq \bar{v}^{n+1}(x_i), \qquad \forall i = 1, \dots, N,$$

and by induction on n we end the proof.

The fact that $\{V_n\}$ and $\{\bar{V}_n\}$ stay in $\mathcal{V}$ immediately follows by the monotonicity. □

Proposition 3.1 clearly imply that V^* is the maximum element of $\mathcal{V}$, as it is the limit of all monotone sequences generated by T_h starting at any $V_0 \in \mathcal{V}$.

The following proposition shows that $\bar{V}_n$ also converges to V^*.

Proposition 3.2. *Let* $\{V_n\}$, $\{\bar{V}_n\}$ *be the sequences respectively defined in* (3.6) *and* (3.8), *the following inequality holds*

$$V_n \le \bar{V}_n \le V^*, \qquad \forall n. \tag{3.9}$$

Proof. The second inequality is trivial. We start by proving that the first inequality in (3.9) holds true for $n=1$. Since $v^1(x_1)=\bar{v}^1(x_1)$ and $v^0(x_1)\le \bar{v}^1(x_1)$ we have

$$\begin{aligned} v^1(x_2) &\le (1-\lambda h)\sum_{j=1}^{N} \lambda_j(a)v^0(x_j)+hf(x_2,a) \\ &\le (1-\lambda h)\lambda_1(a)\bar{v}^1(x_1)+\sum_{j=2}^{N}\lambda_j(a)v^0(x_j)+hf(x_2,a), \qquad \forall a\in A, \end{aligned}$$

that is, $v^1(x_2)\le \bar{v}^1(x_2)$. By iterating we prove that $V_1\le \bar{V}_1$.

Let us now show that $V_n \le \bar{V}_n$ implies $V_{n+1}\le \bar{V}_{n+1}$. In fact, for any i, we have

$$\begin{aligned} v^{n+1}(x_i) &= \operatorname*{Min}_{a\in A}\left[(1-\lambda h)\sum_{j=1}^{N}\lambda_j(a)v^n(x_j)+hf(x_i,a)\right] \\ &\le \operatorname*{Min}_{a\in A}\left[(1-\lambda h)\sum_{j=1}^{N}\lambda_j(a)\bar{v}^n(x_j)+hf(x_i,a)\right] \\ &\le \operatorname*{Min}_{a\in A}\left\{(1-\lambda h)\left[\sum_{j=1}^{i-1}\lambda_j(a)\bar{v}^{n+1}(x_j)+\sum_{j=i}^{N}\lambda_j(a)\bar{v}^n(x_j)\right]+hf(x_i,a)\right\}. \end{aligned}$$

□

Then defining $v_h^{k,n}$ as the affine function obtained by interpolating $\bar{V}_n$ instead of V_n we obtain a better approximation of the viscosity solution v of (HJ). Notice that the error estimates (3.3) and (3.4) remain valid.

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