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A Simple Linear Model for the Optimal Exploitation of Renewable Resources

Colin W. Clark* and John D. De Pree*

Department of Mathematics, University of British Columbia, Vancouver, Canada V6T 1W5

Department of Mathematics, New Mexico State University, Las Cruces, New Mexico, U.S.A. 88001

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Abstract. This paper describes a simple linearized model for the optimal control of a natural resource stock. Applications of the model to fisheries and forestry, as well as to mineral exploration and recovery, are discussed.

1. Introduction

In this article we discuss a simple but quite general deterministic model of renewable resource exploitation. Let $x=x(t)$ be a real variable representing the biomass of a given biological resource at time $t \geqslant 0$. The fundamental equation of our model is

$$\frac{dx}{dt} = F(x,t) - h(t), \tag{1}$$

where $F(x,t)$ is a given function describing the natural growth rate of the resource biomass, and where $h(t) \geqslant 0$ denotes the exploitation rate, or rate of "harvesting." The initial population $x(0)=x_0$ is assumed to be known.

By specializing the function $F(x,t)$ in various ways, we obtain a variety of models that have been used by previous authors. For example, the logistic growth function $F(x,t)=rx(1-x/K)$ leads to the Schaefer model [12], often used in fisheries management. An alternative fisheries model, due to Beverton

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and Holt [2], utilizes a growth function of the form $F(x,t)=g(t)x$. The function $F(x,t)=at^{-b}x^{c}e^{-dx}$, which has been used in forestry [10] will be discussed in detail later in the paper. Finally the case $F(x,t)\equiv 0$ provides a model of exhaustible resources [8].

The question of determining what criterion to utilize in defining a concept of optimal resource exploitation is always a difficult one. Without wishing to get involved in moral or political questions, we shall here adopt the standard economic criterion of maximizing the present value of net revenues.

Let p denote the unit value (price) of the harvested resource, and let c denote unit harvesting cost. We write

$$Q = p - c \tag{2}$$

for the unit net revenue, and allow Q to depend on both the time t and the stock level x at time t:

$$Q = Q(t,x).$$

Our basic assumption, however, is that Q is independent of the harvest rate $h(t)$. As we shall see, this assumption leads to a linear control problem of a particularly simple nature.

If $\delta = \delta(t)$ denotes the instantaneous discount rate in effect at time t, then the appropriate discount factor for discounting revenues back to time zero is clearly equal to

$$\alpha(t) = \exp\left(-\int_0^t \delta(\tau)\,d\tau\right). \tag{3}$$

We can therefore write the present value of net revenues corresponding to a given harvest policy $h(t)$, $t \geqslant 0$, as

$$\text{P.V.} = \int_0^\infty \alpha(t)Q(t,x)h(t)\,dt. \tag{4}$$

We then define an optimal harvest policy to be an admissible function $h^*(t) \geqslant 0$ that maximizes (4) over a certain class of admissible controls, subject to the differential equation (1) and to the obvious constraint

$$x(t) \geqslant 0. \tag{5}$$

By an admissible control we mean any measurable function $h(t)$ defined for $t \geqslant 0$ and satisfying the inequalities

$$0 \leqslant h(t) \leqslant h_{\max} \tag{6}$$

where $h_{\max} = h_{\max}(x,t)$ is a given nonnegative function representing the maximum harvest capacity. (In the final section of the paper we shall extend our model so that $h_{\max}$ also becomes a decision variable.)

We allow the possibility that $h_{\max} = +\infty$, in which case the class of admissible controls must be extended to include impulse (delta-function) controls, which result in discontinuous jumps in the state variable x.

The paper is organized as follows. In the next section we solve the above control problem by a completely elementary argument, under suitable smoothness assumptions and subject to an important uniqueness assumption. We then apply this solution to derive several known results, for models of optimal fishing, capital accumulation and mineral exploration. Next we apply the theory to a model of optimal forest rotation and thinning, and present some numerical results. In the final section we generalize the model to include the problem of optimal investment policy.

2. Solution of the Control Problem

The control variable $h(t)$ can be eliminated from the present-value integral (4) by substitution from Equation (1), leading to an expression of the form

$$J(x) = \int_{t_0}^{t_1} \{ G(x,t) + H(x,t)\dot{x} \} \, dt, \tag{7}$$

where $t_0 = 0$ and $t_1 = \infty$. We shall now assume, however, that t_0 and t_1 are arbitrary, but finite; this assumption is easily relaxed later. The constraints (6) can now be expressed in the form

$$A \leqslant \dot{x}(t) \leqslant B \tag{8}$$

where A and B are given functions of t and x. The functions A, B, G, H are assumed sufficiently smooth—continuity of A, B and continuous differentiability of G, H suffices for our purpose.

We seek to maximize $J(x)$ over the class of piecewise smooth functions $x(t)$ defined for $t_0 \leqslant t \leqslant t_1$ satisfying the constraints (5) and (8) and end-point conditions

$$x(t_0) = x_0, \qquad x(t_1) = x_1. \tag{9}$$

This problem, the so-called singular case of the simplest problem of the calculus of variations, is easily solved by elementary means (cf. [7]). Consider the equation

$$\frac{\partial G}{\partial x} = \frac{\partial H}{\partial t} \tag{10}$$

(which is in fact the Euler-Lagrange equation for our problem). We shall assume that Equation (10) determines a unique function $x = x^*(t)$ for $t_0 \leqslant t \leqslant t_1$, and that x^* is piecewise smooth. We also suppose that $x^*(t)$ satisfies the constraints (5) and (8); problems in which these various conditions are violated can also be handled by the methods given here, with suitable modifications. In the terminology of control theory, $x^*(t)$ is called the singular solution.

Finally we assume that for all t,

$$\frac{\partial G}{\partial x} \gtrless \frac{\partial H}{\partial t} \quad \text{whenever} \quad x \lessgtr x^*(t). \tag{11}$$

By continuity of G, H, and by our uniqueness assumption, if (11) is not valid then the reverse inequalities must hold for all t. In the latter case it can be seen that $x^*(t)$ is a minimizing solution, rather than the desired maximizing solution.

The solution to our variational problem can now be described in the following terms. The singular solution $x^*(t)$ is the *optimum optimorum* level for the variable x. If $x^*(t_0) \neq x_0$, then $x(t)$ should be controlled initially so as to approach x^* as rapidly as possible:

$$\dot{x} = \begin{cases} A(x,t) & \text{whenever} \quad x(t) > x^*(t) \\ B(x,t) & \text{whenever} \quad x(t) < x^*(t). \end{cases} \tag{12}$$

In terms of the original harvesting model this means that the optimal policy $h(t)$ is given by

$$h(t) = \begin{cases} h_{\max} & \text{whenever} \quad x(t) > x^*(t) \\ 0 & \text{whenever} \quad x(t) < x^*(t). \end{cases} \tag{13}$$

Of course the singular control $h^*(t) = F(x^*(t)) - \dot{x}^*(t)$ is required whenever $x(t) = x^*(t)$. Such a control policy is called a bang-bang-singular policy.

A similar terminal adjustment phase is required in case $x^*(t_1) \neq x_1$. An example is illustrated in Figure 1. Here we have $x_0 < x^*(0)$, so that the initial adjustment phase PQ is determined by $\dot{x} = A(x,t)$, i.e. $h(t) = 0$. The terminal adjustment ST in this case also uses the same control limit $A(x,t)$. The optimal solution $x(t)$ is the curve PQRST.

To establish the optimality of this solution, consider an alternative possibility $x_a(t)$, such as the dashed curve of Figure 1. We then have

$$\int_{t_0}^{t_a} \{G(x,t) + H(x,t)\dot{x}\}\,dt - \int_{t_0}^{t_a} \{G(x_a,t) + H(x_a,t)\dot{x}_a\}\,dt$$

$$= \oint_{\mathrm{PQRP}} \{G(x,t)\,dt + H(x,t)\,dx\}$$

$$= -\iint_{\mathrm{PQR}} \left\{ \frac{\partial H}{\partial t} - \frac{\partial G}{\partial x} \right\} dt\,dx > 0$$

by Green's theorem and by hypothesis (11). The same calculation applies for $t_a \leqslant t \leqslant t_1$, and we conclude that $J(x) > J(x_a)$. It is clear how to modify the argument to cover an arbitrary admissible curve $x_a(t) \not\equiv x(t)$. Thus the curve $x(t)$ is indeed the optimal solution to our problem.

For the case $t_1 = \infty$ we require hypotheses ensuring the boundedness of the integral (4). We therefore suppose that $\delta(t) \geqslant \delta_0 > 0$ for all t, that any feasible

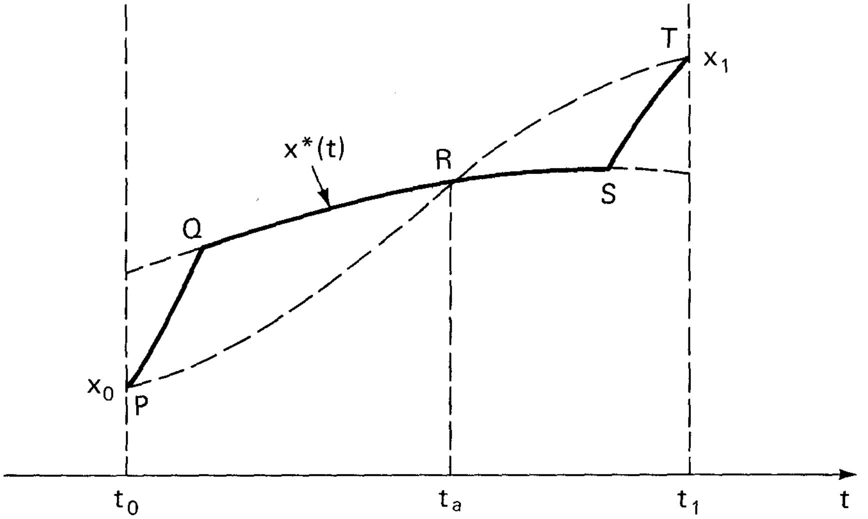

Fig. 1.

trajectory $x(t)$ is bounded, and that $Q(t,x)$ is also bounded. The above argument then shows that the optimal harvest policy consists of an initial bang-bang adjustment, as before, after which the singular path is followed *ad infinitem.*

This definitive characterization of optimal trajectories for the linear infinite time-horizon problem is of some interest in view of the well known difficulties in the nonlinear case. We also observe that the optimal policy in the linear case is expressed in a simple feedback form, another distinctive advantage over the nonlinear case.

3. Some Applications

We now apply our model to derive a number of known resource-exploitation policies. Applying Equation (10) to the present-value integral (4), we obtain the following basic equation for the singular path $x^*(t)$:

$$\frac{\partial F}{\partial x} + \frac{F}{Q}\frac{\partial Q}{\partial x} = \delta(t) - \frac{1}{Q}\frac{\partial Q}{\partial t} \tag{14}$$

where

$$Q(t,x) = p(t) - c(t,x)$$

denotes the net revenue from a unit harvest when the biomass level equals x. A detailed economic analysis of Equation (14) is given in [4].

Maximum Sustained Yield

Suppose that

$$F(x,t) = f(x) \quad \text{where} \quad f(0) = f(K) = 0 \quad \text{and} \quad f''(x) < 0.$$

Take a finite time horizon T, set $\delta(t)\equiv 0$ and $Q(t,x)\equiv 1$. The problem then is simply to maximize the total yield Y over the given time interval:

$$Y = \int_0^T h(t)\,dt.$$

Equation (14) becomes simply $f'(x^*)=0$, i.e. x^* is the population level at which the biological production rate $f(x)$ is maximized. This is referred to as the level of maximum sustained yield.

If $x_0>x^*$, the optimal policy is to harvest the excess stock at the fastest possible rate, subsequently maintaining maximum sustainable yield. If $x_0<x^*$, no harvest should be taken until the stock has grown to the level x^*.

Maximum sustained yield is the most popularly accepted concept of optimal renewable resource harvesting. It is the basis for almost every existing international agreement in marine fisheries. Yet economists have been severely critical of the concept for its failure to consider the costs associated with harvesting. The following model incorporates such costs.

The Gordon-Schaefer Fishery Model [*5*], [*12*]

We now specify

$$f(x) = rx\left(1-\frac{x}{K}\right). \tag{15}$$

Assume a constant price p, and suppose that the cost of a unit harvest is inversely proportional to the stock level x from which the harvest is taken. (The latter assumption stems from a supposition that the fishermen search at random for a uniformly distributed fish stock.) Thus

$$Q(t,x) = Q(x) = p-\frac{c}{x} \qquad \text{where} \quad c = \text{constant}. \tag{16}$$

Gordon [5] suggested that optimal fishery management should result, not in the maximization of biological yield, $f(x)$, but rather in the maximization of net sustained economic yield, $Q(x)f(x)$. This leads to

$$x^* = \frac{K}{2}+\frac{c}{2p}, \tag{17}$$

which is also the singular solution for the integral of economic yield:

$$Y_{\text{ec}} = \int_0^T Q(x)h(t)\,dt. \tag{18}$$

As observed by Gordon, this solution is more conservative than the traditional maximum sustained yield solution, since $x^*>K/2=$level of maximum yield. When discounted yields are considered, however, the conclusion is no longer valid.

The Gordon-Schaefer Model with Discounting [3], [4]

Assume a constant discount rate $\delta > 0$ and consider the present value

$$\text{P.V.} = \int_0^\infty e^{-\delta t} Q(x) h(t)\, dt.$$

The optimal biomass level (i.e. the singular solution) is now given by

$$f'(x^*) + \frac{Q'(x^*)f(x^*)}{Q(x^*)} = \delta, \tag{19}$$

which can be solved for $x^* = x_\delta^*$:

$$x_\delta^* = \frac{K}{4}\left\{\left[\frac{c}{pK} + 1 - \frac{\delta}{r}\right] + \left\{\left[\frac{c}{pK} + 1 - \frac{\delta}{r}\right]^2 + \frac{8c\delta}{pKr}\right\}^{1/2}\right\}.$$

When $\delta = 0$ this reduces to the Gordon solution (17). But x_δ^* is a decreasing function of δ, and large discount rates may thus result in optimal population levels below the maximum yield level. High discount rates thus lead to reduced resource conservation [3].

Optimal Economic Growth

Let $k(t)$ denote the capital-to-labour ratio in a certain economy, and let $c(t)$ denote the rate of consumption. Let $U(c)$ denote utility of consumption, and consider the problem of maximizing total discounted utility

$$J = \int_0^\infty e^{-\delta t} U(c(t))\, dt$$

subject to

$$\frac{dk}{dt} = f(k) - c(t); \qquad k(t) \geqslant 0.$$

This is the celebrated Ramsey-Cass-Samuelson-Solow model of optimal economic growth—see [9]. In the special but rather uninteresting case that $U(c)$ is a linear function, $U(c) = \alpha c$, we may apply our simple linear theory, obtaining a singular solution $k(t) = k^*$ determined by

$$f'(k^*) = \delta. \tag{20}$$

This equation, which also holds for equilibrium ("turnpike") solutions in the nonlinear case, is called the modified golden rule of capital accumulation.

It is clear that our fundamental rule for singular solutions, Equation (14), is merely an extension of the golden rule (20) necessitated by the general nature of our model. Each term in Equation (14) has a straightforward economic interpretation [4].

Mineral Exploration

The following model of mineral exploration has been proposed by R. Uhler [14]. Let $z = z(t)$ denote the cumulative effort devoted to exploration for petroleum (for example) over a given geographical area. Empirical studies suggest that the rate of discovery varies with z, increasing at first as geological structures become better known, but ultimately decreasing as undiscovered reserves become more scarce. We suppose that the net economic yield from exploration is given by

$$pD(z)v(t) - cv(t) = Q_1(z)v(t)$$

where $v = \dot{z}$ is the rate of exploration, where $D(z)v(t)$ is the rate of discovery, and where $p =$ unit value of discovered reserves and $c =$ unit cost of exploration. The objective is to maximize

$$\text{P.V.} = \int_0^\infty e^{-\delta t} Q_1(z)v(t)\,dt$$

subject to $\dot{z} = v$.

Equation (10) for the singular solution z^* is simply

$$Q_1(z^*) = pD(z^*) - c = 0. \tag{21}$$

This seems rather banal: exploration should proceed (at the maximum rate $v_{\max}$) until the revenues no longer repay the costs of exploration. An interesting complication arises, however, in the case that Equation (21) possesses two solutions $z_1^* < z_2^*$ (Fig. 2). In this case, the initial exploration incurs a loss, but leads to the accumulation of knowledge from which later benefits accrue. Clearly exploration is advisable only if the present value of the ultimate benefits exceeds the present value of the initial exploration costs. Thus the optimal level of cumulative effort is either $z = 0$ or $z = z_2^*$, depending on the circumstances.

We have included this simple example to show that the linear model can be applied in many cases where the hypotheses listed in Section 2 fail. Typically a finite number of candidates for optimal solution can be easily identified, and the choice of the *optimum optimorum* is then reduced to a numerical problem.

Mining

The following model for the mining of known reserves originates with Hotelling [8]; see also [6], [13]. Let $x(t)$ denote the reserves remaining at time t, and let $h(t) \geqslant 0$ denote the rate of production. Then

$$\frac{dx}{dt} = -h(t)$$

and the objective functional is

$$\text{P.V.} = \int_0^T e^{-\delta t}\{p(t) - c(x)\}h(t)\,dt; \qquad T \leqslant +\infty,$$

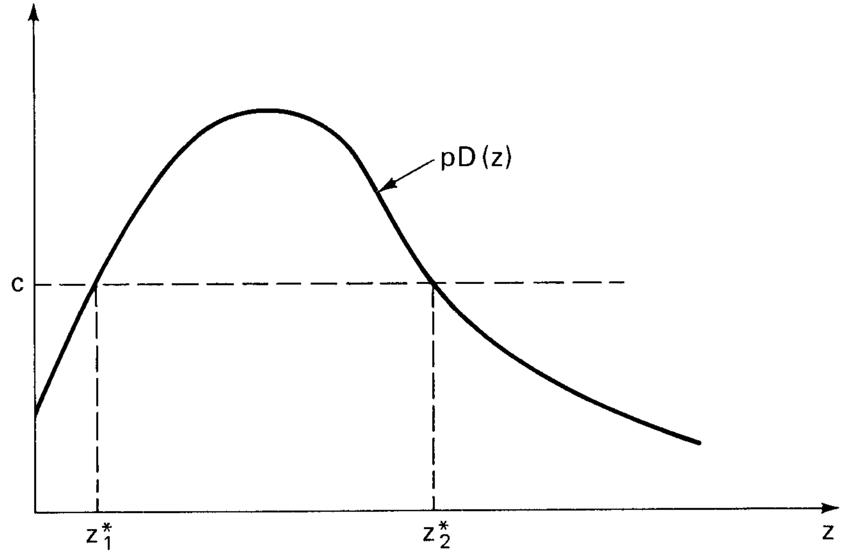

Fig. 2.

where $p(t)$ is the anticipated price level at time t, and $c(x)$ denotes the unit production cost. There is an additional constraint of the form

$$x(T) \geqslant 0, \qquad \text{i.e.} \qquad \int_0^T h(t) \leqslant K = x(0).$$

Equation (14) reduces to

$$\frac{\dot{p}(t)}{p(t)-c(x)} = \delta.$$

If the unit production cost $c(x)$ is a strictly decreasing function of x, this equation defines a unique singular path $x = x^*(t)$. If $c(x)$ is a constant independent of x, however, then all production takes place at the instant T at which $\dot{p}(T)/\{p(T)-c\} = \delta$.

4. Forestry Management: Optimal Thinning and Rotation

In a study of optimal thinning and rotation of pine forests in Finland, Kilkki and Vaisanen [10] developed a nonlinear discrete-time model that could be analyzed numerically by standard techniques of dynamic programming. In this section we shall describe a linearized continuous-time version of the Kilkki-Vaisanen model, which can then be studied by means of the methods described above. We also take the opportunity to correct an apparent error in the Kilkki-Vaisanen work.

Let $x(t)$ denote the volume of salable timber in a given forest stand of age

$t \geq t_0 \geq 0$. Assume a forest-growth equation of the form

$$\frac{dx}{dt} = g(t)f(x) \qquad (t \geq t_0), \tag{22}$$

where $g(t)$ is a positive decreasing function of t, and $f(x)$ is a positive function with a unique maximum at $x = x_{max}$. In the Kilkki-Vaisanen model these functions are of the form

$$g(t) = at^{-b}; \qquad f(x) = xe^{-cx} \tag{23}$$

where a, b, c are positive constants.

Thinning is a process that simply reduces the volume $x(t)$ of standing timber. (In the forestry literature this is called "thinning from above," and contrasts with "thinning from below," which is a selective process of removing low-value trees, thereby enhancing the value of the remaining stand.) If $h(t) \geq 0$ denotes the rate of thinning, Equation (22) becomes

$$\frac{dx}{dt} = g(t)f(x) - h(t) \qquad (t \geq t_0). \tag{24}$$

The forest is assumed to be thinned at a rate $h(t) \geq 0$ (to be determined) for $t_0 \leq t \leq T$, and the remaining stand is then clearcut at age T, when a new rotation commences. Both the thinning policy $h(t)$ and the rotation period T are considered as decision variables. The net present value of a single rotation, discounted to time $t = 0$, is given by

$$P_1(h) = \int_{t_0}^{T} e^{-\delta t} R(t) h(t)\, dt + e^{-\delta T} q(T) x(T) \tag{25}$$

where

$R(t)$ = unit net revenue from thinning

$q(T)$ = unit net value of timber at age T.

We assume in general that, since clearcutting costs may be different from thinning costs, $q(T) \neq R(T)$.

We shall suppose that the second, and all subsequent, rotations are characterized by the same growth functions, and the same economic relations, as the first rotation. Maximization of the present value of all future rotations then implies that the optimal thinning and rotation policy will be the same for each rotation. Thus the total present value is given by

$$\text{P.V.} = \sum_{k=0}^{\infty} P_1 e^{-k\delta T} = \frac{P_1}{1 - e^{-\delta T}}. \tag{26}$$

In order to determine the optimal policy, we first treat T as a parameter, and determine the optimal thinning policy $h^*(t)$ for $t_0 \leq t \leq T$. If $P_1^*(T)$ denotes $\max P_1(h)$, we then maximize $P_1^*(T)$ with respect to T.

The singular solution $x^*(t)$ for (24), (25) satisfies

$$f'(x^*) = \frac{1}{g(t)}\left[\delta - \frac{R'(t)}{R(t)}\right]. \tag{27}$$

If $f'(x)$ is monotone, this equation determines a unique singular path $x^*(t)$. The optimal thinning policy then consists of an initial bang-bang adjustment beginning at $t = t_0$, followed by a singular control, followed in turn by a terminal bang-bang phase.

The latter phase is not determined by a given terminal value $x(T) = x_T$ as in Equation (9), but is instead characterized by an appropriate transversality condition. Intuitively it is clear that if (as we shall henceforth assume)

$$q_T > R(T)$$

then thinning should not be carried out for t near T because greater revenues can be obtained by clearcutting. Thus we will have

$$h^*(t) = 0 \qquad \text{for} \quad t_1 \leqslant t \leqslant T.$$

See Figure 3.

The information obtained so far is readily translated into an effective numerical scheme for determining the optimal thinning and rotation policy. First the singular path $x^*(t)$ is computed. Assuming next that $x(t_0) = x^*(t_0)$, we

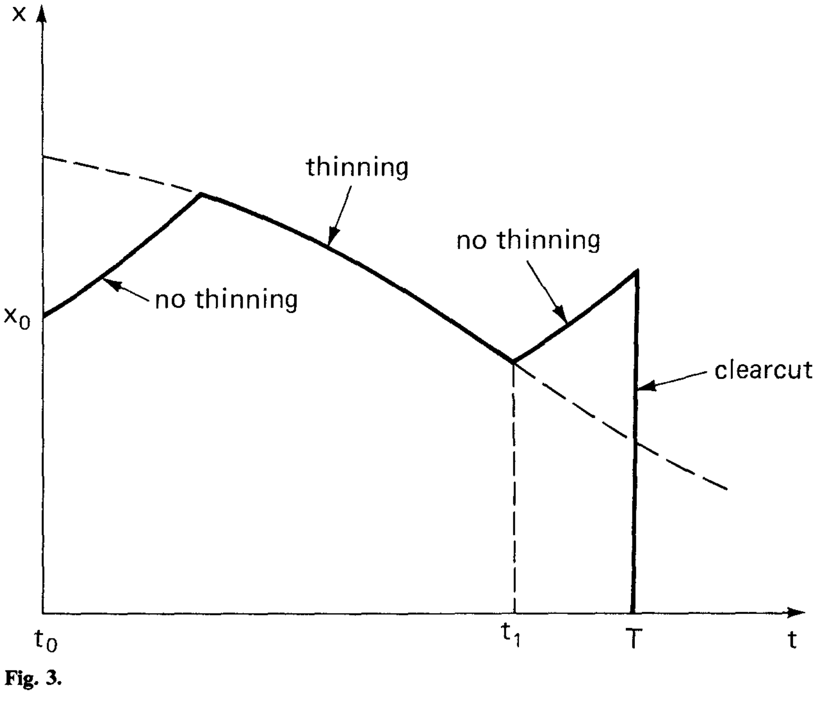

Fig. 3.

then determine t_1 so that $P_1(h)$ is maximized, with $h(t)=h^*(t)$ for $t<t_1$ and $h(t)=0$ for $t_1 \leqslant t \leqslant T$. At this stage, $P_1^*(T)$ is known, so that T^*, the time of clearcutting that maximizes $P_1^*(T)$, can then be determined.

If $x(t_0) \neq x^*(t_0)$, this procedure may require modification if the control constraints prevent $x(t)$ from reaching the singular path before $t=t_1$. Of practical importance is the case in which $x(t_0)$ is below $x^*(t_0)$ and $x(t_1)$ is still below $x^*(t_1)$. Clearly an optimal policy in this case does not involve any thinning at all, so the only problem is to determine the optimal age of rotation T^*. For this, the well known Faustmann formula suffices:

$$\frac{V'(T^*)}{V(T^*)} = \frac{\delta}{1-e^{-\delta T^*}}$$

where $V(T)=q(T)x(T)$ denotes the net revenue obtained from clearcutting at age T.

The above method was used to determine an optimal thinning and rotation policy based on the following model used by Kilkki and Vaisanen [10]:

$$\frac{dx}{dt} = 56.9t^{-1.23}xe^{.003x}.$$

Here x denotes the volume of usable timber in a one-hectare stand of Scotch pine of age t years. A linear approximation to the cost and price data used by these authors yielded a net revenue function

$$R(t) = .337t + 5.06 \qquad (Fmk/m^3)$$

Following Kilkki and Vaisanen, we supposed that thinning and clearcutting could occur only at five-year intervals, $t=50, 55, 60, \ldots$. We thus discretized our continuous-time model accordingly.

The results, obtained on an IBM 370 computer, are shown in the tables. Table I gives the singular paths $x^*(t)$ corresponding to a range of discount rates δ from 0 to 8% per annum. The sensitivity of these results to the values of δ is noteworthy, and is typical of optimal forest management policies.

Table I. Singular path $x^*(t)$ (m^3/hectare).

Discount Rate δ	Age (t), years															
	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125
.01	397	389	380	371	362	352	343	333	324	315	306	297	288	280	271	263
.02	293	280	266	254	241	229	217	205	194	183	173	163	153	143	134	125
.03	229	213	197	183	169	155	142	129	117	106	94	84	73	63	53	44
.04	181	163	147	131	116	101	87	74	61	49	37	26	15	5	—	—
.05	142	124	106	89	74	58	44	30	17	4	—	—	—	—	—	—
.06	110	90	72	55	38	23	8	—	—	—	—	—	—	—	—	—
.07	82	62	43	25	8	—	—	—	—	—	—	—	—	—	—	—
.08	57	36	17	—	—	—	—	—	—	—	—	—	—	—	—	—

Table II. Optimal rotation age, T^* (years) and optimal age, t_1, of final thinning (years). (A blank implies no thinning.)

Discount Rate	Cost Ratio (ρ)					
	1.0		1.1		$\geqslant 1.2$	
δ	t_1	T^*	t_1	T^*	t_1	T^*
.01	95	100	60	85	—	80
.02	85	90	60	75	—	70
.03	75	80	55	70	—	65
.04	70	75	—	60	—	60
.05	60	65	—	55	—	55
.06	55	60	—	55	—	55
.07	—	50	—	50	—	50
.08	—	50	—	50	—	50

Table II shows the optimal rotation age T^*, and the age t_1 of final thinning, for the same range of discount rates δ. In this table the parameter ρ represents the ratio of thinning costs to clearcutting costs. For example, when thinning and clearcutting costs are equal ($\rho = 1.0$), optimal thinning continues until the last period prior to clearcutting: $t_1 = T^* - 5$. When $\rho \geqslant 1.2$, on the other hand, thinning is never profitable. Notice that, when thinning is profitable, it has the effect of increasing the length of the optimal rotation period T^*. As in Table I, the sensitivity of the optimal thinning and rotation policy to the discount rate δ is apparent.

While somewhat simpler to interpret, our numerical results cannot be compared directly with those of Kilkki and Vaisanen, because of the linearization assumed in our treatment. Moreover, our method also differs in the choice of the objective functional (26), representing total discounted revenues over an infinite time horizon. Instead of this, Kilkki and Vaisanen assumed a fixed "site-value" S, and maximized the single rotation value

$$P_1(h) + Se^{-\delta T}$$

where $P_1(h)$ is given by (25). Since the value of a forest site depends (critically) upon the proposed utilization, this procedure appears to be incorrect.

5. Optimal investment policy

Another significant aspect of resource exploitation that can be studied on the basis of our linear model is the question of optimal investment in harvesting equipment. We shall discuss this problem briefly on the basis of the Gordon-Schaefer fishery model described in Section 3.

Our analysis leads to the standard rule of investment planning: the optimal level of investment is such that the marginal present value of returns equals the marginal cost of investment. In renewable resource management, however, this rule has a perhaps unexpected corollary: the optimal level of investment during the initial harvest stage may be *greater* than the level required for optimal sustained yield. If investment is irreversible, the optimizing manager may thus wind up with an excess of harvesting capacity.

Let $E = E(t)$ denote the rate of fishing "effort" utilized in a certain fishery. It is normal to assume that the resulting rate of harvest $h(t)$ is given by

$$h(t) = qE(t)x(t) \tag{28}$$

where q is a positive constant. Considering now $E(t)$ as the basic control variable, we may express the optimization problem in the form

$$\text{maximize} \qquad \int_0^\infty e^{-\delta t}\{px(t) - c\}E(t)\,dt$$

$$\text{subject to} \qquad \frac{dx}{dt} = f(x) - qEx \qquad (t \geqslant 0)$$

$$x(0) = x_0;\ x(t) \geqslant 0$$

$$0 \leqslant E(t) \leqslant E_{\max}.$$

Now let $K = K(t)$ denote the level of capital invested in fishing equipment. We shall assume that the maximum rate of effort, $E_{\max}$, is proportional to K:

$$E_{\max} = \alpha K \tag{29}$$

where α is a positive constant, called the coefficient of capital effectiveness. We now consider K, and concomitantly $E_{\max}$, as an additional decision variable. Essentially this allows for a separate treatment of fixed costs and variable costs in our fishery model.

Irreversibility of invested capital seems to be a characteristic of resource industries. (See Arrow [1] for a general treatment of irreversible investment.) In many cases, rapid acquisition of equipment may be feasible, but disinvestment may be difficult since unwanted equipment may have no value other than as scrap. We shall model these observations in an extreme form by assuming that

$$0 \leqslant \frac{dK}{dt} = I(t) \leqslant +\infty. \tag{30}$$

where $I(t)$ denotes the rate of investment.

The net present value corresponding to a given investment schedule $I(t)$, and given harvest policy $h(t) = qx(t)E(t)$ is given by

$$NPV = \int_0^\infty e^{-\delta t}\{(px(t) - c)E(t) - I(t)\}\,dt \tag{31}$$

It is intuitively clear that the optimal investment policy is a bang-bang policy, and this can easily be verified via the Pontrjagin maximum principle [11]. Let x^* denote the optimal population level as determined by Equation (19). If $x(0) > x^*$ it can be seen that the optimal investment policy utilizes an impulse control at time $t=0$ to adjust $K(0)$ to some optimal level $K^* \geqslant K(0)$. The problem thus reduces to the determination of K^*.

Let $PV(E_{max})$ denote the present value of the optimal harvest policy, given that $E(t) \leqslant E_{max}$. Then (given that $x(0) > x^*$) $PV(E_{max})$ is a strictly increasing function of E_{max}, and is bounded by $PV(+\infty) < +\infty$. Given K^* we have $E_{max} = \alpha K^*$ and hence

$$NPV = PV(E_{max}) - \frac{1}{\alpha} E_{max} + K(0).$$

The problem is then to choose $K^* = \alpha^{-1} E_{max}$ so as to maximize this expression. Assuming differentiability, we see that the solution satisfies

$$\frac{dPV}{dE_{max}} = \frac{1}{\alpha}, \quad \text{or} \quad \frac{dPV}{dK} = 1 \tag{32}$$

i.e. the optimal value of K is such that marginal increase in present value equals marginal cost of investment.

The solution now falls into three distinct cases, depending on the capital-effectiveness coefficient α. If α is small, Equation (32) will have no solution, so that $K^* = 0$ is the optimum investment level. If $K(0) = 0$, the fishery will not be exploited at all because fixed costs are too high.

If α is of moderate size, then (32) will have a solution $E_{max} = E^*_{max}$ where $0 < E^*_{max} < E^* = f(x^*)/qx^*$. If $K(0) \leqslant \alpha^{-1} E^*_{max}$, the fishery will be exploited, but at a lower level than would occur if only variable costs were relevant. Of course if $K(0)$ is sufficiently large, i.e. if $K(0) \geqslant \alpha^{-1} E^*$, then the fishery will be exploited at the level E^*.

Finally, when capital effectiveness is high, we obtain $E^*_{max} > E^*$. The fishery is exploited initially at an effort level in excess of the optimal sustained level. Once the population has been reduced to x^*, the level of effort must also be reduced from E^*_{max} to E^*. This is true even though the excess capital

$$K_e = \frac{1}{\alpha}(E^*_{max} - E^*)$$

cannot be reduced. In other words, an optimally managed fishery in which investment is irreversible may reach an equilibrium position with excess equipment that should not be utilized.

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