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Isoparametric families on projective spaces

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Introduction

Many authors investigated real hypersurfaces of $\mathbb{C}P^m$. In particular, Takagi classified homogeneous real hypersurfaces in $\mathbb{C}P^m$ and proved that the number of distinct principal curvatures of a homogeneous real hypersurface is 2, 3 or 5 [14]. These hypersurfaces are special cases of Wang's isoparametric hypersurface [16]. In fact, there is no generally accepted notion of isoparametric hypersurfaces in spaces other than space forms. Wang's definition can be characterized by a transnormal system (cf. Sect. 2).

The aim of this paper is to investigate the number of distinct principal curvatures and their multiplicities of hypersurfaces in transnormal systems on $\mathbb{F}P^m$, where $\mathbb{F}=\mathbb{C}$ or $\mathbb{H}$. In particular, we obtain the following results.

Theorem A. *Let M be a connected hypersurface in a transnormal system on $\mathbb{F}P^m$. Then*

(1) *The number g_M of principal curvatures of M is constant and can take only the values* 2, 3, 5 *or* 7.

(2) *There are at most* 3 *different multiplicities and one of them is* 1 *when* $\mathbb{F}=\mathbb{C}$ (3 *when* $\mathbb{F}=\mathbb{H}$).

The method we use to obtain these results is based on the observation that $\tilde{M}=\pi^{-1}(M)$ is an isoparametric hypersurface in $\mathbb{S}^n$, where π is the Hopf fibration $\mathbb{S}^n \to \mathbb{F}P^m$. We combine this with infinitesimal, as well as global geometric and topological arguments to obtain our results. One of the key ingredients of our consideration is the number of non-horizontal eigenspaces of the Weingarten map on $\tilde{M}$. In fact, we have the following results.

Theorem B. *Let k be the number of non-horizontal eigenspaces of the Weingarten map on $\tilde{M}$. Then*

(1) *k is constant on $\tilde{M}$ and takes only the values* 2, 4 *or* 6.

(2) *$\tilde{M}$ has constant principal curvatures if and only if $k=2$.*

Let m_M($m_{\bar{M}}$, resp.) be the possible multiplicities of principal curvatures of M($\bar{M}$, resp.) and g_M($g_{\bar{M}}$, resp.) the number of distinct principal curvatures of M($\bar{M}$, resp.). With this notation we can make the arguments in the above theorems more precise. In fact, the following tables yield all the possibilities of k, g_M and m_M.

Table 1. ($\mathbb{F}=\mathbb{C}$)

k	$g_{\bar{M}}$	g_M	$\dim M$	$m_{\bar{M}}$	m_M
2	2	2	$2p+1$	$1, 2p+1$	$1, 2p$
		3	$2p+2q+1$	$2p+1, 2q+1$	$1, 2p, 2q$
	4	3	$2p+1$	$1, 1, p, p$	$1, p, p$
		5	$4p+5$	$2, 2, 2p+1, 2p+1$	$1, 2, 2, 2p, 2p$
		5	17	4, 4, 5, 5	1, 4, 4, 4, 4
		5	29	6, 6, 9, 9	1, 6, 6, 8, 8
*4	4	3	3	1, 1, 1, 1	1, 1, 1
		5	$2p+3$	$1, 1, p+1, p+1$	$1, 1, 1, p, p$
		7	$2p+2q+3$	$p+1, p+1, q+1, q+1$	$1, 1, 1, p, p, q, q$
6	6	5	5	1, 1, 1, 1, 1, 1	1, 1, 1, 1, 1

Table 2. ($\mathbb{F}=\mathbb{H}$)

k	$g_{\bar{M}}$	g_M	$\dim M$	$m_{\bar{M}}$	m_M
2	2	2	$4p+3$	$3, 4p+3$	$3, 4p$
		3	$4p+4q+3$	$4p+3, 4q+3$	$3, 4p, 4q$
	4	3	7	3, 3, 2, 2	3, 2, 2
		5	$4p+7$	$2, 2, 2p+3, 2p+3$	$3, 2, 2, 2p, 2p$
		5	27	6, 6, 9, 9	3, 6, 6, 6, 6
*4	4	5	$4p+7$	$3, 3, 2p+2, 2p+2$	$3, 3, 3, 2p-1, 2p-1$
		7	$4p+4q+7$	$2p+3, 2p+3, 2q+2, 2q+2$	$3, 3, 3, 2p, 2p, 2q-1, 2q-1$

* The cases represent possible values, not the existence

Some of the case $k=4$ are illustrated by examples. In particular, we point out the existence of non-isometric transnormal systems in $\mathbb{C}P^m$ whose corresponding isoparametric families are isometric.

2. Preliminaries

A *transnormal system* on a complete connected Riemannian manifold N is a partition of N into nonempty connected submanifolds such that any geodesic of N cuts these submanifolds orthogonally at none or all of its points. A nonconstant real-valued function f on a space of constant curvature is called *isoparametric* if $|f|^2$

and Δf are functions of f. The level sets of such a function give an isoparametric family which is a special case of transnormal systems.

Note that the Hopf fibration $\pi: \mathbb{S}^n \to \mathbb{FP}^m$ is a *Riemannian submersion* with totally geodesic fibers, i.e., each fiber is totally geodesic in $\mathbb{S}^n$ and at each point z of $\mathbb{S}^n$ the differential $(\pi_*)_z$ preserves the lengths of *horizontal* vectors, i.e., vectors which are orthogonal to the fiber $\pi^{-1}(x)$, where $\pi(z)=x$. A vector at z is called *vertical* if it is tangent to the fiber $\pi^{-1}(x)$. For each tangent vector $\bar{X}$ on $\mathbb{S}^n$, $v\bar{X}$ and $h\bar{X}$ denote the vertical and horizontal components of $\bar{X}$, respectively. To each tangent vector field X on $\mathbb{FP}^m$, there exists a unique horizontal vector field on $\bar{X}$ on $\mathbb{S}^n$ such that $(\pi_*)_z\bar{X}=X_x$ for all z in $\mathbb{S}^n$.

Let $\bar{D}$ and D be the Riemannian connections of $\mathbb{S}^n$ and $\mathbb{FP}^m$, respectively. If X and Y are tangent vector fields on $\mathbb{FP}^m$, then

$$h(\bar{D}_{\bar{X}}\bar{Y})=\overline{(D_X Y)}\ , \tag{2.1}$$

where $\bar{X}, \bar{Y}, \overline{D_X Y}$ mean their horizontal lifts [14]. In fact, (2.1) is true for any Riemannian submersion with totally geodesic fibers. Let Σ be a transnormal system on $\mathbb{FP}^m$ containing a hypersurface M, and $\bar{M}=\pi^{-1}(M)$. Let $\tilde{n}$ be a unit normal vector field on $\bar{M}$. Then the relationship between the two shape operatiors $A_{\tilde{n}}$ and A_n is given by

$$h(A_{\tilde{n}}\bar{X})=\overline{A_n X}\ , \tag{2.2}$$

where $n=\pi_*(\tilde{n})$ and X is a tangent vector on M.

Proposition 2.1. *$\pi^{-1}(\Sigma)$ is an isoparametric family, and hence $\bar{M}$ is an isoparametric hypersurface on $\mathbb{S}^n$.*

Proof. Let γ be a geodesic which is normal to $\bar{M}$ at a point z. Then $\pi\circ\gamma$ is a geodesic of $\mathbb{FP}^m$ (cf. O'Neill [10]) and normal to M at $\pi(z)$. Since Σ is a transnormal system, $\pi\circ\gamma$ is normal to members of Σ at all of its points. Thus γ is normal to members of $\pi^{-1}(\Sigma)$, and hence $\pi^{-1}(\Sigma)$ is transnormal. By [17], it is an isoparametric family on $\mathbb{S}^n$.

Proposition 2.2 (Münzner [7, 8], Abresch [1]).

(i) *The number g of distinct principal curvatures of $\bar{M}$ is* 1, 2, 3, 4 *or* 6. *Let $p_1,\dots,p_g$ be the distinct principal curvatures with multiplicities $m_1,\dots,m_g$. Assume that $p_i=\cot t_i$ and $0<t_1<\dots<t_g<\pi$. Then*

(ii) $m_i=m_{i+2}$ (*subscripts* mod g).

(iii) $t_k=t_1+(k-1)\pi/g$, $1\leqq k<g$.

Thus there are at most two different multiplicities, say m_+ and m_-. Moreover, $m_+=m_-$ for odd g.

(iv) *$\bar{M}$ has exactly two focal submanifolds $\bar{F}_\pm$ of codimensions $m_\pm+1$.*

(v) *If $g=3$, then $m_+=m_-=1,2,4$ or 8. If $g=6$, then $m_+=m_-=1$ or 2.*

(vi) *$\mathbb{S}^n$ is divided into two sphere bundles $B_\pm$ over the focal submanifolds $\bar{F}_\pm$ with common boundary along $\bar{M}$. Moreover,*

$$H^q(\bar{F}_\pm;Z_2)=\begin{cases} Z_2 & \textit{for}\quad q\equiv 0, m_\mp \,(\operatorname{mod} m_+ + m_-) \quad \textit{and}\quad 0\leqq q<n \\ 0 & \textit{otherwise}\ . \end{cases}$$

In Sects. 3 and 4, we consider $\mathbb{FP}^m$ for $m\geqq 2$.

3. Complex projective space

In this section, we assume that $n=2m+1$ and $\mathbb{F}=\mathbb{C}$. Let $\bar{J}$ be the canonical complex structure on $\mathbb{C}^{m+1}$. Let V denote the canonical vertical vector field on $\bar{M}$, i.e., $V_z=iz$ for all z in $\bar{M}$. Note that $\bar{M}$ is an isoparametric hypersurface of $\mathbb{S}^{2m+1}$. Let $p_1,\dots,p_g$ be the distinct principal curvatures of $\bar{M}$ and $T(p_i)$ denote the eigenspace corresponding to p_i. If $T(p_i)$ contains a non-horozontal vector, then we may choose an orthogonal basis for $T(p_i)$ of the form $\{W_{i_1},\dots, W_{ir_i}, U_i+V\}$, where W_{ik} and U_i are horizontal. If $i\neq j$, then

$$\langle U_i, U_j\rangle=-1 \ , \quad \text{and hence } \measuredangle(U_i, U_j)>\pi/2 \ . \tag{3.1}$$

We will use the following identities which follows from $\bar{D}_V V=0$ and $\bar{D}_V\tilde{n}-\bar{D}_{\tilde{n}}V=0$ (cf. [9]).

$$\langle \bar{A}(V), \bar{X}\rangle=\langle Jn, X\rangle \quad \text{for} \quad X\in TM \ , \tag{3.2}$$

$$\langle \bar{A}(V), V\rangle=0 \ , \tag{3.3}$$

$$|\bar{A}(V)|=1 \ , \tag{3.4}$$

where $\bar{A}=A_{\tilde{n}}$, $A=A_n$ and J is the complex structure induced by $\bar{J}$.

Assume that $T(p_1),\dots,T(p_k)$ contain non-horizontal vectors and that $T(p_{k+1}),\dots,T(p_g)$ consist of horizontal vectors. We may assume that $U_1,\dots,U_{k-1}$ are linearly independent. Since $\bar{A}(U_i+V)$ is horizontal by (3.3), we can express $\bar{A}(U_i)$ and $\bar{A}(V)$ as follows:

$$\begin{aligned}\bar{A}(U_i)&=a_1U_1+\dots+(a_i+p_i)U_i+\dots+a_{k-1}U_{k-1}+p_iV\\ \bar{A}(V)&=-a_1U_1-a_2U_2-\dots-a_{k-1}U_{k-1} \ , \quad 1\leqq i\leqq k-1 \ .\end{aligned} \tag{3.5}$$

Then A is represented by the matrix

$$\left(\begin{array}{c|c} D & 0 \\ \hline 0 & B\end{array}\right), \quad \text{where } D \text{ is diagonal and}$$

$$B=\begin{pmatrix} a_1+p_1 & a_1 & \dots & a_1 \\ a_2 & a_2+p_2 & \dots & a_2 \\ \dots & \dots & \dots & \dots \\ a_{k-1} & a_{k-1} & \dots & a_{k-1}+p_{k-1}\end{pmatrix} . \tag{3.6}$$

Since each fiber is totally geodesic, *trace* $\bar{A}=$ *trace* A. Hence we have

$$a_1+a_2+\dots+a_{k-1}=p_k \ . \tag{3.7}$$

Proposition 3.1. $k\geqq 2$, *i.e., there are at least two eigenspaces which contain non-horizontal vectors.*

Proof. Suppose that $k=0$. Then there are no vertical vectors, a contradiction. Suppose that $k=1$. Then U_1+V generates a 1-dimensional space containing V. Then $U_1=0$, and hence V is an eigenvector. This implies that *trace* $A=$ *trace* $\bar{A}-p_1$, a contradiction.

Thus $g \geqq 2$. Let $F_\pm = \pi(\bar{F}_\pm)$. Suppose that both m_+ and m_- are even, then $m_+ + m_-$ is even. By (2.3), we have $H^q(\bar{F}_+; Z_2) = 0$ for odd q. From the fibration $S^1 \to \bar{F}_+ \to F_+$, we have the Gysin exact sequence

$$\ldots \to H^q(F_+) \to H^q(\bar{F}_+) \to H^{q-1}(F_+) \to H^{q+1}(F_+) \to \ldots .$$

Then $H^q(F_+) = 0$ for odd q. But $\dim F_+ = $ odd, a contradiction. This means that we can exclude the case $g=3$ altogether and the case $g=6$ with $m_+ = m_- = 2$. Therefore the possible values of g are 2, 4 or 6.

Case 1. $g=2$.

Since $2 \leqq k \leqq g$, we have $k=2$. Thus both eigenspaces contain non-horizontal vectors. From (3.3) and (3.5), we have $\bar{J}\bar{n} = -a_1 U_1$ and $a_1 = p_2$. Then Jn is an eigenvector with the eigenvalue $p_1 + p_2$. Thus we have

Proposition 3.2. *Assume that* $g=2$.

(i) *If* $m_- = 1$, *then* $m_+ \neq 1$ *and* M *has* 2 *constant principal curvatures* p_1 *and* $p_1 + p_2$ *with multiplicities* $m_+ - 1$ *and* 1.

(ii) *If* $m_+, m_- > 1$, *then* M *has* 3 *constant principal curvatures* p_1, p_2 *and* $p_1 + p_2$ *with multiplicities* $m_+ - 1$, $m_- - 1$, 1.

Remark 3.3. Note that $\bar{M}$ is a product of two spheres and its focal submanifolds are also spheres. Thus the two focal submanifolds of M are complex projective spaces, and hence M lies in a tube over a complex projective space.

Case 2. $g=4$.

Let $K_i = \langle U_i, U_i \rangle$. Suppose that $k=3$ for some point of $\bar{M}$. By (3.5) and (3.7), we have $a_1 + a_2 = p_3$ and $\bar{A}(V) = -a_1 U_1 - a_2 U_2$. (3.5) implies

$$a_1 p_1 + a_2 p_2 = -1 . \tag{3.8}$$

Thus we have

$$p_1 = \langle \bar{A}(U_1), V \rangle = \langle U_1, \bar{A}(V) \rangle = -a_1 K_1 + a_2 ,$$

$$p_2 = -a_2 K_2 + a_1 \quad \text{and}$$

$$a_i(K_i + 1) = p_3 - p_i . \tag{3.9}$$

Assume that $p_1 > p_2 > p_3$, then $p_1 p_3 = -1$ or $p_2 p_3 = -1$. We may assume that $p_1 p_3 = -1$. Then, by (3.8),

$$-1 = (p_3 - a_2) p_1 + a_2 p_2 = a_2 (p_2 - p_1) - 1 .$$

Thus we have $a_2(p_2 - p_1) = 0$ and hence $a_2 = 0$, a contradiction. Thus we have

Proposition 3.4. *If* $g=4$, *then* $k=2$ *or* 4.

Remark 3.5. The formulas (3.8) and (3.9) are true for any k.

The number k may depend on the points of $\bar{M}$. In fact, k is constant on $\bar{M}$. At this moment we assume that k is constant on $\bar{M}$. We will prove this later (Proposition 3.12).

(1) $k=2$ on $\bar{M}$, i.e., $T(p_1)$ and $T(p_2)$ are non-horizontal and $T(p_3)$ and $T(p_4)$ are horizontal. As in the case $g=2$, Jn is an eigenvector with the eigenvalue $p_1 + p_2$.

Proposition 3.6. *Assume that $g=4$ and $k=2$.*

(i) *If $m_-=1$, then M has 3 constant principal curvatures p_3, p_4, p_1+p_2 with multiplicities m_+, m_+, 1.*

(ii) *If m_+, $m_->1$, then M has 5 constant principal curvatures p_1, p_2, p_3, p_4, p_1+p_2 with multiplicities m_--1, m_--1, m_+, m_+, 1.*

Remark 3.7. In Proposition 3.6, M lies in a tube over a complex submanifold. Note that non-horizontal eigenspaces have odd dimensions [4, 15].

(2) $$k=4 \text{ on } \tilde{M} \ .$$

Thoughout this case we assume that $p_1>p_2>p_3>p_4$. Note that any three of U_i's are linearly independent, since $\measuredangle(U_i, U_j)>\pi/2$ for $i\neq j$. By (3.9) and Remark 3.5, we have $a_i<0$, $(i=1,2,3)$. The characteristic polynomial $f(x)$ of B is given by

$$\begin{aligned} f(x) = & -x^3+(p_1+p_2+p_3+p_4)x^2-[(p_1+p_3)(p_2+p_4)-1]x \\ & +[a_1(p_2p_3+1)+a_3(p_1p_2+1)]-(p_2+p_4) \end{aligned} \tag{3.10}$$

[cf. (3.6), (3.7), (3.8) and Remark 3.5]. On the other hand, if we replace U_1, U_2, U_3, a_1, a_2, a_3 by U_2, U_3, U_4, c_2, c_3, c_4 in (3.5), then $c_i>0$ $(i=2,3,4)$ and

$$\begin{aligned} f(x) = & -x^3+(p_1+p_2+p_3+p_4)x^2-[(p_1+p_3)(p_2+p_4)-1]x \\ & +[c_2(p_3p_4+1)+c_4(p_2p_3+1)]-(p_1+p_3) \ . \end{aligned} \tag{3.11}$$

If the constant term is equal to $-(p_1+p_3)[-(p_2+p_4)$. resp.], then $f(x)$ has 3 distinct roots p_1+p_3, p_2, p_4 [p_1, p_3, p_2+p_4, resp.]. From the properties of a_i, c_i, p_i we obtain that the constant term of $-f(x)$ is between p_1+p_3 and p_2+p_4. Thus the graph of $-f(x)$ must lie between the two parallel curves in Fig. 1 and be parallel to them. Then $f(x)$ has 3 distinct roots q_1, q_2, q_3 which are different from p_1, p_2, p_3, p_4, p_1+p_3, p_2+p_4. Moreover, they are nonconstant.

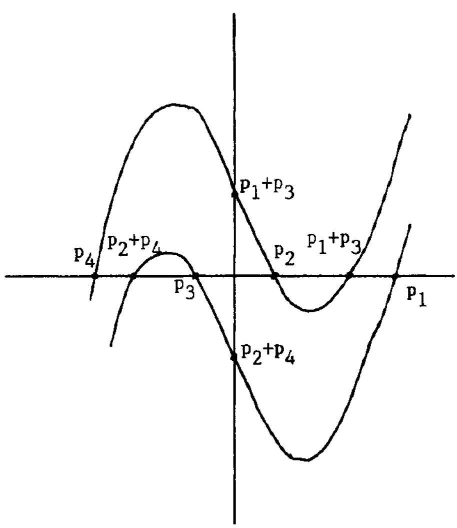

Fig. 1

Proposition 3.8. *The q_i's are nonconstant.*

Proof. Suppose that q_1 is constant then q_2 and q_3 are constant. Thus all principal curvatures are constant. Then Jn is principal (cf. [14]). Then one of q_i's is equal to p_1+p_3 or p_2+p_4 [4], a contradiction.

Proposition 3.9. *Assume that $g=4$ and $k=4$.*

(i) *If $m_-=1$ and $m_+\neq 1$, then M has 2 constant principal curvatures p_2, p_4 with the same multiplicity m_+-1 and 3 nonconstant principal curvatures with the same multiplicity 1.*

(ii) *If $m_+, m_->1$, then M has 4 constant principal curvatures p_1, p_2, p_3, p_4 with multiplicities $m_--1, m_+-1, m_--1, m_+-1$ and 3 nonconstant principal curvatures with the same multiplicity 1.*

(iii) *If $m_-=m_+=1$, then M has 3 nonconstant principal curvatures with the same multiplicity 1.*

Remark 3.10. We have examples for (i) and (ii). But we don't know if the case (iii) exists..

Now we are going to prove that k is constant. To prove this, let $\phi_t: \bar{M}\to\mathbb{S}^{2m+1}$ be the normal exponential map, i.e., $\phi_t(z)=\cos tz+\sin t\tilde{n}$. Then $\phi_t(\bar{M})$ has constant principal curvatures $\cot(t_i-t)$ with the corresponding principal distributions $T(p_i)$ unless $t=t_i$ for some i [7].

Proposition 3.11. *If W is a horizontal eigenvector on $\bar{M}$, then it is also a horizontal eigenvector on $\bar{M}_t=\phi_t(\bar{M})$ unless $t=t_i$ for some i.*

Proof. Let $V(t)$ be the canonical unit vertical vector field on $\bar{M}_t$. Then $V(t)=i(\cos tz+\sin t\tilde{n})=\cos tV+\sin t\bar{A}(V)$. Since W is orthogonal to $\bar{A}(V)$, W is horizontal.

We want to express U_i+V in terms of $U_i(t)+V(t)$, the non-horizontal eigenvector with the eigenvalue $p_i(t)$ at $\phi_t(z)$. Since $\langle U_i+V, V(t)\rangle=p_i\sin t+\cos t$, we have

$$U_i(t)+V(t)=(U_i+V)/(p_i\sin t+\cos t)\ . \tag{3.12}$$

Proposition 3.12. *Assume that $g=4$, then k is constant on $\bar{M}$.*

Proof. Suppose that $k=2$ on a nonempty set S and $k=4$ on a nonempty set S'. Let d be the distance function on $\mathbb{S}^{2m+1}$. If $d(z,S)\to 0$, then the constant term of $-f(x)$ converges to p_2+p_4 or p_1+p_3. We may assume that it converges to p_2+p_4. Then a_1 and a_3 converges to 0. Note that $a_i(K_i+1)=p_4-p_i$. Thus K_1 and K_3 are unbounded if $d(z,S)\to 0$. Clearly K_4 is unbounded if $t_1\to\pi/4$. From (3.12) we obtain that K_2 is bounded below. Then U_1, U_2, U_3, U_4 are almost orthogonal to each other if $t_1\to\pi/4$. But they generate a 3-dimensional space, a contradiction to (3.1).

Case 3. $g=6$ and $m_+=m_-=1$.

Throughout this case we assume that $p_1>p_2>\ldots>p_6$.

(1) Suppose that $k=2$. Then Jn is principal as in the case $g=2$. We may assume that $T(p_2)$ and $T(p_5)$ are the two non-horizontal eigenspaces. Note that $p_2+p_5=p_3$ for some t_1. Then a focal submanifold of M must have dimension 3 [4], a contradiction.

(2) Suppose that $k=3$, 4 or 5. Then we can obtain similar contradictions. For example, if $k=4$ and $T(p_5)$, $T(p_6)$ are the two horizontal eigenspaces, then

$$a_2(p_2-p_1)+a_3(p_3-p_1)=-(1+p_1p_4)=0 .$$

But a_2, $a_3<0$, a contradiction.

Thus we have $k=6$, i.e., there are no horizontal eigenspaces. If a principal curvature p of M has multiplicity >1, let X_1 and X_2 be orthogonal eigenvectors with the same eigenvalue p. Let $\bar{X}_i$ be their horizontal lifts and $\bar{A}(\bar{X}_i)=p\bar{X}_i+b_iV$. If $b_2\neq 0$, then the vector $\bar{X}_1-(b_1/b_2)\bar{X}_2$ is a horizontal eigenvector, a contradiction.

Proposition 3.13. *If $g=6$, then $\bar{M}$ has no horizontal eigenspaces and M has 5 principal curvatures with the same multiplicity 1. Moreover, at least one of them is nonconstant.*

For the existence of nonconstant principal curvatures, see [14].

4. Quaternionic projective space

In this section, we assume that $n=4m+3$ and $\mathbb{F}=\mathbb{H}$. Let $\bar{J}_1, \bar{J}_2, \bar{J}_3$ be the canonical complex structures on $\mathbb{H}^{m+1}$. Note that $\bar{M}$ is an isoparametric hypersurface of $\mathbb{S}^{4m+3}$ and is also invariant under the canonical $\mathbb{S}^1$-actions given by $\bar{J}_i$'s. Thus the possible values of g are 2 and 4, as shown in Sect. 3.

Case 1. $g=2$.

Clearly $\dim T(p_i)\geqq 3$. Choose orthogonal bases

$$\mathscr{B}_i\cup\{U_i+V_1, U_i'+V_2, U_i''+V_3\}$$

for $T(p_i)$, where $\mathscr{B}_i$, U_i, U_i', U_i'' are horizontal, and V_i's are orthonormal vertical vectors ($i=1,2,3$). As in Sect. 3, $\pi_*(U_1)$, $\pi_*(U_1')$, $\pi_*(U_1'')$ are eigenvectors of the shape operator $A=A_n$ with the same eigenvalue p_1+p_2.

Proposition 4.1. *Suppose that $g=2$.*

(i) *If $m_-=3$, then $m_+>3$ and M has 2 constant principal curvatures p_1+p_2, p_2 with multiplicities* 3, m_+-3.

(ii) *If m_-, $m_+>3$, then M has 3 constant principal curvatures p_1+p_2, p_1, p_2 with multiplicities* 3, m_--3, m_+-3.

Since $\bar{M}$ is a product of two spheres and the two focal submanifolds are spheres, the two focal submanifolds of Σ are quaternionic projective spaces. Thus M lies in a tube over a quaternionic projective space.

Case 2. $g=4$.

Since $2(m_++m_-)=4m+2$, we may assume that m_- is odd and that m_+ is even. Assume that the two principal curvatures p_1 and p_2 have the same multiplicity m_-. Let $V_i=\bar{J}_iz$ be the canonical vertical vectors. Let $\pi_i:\mathbb{S}^{4m+3}\to\mathbb{CP}^{2m+1}$ be the Riemannian submersion obtained by taking V_i as a unit vertical vector.

Proposition 4.2. *Assume that $g=4$. If J_1n is an eigenvector with respect to π_1, then J_2n and J_3n are also eigenvectors with respect to π_2 and π_3. (J_i denotes the canonical complex structure on $\mathbb{CP}^{2m+1}$ induced by $\bar{J}_i$).*

Proof. Note that $T(p_3)$ and $T(p_4)$ are horizontal with respect to V_1, i.e., the members of them have no components of V_1. Consider the vertical vectors $V_1 \pm V_2$. $T(p_3)$ and $T(p_4)$ are horizontal with respect to these vertical vectors. This implies that the members of them have no components of V_2. Thus $J_2 n$ is an eigenvector with respect to π_2.

Therefore we have two possible cases.

(1) $J_1 n$ is an eigenvector with respect to π_1.
Note that $T(p_3)$ and $T(p_4)$ are horizontal, but $T(p_1)$ and $T(p_2)$ are not horizontal. Clearly, $m_- \geqq 3$. As in the case $g=2$, choose orthogonal bases

$$\mathscr{B}_i \cup \{U_i + V_1, U_i' + V_2, U_i'' + V_3\}$$

for $T(p_i)$, where $\mathscr{B}_i$, U_i, U_i', U_i'' are horizontal $(i=1,2)$.

Proposition 4.3. *Assume that $g=4$ and that $J_1 n$ is principal with respect to π_1.*

(i) *If $m_- = 3$, then $m_+ = 2$ and M has 3 constant principal curvatures p_3, p_4, $p_1 + p_2$ with multiplicities 2, 2, 3, 3.*

(ii) *If $m_- \neq 3$, then M has 5 constant principal curvatures $p_1, p_2, p_3, p_4, p_1 + p_2$ with multiplicities $m_- - 3$, $m_- - 3$, m_+, m_+, 3.*

(2) $J_1 n$ is not an eigenvector with respect to π_1.

Lemma 4.4. $m_- \neq 1$ *and* $m_+ \neq 2$.

Proof. Suppose that $m_- = 1$. Since we assume that $m \geqq 2$ and m_+ is even, $m_+ \geqq 4$. We choose orthogonal bases

$$\mathscr{B}_i \cup \{U_i + V_1, U_i' + V_2, U_i'' + V_3\}$$

for $T(p_i)$, where $\mathscr{B}_i$, U_i, U_i', U_i'' are horizontal, $(i=3,4)$. Note that

$$U_3, U_4 \perp U_3', U_4' \perp U_3'', U_4'' .$$

Therefore they are linearly independent. This means that we have 9 independent vectors from an 8-dimensional space, a contradiction. By a similar argument, we have $m_+ \neq 2$.

Now we may assume that $m_- \geqq 3$ and $m_+ \geqq 4$. We choose orthogonal bases

$$\mathscr{B}_i \cup \{U_i + V_1, U_i' + V_2, U_i'' + V_3\}$$

for $T(p_i)$, where $\mathscr{B}_i$, U_i, U_i', U_i'' are horizontal, $(i=1,2,3,4)$. As in Sect. 3, we have three 3×3 matrices B, B', B'' which correspond to U_i, U_i', U_i''. In fact, they are equal. Thus A is represented by the matrix

$$\left(\begin{array}{c|ccc} D & & 0 & \\ \hline & B & 0 & 0 \\ 0 & 0 & B & 0 \\ & 0 & 0 & B \end{array}\right), \quad \text{where } D \text{ is diagonal and}$$

$$B = \begin{pmatrix} p_1 + a_1 & a_1 & a_1 \\ a_2 & p_2 + a_2 & a_2 \\ a_3 & a_3 & p_3 + a_3 \end{pmatrix}.$$

Proposition 4.5. *Assume that $g=4$ and that $J_1 n$ is not principal with respect to π_1.*

(i) *If $m_-=2$, then M has 2 constant principal curvatures p_3, p_4 with the same multiplicity m_+-3 and 3 nonconstant principal curvatures with the same multiplicity* 3.

(ii) *If $m_-\neq 3$, then M has 4 constant principal curvatures p_1, p_2, p_3, p_4 with multiplicities m_--3, m_--3, m_+-3, m_+-3 and 3 nonconstant principal curvatures with the same multiplicity* 3.

5. Examples

In this section, we give explicit examples on cases which have been handled in previous sections.

Example 5.1. Consider the fibration $\pi: \mathbb{S}^7 \to \mathbb{HP}^1$. On $\mathbb{HP}^1 = \mathbb{S}^4$, there are 3 different isoparametric hypersurfaces

(1) M_1 with 1 principal curvature.

$\pi^{-1}(M_1)$ is an isoparametric hypersurface with $g=2$ and $(m_-, m_+)=(3,3)$. This gives an example for Proposition 3.2 (i).

(2) M_2 with 2 principal curvatures.

$\bar{M}_2 = \pi^{-1}(M_2)$ is an isoparametric hypersurface with $g=4$ and $(m_-, m_+) = (1,2)$. Since the multiplicities are less than 3, $J_1 n$ can not be principal. Thus its image M_2' under the fibration $\mathbb{S}^7 \to \mathbb{CP}^3$ has nonconstant principal curvatures. This gives an example for Proposition 3.9 (i).

(3) M_3 with 3 principal curvatures.

$\bar{M}_3 = \pi^{-1}(M_3)$ is an isoparametric hypersurface with $g=6$ and $(m_-, m_+) = (1,1)$. Accidentally we obtained an example $g=6$. We know that its image under the fibration $\mathbb{S}^7 \to \mathbb{CP}^3$ has a nonconstant principal curvature. But M_3 has constant principal curvatures.

Remark 5.2. From Takagi's table we have a homogeneous hypersurface M_2'' which is the image of an isoparametric hypersurface in $\mathbb{S}^7$ with $g=4$ and $(m_-, m_+)=(1,2)$, [14]. But all isoparametric hypersurface with $g=4$ and $(m_-, m_+)=(1,2)$ are congruent. Thus we have two non-isometric hypersurfaces M_2' and M_2'' which correspond to isometric hypersurfaces in $\mathbb{S}^7$. These are examples which were mentioned in Sect. 1.

Example 5.3. The inhomogeneous examples of Ozeki and Takeuchi [11] are invariant under the canonical $\mathbb{S}^3$-action (and hence under the canonical $\mathbb{S}^1$-action). These give examples for Proposition 3.9 and Proposition 4.5.

Example 5.4. The hypersurfaces of type **C** and type **E** in Takagi's table [14] are invariant under the canonical $\mathbb{S}^3$-action. These give example for Proposition 4.3.

We proved that the example of type **B** in Takagi's example can not be invariant under the canonical $\mathbb{S}^3$-action (cf. Lemma 4.4). The remaining case in his table is the examples of type **D**. Let $\bar{M}$ be an isoparametric hypersurface which is obtained from a hypersurface of type **D**. Then $g=4$ and $(m_-, m_+)=(5,4)$. Suppose that $\bar{M}$ is $\mathbb{S}^3$-

invariant. Note that $\dim \bar{F}_- = 13$ and hence $\dim F_- = 10$. From the Gysin sequence

$$\ldots \to H^q(F_-) \to H^q(\bar{F}_-) \to H^{q-3}(F_-) \to H^{q+1}(F_-) \to \ldots$$

we have $H^4(F_-) \neq 0$ and $H^6(F_-) = 0$, a contradiction. Thus $\bar{M}$ can not be invariant under the $\mathbb{S}^3$-action.

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