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Exact Controllability of Semilinear Abstract Systems with Application to Waves and Plates Boundary Control Problems*

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Abstract. This paper studies (global) exact controllability of abstract semilinear equations. Applications include boundary control problems for wave and plate equations on the explicitly identified spaces of exact controllability of the corresponding linear systems.

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1. Motivating Examples, Corresponding Results, Literature

1.1. Motivating Examples and Corresponding Results

Throughout this paper we let Ω be an open bounded domain of R^n with sufficiently smooth boundary Γ. For the sake of simplicity of notation, boundary controls are applied to the entire boundary Γ.

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Wave Equation with Dirichlet Controls. In Ω we consider the following semilinear problem for the wave equation in the solution $w(t, x)$:

$$\begin{cases} w_{tt} = \Delta w + f(w) & \text{in } (0, T] \times \Omega = Q, \quad (1.1a) \\ w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x) & \text{in } \Omega, \quad (1.1b) \\ w|_\Sigma \equiv u & \text{in } (0, T] \times \Gamma = \Sigma, \quad (1.1c) \end{cases}$$

with control function u based on Σ. The assumption on the nonlinearity f is as follows:

$f\colon \Re \to \Re$ is an absolutely continuous function with first derivative f' a.e. (almost everywhere) which is a.e. uniformly bounded on $\Re$;

$$|f'(r)| \le \text{const} \qquad \text{for a.e.} \quad r \in \Re. \tag{1.2}$$

The linear problem (1.1) with $f \equiv 0$ is exactly controllable over $[0, T]$ on the space $L_2(\Omega) \times H^{-1}(\Omega)$ within the class of controls $u \in L_2(0, T; L_2(\Gamma))$, provided that $T > 0$ is sufficiently large [LT4], [L5], [H2], [T3], [BLR]; equivalently, on the space $H_0^1(\Omega) \times L_2(\Omega)$ within the class of controls $u \in H_0^1(0, T; L_2(\Gamma))$, see Theorem 3.1 below. In turn, equivalently, on the space $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$, $0 < \gamma < 1$, $\gamma \neq \frac{1}{2}$ (resp. $H_{00}^{1/2}(\Omega) \times [H_{00}^{1/2}(\Omega)]'$, if $\gamma = \frac{1}{2}$ within the class of controls $u \in H_0^\gamma(0, T; L_2(\Gamma))$ (resp. $u \in H_{00}^{1/2}(0, T; L_2(\Gamma))$ if $\gamma = \frac{1}{2}$), see Corollary A.3 in Appendix A below. One of the contributions of this paper is to extend the same exact controllability property to the semilinear problem (1.1) subject to (1.2) over the same time interval.

Theorem 1.1. *Let $T > 0$ be a time for which exact controllability of the linear problem with $f \equiv 0$ holds true in any one of the equivalent statements above. Let f satisfy assumption* (1.2). *Then a similar exact controllability result holds true for the original problem* (1.1) *for the same $T > 0$: for any pair $\{w_0, w_1\} \in H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$, $0 \le \gamma \le 1$, $\gamma \neq \frac{1}{2}$ (resp. $\{w_0, w_1\} \in H_{00}^{1/2}(\Omega) \times [H_{00}^{1/2}(\Omega)]'$ for $\gamma = \frac{1}{2}$), there exists a suitable control function $u \in H_0^\gamma(0, T; L_2(\Gamma))$ (resp. $u \in H_{00}^{1/2}(0, T; L_2(\Gamma))$ for $\gamma = \frac{1}{2}$ such that the corresponding solution of problem* (1.2) *satisfies $w(T, \cdot) = w_t(T, \cdot) = 0$.*

Theorem 1.1 is a specialization (see Section 4) of an abstract result (Theorem 2.1) given in Section 2.

Wave Equation with Neumann Control. In Ω we consider the problem

$$\begin{cases} w_{tt} = \Delta w & \text{in } (0, T] \times \Omega = Q, \quad (1.3a) \\ w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x) & \text{in } \Omega, \quad (1.3b) \\ \left.\dfrac{\partial w}{\partial \nu}\right|_\Sigma = g(w|_\Gamma) + u & \text{in } (0, T] \times \Gamma = \Sigma, \quad (1.3c) \end{cases}$$

where ν is a unit normal outward vector. Moreover, on the basis of the regularity results as in [LT9], the scalar function g satisfies the following assumption when $\dim \Omega \geq 2$:

$$g\colon \text{continuous } H^{\beta}(\Sigma) \to L_2(\Sigma), \tag{1.4}$$

$$\beta = \begin{cases} \frac{1}{5} - \varepsilon, & \varepsilon > 0 \text{ arbitrary}, \quad \text{if } \Omega \text{ is a general smooth domain}, \\ \frac{1}{4} - \varepsilon, & \varepsilon > 0 \text{ arbitrary}, \quad \text{if } \Omega \text{ is a parallelepiped}, \\ \frac{1}{3} & \text{if } \Omega \text{ is a sphere}. \end{cases} \tag{1.5}$$

For $\dim \Omega = 1$, take $\beta = 1$.

Theorem 1.2. *Let μ be an $L_2(\Sigma)$-control function that steers the origin $\{0, 0\}$ to the state $\{v(T, \cdot), v_t(T, \cdot)\} \in H^1(\Omega) \times L_2(\Omega)$ at time T along the solution of the linear problem*

$$\begin{cases} v_{tt} = \Delta v & \text{in } (0, T] \times \Omega = Q, & (1.6a) \\ v(0, x) = v_t(0, x) \equiv 0 & \text{in } \Omega, & (1.6b) \\ \left.\dfrac{\partial v}{\partial \nu}\right|_{\Sigma} = \mu \in L_2(\Sigma) & \text{in } (0, T] \times \Gamma = \Sigma. & (1.6c) \end{cases}$$

Then the control function

$$u = \mu - g(v|_{\Sigma}) \in L_2(\Sigma) \tag{1.7}$$

used in (1.3c) *of the nonlinear problem* (1.3) *with $w_0 = w_1 = 0$ produces the same solution*:

$$w(t) \equiv v(t), \qquad w_t(t) \equiv v_t(t), \qquad 0 \leq t \leq T. \tag{1.8}$$

In particular, $\{w(T, \cdot), w_t(T, \cdot)\} = \{v(T, \cdot), v_t(T, \cdot)\}$. Thus, problem (1.3) *is exactly controllable on $H^1(\Omega) \times L_2(\Omega)$ at time T, whenever problem* (1.6) *is.* (*Notice that* (1.4) *is satisfied if, for instance,*

$$|g(s)| \leq a + b|s|^k, \qquad k < \frac{n}{n - 2\beta}.$$

Hence, g can be superlinear.)

Theorem 1.2 is proved in Section 5. For exact controllability results for problem (1.6) see the recent direct approaches in [L5], [L7], and [LT5], which followed the original results via uniform stabilization in [C] and [L1].

Remark 1.1. Theorem 1.2 with $\beta = \frac{1}{5} - \varepsilon$ in a general smooth domain (see (1.5)) remains true if $(-\Delta)$ in (1.6a) is replaced by a general second-order uniformly elliptic operator with smooth coefficients depending on the space variable (but not on the time variable). This is so because the sharp trace regularity theory [LT9], which provides the key estimates for the validity of the proof of Theorem 1.2, remains true in this more general context.

Remark. 1.2. By a similar argument, we may give the following result. Consider the nonlinear problem

$$\begin{cases} y_{tt} = \Delta y + h(y|_\Sigma) + u & \text{in } Q, & (1.9a) \\ y(0, x) = y_0, \quad y_t(0, x) = y_1 & \text{in } \Omega, & (1.9b) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_\Sigma = 0 & \text{in } \Sigma, & (1.9c) \end{cases}$$

where the scalar function h satisfies the following conditions when $\dim \Omega \geq 2$:

$$h\text{: continuous } H^\alpha(\Sigma) \to L_2(Q), \tag{1.10}$$

$$\alpha = \begin{cases} \frac{3}{5} & \text{if } \Omega \text{ is a general smooth domain,} \\ \frac{3}{4} - \varepsilon & \text{if } \Omega \text{ is a parallelepiped,} \quad \varepsilon > \text{arbitrary,} \\ \frac{2}{3} & \text{if } \Omega \text{ is a sphere.} \end{cases} \tag{1.11}$$

Let μ be an $L_2(Q)$-control function that steers the origin $\{0, 0\}$ to the state $\{\eta(T, \cdot), \eta_t(T, \cdot)\} \in H^1(\Omega) \times L_2(\Omega)$ at time T, along the solution of the linear problem

$$\begin{cases} \eta_{tt} = \Delta \eta + \mu & \text{in } Q, \\ \eta(0, \cdot) = \eta_t(0, \cdot) = 0 & \text{in } \Omega, \\ \left.\dfrac{\partial \eta}{\partial \nu}\right|_\Sigma = 0 & \text{in } \Sigma. \end{cases} \tag{1.12}$$

Then the control function

$$u = \mu - h(\eta|_\Sigma) \in L_2(Q) \tag{1.13}$$

used in (1.9a) of the nonlinear problem (1.9) with $y_0 = y_1 = 0$ produces the same solution $y(t) \equiv \eta(t)$, $y_t(t) = \eta_t(t)$, $0 \leq t \leq T$. Thus, in particular, problem (1.9) is exactly controllable on $H^1(\Omega) \times L_2(\Omega)$ at any time T, since problem (1.12) is also [T1].

A remark, similar to Remark 1.1, that $(-\Delta)$ in (1.9a) may be replaced by a general second-order uniformly elliptic operator with space-depending smooth coefficient, also holds true for (1.9a). We note explicitly that in the dynamics (1.3) and (1.9), the nonlinearity and the control appear as additive terms.

Euler–Bernoulli Equation with Controls on $w|_\Sigma$ and $\Delta w|_\Sigma$. In Ω we consider the semilinear problem

$$\begin{cases} w_{tt} + \Delta^2 w = f(w) & \text{in } (0, T] \times \Omega = Q, & (1.14a) \\ w(0, x) = w_0, \quad w_t(0, x) = w_1 & \text{in } \Omega, & (1.14b) \\ w|_\Sigma = u_1 & \text{in } (0, T] \times \Gamma = \Sigma, & (1.14c) \\ \Delta w|_\Sigma = u_2 & \text{in } \Sigma, & (1.14d) \end{cases}$$

with nonlinearity $f(t)\colon \mathfrak{R} \to \mathfrak{R}$ satisfying the following assumptions: f' is absolutely continuous and

$$|f'(r)| + |f''(r)| \leq \text{const} \qquad \text{for a.e.} \quad t \in \mathfrak{R}. \tag{1.15}$$

other point. We shall prove that in equilibrium, the relative amounts of time spent in different states are *independent of the topology.*

At the ith step, call the current solution x_i, call the control parameter (a nonnegative number) c_i, and write $f_i = f(x_i)$. Start with an initial solution x_0, possibly selected randomly, and assume we are given values c_0, dc_i and c_f of the control parameter, obtained as discussed later.

The algorithm repeats the following steps until $c_i \leq c_f$:

1. Generate a neighbour x_p of the current solution x_i. This is a potential candidate for x_{i+1}.

2. Set x_{i+1} to x_p with probability $\min\{1, \exp((f_i - f_p)/c_i)\}$, and to x_i with the complementary probability.

3. Set c_{i+1} to $c_i - dc_i$ and replace i with $i+1$.

Note that we may set $dc_i = 0$; that is, we may hold the system at a fixed control value for many iterations. Also, the above choice of acceptance probability is natural but is not the only possibility, as we shall see later. The effect is that downhill steps are always accepted while uphill steps are more likely to be accepted if they are small than if they are large.

As an illustration of this effect, Fig. 2.1 shows a series of steps taken for each of two separate fixed values of c, in a two dimensional problem with two local minima. (The space has been discretised in a simple way.) A deterministic descent method will always go to the minimum in whose basin it finds itself, whereas the annealing method can climb out of one basin into the next. In the long run, the annealing method will spend a proportion of its time in each basin determined by the function values there, together with the current value of the control parameter. As the control parameter decreases, the proportion of time spent at the global optimum increases: if we run for long enough at a small enough value of c, the global optimum will consume almost all of the steps.

Because of its ability to make uphill steps, the annealing method avoids being trapped in local optima. However, it is far from apparent at first that one could not always do at least as well by running a deterministic algorithm many times, with randomly chosen starting points. The following example shows that cases exist where the annealing algorithm performs much better than such a method.

Let $X \subset R^2$ be the square centred at 0 and of side $2N$ (see Fig. 2.2a); discretise X on a grid of mesh size δ, centre 0. Let $r = \max\{|x_1|, |x_2|\}$ be the l_∞ distance of any point $x \in X$ from 0; and let $f: X \to R$, a function of r only, be as shown in Fig. 2.2b. In fact, $f(x) = r$ except when $r = n + \delta$ where n is any positive integer; in that case, $f = n - \varepsilon$ for some $\varepsilon \ll \delta$. The problem is to find the minimum of f on X, which of course lies at 0.

We describe two search techniques to do the minimization. They are a version of downhill search with random starting point, and the annealing method. To provide a realistic comparison with combinatorial problems, we assume that the downhill search takes steps of size δ and does not have access to gradient information, so it locates a local optimum in a time which is much the same as the annealing method

The approach of using a fixed-point theorem is a well-established strategy that goes back to a 1965 paper [H1] in the case of finite-dimensional systems, see the review article [CQ]. In carrying out this strategy, [Z] relied on the so-called H.U.M. method [L5]. The final result in [Z] is an exact controllability statement for the wave problem (1.1) on the same state space $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$, within the class of $H_0^\gamma(0, T; L_2(\Gamma))$-boundary controls, as in our Theorem 1.1 above, however, only in the range $0 < \gamma < 1$. The limit cases $\gamma = 0$, are explicitly excluded from [Z], as the requirement $0 < \gamma < 1$ is essential to its treatment; the case $\gamma = 0$ is excluded because of lack of the compactness property which is instead required by Schauder fixed-point theorem; the case $\gamma = 1$ is likewise explicitly excluded in [Z], as the requirement $\gamma < 1$ is "essential" [Z, above (2.54)] in that treatment. On the other hand, the limit cases $\gamma = 0$, 1 are the most interesting and natural cases in applications. Thus, the present paper takes these two limit cases as a motivation to restudy the problem over the entire range $0 \le \gamma \le 1$.

In order to overcome the difficulties in the limit cases $\gamma = 0$ and $\gamma = 1$ encountered by the approach in [Z] and thus solve the exact controllability for the wave equation (1.1) for all $0 \le \gamma \le 1$, the issue of exact controllability of semilinear waves and plates is taken up anew in this paper through a direct approach, which is quite different from the one in [Z].

The main technical differences are:

(i) Instead of using H.U.M., this paper uses a direct approach based on the explicit construction of the controllability map.

(ii) Instead of applying Schauder fixed point, we use a global inversion theorem (implicit function theorem) which requires the uniform bound (2.46) or (2.52) below and which dispenses with the need of compactness present in the Schauder approach: this way the case $\gamma = 0$ is also included, where the compactness required by Schauder fixed point simply does not hold true.

(iii) In establishing the required uniform bound (2.46) below, we use in a crucial way that certain families of operators which enter into the description of the problem are collectively compact, a concept already used by the authors in the study of other boundary control problems for second-order hyperbolic mixed problems [LT3, Lemma 3.12]. (If we use Schauder fixed point instead of the global inversion theorem, as in the first version of our paper, then the uniform estimate (2.46) is still the main technical difficulty, which esablishes that the fixed-point map takes the whole space where a fixed point is sought into a ball of finite radius; this way we get the desired controllability result for $0 < \gamma \le 1$, but not for $\gamma = 0$ because of the compactness required by Schauder.) Our aim is to single out the general and essential features of the problem, which are common to various waves and plates equations. This leads to the abstract Theorem 2.1. Specialization of Theorem 2.1 to the wave problem (1.1) produces the new Theorem 1.1 for $\gamma = 0$ and $\gamma = 1$, but also recovers the exact controllability result in [Z] for $0 < \gamma < 1$. However, in Section 3, we explicitly treat only the most demanding cases $\gamma = 1$ and $\gamma = 0$. Application of the present abstract Theorem 2.1 to waves and plates problems relies, as usual, on a uniqueness property of the corresponding linearized homogeneous problem, which is presently known only under certain assumptions of the "potential" function [H4], [H5], [I], [KRS], [PS], [R]. We have several other examples of plates problems where all abstract assumptions of our Theorem 2.1 are already verified save for the uniqueness assumption (C.2), see Remark 4.1. Progress in this uniqueness question [L8] will enlarge the range of applicability of the abstract Theorem 2.1.

2. Abstract Formulation. Statement of Main Result. Proof

In this section we study the question of exact controllability of an abstract semilinear operator equation, subject to certain assumptions. In later sections we verify that these assumptions are natural for, and in fact automatically satisfied by, the dynamics of our interest: wave equations and plate equations.

2.1. *Abstract Formulation. Exact Controllability Problem*

Well-posedness. Let Y and U be two Hilbert spaces. Our basic operator model is the equation

$$\dot{y} = Ay + F(y) + Bu, \qquad y(0) = y_0 \in Y, \tag{2.1}$$

to be interpreted as specified below. Standing assumptions on (2.1) are as follows:

A: $Y \supset \mathscr{D}(A) \to Y$ is the infinitesimal generator of a strongly continuous semigroup on Y, denoted by e^{At}, $t \geq 0$;

B: $B \in L(U; [\mathscr{D}(A^*)]')$, so that[1] $A^{-1}B \in L(U, Y)$, where A^* is the adjoint of A in Y, $[\mathscr{D}(A^*)]'$ is the dual space of $\mathscr{D}(A^*)$ with respect to the Y-topology;

F: $Y \to Y$ is a nonlinear operator, continuous on Y, with Frechet derivative $F'[y] \in L(Y)$ at the point $y \in Y$, satisfying

$$\| F'[y] \|_{L(Y)} \leq \text{const}, \qquad \text{uniformly in} \quad y \in Y. \tag{2.2}$$

Instead of the differential version (2.1) on, say, $[\mathscr{D}(A^*)]'$, we consider its variation of parameter version

$$\begin{cases} y(t) = e^{At}y_0 + (\mathscr{L}u)(t) + (\mathscr{R}\mathscr{F}y)(t), & (2.3) \\ y(T) = e^{AT}y_0 + \mathscr{L}_T u + \mathscr{R}_T\mathscr{F}y, & (2.4) \\ (\mathscr{F}y)(t) = F(y(t)); & (2.5) \end{cases}$$

to be interpreted under some minimal requirement of well-posedness as follows. There exist two Hilbert spaces:

a Hilbert space $\tilde{\mathscr{U}}_T$, based on $[0, T] \times U$, dense in, say $L_2(0, T; U)$, (2.6)

and

a Hilbert space $H \subset Y$ (H will be the space of exact controllability at $t = T$), (2.7)

[1] Without loss of generality for the problem here considered, we may take that A^{-1} is well defined as a bounded operator on all of Y.

such that

$$(\mathscr{L}u)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau\text{: continuous } \tilde{\mathscr{U}}_T \to C([0, T]; Y), \tag{2.8}$$

$$\mathscr{L}_T u = \int_0^T e^{A(T-t)}Bu(t)\, dt\text{: continuous } \tilde{\mathscr{U}}_T \to H. \tag{2.9}$$

Moreover, in (2.3) and (2.4) we have set

$$(\mathscr{R}g)(t) = \int_0^t e^{A(t-\tau)}g(\tau)\, d\tau\text{: continuous } L_1(0, T; Y) \to C([0, T]; Y), \tag{2.10}$$

$$\mathscr{R}_T g = \int_0^T e^{A(T-t)}g(t)\, dt\text{: continuous } L_1(0, T; Y) \to Y. \tag{2.11}$$

By (2.2)–(2.11), a fixed-point solution $y \in C([0, T]; Y)$ of (2.3), i.e., of (2.1), exists for $u \in \tilde{\mathscr{U}}_T$. The space $\tilde{\mathscr{U}}_T$ is invoked only for the well-posedness of (2.1) and is not needed in subsequent sections.

Exact Controllability Problem on the Space H, at Time T, Within the Class of $\mathscr{U}_T$-Controls. We now let $\mathscr{U}_T$ be another Hilbert space $\mathscr{U}_T \subset \tilde{\mathscr{U}}_T$ (the space $\tilde{\mathscr{U}}_T$ is not used anymore in this paper). Given $y_0 \in H$ (resp. $y_T \in H$), we seek, if possible, $u \in \mathscr{U}_T$ such that the corresponding solution of (2.3), (2.4) (resp. with initial condition $y_0 = 0$) satisfies $y(T) = 0$ (resp. $y(T) = y_T$). The two formulations are equivalent in the cases of our interest, which involve time reversible dynamics, see Sections 3 and 4. We consider the following linearized version of (2.1):

$$\dot{z} = Az + F'[\eta]z + Bu + F(0), \qquad z(0) = z_0 \in Y, \tag{2.12}$$

for a fixed arbitrary element $\eta \in Y$. The corresponding variation of parameter version of (2.12) is

$$\begin{cases} z(t) = e^{At}z_0 + (\mathscr{L}u)(t) + (\mathscr{K}[\eta]z)(t) + (\mathscr{R}F(0))(t), & (2.13)\\ z(T) = e^{AT}z_0 + \mathscr{L}_T u + \mathscr{K}_T[\eta]z + \mathscr{R}_T F(0), & (2.14) \end{cases}$$

where we have set, recalling (2.10), (2.11),

$$\mathscr{K}[\eta] = \mathscr{R}F'[\eta], \qquad (\mathscr{K}[\eta]g)(t) = \int_0^t e^{A(t-\tau)}F'[\eta]g(\tau)\, d\tau, \tag{2.15}$$

$$\mathscr{K}_T[\eta] = \mathscr{R}_T F'[\eta], \qquad \mathscr{K}_T[\eta]g = \int_0^T e^{A(T-t)}F'[\eta]g(t)\, dt. \tag{2.16}$$

For each $u \in \mathscr{U}_T$, (2.13), (2.14) define a corresponding solution $z \in C([0, T]; Y)$.

2.2. Assumptions and Statement of Main Result

We make two sets of assumptions throughout: structural assumptions (A.1)–(A.5) on the operators describing model (2.1); and controllability assumptions (C.1) and (C.2) on the linear and linearized versions of problem (2.1).

Structural Assumptions (A.1)–(A.5)

(A.1) *Assumption on the operator $\mathscr{L}$ defined by* (2.8). There exists a Hilbert space $\mathscr{E}_T \supset C([0, T]; Y)$ such that[2]

$$\mathscr{L}: \mathscr{U}_T \to \mathscr{E}_T \text{ is continuous.} \tag{2.17}$$

Moreover, either

$$\mathscr{L}: \mathscr{U}_T \to \mathscr{E}_T \qquad \text{is compact} \tag{2.18a}$$

or else

$$\mathscr{R}_T: \bigcup_{\eta \in \mathscr{E}_T} F'[\eta]g \to H \qquad \text{is compact,} \quad \text{for each } g \in \mathscr{E}_T \text{ fixed.} \tag{2.18b}$$

(A.2) *Assumptions on the family $\mathscr{K}[\eta] = \mathscr{R}F'[\eta]$ defined by* (2.15).

(a) The following family of operators is collectively compact in the parameter $\eta \in \mathscr{E}_T$:

$$\mathscr{K}[\eta] = \mathscr{R}F'[\eta]: \mathscr{E}_T \to \mathscr{E}_T. \tag{2.19}$$

This means, explicitly, that the following two properties hold true:

1. For each mixed $\eta \in \mathscr{E}_T$, the operator $\mathscr{K}[\eta]: \mathscr{E}_T \to \mathscr{E}_T$ is compact. (2.20a)

2. The set union $\bigcup_{\eta \in \mathscr{E}_T} \mathscr{K}[\eta]$ (unit ball of $\mathscr{E}_T$) is a precompact set in $\mathscr{E}_T$, (2.20b)

   where the union of the image under $\mathscr{K}[\eta]$ of the unit ball in $\mathscr{E}_T$ is taken over all η running in $\mathscr{E}_T$.

(b) For any sequence $\eta_n \in \mathscr{E}_T$, $n = 1, 2, \ldots$, we can extract a subsequence η_{n_k}, $k = 1, 2, \ldots$, such that

$$\mathscr{K}[\eta_{n_k}] = \mathscr{R}F'[\eta_{n_k}] \to \mathscr{K}^0 = \mathscr{R}F_0 \qquad \text{strongly in } \mathscr{E}_T \tag{2.21}$$

for a suitable operator $F_0 \in L(Y)$, which depends on the subsequence. As a consequence of (2.21) and (2.20), we obtain that $\mathscr{K}^0$ is compact [A1, p. 5] and that

$$\|\mathscr{K}[\eta]\|_{L(\mathscr{E}_T)} \leq \text{const} \qquad \text{uniformly in} \quad \eta \in \mathscr{E}_T. \tag{2.22}$$

(A.3) *Assumption on the family $\mathscr{K}_T[\eta] = \mathscr{R}_T F'[\eta]$ defined by* (2.16). The family of operators

$$\mathscr{K}_T[\eta] = \mathscr{R}_T F'[\eta]: \mathscr{E}_T \to H \tag{2.23}$$

has the property that, for any sequence $\eta_n \in \mathscr{E}_T$, $n = 1, 2, \ldots$, we can extract a subsequence η_{n_k}, $k = 1, 2, \ldots$, such that

$$\mathscr{K}_T[\eta_{n_k}] = \mathscr{R}_T F'[\eta_{n_k}] \to \text{some } \mathscr{K}_T^0 = \mathscr{R}_T F_0 \qquad \text{weakly in } H \text{ from } \mathscr{E}_T, \tag{2.24}$$

[2] In applications there is much flexibility in the choice of $\mathscr{E}_T$.

i.e., $(\mathscr{K}_T[\eta_{n_k}]g, h)_H \to (\mathscr{K}_T^0 g, h)_H$, $\forall g \in \mathscr{E}_T$, $\forall h \in H$, for a suitable operator $F_0 \in L(Y)$, which depends on the subsequence. As a consequence of (2.24) we obtain

$$\| \mathscr{K}_T[\eta] \|_{L(\mathscr{E}_T, H)} \leq \text{const}_T \qquad \text{uniformly in} \quad \eta \in \mathscr{E}_T. \tag{2.25}$$

The above assumption is used in the first alternative of (A.1), i.e., when $\mathscr{L}$ is compact as in (2.18a). On the other hand, in the second alternative of (A.1), i.e., when $\mathscr{R}_T$ is compact as in (2.18b), then the convergence in (2.24) is strongly in H in view of (2.2): for each $g \in \mathscr{E}_T$ and any sequence $\{\eta_n\} \in \mathscr{E}_T$ we can, by (2.2), extract a subsequence $\{\eta_{n_k}\}$ such that $F'[\eta_{n_k}]g$ is weakly convergent in $\mathscr{E}_T$ and so, by (2.18b), we then have that $\mathscr{R}_T F'[\eta_{n_k}]g$ is strongly convergent in H.

(A.4) *Assumption on the trajectory* $(\mathscr{R}F(0))(t)$, $0 \leq t \leq T$. We have

$$\text{trajectory } \{(\mathscr{R}F(0))(t), 0 \leq t \leq T\} \subset \text{compact set of } \mathscr{E}_T. \tag{2.26}$$

(A.5) *Assumption on the point* $\mathscr{R}_T F(0)$. We have

$$\mathscr{R}_T F(0) \in H. \tag{2.27}$$

Before formulating our controllability assumptions, we need the following considerations. As a consequence of our assumption (2.20a), it follows that, for any $\eta \in \mathscr{E}_T$,

$$(I - \mathscr{K}[\eta])^{-1} \in L(\mathscr{E}_T). \tag{2.28}$$

Indeed, by the compactness property in (2.20a), it suffices (and is equivalent) to show the following injectivity statement:

$$(I - \mathscr{K}[\eta])f = 0, \text{ say } f \in L_2(0, T; Y), \text{ implies } f = 0. \tag{$*$}$$

By (2.15), we have that identity (*) implies $\dot{f} = Af + F'[\eta]f$, $f(0) = 0$ and so $f = \exp\{(A + F'[\eta])t\}f(0) \equiv 0$, and (2.28) is proved.

Linearized Problem. Returning to (2.13) and using (2.28), we have, by (2.17) and (2.26) with $z_0 \in Y$,

$$z(t) = (I - \mathscr{K}[\eta])^{-1}[e^{A\cdot} z_0 + \mathscr{L}u + \mathscr{R}F(0)] \in \mathscr{E}_T, \tag{2.29}$$

which used in (2.14) yields, for $u \in \mathscr{U}_T$,

$$z(T) = e^{AT} z_0 + \{\mathscr{L}_T + \mathscr{K}_T[\eta](I - \mathscr{K}[\eta])^{-1} \mathscr{L}\}u + \zeta, \tag{2.30}$$

where ζ is a fixed vector of H (see (2.23), (2.28), (2.26), and (2.27)) and

$$\zeta = \mathscr{K}_T[\eta](I - \mathscr{K}[\eta])^{-1}[e^{A\cdot} z_0 + \mathscr{R}F(0)] + \mathscr{R}_T F(0) \in H. \tag{2.31}$$

Equations (2.29)–(2.31) give explicitly the map $u \to z$ of the linearized problem (2.13), (2.14). Note that $\zeta = 0$, if $z_0 = 0$ and $F(0) = 0$. Moreover, if $z_0 = 0$, then $z(T) \in H$, by (2.17), (2.23), (2.28), (2.9), and (2.31).

Controllability Assumptions (C.1) and (C.2)

(C.1) (Exact controllability from the origin on the space H, at time T, of the linear problem (2.1) with $y_0 = 0$, $F = 0$, within the class of $\mathscr{U}_T$-controls.)

$$\mathscr{L}_T\colon \mathscr{U}_T \to H \qquad \text{is surjective (onto).} \tag{2.32}$$

(C.2) (Approximate controllability from the origin of the linearized problem (2.13), (2.14), and its limit version in the sense of (2.21), (2.24), with $z_0 = 0$, $F(0) = 0$.) With reference to (2.30) with $z_0 = 0$, $F(0) = 0$, hence $\zeta = 0$, we assume that:

(a) For each fixed $\eta \in \mathscr{E}_T$, the map

$$\mathscr{M}_T[\eta]\colon u \to z(T) = \mathscr{M}_T[\eta]u$$
$$= \{\mathscr{L}_T + \mathscr{K}_T[\eta](I - \mathscr{K}[\eta])^{-1}\mathscr{L}\}u\colon \mathscr{U}_T \to H \tag{2.33}$$

has range dense in H (in the topology of H).

(b) Also, let $\mathscr{K}^0$ and $\mathscr{K}_T^0$ be any of the limit operators obtained in (2.21) and (2.24). We likewise assume that the operator (which is well defined, see Lemma 2.2 below)

$$\mathscr{M}_T^0 \equiv \mathscr{L}_T + \mathscr{K}_T^0(I - \mathscr{K}^0)^{-1}\mathscr{L}\colon \mathscr{U}_T \to H \tag{2.34}$$

has range dense in H (in the topology of H). An equivalent formulation of (2.33), (2.34) is that the Hilbert adjoint map/operator $\mathscr{M}_T^*\colon H \to \mathscr{U}_T$ in the sense

$$(\mathscr{M}_T u, y)_H = (u, \mathscr{M}_T^* y)_{\mathscr{U}_T},$$

want $\mathscr{M}_T$ either $\mathscr{M}_T[\eta]$ or $\mathscr{M}_T^0$, has trivial null space $\mathscr{N}$ in H; i.e., for each $\eta \in \mathscr{E}_T$,

$$\mathscr{N}\{\mathscr{M}_T^*[\eta]\} = \mathscr{N}\{\mathscr{L}_T^* + \mathscr{L}^*(I - \mathscr{K}^*[\eta])^{-1}\mathscr{K}_T^*[\eta]\} = \{0\} \qquad \text{in } H, \tag{2.35}$$

equivalent to (2.33), and likewise

$$\mathscr{N}\{(\mathscr{M}_T^0)^*\} = \mathscr{N}\{\mathscr{L}_T^* + \mathscr{L}^*[I - (\mathscr{K}^0)^*]^{-1}(\mathscr{K}_T^0)^*\} = \{0\} \qquad \text{in } H, \tag{2.36}$$

equivalent to (2.34).

Our main exact controllability result (from the origin) for problem (2.1) is as follows.

Theorem 2.1. *Assume* (A.1)–(A.5), (C.1), *and* (C.2). *Then, for any* $y_T \in H$, *there exists* $u \in \mathscr{U}_T$, *such that the corresponding solution* y *of* (2.1) (*or* (2.3), (2.4)) *with* $y_0 = 0$ *satisfies* $y(T) = y_T$.

2.3. *Proof of Theorem 2.1*

The proof employs a global inversion theorem.

Step 1. Let $y_T \in H$ be assigned. By (C.1) = (2.32), there exists $v_T^0 \in \mathscr{U}_T$ (constructed below) such that

$$\mathscr{L}_T v_T^0 = y_T \quad \text{and, in fact,} \quad v_T^0 = \mathscr{L}_T^\# y_T \in [\mathscr{N}(\mathscr{L}_T)]^\perp \subset \mathscr{U}_T, \tag{2.37}$$

where $\mathscr{L}_T^\#$ is the pseudoinverse of $\mathscr{L}_T\colon \mathscr{U}_T \to H$. This means that if

$$\mathscr{U}_T = \mathscr{N}(\mathscr{L}_T) + [\mathscr{N}(\mathscr{L}_T)]^\perp \tag{2.38}$$

denotes the orthogonal decomposition of the Hilbert space $\mathscr{U}_T$, in terms of the null space of $\mathscr{L}_T$ and its orthogonal complement, then

$$\mathscr{L}_T^\# = (\mathscr{L}_T|_{[\mathscr{N}(\mathscr{L}_T)]^\perp})^{-1}\colon H \to [\mathscr{N}(\mathscr{L}_T)]^\perp, \tag{2.39}$$

where

$$\begin{cases} \mathscr{L}_T|_{[\mathscr{N}(\mathscr{L}_T)]^\perp} = \mathscr{L}_T \text{ restricted over } [\mathscr{N}(\mathscr{L}_T)]^\perp, & (2.40) \\ \mathscr{L}_T \mathscr{L}_T^\# = \text{identity on } H, & (2.41) \\ \mathscr{L}_T^\# \mathscr{L}_T = \Pi_T = \text{orthogonal projection } H \text{ onto } [\mathscr{N}(\mathscr{L}_T)]^\perp, & (2.42a) \\ \mathscr{L}_T^\# \mathscr{L}_T = \text{identity over } [\mathscr{N}(\mathscr{L}_T)]^\perp. & (2.42b) \end{cases}$$

Accordingly, we henceforth restrict our search to control functions $\mu \in [\mathscr{N}(\mathscr{L}_T)]^\perp$ such that, replacing u in (2.4) with such μ, the solution map $\mu \to y(T)$ of problem (2.4) with $y_0 = 0$ satisfies $y(T) = y_T$. Once such a μ is found, any other $u \in \mathscr{U}_T$ with projection $\Pi_T u = \mu$ will also yield $y(T) = y_T$.

Applying $\mathscr{L}_T^\#$ to (2.4) with u there replaced by μ now and with $y_0 = 0$ and using (2.37) and (2.42) yields

$$v_T^0 = \mu + \mathscr{L}_T^\# \mathscr{R}_T F(y(\mu)) = \mu + \Lambda[\mu] \in [\mathscr{N}(\mathscr{L}_T)]^\perp \subset \mathscr{U}_T, \tag{2.43a}$$

where the operator Λ is defined by

$$\Lambda[\mu] = \mathscr{L}_T^\# \mathscr{R}_T F(y(\mu)). \tag{2.43b}$$

Step 2. Our final objective is to show that the C^1-map

$$g[\mu] = \mu + \Lambda[\mu]\colon [\mathscr{N}(\mathscr{L}_T)]^\perp \to \text{itself} \tag{2.44}$$

is, in fact, surjective (onto), hence a homeomorphism of $[\mathscr{N}(\mathscr{L}_T)]^\perp$ onto itself. To this end, we invoke a global inversion theorem: the map g in (2.44) is a homeomorphism of $[\mathscr{N}(\mathscr{L}_T)]^\perp$ onto itself, provided that its Frechet derivative

$$g'[\mu] = I + \Lambda'[\mu] \tag{2.45}$$

is a boundedly invertible operator on all of $[\mathscr{N}(\mathscr{L}_T)]^\perp$ at each μ, with inverse uniformly bounded in μ, i.e., provided that

$$\|(g'[\eta])^{-1}\| = \|(I + \Lambda'[\mu])^{-1}\| \le \text{const} < \infty, \qquad \text{uniformly in } \mu, \tag{2.46}$$

where the norm in (2.46) is the (uniform) norm of $L([\mathscr{N}(\mathscr{L}_T)]^\perp)$, see, e.g., [S1, Theorem 1.22, p. 16], [B], and [D, Section 15.2, p. 152].

Step 3. **Claim.** *Recalling* (2.8), (2.9) (2.15), *and* (2.16) *we have*

$$\Lambda'[\mu] = \mathscr{L}_T^{\#}\mathscr{K}_T[\mu](I - \mathscr{K}[\eta])^{-1}\mathscr{L}, \tag{2.47}$$

where the inverse in (2.47) *is well defined by* (2.28).

Proof of Claim. By (2.43b) we obtain via (2.16), for each μ,

$$\Lambda'[\mu] = \mathscr{L}_T^{\#}\mathscr{R}_T F'(y(\mu))\frac{dy}{d\mu} = \mathscr{L}_T^{\#}\mathscr{K}_T[\eta]\frac{dy}{d\mu}. \tag{2.48}$$

On the other hand, from (2.3) with ($y_0 = 0$ and) u replaced by μ we obtain via (2.15)

$$\frac{dy}{d\mu} = \mathscr{L} + \mathscr{R}F'(y(\mu))\frac{dy}{d\mu} = \mathscr{L} + \mathscr{K}[\eta]\frac{dy}{d\mu},$$

or

$$\frac{dy}{d\mu} = (I - \mathscr{K}[\eta])^{-1}\mathscr{L}, \tag{2.49}$$

where the inverse is well defined by (2.28). Inserting (2.49) into (2.48) yields the desired conclusion (2.47). □

In view of (2.46), (2.47), we redefine, for each $\eta \in \mathscr{E}_T$, the operator $\Lambda'[\mu]$ by calling it $C_T[\eta]$ (since it will be related to the approximate controllability property (C.2)), i.e., we set

$$\Lambda'[\eta] = C_T[\eta] = \mathscr{L}_T^{\#}\mathscr{K}_T[\eta](I - \mathscr{K}[\eta])^{-1}\mathscr{L}\colon [\mathscr{N}(\mathscr{L}_T)]^{\perp} \to \text{itself}. \tag{2.50}$$

In addition, if $\mathscr{K}_0$ and $\mathscr{K}_T^0$ is any of the strong, respectively weak, limits in (2.21), (2.24) we define, likewise,

$$C_T^0 = \mathscr{L}_T^{\#}\mathscr{K}_T^0(I - \mathscr{K}^0)^{-1}\mathscr{L}\colon [\mathscr{N}(\mathscr{L}_T)]^{\perp} \to \text{itself}. \tag{2.51}$$

Step 4. Thus by (2.46), (2.50), our final goal will be to show that the operator $g'[\mu] = I + \Lambda'[\eta] = I + C_T[\eta]$ has an inverse in $L[\mathscr{N}(\mathscr{L}_T)]^{\perp}$ which is uniformly bounded here in μ:

$$\begin{aligned}\|(I + \Lambda'[\mu])^{-1}\| &= \|(I + C_T[\eta])^{-1}\|\\ &= \|\{I + \mathscr{L}_T^{\#}\mathscr{K}_T[\eta](I - \mathscr{K}[\eta])^{-1}\mathscr{L}\}^{-1}\| \le \text{const} < \infty,\\ &\qquad \text{uniformly in } \mu,\end{aligned} \tag{2.52}$$

where the norm in (2.52) is the (uniform) norm of $[\mathscr{N}(\mathscr{L}_T)]^{\perp}$. Two inversions are involved in (2.52) and both inverses have to be shown uniformly bounded in μ. The first task is accomplished in Lemma 2.2, equation (2.53) below; the second and conclusive task is accomplished in Lemma 2.5, equation (2.77) below.

Step 5. **Lemma 2.2.** *Not only do we have* $(I - \mathscr{K}[\eta])^{-1} \in L(\mathscr{E}_T)$ *for each* $\eta \in \mathscr{E}_T$ *as was shown in* (2.28), *but moreover*:

(a_1) $\|(I - \mathscr{K}[\eta])^{-1}\|_{L(\mathscr{E}_T)} \le \text{const}$, *uniformly in* $\eta \in \mathscr{E}_T$. (2.53) (a$_2$) *Let η_{n_k} be a subsequence of a given arbitrary sequence $\eta_n \in \mathscr{E}_T$, such that*

$$\mathscr{K}[\eta_{n_k}] \to \text{some } \mathscr{K}^0 = \mathscr{R}F_0, \qquad \text{strongly in } \mathscr{E}_T, \tag{2.54}$$

for some $F_0 \in \boldsymbol{L}(Y)$, as guaranteed by assumption (A.2b) = (2.21); *then, as noted below* (2.21), *$\mathscr{K}^0$ is compact in $\mathscr{E}_T$*[A1, *p.* 5]; *and moreover $(I - \mathscr{K}^0)^{-1} \in \boldsymbol{L}(\mathscr{E}_T)$ and*

$$(I - \mathscr{K}[\eta_{n_k}])^{-1} \to (I - \mathscr{K}^0)^{-1} \qquad \text{strongly in } \mathscr{E}_T. \tag{2.55}$$

Proof of Lemma 2.2. Property (a$_2$) implies property (a$_1$). We show at once the bound (2.53) with $\eta = \eta_{n_k}$ uniformly in k, as well as the strong convergence in (2.55). We have already noted that $\mathscr{K}^0$ is compact on $\mathscr{E}_T$ [A1, p. 5] because of (A.2); thus the argument which showed (2.28) now applies to $\mathscr{K}^0$ of the form $\mathscr{K}^0 = \mathscr{R}F_0$ by use of (2.15), yielding $(I - \mathscr{K}^0)^{-1} \in \boldsymbol{L}(\mathscr{E}_T)$. Then, assumption (A.2) that $\mathscr{K}[\eta_{n_k}]$ is collectively compact and strongly convergent implies, via Theorem 1.6, p. 8, of [A1], that (2.53) holds true for $\eta = \eta_{n_k}$, in which case (2.55) follows as well. □

Lemma 2.3.

(a) *The operators $\boldsymbol{C}_T[\eta]$ in* (2.50) *for each $\eta \in \mathscr{E}_T$ and the operator $\boldsymbol{C}_T^0$ in* (2.51) *are all compact.*

(b) *By virtue of the approximate controllability assumptions* (C.2), (*i.e.*, (2.33) *and* (2.34), *or equivalently* (2.35) *and* (2.36)), *the operators*

$$I + \boldsymbol{C}_T[\eta] = I + \mathscr{L}_T^{\#}\mathscr{K}_T[\eta](I - \mathscr{K}[\eta])^{-1}\mathscr{L}\colon$$

$$[\mathscr{N}(\mathscr{L}_T)]^{\perp} \to \text{itself}, \qquad \text{for each} \quad \eta \in \mathscr{E}_T, \tag{2.56}$$

$$I + \boldsymbol{C}_T^0 = I + \mathscr{L}_T^{\#}\mathscr{K}_T^0(I - \mathscr{K}^0)^{-1}\mathscr{L}\colon [\mathscr{N}(\mathscr{L}_T)]^{\perp} \to \text{itself} \tag{2.57}$$

are all boundedly invertible:

$$(I + \boldsymbol{C}_T[\eta])^{-1} \in \boldsymbol{L}([\mathscr{N}(\mathscr{L}_T)]^{\perp}), \tag{2.58}$$

$$(I + \boldsymbol{C}_T^0)^{-1} \in \boldsymbol{L}([\mathscr{N}(\mathscr{L}_T)]^{\perp}). \tag{2.59}$$

Proof. (a) Compactness of $\boldsymbol{C}_T[\eta]$, $\boldsymbol{C}_T^0$ is a consequence of either compactness of $\mathscr{L}$ in (2.18a), or else of compactness of $\mathscr{R}_T$ in (2.18b), in assumption (A.1).

(b) Thus to show, say, (2.58), it is equivalent to show that

$$(I + \boldsymbol{C}_T^*[\eta])^{-1} \in \boldsymbol{L}([\mathscr{N}(\mathscr{L}_T)]^{\perp}), \tag{2.60}$$

where the adjoint is taken in $[\mathscr{N}(\mathscr{L}_T)]^{\perp} \subset \mathscr{U}_T$, and, in fact, because of compactness of $\boldsymbol{C}_T^*[\eta]$, it suffices (and is equivalent) to show that $I + \boldsymbol{C}_T^*[\eta]$ is injective on $[\mathscr{N}(\mathscr{L}_T)]^{\perp}$. But, by (2.56), (2.42b), and (2.41) on $[\mathscr{N}(\mathscr{L}_T)]^{\perp}$:

$$\begin{aligned} I + \boldsymbol{C}_T^*[\eta] &= I + \mathscr{L}^*(I - \mathscr{K}^*[\eta])^{-1}\mathscr{K}_T^*[\eta](\mathscr{L}_T^{\#})^* \\ &= \mathscr{L}_T^*(\mathscr{L}_T^{\#})^* + \mathscr{L}^*(I - \mathscr{K}^*[\eta])^{-1}\mathscr{K}_T^*[\eta](\mathscr{L}_T^{\#})^* \\ &= \{\mathscr{L}_T^* + \mathscr{L}^*(I - \mathscr{K}^*[\eta])^{-1}\mathscr{K}_T^*[\eta]\}(\mathscr{L}_T^{\#})^*, \end{aligned} \tag{2.61}$$

where * denotes Hilbert space adjoints;

$$(\mathscr{L}_T^{\#})^*: [\mathscr{N}(\mathscr{L}_T)]^{\perp} \to H, \qquad \mathscr{K}_T^*: H \to \mathscr{E}_T, \qquad \mathscr{L}^*: \mathscr{E}_T \to \mathscr{U}_T,$$
$$\mathscr{K}^*: \mathscr{E}_T \to \mathscr{E}_T, \qquad \mathscr{L}_T^*: H \to \mathscr{U}_T.$$

But $\mathscr{L}_T^{\#}$ is surjective (by definition (2.39)) and so $(\mathscr{L}_T^{\#})^*$ is injective. Thus, (2.61) says that $I + C_T^*[\eta]$ is injective provided that the operator { } in (2.61) is injective in $[\mathscr{N}(\mathscr{L}_T)]^{\perp}$; and this is the case by assumption (2.35). The proof is identical for (2.59) using assumption (2.36). □

Step 6. To show (2.52) we need a specialization of the following lemma, which at no extra effort we put in a general framework.

Lemma 2.4. *Let Z_i be two Banach spaces, $i = 1, 2$, let Q be a compact operator $Z_1 \to Z_2$, and let $W(p)$ be a family of bounded operators $Z_2 \to Z_1$ depending on the parameter $p \in \mathscr{P}$, such that $W(p_n) \to W^0$ weakly for any sequence p_n, with W^0: $Z_2 \to Z_1$ depending on the sequence. Assume further that the operators $I + W(p)Q$ and $I + W^0Q$ are all injective on Z_1, and hence boundedly invertible on Z_1. Then*

$$\| [I + W(p)Q]^{-1} \|_{L(Z_1)} \leq \text{const}, \qquad \textit{uniformly in} \quad p \in \mathscr{P}, \tag{2.62}$$

and the weak convergence as in (2.74) *below holds true.*

First proof. We must show that there exists a constant $C > 0$ such that, for all $z \in Z_1$, we have

$$\| z + W(p)Qz \| \geq C \| z \|, \qquad \text{uniformly in} \quad p \in \mathscr{P}, \tag{2.63}$$

in the norms of Z_1. Suppose not. Then there exist sequences $\{z_n\}$ in Z_1 and $\{p_n\}$ in $\mathscr{P}$ such that

$$\| z_n \| \equiv 1 \quad \text{yet} \quad z_n + W(p_n)Qz_n \to 0 \qquad (\text{strongly in } Z_1). \tag{2.64}$$

Thus, we can extract a subsequence, still denoted by z_n, such that

$$z_n \to \text{some } z \qquad \text{weakly in } Z_1 \tag{2.65}$$

and hence, by the assumed compactness of Q,

$$Qz_n \to Qz \qquad \text{strongly in } Z_2. \tag{2.66}$$

The strong convergence in (2.66) and the weak convergence $W(p_n) \to W^0$ easily imply [K1, p 151]

$$W(p_n)Qz_n \to W^0Qz \qquad \text{weakly in } Z_1. \tag{2.67}$$

By (2.64), (2.65), and (2.67) we have the weak limit equals the strong limit and

$$z + W^0Qz = 0. \tag{2.68}$$

But, by assumption, (2.68) implies $z = 0$. We now conclude the proof by establishing a contradiction between the property that $z = 0$ just obtained and $\| z_n \| \equiv 1$ as in (2.64) left. Indeed, since $z = 0$ we have $Qz_n \to 0$ strongly in Z_2 by (2.66), and since $\| W(p_n) \|$ is uniformly bounded in n from the assumption, we have

$$\| W(p_n)Qz_n \| \leq \text{const} \| Qz_n \| \to 0 \quad \text{or} \quad W(p_n)Qz_n \to 0 \qquad \text{(strongly in } Z_1). \tag{2.69}$$

Then (2.69) and (2.64) (right) imply that $z_n \to 0$ strongly in Z_1 and this contradicts $\| z_n \| \equiv 1$. □

Second proof. (It provides more information about the adjoint family.) By the assumptions, Q^* is compact and $W^*(p_n) \to (W^0)^*$ weakly for any sequence $\{p_n\}$ in $\mathscr{P}$, and hence

$$Q^*W^*(p_n) \to Q^*(W^0)^* \qquad \text{strongly in } Z_1, \tag{2.70}$$

$$Q^*W^*(p) \text{ is a collectively compact family in } Z_1 \text{ in the parameter } p \in \mathscr{P}. \tag{2.71}$$

(Indeed, for any sequence $\{p_n\}$ and any $\{z_n\}$, $\| z_n \| \leq 1$, we can extract a convergent subsequence from $\{Q^*W^*(p_n)z_n\}$ since $\| W^*(p) \|$ is uniformly bounded in p and Q^* is compact and p. 12 of [A1] applies.) By virtue of (2.70) and (2.71), as well as by virtue that $[I + Q^*(W^0)^*]^{-1}$ is a bounded operator in all of Z_1, from the assumptions we can appeal again to Theorem 1.6, p. 8 of [A1] as was done in the proof of Lemma 2.2. As a result we obtain, for any sequence $\{p_n\} \in \mathscr{P}$,

$$\begin{aligned} &[I + Q^*W^*(p_n)]^{-1} \to [I + Q^*(W^0)^*]^{-1} \qquad \text{strongly in } Z_1, \\ &\qquad \| [I + Q^*W^*(p_n)]^{-1} \| \leq \text{const}, \qquad \text{uniformly in } n, \end{aligned} \tag{2.72}$$

and hence

$$\| [I + W(p)Q]^{-1} \| = \| [I + Q^*W^*(p)]^{-1} \| \leq \text{const}, \qquad \text{uniformly in } \quad p \in \mathscr{P}, \tag{2.73}$$

as desired, from which it follows via (2.72) that, for any $\{p_n\} \in \mathscr{P}$, we have

$$[I + W(p_n)Q]^{-1} \to [I + W^0Q]^{-1} \qquad \text{weakly in } Z_1. \quad \square \tag{2.74}$$

The abstract Lemma 2.4 is specialized in the next result, part (ii) (at least in the more demanding case where $\mathscr{L}$ is compact as in (2.18a), but $\mathscr{R}_T$ is not compact as in (2.18b)).

Lemma 2.5.

(i) *Let η_{n_k} be a subsequence of a given arbitrary sequence $\eta_n \in \mathscr{E}_T$, such that* (2.54) *holds true as well as*

$$\mathscr{K}_T[\eta_{n_k}] \to \mathscr{K}_T^0 = \mathscr{R}_T F_0 \qquad \textit{weakly in } H \tag{2.75}$$

as guaranteed by (2.24) *in assumption* (A.3). *Then, see* (2.50),

$$C_T[\eta_{n_k}] \textit{ converges weakly to } \quad C_T^0 = \mathscr{L}_T^{\#} \mathscr{K}_T^0 (I - \mathscr{K}^0)^{-1} \mathscr{L} \quad \textit{in } [\mathscr{N}(\mathscr{L}_T)]^{\perp}, \tag{2.76}$$

while convergence in both (2.75) *and* (2.76) *is "strong" if* $\mathcal{R}_T$ *is compact as in* (2.18b).

(ii) *Not only do we have* $(I + \boldsymbol{C}_T[\eta])^{-1} \in \boldsymbol{L}([\mathcal{N}(\mathcal{L}_T)]^\perp)$ *for each* $\eta \in \mathscr{E}_T$ *as in* (2.58), *but in fact*

$$\|(I + \boldsymbol{C}_T[\eta])^{-1}\|_{\boldsymbol{L}([\mathcal{N}(\mathcal{L}_T)]^\perp)} \le \text{const}, \qquad \textit{uniformly in} \quad \eta \in \mathscr{E}_T. \tag{2.77}$$

Proof. (i) Properties (2.55) and (2.75) imply that the operator family

$$W_T[\eta] \stackrel{\text{def}}{=} \mathcal{L}_T^\# \mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1} \tag{2.78}$$

satisfies

$$W_T[\eta_{n_k}] \to \mathcal{L}_T^\# \mathcal{K}_T^0(I - \mathcal{K}^0)^{-1} \qquad \text{weakly in} \quad \mathscr{E}_T \to [\mathcal{N}(\mathcal{L}_T)]^\perp \tag{2.79}$$

from which it follows that (see (2.50) and (2.51)) $\boldsymbol{C}_T[\eta_{n_k}] = W_T[\eta_{n_k}]\mathcal{L} \to \boldsymbol{C}_T^0$ weakly as desired, and (2.76) is proved.

(ii) First assume the first alternative in (A.1) that $\mathcal{L}$ is compact as in (2.18a). Then this part is merely a specialization of Lemma 2.4 with the spaces Z_1, Z_2 there given by the spaces $\mathcal{U}_T$, $\mathscr{E}_T$ now, respectively; the operator Q there given by the operator $\mathcal{L}$ now, which is compact by assumption (2.18a); the parameter $p \in \mathcal{P}$ there given by the parameter $\eta \in \mathscr{E}_T$ now; the family $W(p)$ there given by the family $W_T[\eta]$ in (2.78) now, which is weakly convergent to W^0 there given by $\mathcal{L}_T^\# \mathcal{K}_T^0$ $(I - \mathcal{K}^0)^{-1}$ now via (2.79); the operators $I + W(p)Q$ and $I + W^0Q$ there given by the operators $I + \boldsymbol{C}_T[\eta]$ and $I + \boldsymbol{C}_T^0$ now, which are injective with bounded inverse by (2.58), (2.59). Thus, the uniform bound (2.62) there specializes to the uniform bound (2.77) now, at least under the assumption (2.18a) that $\mathcal{L}$ be compact.

Next, suppose that $\mathcal{L}$ is merely bounded as in (2.17) while now the second alternative in (A.1) holds true that $\mathcal{R}_T$ is compact as in (2.18b). This case is far simpler: now the convergence in (2.75) and in (2.76) is strong (in view of (2.2), (2.24), and (2.50) as noted below (2.25) in the first case); while $\boldsymbol{C}_T[\eta]$ is a collectively compact family on $[\mathcal{N}(\mathcal{L}_T)]^\perp$ in the parameter $\eta \in \mathscr{E}_T$. Thus, since $(I + \boldsymbol{C}_T^0)$ is injective with bounded inverse by (2.59), we can invoke directly Theorem 1.6, p. 8, of [A1] and conclude with the uniform bound (2.77). (That $\boldsymbol{C}_T[\eta]$ is collectively compact follows from the fact that, for any sequence $\{\eta_n\}$ and any bounded sequence $\{z_n\}$, we can extract a convergence subsequence from $\{\boldsymbol{C}_T[\eta_n]z_n\}$, see p. 12 of [A1]). □

Thus, the uniform bound (2.52) is proved, and so is Theorem 2.1 □

Remark 2.1. We write explicitly the specialization of the additional information obtained in the second proof of Lemma 2.4 in (2.70), (2.71), (2.72), (2.73), and (2.74) respectively (even if we do not use it in the following), which applies to the first alternative when $\mathscr{L}_T$ is compact as in (2.18a), while $\mathscr{R}_T$ is not compact as in (2.18b):

$$C_T^*[\eta] \to (C_T^0)^* \qquad \text{strongly in } [\mathcal{N}(\mathscr{L}_T)]^\perp, \tag{2.80}$$

$C_T^*[\eta]$ is a collectively compact family on $[\mathcal{N}(\mathscr{L}_T)]^\perp$ in the parameter $\eta \in \mathscr{E}_T$. (2.81)

$$(I + C_T^*[\eta_{n_k}])^{-1} \to (I + (C_T^0)^*)^{-1} \qquad \text{strongly in } [\mathcal{N}(\mathscr{L}_T)]^\perp, \tag{2.82}$$

$$(I + C_T[\eta_{n_k}])^{-1} \to (I + C_T^0)^{-1} \qquad \text{weakly in } [\mathcal{N}(\mathscr{L}_T)]^\perp \tag{2.83}$$

for any sequence $\eta_{n_k} \in \mathscr{E}_T$ such that the strong limit (2.54) and the weak limit (2.75) hold true.

3. Application: A Semilinear Wave Equation with Dirichlet Boundary Control. Problem (1.1)

The goal of this section is to show that the semilinear wave problem (1.1) subject to (1.2) fits automatically the abstract model of Section 2 on appropriate spaces. As a consequence, Theorem 1.1 is nothing but a specialization of the abstract Theorem 2.1. As mentioned in Section 1, the procedure of this section is readily adapted to obtain exact controllability results for the wave problem (1.1) on any state space $H = H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$ using the control space $\mathscr{U}_T = H_0^\gamma(0, T; L_2(\Gamma))$, $0 \le \gamma \le 1$, $\gamma \neq \frac{1}{2}$, as well as the special case $\gamma = \frac{1}{2}$. However, in this section we explicitly treat only the most demanding and most desirable cases $\gamma = 1$ (in Subsection 3.1) and $\gamma = 0$ (in Subsection 3.2), which are not covered by the methods in [Z].

3.1. *The Case $\gamma = 1$ in Theorem 1.1 for Problem (1.1)*

With reference to the setting of Section 2, the following is the relevant specialization for the wave problem (1.1) in the case $\gamma = 1$ of Theorem 1.1:

$$Y = L_2(\Omega) \times H^{-1}(\Omega), \qquad H = H_0^1(\Omega) \times L_2(\Omega), \tag{3.1}$$

$$U = L_2(\Gamma), \qquad \mathscr{U}_T = H_0^1(0, T; L_2(\Gamma)), \tag{3.2}$$

$$\mathscr{E}_T = L_2(0, T; L_2(\Omega) \times H^{-1}(\Omega)). \tag{3.3}$$

3.1.1. Verification of Assumption (C.1): Exact Controllability of Linear System. We verify assumption (C.1) of exact controllability of the linear problem (1.1) with $f \equiv 0$.

Theorem 3.1. *Let $f \equiv 0$ in* (1.1) *and let $T > 0$ be sufficiently large as in Appendix* A. *Then, for any given pair $\{w_0, w_1\} \in H_0^1(\Omega) \times L_2(\Omega)$, there exists a suitable control function $u \in H_0^1(0, T; L_2(\Gamma))$ such that the corresponding solution of problem* (1.1) *with $f \equiv 0$ satisfies*

$$w(T, \cdot) = w_t(T, \cdot) = 0 \quad \textit{and} \quad \{w, w_t\} \in C([0, T]; H^{1/2}(\Omega) \times L_2(\Omega)). \tag{3.4}$$

By time reversibility, the origin $\{0, 0\}$ *can be steered to all of* $H_0^1(\Omega) \times L_2(\Omega)$ *at time* $t = T$, *by using* $H_0^1(0, T; L_2(\Gamma))$*-control functions. Thus, for such* T,

$$\mathscr{L}_T\text{: continuous operator } H_0^1(0, T; L_2(\Gamma)) \text{ onto } H_0^1(\Omega) \times L_2(\Omega). \quad \square \tag{3.5}$$

A sketch of the proof of this result is given in Appendix A, using a direct approach in the style of [LT2], [LT5], and [T3]. A different approach is given in [L5].

3.1.2. Abstract Setting for Problem (1.1). We introduce the operators A, B, and F in model (2.1) corresponding to problem (1.1). We follow the treatment introduced in [LT1] and [T2]. Thus, details are omitted. See [DLT1] and Appendix A of [FLT]. Let $\mathscr{A}: L_2(\Omega) \supset \mathscr{D}(\mathscr{A}) \to L_2(\Omega)$ be the (positive self-adjoint) operator defined by $\mathscr{A}h = -\Delta h$, $\mathscr{D}(\mathscr{A}) = H^2(\Omega) \cap H_0^1(\Omega)$. Then, $-\mathscr{A}$ generates a strongly continuous (s.c.) cosine operator $C(t)$ on $L_2(\Omega)$ with $S(t) = \int_0^t C(\tau)\, d\tau$. The operator A in model (2.1) is given by

$$A = \begin{vmatrix} 0 & I \\ -\mathscr{A} & 0 \end{vmatrix}, \mathscr{D}(A) = \mathscr{D}(\mathscr{A}) \times \mathscr{D}(\mathscr{A}^{1/2}), \tag{3.6}$$

$$\mathscr{D}(\mathscr{A}^{1/2}) = H_0^1(\Omega), \qquad [\mathscr{D}(\mathscr{A}^{1/2})]' = H^{-1}(\Omega)$$
$$\text{(set theoretically and topologically)} \tag{3.7}$$

which generates the unitary s.c. group e^{At} given by

$$e^{At} = \begin{vmatrix} C(t) & S(t) \\ -\mathscr{A}S(t) & C(t) \end{vmatrix}, \qquad \| C(t) \|_{L(L_2(\Omega))} + \| \mathscr{A}^{1/2} S(t) \|_{L(L_2(\Omega))} \leq \text{const}, \tag{3.8}$$

on either of the spaces

$$\mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega) \quad \text{or} \quad L_2(\Omega) \times [\mathscr{D}(\mathscr{A}^{1/2})]' \tag{3.9}$$

topologically equivalent to, respectively

$$H = H_0^1(\Omega) \times L_2(\Omega), \qquad Y = L_2(\Omega) \times H^{-1}(\Omega). \tag{3.10}$$

Next, let D be the Dirichlet map (harmonic extension of boundary data) defined by

$$Dg = h \quad \Leftrightarrow \quad \begin{cases} \Delta h = 0 & \text{in } \Omega, \\ h = g & \text{in } \Gamma, \end{cases} \tag{3.11}$$

$$D\text{: continuous } L_2(\Gamma) \to H^{1/2}(\Omega) \subset H^{1/2 - 2\varepsilon}(\Omega) = D(\mathscr{A}^{1/4 - \varepsilon}), \quad \forall \varepsilon > 0. \tag{3.12}$$

Then with $U = L_2(\Gamma)$ the operator B in model (2.1) is

$$Bu = \begin{vmatrix} 0 \\ \mathscr{A}Du \end{vmatrix}, \qquad A^{-1}Bu = \begin{vmatrix} -Du \\ 0 \end{vmatrix}, \qquad A^{-1}B \in L(U, Y), \tag{3.13}$$

where $\mathscr{A}$ in $\mathscr{A}Du$ is actually the isomorphic extension, say $L_2(\Omega) \to [\mathscr{D}(\mathscr{A})]'$, of the original operator $\mathscr{A}$ defined above (3.6). Finally, the operator F in model (2.1) is given by

$$F(y) = \begin{vmatrix} 0 \\ f(y_1(\cdot)) \end{vmatrix}, \qquad F'[\eta]y = \begin{vmatrix} 0 \\ f'(\eta_1(\cdot))y_1(\cdot) \end{vmatrix}, \qquad F(0) = \begin{vmatrix} 0 \\ f(0) \end{vmatrix}, \tag{3.14}$$

$y = [y_1, y_2] \in Y$, $\eta = [\eta_1, \eta_2] \in Y$, so that assumption (1.2) on f' becomes assumption (2.2) on $F'[\eta]$. The operator $\mathscr{L}$ in (2.8) is explicitly

$$(\mathscr{L}u)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{3.15a}$$

$$= \begin{vmatrix} \mathscr{A}\displaystyle\int_0^t S(t-\tau)Du(\tau)\,d\tau \\ \mathscr{A}\displaystyle\int_0^t C(t-\tau)Du(\tau)\,d\tau \end{vmatrix}$$

$$: \text{continuous} \begin{cases} L_2(0, T; L_2(\Gamma)) \to C([0, T]; L_2(\Omega) \times H^{-1}(\Omega)) \\ \qquad \text{(see [LT2] and [LLT]),} \\ H_0^1(0, T; L_2(\Gamma)) \to C([0, T]; H^{1/2}(\Omega) \times L_2(\Omega)) \\ \qquad \text{(see Appendix A),} \end{cases} \tag{3.15b}$$

while the operator $\mathscr{L}_T$ in (2.9) is

$$\mathscr{L}_T u = \int_0^T e^{A(T-t)}Bu(t)\,dt$$

$$= \begin{vmatrix} \mathscr{A}\displaystyle\int_0^T S(T-t)Du(t)\,dt \\ \mathscr{A}\displaystyle\int_0^T C(T-t)Du(t)\,dt \end{vmatrix}$$

$$: \text{continuous} \begin{cases} L_2(0, T; L_2(\Gamma)) \to Y = L_2(\Omega) \times H^{-1}(\Omega), \\ H_0^1(0, T; L_2(\Gamma)) \to H = H_0^1(\Omega) \times L_2(\Omega)) \end{cases} \tag{3.16}$$
$$\text{(see Appendix A),}$$

so that as the space $\tilde{\mathscr{U}}_T$ in (2.8), (2.9) we may take $\tilde{\mathscr{U}}_T = \mathscr{U}_T = H_0^1(0, T; L_2(\Gamma))$.

The operator $\mathscr{K}[\eta]$ and $\mathscr{K}_T[\eta]$ in (2.15) and (2.16) are explicitly obtained via (3.8), (3.14) as follows: let $\eta = [\eta_1, \eta_2] \in \mathscr{E}_T$, see (3.3), then

$$(\mathscr{K}[\eta]g)(t) = (\mathscr{R}F'[\eta]g)(t) = \int_0^t e^{A(t-\tau)}F'[\eta]g(\tau)\,d\tau$$

$$= \begin{vmatrix} \displaystyle\int_0^t S(t-\tau)f'(\eta_1(\cdot))g_1(\tau, \cdot)\,d\tau \\ \displaystyle\int_0^t C(t-\tau)f'(\eta_1(\cdot))g_1(\tau, \cdot)\,d\tau \end{vmatrix} \tag{3.17}$$

$$: \text{continuous } L_1(0, T; L_2(\Omega)) \to C([0, T]; H),$$
$$H = \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega),$$

$$\mathcal{K}_T[n]g = \mathcal{R}_T F'[\eta]g = \int_0^T e^{A(T-t)}F'[\eta]g(t)\,dt$$

$$= \left|\begin{array}{l} \int_0^T S(T-t)f'(\eta_1(\cdot))g_1(t,\cdot)\,dt \\ \int_0^T C(T-t)f'(\eta_1(\cdot))g_1(t,\cdot)\,dt \end{array}\right| \tag{3.18}$$

$$: \text{continuous } L_1(0,T;L_2(\Omega)) \to H = H_0^1(\Omega) \times L_2(\Omega).$$

3.1.3. Verification of Assumptions (A.1)–(A.5)

Proposition 3.2 (Verification of (A.1), alternative (2.18a)). *The operator $\mathcal{L}$ in (3.16) satisfies the property*

$$\mathcal{L}: \mathcal{U}_T = H_0^1(0,T;L_2(\Gamma)) \to \mathcal{E}_T = L_2(0,T;L_2(\Omega) \times H^{-1}(\Omega)) \text{ is compact.} \tag{3.19}$$

Proof. We use Aubin's Compactness Lemma [A2]. In view of (3.15b), it suffices to show

$$\frac{d\mathcal{L}}{dt}: \text{continuous } H_0^1(0,T;L_2(\Gamma)) \to L_2(0,T;X), \tag{3.20}$$

where X is an Hilbert space satisfying $L_2(\Omega) \times H^{-1}(\Omega) \subset X$. To show (3.20), we first note that if $u \in H_0^1(0,T;L_2(\Gamma))$, then by problem (1.1) with $f = 0$ we obtain

$$\frac{d^2\mathcal{L}u}{dt^2} = \Delta\mathcal{L}u \in L_2(0,T;H^{-3/2-\varepsilon}(\Omega) \times H^{-2}(\Omega)) \tag{3.21}$$

upon applying p. 85 of [LM] to the regularity (3.15b). Thus, application of the intermediate derivative theorem [LM, p. 15] between (3.15b) and (3.21) yields (3.20) with $X = H^{-1/2-\varepsilon/2}(\Omega) \times H^{-1}(\Omega)$, as desired. □

Proposition 3.3 (Verification of (A.2)). *The family of operators $\mathcal{K}[\eta]$ defined by (3.17) satisfies both*

(a) *the assumption of collective compactness* (2.19) *on the space* $\mathcal{E}_T = L_2(0,T;L_2(\Omega) \times H^{-1}(\Omega))$, *and*
(b) *the assumption of strong convergence* (2.21).

Proof. (a) From (3.17) and (1.2) we obtain by (3.7)

$$\|\mathcal{K}[\eta]g\|_{C([0,T];H_0^1(\Omega)\times L_2(\Omega))} \le \text{const}_T\|g_1\|_{L_1(0,T;L_2(\Omega))} \quad \text{uniformly in } \eta \in \mathcal{E}_T. \tag{3.22}$$

Moreover, from (3.17) and (3.7) we obtain, since $H^{-1}(\Omega) = [\mathscr{D}(\mathscr{A}^{1/2})]'$, in (3.3)

$$\left(\frac{d\mathscr{K}[\eta]g}{dt}\right)(t) = \begin{vmatrix} \int_0^t C(t-\tau)f'(\eta_1(\cdot))g_1(\tau,\cdot)\,d\tau \\ f'(\eta_1(\cdot))g_1(t,\cdot) - \mathscr{A}\int_0^t S(t-\tau)f'(\eta_1(\cdot))g_1(\tau,\cdot)\,d\tau \end{vmatrix}$$

$$\in C\left([0,T];\begin{vmatrix} L_2(\Omega) \\ H^{-1}(\Omega)\end{vmatrix}\right) \tag{3.23a}$$

with norm uniform in $\eta \in \mathscr{E}_T$

$$\left\| \frac{d\mathscr{K}[\eta]g}{dt} \right\|_{C([0,T];L_2(\Omega)\times H^{-1}(\Omega))} \le \text{const}_T \|g_1\|_{L_1(0,T;L_2(\Omega))} \qquad \text{uniformly in} \quad \eta \in \mathscr{E}_T. \tag{3.23b}$$

Thus, application of Aubin's Compactness Lemma [A2] to (3.22) and (3.23) yields at once that $\mathscr{K}[\eta]$ is compact on $\mathscr{E}_T$, i.e., property $(a_1) = (2.20a)$; and, indeed, because of the uniform bounds in (3.22) and (3.23b), then property $(a_2) = (2.20b)$ on collective compactness attains.

(b) With η, $g \in \mathscr{E}_T$, i.e., $\eta_1, g_1 \in L_2(0, T; L_2(\Omega))$, we consider from (3.17)

$$\left(\begin{vmatrix} \mathscr{A}^{1/2} & 0 \\ 0 & \mathscr{A}^{1/2} \end{vmatrix} \mathscr{K}[\eta]g\right)(t) = \begin{vmatrix} \int_0^t \mathscr{A}^{1/2}S(t-\tau)f'(\eta_1(\cdot))g_1(\tau,\cdot)\,d\tau \\ \mathscr{A}^{1/2}\int_0^t C(t-\tau)f'(\eta_1(\cdot))g_1(\tau,\cdot)\,d\tau \end{vmatrix}. \tag{3.24}$$

Next, by the uniform bound (2.2) we have $\eta_1 \to f'(\eta_1)$: continuous $L_2(0, T; L_2(\Omega)) \to$ bounded sphere of $L^\infty(R)$. Thus, by Alaoglou's theorem, there exists a sequence $\eta_{1n} \in L_2(0, T; L_2(\Omega))$ such that $f'(\eta_{1n})$ converges to some $f_0 \in L^\infty(R)$ weak star. Define the operator $F_0 \in L(Y)$ by $F_0 y = [0, f_0(\cdot)y_1(\cdot)]$ for $y = [y_1, y_2] \in Y$. Then we have

$$\begin{vmatrix} \mathscr{A}^{1/2} & 0 \\ 0 & \mathscr{A}^{1/2} \end{vmatrix} K[\eta_n] \to \begin{vmatrix} \mathscr{A}^{1/2} & 0 \\ 0 & \mathscr{A}^{1/2} \end{vmatrix} \mathscr{R}F_0, \qquad \text{weakly in } \mathscr{E}_T, \tag{3.25}$$

i.e., if $g = [g_1, g_2]$, $h = [h_1, h_2] \in \mathscr{E}_T$, and so $g_1, h_1, \mathscr{A}^{-1/2}h_2 \in L_2(0, T; L_2(\Omega))$. then we have by (3.24) that

$$\left(\begin{vmatrix} \mathscr{A}^{1/2} & 0 \\ 0 & \mathscr{A}^{1/2} \end{vmatrix} \mathscr{K}[\eta_n]g, h\right)_{\mathscr{E}_T}$$
$$= \int_0^T \int_0^t (f'(\eta_{1n}(\cdot))g_1(\tau,\cdot), \mathscr{A}^{1/2}S(t-\tau)h_1(t))_\Omega\,d\tau\,dt$$
$$+ \int_0^T \int_0^t (f'(\eta_{1n}(\cdot))g_1(\tau,\cdot), C(t-\tau)\mathscr{A}^{-1/2}h_2(t))_\Omega\,d\tau\,dt \tag{3.26}$$

converges to

$$\left(\begin{vmatrix} \mathscr{A}^{1/2} & 0 \\ 0 & \mathscr{A}^{1/2} \end{vmatrix} \mathscr{R}F_0 g, h\right)_{\mathscr{E}_T}$$

$$= \int_0^T \int_0^t (f_0(\cdot)g_1(\tau,\cdot), \mathscr{A}^{1/2}S(t-\tau)h_1(t))_\Omega \, d\tau \, dt$$

$$+ \int_0^T \int_0^t (f_0(\cdot)g_1(\tau,\cdot), C(t-\tau)\mathscr{A}^{-1/2}h_2(t))_\Omega \, d\tau \, dt \tag{3.27}$$

as it follows by the Lebesgue dominated convergence theorem, using (1.2), weak star convergence and the uniform bound in (3.8). Moreover,

$$\frac{d\mathscr{K}[\eta_n]}{dt} \to \frac{d\mathscr{R}F_0}{dt} \qquad \text{weakly in } \mathscr{E}_T \tag{3.28}$$

since, from (3.17),

$$\left(\frac{d\mathscr{K}[\eta_n]}{dt} g\right)(t) = \begin{vmatrix} \int_0^t C(t-\tau)f'(\eta_{1n}(\cdot))g_1(\tau,\cdot)\, d\tau \\ f'(\eta_{1n}(\cdot))g_1(t,\cdot) - \mathscr{A}^{1/2}\int_0^t \mathscr{A}^{1/2}S(t-\tau)f'(\eta_{1n}(\cdot))g_1(\tau,\cdot)\, d\tau \end{vmatrix} \tag{3.29}$$

and essentially the same computations as in (3.25)–(3.27) apply now to show (3.28) using (3.29). As a consequence of the weak convergence in (3.25) and (3.28) and of compactness of $\mathscr{A}^{-1/2}$ on $\mathscr{L}_2(\Omega)$, we deduce that

$$\mathscr{K}[\eta_n] \to \mathscr{R}F_0 \qquad \text{strongly on } \mathscr{E}_T \tag{3.30}$$

as desired. Thus property (2.21) has been verified. □

Proposition 3.4 (Verification of (A.3)). *The family of operators* $\mathscr{K}_T[\eta]$ *defined by* (3.18) *satisfies* (2.23) *and the assumption of weak convergence* (2.24).

Proof. Property (2.23) is already contained in (3.18). Property (2.24) follows through an argument similar to the one of part (b) of Proposition 3.3. As in (3.25) we obtain

$$\begin{vmatrix} \mathscr{A}^{1/2} & 0 \\ 0 & \mathscr{A}^{1/2} \end{vmatrix} \mathscr{K}_T[\eta_n]$$

$$\to \begin{vmatrix} \mathscr{A}^{1/2} & 0 \\ 0 & \mathscr{A}^{1/2} \end{vmatrix} \mathscr{R}_T F_0, \qquad \text{weakly in} \quad L_2(\Omega) \times [\mathscr{D}(\mathscr{A}^{1/2})]' \tag{3.31}$$

which is equivalent to $\mathscr{K}_T[\eta_n] \to \mathscr{R}_T F_0$ weakly in $\mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)$. □

Proposition 3.5 (Verification of (A.4) and (A.5)). *We have*

(i) *trajectory* $\{(\mathscr{R}F(0))(t), 0 \le t \le T\} \subset$ *compact set of* $\mathscr{E}_T = L_2(0, T; L_2(\Omega) \times H^{-1}(\Omega))$,

(ii) $\mathscr{R}_T F(0) \in H = \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)$.

Proof. (i)

$$(\mathscr{R}F(0))(t) = \int_0^t e^{A(t-\tau)}F(0)\,d\tau$$

$$= \left| \begin{matrix} \displaystyle\int_0^t S(t-\tau)\{f(0)\}(\cdot)\,d\tau \\ \displaystyle\int_0^t C(t-\tau)\{f(0)\}(\cdot)\,d\tau \end{matrix} \right| \in C([0, T]; \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)) \quad (3.32)$$

and part (i) follows. Part (ii) is then contained in (3.32) with $t = T$. □

3.1.4. Verification of assumption (C.2). It remains to verify assumption (C.2) on the approximate controllability (2.33) and its limit version (2.34) in the sense of (2.21) and (2.24). These two approximate controllability properties amount to the same "uniqueness property" ("observability" in the terminology of standard control theory) as explained below. To verify (2.33), equivalently (2.35), we consider the problem

$$\begin{cases} \zeta_{tt} = \Delta\zeta + f'(\eta_1)\zeta & \text{in } (0, T] \times \Omega = Q, \quad (3.33a) \\ \zeta|_{t=0} = 0, \quad \zeta_t|_{t=0} = 0 & \text{in } \Omega \quad (3.33b) \\ \zeta|_\Sigma = u & \text{in } (0, T] \times \Gamma = \Sigma, \quad (3.33c) \end{cases}$$

with η_1 a fixed element of $L_2(0, T; L_2(\Omega))$, which corresponds to the linearized abstract version (2.12) with $F(0) = 0$ and with $z(t) = [\zeta(t), \zeta_t(t)]$. We seek the dual $\mathscr{M}_T^*[\eta]$ of the map $\mathscr{M}_T[\eta]$ in (2.33) as applied to the present case:

$$\mathscr{M}_T[\eta]: u \to z(T) = (\zeta(T), \zeta_t(T)): \mathscr{U}_T = H_0^1(0, T; L_2(\Gamma)) \\ \to H = \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega). \quad (3.34)$$

Let φ be the solution of the corresponding homogeneous problem backward in time:

$$\begin{cases} \varphi_{tt} = \Delta\varphi + f'(\eta_1)\varphi & \text{on } Q, \quad (3.35a) \\ \varphi|_{t=T} = \varphi_0 = y_1 \in L_2(\Omega), & \\ \varphi_t|_{t=T} = \varphi_1 = -\mathscr{A}y_0 \in [\mathscr{D}(\mathscr{A}^{1/2})]' = H^{-1}(\Omega) & \text{in } \Omega, \quad (3.35b) \\ \varphi|_\Sigma = 0 & \text{on } \Sigma. \quad (3.35c) \end{cases}$$

Then, for $[y_0, y_1] \in H = \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)$, $u \in H_0^1(0, T; L_2(\Gamma))$, we have via (3.35b)

$$\left(\mathscr{M}_T[\eta]u, \left|\begin{matrix} y_0 \\ y_1 \end{matrix}\right|\right)_H = \left(\left|\begin{matrix} \zeta(T) \\ \zeta_t(T) \end{matrix}\right|, \left|\begin{matrix} y_0 \\ y_1 \end{matrix}\right|\right)_H$$

$$= (\zeta_t(T), \varphi(T))_\Omega - (\zeta(T), \varphi_t(T))_\Omega$$

$$= -\left(u, \frac{\partial\varphi}{\partial\nu}\right)_\Sigma = \left(u, \mathscr{M}^*_T[\eta]\left|\begin{matrix} y_0 \\ y_1 \end{matrix}\right|\right)_{H_0^1(0, T; L_2(\Gamma))} \quad (3.36)$$

with L_2 inner products over Ω and Σ, where the identity in the middle of (3.36) can be verified, as usual, by multiplying problem (3.33) by φ and problem (3.35) by ζ

and integrating by parts (this is the counterpart of the operator version in (A.8) and (A.9) in Appendix A when $f' \equiv 0$). As in (A.9) we have in norm equivalence

$$\left(u, \mathcal{M}_T^*[\eta]\begin{vmatrix} y_0 \\ y_1 \end{vmatrix}\right)_{H_0^1(0,T;L_2(\Gamma))} = \int_0^T \left(u_t, \frac{d}{dt}\mathcal{M}_T^*[\eta]\begin{vmatrix} y_0 \\ y_1 \end{vmatrix}\right)_{L_2(\Gamma)} dt$$

$$= -\left(u, \frac{d^2}{dt^2}\mathcal{M}_T^*[\eta]\begin{vmatrix} y_0 \\ y_1 \end{vmatrix}\right)_{L_2(\Sigma)} \tag{3.37}$$

after integration by parts in t using $u = 0$ at $t = 0$ and $t = T$. Comparing (3.37) with (3.36) yields

$$\frac{d^2}{dt^2}\mathcal{M}_T^*[\eta]\begin{vmatrix} y_0 \\ y_1 \end{vmatrix} = \frac{\partial \varphi}{\partial \nu}(t, \varphi_0, \varphi_1), \qquad \varphi_0 = y_1, \quad \varphi_1 = -\mathcal{A}y_0. \tag{3.38}$$

To test the injectivity condition (2.35) on $\mathcal{M}_T^*[\eta]$ of assumption (C.2) we let

$$0 \equiv \mathcal{M}_T^*[\eta]\begin{vmatrix} y_0 \\ y_1 \end{vmatrix}, \qquad [y_0, y_1] \in H, \quad \eta \in \mathcal{E}_T, \tag{3.39}$$

and we want to deduce that in fact $[y_0, y_1] = 0$. Now (3.40) implies by (3.38)

$$\frac{d^2\mathcal{M}_T^*[\eta]}{dt^2}\begin{vmatrix} y_0 \\ y_1 \end{vmatrix} = \frac{\partial \varphi}{\partial \nu}\Big|_\Sigma \equiv 0 \tag{3.40}$$

for the solution φ of (3.35). But the initial data $\{\varphi_0, \varphi_1\} \in L_2(\Omega) \times H^{-1}(\Omega)$ in (3.35b) yield the *a- priori* regularity $\varphi \in L^\infty(0, T; L_2(\Omega))$. To conclude that, in fact $[y_0, y_1] = 0$, we need the following uniqueness result.

Theorem 3.6. *Consider the problem*

$$\begin{cases} \varphi_{tt} = \Delta\varphi + p(t,x)\varphi & \text{in } Q, \quad (3.41\text{a}) \\ \varphi|_{t=0} = \varphi_0 \in L_2(\Omega), \quad \varphi_t|_{t=0} = \varphi_1 \in H^{-1}(\Omega) & \text{in } \Omega. \quad (3.41\text{b}) \\ \varphi\Big|_\Sigma = \dfrac{\partial\varphi}{\partial\nu}\Big|_\Sigma \equiv 0 & \text{in } \Sigma \quad (3.41\text{c}) \end{cases}$$

with $p \in L^\infty(Q)$. *Let* $T > T(x^0)$, *defined in Appendix* A, *below* (A.23c). *Then, in fact,* $\varphi_0 = \varphi_1 = 0$.

Proof of Theorem 3.6. Step 1. Equation (3.41b) yields the *a priori* regularity $\varphi \in L^\infty(0, T; L_2(\Omega))$. Using this information, we now boost the regularity of the solution to (3.41) in the sense that, for $T > T(x^0)$, we have

$$\infty > \|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))} \geq C_T(T - T(x^0))\|\{\varphi_0, \varphi_1\}\|^2_{H_0^1(\Omega)\times L_2(\Omega)}. \tag{3.42}$$

Thus, with $H_0^1(\Omega) = \mathcal{D}(\mathcal{A}^{1/2})$, we actually have that the solution to (3.41) satisfies

$$\{\varphi_0, \varphi_1\} \in H_0^1(\Omega) \times L_2(\Omega) \quad \text{and} \quad \{\varphi, \varphi_t\} \in L^\infty(0, T; H_0^1(\Omega) \times L_2(\Omega)). \tag{3.43}$$

To prove (3.42), we use the multipliers $h \cdot \nabla\varphi$, φ, and φ_t, with h the radial field $h(x) = x - x_0$ which defines $T(x^0)$ in Appendix A. By applying the first two multipliers to (3.41a) and using (3.41c) we obtain (see, e.g., [T3], (2.20) combined with (2.25) and (2.26) when $p \equiv 0$) with $n = \dim \Omega$

$$\int_0^T E(t)\,dt = 2\int_Q p\varphi h\cdot\nabla\varphi\,dQ + (n-1)\int_Q p\varphi^2\,dQ + 2b(T) - 2b(0), \tag{3.44}$$

$$b(t) = -\frac{(n-1)}{2}(\varphi_t, \varphi)_\Omega + (\varphi_t, h\cdot\nabla\varphi)_\Omega, \tag{3.45a}$$

$$E(t) = \int_Q |\nabla\varphi|^2 + |\varphi_t|^2\,d\Omega. \tag{3.45b}$$

Using Komornik's estimate of (3.45a) (e.g., Section 5 of [L5] and [K2])

$$2|b(t)| \le R(x^0)E(t) \qquad \text{so that} \quad 2(b(T) - b(0)) \le R(x^0)[E(T) + E(0)], \tag{3.46}$$

we obtain, from (3.44) for any $\varepsilon_1 > 0$ since $p \in L^\infty(Q)$,

$$(1-\varepsilon_1)\int_0^T E(t)\,dt = \mathcal{O}(T\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}) + R(x^0)[E(T) + E(0)]. \tag{3.47}$$

Next, the multiplier φ_t applied to (3.41a) yields

$$E(t) = E(0) + 2\int_0^t\int_\Omega p\varphi\varphi_t\,d\Omega\,d\tau \tag{3.48}$$

from which since $p \in L^\infty(Q)$

$$E(t) \le E(0) + \varepsilon T\|\varphi_t\|^2_{L^\infty(0,T;L_2(\Omega))} + \frac{C_p}{\varepsilon}T\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}. \tag{3.49}$$

Selecting $\varepsilon = \varepsilon_2/T$, $\varepsilon_2 > 0$ preassigned, we obtain

$$(1-\varepsilon_2)\sup_{0\le t\le T} E(t) \le E(0) + \mathcal{O}_T(\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}), \tag{3.50}$$

where $\mathcal{O}_T$ means that the constant of the upper bound may depend on T. Inserting (3.48) into the left-hand side of (3.47) and using (3.50) for the term $E(T)$ on the right-hand side of (3.47), we obtain

$$\begin{aligned}(1-\varepsilon_1)TE(0) = R(x^0)\left(\frac{1}{1-\varepsilon_2} + 1\right)E(0)& \\ + \mathcal{O}_T(\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}) - 2(1-\varepsilon_1)\alpha_T&,\end{aligned} \tag{3.51}$$

$$\begin{aligned}|\alpha_T| &= \left|\int_0^T\int_0^t\int_\Omega p\varphi\varphi_t\,d\Omega\,d\tau\,dt\right| \\ &\le \varepsilon\,T^2\|\varphi_t\|^2_{L^\infty(0,T;L_2(\Omega))} + \frac{c_p}{\varepsilon}T^2\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}\end{aligned}$$

where we select $\varepsilon = \varepsilon_3/T^2$, $\varepsilon_3 > 0$ preassigned, so that by (3.50))

$$\le \frac{\varepsilon_3}{1-\varepsilon_2}E(0) + \mathcal{O}_T(\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}). \tag{3.52}$$

Using (3.52) into (3.51) yields

$$(1-\varepsilon_1)T - \left[R(x^0)\left(\frac{1}{1-\varepsilon_2}+1\right) - \frac{\varepsilon_3}{1-\varepsilon_2}\right]E(0) = \mathcal{O}_T(\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}), \tag{3.53}$$

where we may take $\mathcal{O}_T = T^4\mathcal{O}$, and (3.42) follows for $T > T(x^0) = 2R(x^0)$, with $c_T = c/T^4$.

Step 2. We return to (3.41) with boosted regularity as described by (3.43). We then invoke the uniqueness theorem [H5], [KRS], [R] and obtain $\varphi_0 = \varphi_1 = 0$. □

In the case of assumption (2.35), we have $p = f'(\eta_1)$, $\eta_1 \in L_2(0, T; L_2(\Omega))$ and, in the case of (2.36), we have $p = f_0$, with f_0 any of the $L^\infty(Q)$ functions obtained as a limit above (3.25). Assumption (C.2) is verified. □

3.2. *The Case $\gamma = 0$ in Theorem 1.1 for Problem (1.1)*

We give only a brief sketch of the case $\gamma = 0$ in Theorem 1.1. With reference to the setting of Section 2, the following is the relevant specialization for the wave problem (1.1) in the case $\gamma = 0$ of Theorem 1.1:

$$Y = L_2(\Omega) \times H^{-1}(\Omega), \qquad H = L_2(\Omega) \times H^{-1}(\Omega) \tag{3.54}$$

$$U = L_2(\Gamma), \qquad \mathcal{U}_T = L_2(0, T; L_2(\Gamma)), \tag{3.55}$$

$$\mathcal{E}_T = L_2(0, T; L_2(\Omega) \times H^{-1}(\Omega)). \tag{3.56}$$

Assumption (A.1) in the alternative (2.18b). First we note that the continuity requirement (2.17) with $\mathcal{U}_T$ and $\mathcal{E}_T$ as in (3.55), (3.56) is *a fortiori* true [LT2], [L5], [LLT]. Next, recalling (3.14), we have $F'[\eta]y = [0, f'(\eta_1(\cdot))y_1(\cdot)]$, *where* $\eta_1 \in L_2(0, T; L_2(\Omega))$ and $y_1 \in L_2(\Omega)$. Thus, in view of its definition (2.11). assumption (2.18b) on $\mathcal{R}_T$ is *a fortiori* satisfied provided that the operator

$$\mathcal{R}_T \begin{vmatrix} 0 \\ g_2 \end{vmatrix} = \int_0^T e^{A(T-t)} \begin{vmatrix} 0 \\ g_2(t) \end{vmatrix} dt = \begin{vmatrix} \int_0^T S(T-t)g_2(t)\,dt \\ \int_0^T C(T-t)g_2(t)\,dt \end{vmatrix} \tag{3.57}$$

is compact: $g_2 \in L_1(0, T; L_2(\Omega)) \to H = L_2(\Omega) \times H^{-1}(\Omega)$, which is certainly true since, in fact,

$$\mathcal{R}_T \begin{vmatrix} 0 \\ g_2 \end{vmatrix} \in H_0^1(\Omega) \times L_2(\Omega).$$

Assumption (A.2). This is the same as in the case $\gamma = 1$ verified in Subsection 3.1.3, since this assumption does not depend on the space H, but on the space $\mathcal{E}_T$ which is the same as before, see (3.3) and (3.56).

Assumption (A.3). As noted just below (2.25), the convergence in (2.24) is actually strong, if $\mathscr{R}_T$ satisfies assumption (2.18b) as verified above.

Assumption (A.4). Same as in the case $\gamma = 1$ verified in Subsection 3.1.3, since this assumption does not depend on the space H, but on the space $\mathscr{E}_T$ which is the same as before.

Assumption (A.5). *A fortiori* true from (3.32) with H as in (3.54).

Assumption (C.1). Exact controllability of the linear problem (1.1) with $f \equiv 0$ and $\mathscr{U}_T$ and H as in (3.55), (3.54) holds true, as recalled just below (1.2).

Assumption (C.2). With H and $\mathscr{U}_T$ as in (3.54), (3.55), the procedure of Subsection 3.1.4 starts from

$$\mathscr{M}_T[\eta]: u \to z(T): \mathscr{U}_T = L_2(0, T; L_2(\Gamma)) \to H = L_2(\Omega) \times H^{-1}(\Omega) \tag{3.58}$$

(counterpart of (3.34)) and now yields through the counterpart relations of (3.36)

$$\mathscr{M}_T^*[\eta] \begin{vmatrix} y_0 \\ y_1 \end{vmatrix} = \frac{\partial \varphi(t)}{\partial \nu}, \tag{3.59}$$

where φ satisfies the same problem (3.35) as before, except that now $\{\varphi_0, \varphi_1\} \in H_0^1(\Omega) \times L_2(\Omega)$, smoother than in (3.35b). Thus, we are now at the level of step 2, proof of Theorem 3.6, and, as in that step, the references mentioned there imply the required uniqueness $\varphi_0 = \varphi_1 = 0$.

4. Application: A semilinear Euler-Bernoulli Equation with Boundary Controls. Problem (1.14)

The goal of this section is to show that the semilinear Euler-Bernoulli's problem (1.14) subject to condition (1.15) fits automatically into the abstract model of Section 2 on appropriate spaces. As a consequence, Theorem 1.3 is nothing but a specialization of the abstract Theorem 2.1. With reference to the setting of Section 2, the following is the relevant specialization for problem (1.14). Throughout this section we let $\mathscr{A}: L_2(\Omega) \supset \mathscr{D}(\mathscr{A}) \to L_2(\Omega)$ be the (positive self-adjoint) operator defined by

$$\mathscr{A}h = \Delta^2 h, \qquad \mathscr{D}(\mathscr{A}) = \{h \in H^4(\Omega): h|_\Gamma = \Delta h|_\Gamma = 0\}. \tag{4.1}$$

Then we take

$$\begin{aligned} Y &= L_2(\Omega) \times [\mathscr{D}(\mathscr{A}^{1/2})]', \qquad H = \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega), \\ \mathscr{D}(\mathscr{A}^{1/2}) &= H^2(\Omega) \cap H_0^1(\Omega), \end{aligned} \tag{4.2}$$

$U = H^m(\Gamma) \times L_2(\Gamma)$, $m > 0$ fixed but arbitrary (for exact controllability purposes we may take henceforth m as "large" as desired), and

$$\mathscr{U}_T = H_0^m(\Sigma) \times H^{1/4}(0, T; L_2(\Gamma)), \qquad \mathscr{E}_T = L_2(0, T; H^1(\Omega) \times H^{-1}(\Omega)). \tag{4.3}$$

4.1. *Verification of Assumption (C.1): Exact Controllability of the Linear System*

We verify assumption (C.1) of exact controllability of the linear problem (1.14) with $f \equiv 0$, $u_1 \equiv 0$, and $u_2 \in H^{1/4}(0, T; L_2(\Gamma))$ [L2], or $u_2 \in L_2(0, T; H^{1/2}(\Gamma))$ [LT11].

Theorem 4.1. *Let* $f \equiv 0$ *in* (1.14) *and let* $T > 0$ *be arbitrary. Then, for any given pair* $\{w_0, w_1\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega)$, *there exists a suitable control* $u_2 \in H^{1/4}(0, T; L_2(\Gamma))$, *such that the solution of* (1.14) *corresponding to such* u_2 and $u_1 = 0$ *satisfies* $w(T, \cdot) = w_t(T, \cdot) = 0$.

4.2. *Abstract Setting for Problem (1.14)*

We follow our previous operator treatement, e.g., [LT7] and [LT8]. With $\mathscr{A}$ defined in (4.1) above, let again $C(t)$ and $S(t)$ denote the corresponding cosine and sine operators. Then the operator A which appears in the abstract model (2.1) is given by the same expression as in (3.6) with corresponding s.c. group given by (3.8) on the spaces of (3.9). In place of the Dirichlet map D in (3.11), we now define two corresponding (Green) operators G_1 and G_2:

$$G_1 g_1 \equiv h, \qquad G_1\colon \text{continuous } H^s(\Gamma) \to H^{s+1/2}(\Omega), \qquad \text{[LM, pp. 188–189]}, \tag{4.4}$$

$$G_2 g_2 = y, \qquad G_2\colon \text{continuous } H^s(\Gamma) \to H^{s+5/2}(\Omega), \tag{4.5}$$

$$\begin{cases} \Delta^2 h = 0 & \text{in } \Omega \\ h = g_1 & \text{on } \Gamma, \\ \Delta h = 0 & \text{on } \Gamma, \end{cases} \qquad \begin{cases} \Delta^2 y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \\ \Delta y = g_2 & \text{on } \Gamma \end{cases} \qquad \begin{matrix} (4.6a) \\ (4.6b) \\ (4.6c) \end{matrix}$$

(we note that $G_1 = D$, D being the operator in (3.12), see [LT8]). Thus, if $U = H^m(\Gamma) \times L_2(\Gamma)$, we have that the operator B which appears in the abstract model (2.1) is given by

$$B \begin{vmatrix} u_1 \\ u_2 \end{vmatrix} = \begin{vmatrix} 0 \\ \mathscr{A}(G_1 u_1 + u_2) \end{vmatrix}, \qquad A^{-1}B = \begin{vmatrix} -G_1 u_1 - G_2 u_2 \\ 0 \end{vmatrix}, \tag{4.7}$$

and $A^{-1}B \in L(U, Y)$, Y as in (4.2). The definition of the nonlinear operator F is the same as the one in (3.14). Now, the counterpart of (3.15) for $u = [u_1, u_2]$ is

$$(\mathscr{L}u)(t) = \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau = (\mathscr{L}_1 u_1)(t) + (\mathscr{L}_2 u_2)(t), \tag{4.8a}$$

$$(\mathscr{L}_i u_i)(t) = \begin{vmatrix} \mathscr{A} \displaystyle\int_0^t S(t-\tau) G_i u_i(\tau)\, d\tau \\ \mathscr{A} \displaystyle\int_0^t C(t-\tau) G_i u_i(\tau)\, d\tau \end{vmatrix}, \qquad \begin{matrix} (4.8b) \\ (4.8c) \end{matrix}$$

where, for instance (see Appendix B, for sharper results),

$$\mathscr{L}_1\colon \text{continuous } H_0^2(0, T; H^{10/3}(\Gamma)) \to C([0, T]; H^3(\Omega) \times H^1(\Omega)), \tag{4.9a}$$

$$\mathscr{L}_{1T}\colon \text{continuous } H_0^2(0, T; H^{10/3}(\Gamma)) \to \mathscr{D}(\mathscr{A}^{3/4}) \times \mathscr{D}(\mathscr{A}^{1/4}), \tag{4.9b}$$

$$\mathscr{L}_2\colon \begin{cases} \text{continuous } L_2(\Sigma) \to C\left([0, T]; \begin{vmatrix} H_0^1(\Omega) \\ H^{-1}(\Omega) \end{vmatrix}\right) \quad \text{[LT7, Theorem 1.3]}, & (4.10a) \\ \text{continuous } H_0^1(0, T; L_2(\Gamma)) \to C\left([0, T]; \begin{vmatrix} H^{5/2}(\Omega) \\ H_0^1(\Omega) \end{vmatrix}\right), & (4.10b) \\ \text{continuous } H^{1/4}(0, T; L_2(\Gamma)) \to C\left([0, T]; \begin{vmatrix} H^{11/8}(\Omega) \cap H_0^{11/8-\varepsilon}(\Omega) \\ [\mathscr{D}(\mathscr{A}^{1/8})]' \end{vmatrix}\right) & (4.10c) \end{cases}$$

for any $\varepsilon > 0$, see Appendix B. Note that (4.10c) follows by interpolation from (4.11a) and (4.11b) [LM, Theorem 14.2, p. 95], see Appendix B.

4.3. *Verification of Assumptions (A.1)–(A.5)*

Proposition 4.2 (Verification of (A.1), alternative (2.18a)). *The operator $\mathscr{L}$ in* (4.8) *satisfies, for, say, $m \geq 4$,*

$$\mathscr{L}\colon \mathscr{U}_T = H_0^m(\Sigma) \times H^{1/4}(0, T; L_2(\Gamma)) \to \mathscr{E}_T = L_2(0, T; H^1(\Omega) \times H^{-1}(\Omega)) \quad \textit{is compact.} \tag{4.11}$$

Proof. The regularity results (4.9a) and (4.10c) give compactness in the space variable into $\to H^1(\Omega) \times [\mathscr{D}(\mathscr{A}^{1/4})]'$. Next, we use Aubin's Lemma as in Proposition 3.2 to obtain the full statement of compactness in time and space, as required by (4.11). Details are omitted. □

Verification of Assumptions (A.2)–(A.5) proceeds as in Section 3, *mutatis mutandis*. Details are omitted, but we point out, however, that from the present version of (3.17) and (3.23a), we obtain the following bounds, uniformly in $\eta \in \mathscr{E}_T$:

$$\|\mathscr{K}[\eta]g\|_{C([0,T];\mathscr{D}(\mathscr{A}^{1/2})\times L_2(\Omega))} \leq \text{const}_T \|g_1\|_{L_1(0,T;L_2(\Omega))}, \tag{4.12a}$$

$$\left\|\frac{d\mathscr{K}[\eta]g}{dt}\right\|_{C([0,T];L_2(\Omega)\times[\mathscr{D}(\mathscr{A}^{1/2})]')} \leq \text{const}_T \|g_1\|_{L_1(0,T;L_2(\Omega))}, \tag{4.12b}$$

conterparts of (3.22), (3.23b) in Proposition 3.3. These results are then used to verify assumption (A.2).

4.4. *Verification of Assumption (C.2)*

It remains to verify assumption (C.2) on the approximate controllability (2.33) and its limit version (2.34) in the sense of (2.21) and (2.24). These two approximate controllability properties amount to the same "uniqueness property" as described below. To verify (2.33), equivalently (2.35), we consider the homogeneous problem

$$\begin{cases} \zeta_{tt} + \Delta^2\zeta = f'(\eta_1)\zeta & \text{in } (0, T] \times \Omega = Q, & (4.13a) \\ \zeta|_{t=0} = 0, \quad \zeta_t|_{t=0} = 0 & \text{in } \Omega, & (4.13b) \\ \zeta|_\Sigma = u_1 & \text{in } (0, T] \times \Gamma = \Sigma, & (4.13c) \\ \Delta\zeta|_\Sigma = u_2 & \text{in } \Sigma, & (4.13d) \end{cases}$$

with η_1 fixed element of $L_2(0, T; H^1(\Omega))$ (the first component of the space $\mathscr{E}_T$ in (4.3), which corresponds to the abstract linearized version (2.13) with $z(t) = [\zeta(t), \zeta_t(t)]$ and $F(0)) = 0$). With

$$\mathscr{M}_T[\eta]: u = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix} \to z(T) = \begin{vmatrix} \zeta(T) \\ \zeta_t(T) \end{vmatrix}$$

$$:\text{continuous } H_0^m(\Sigma) \times H^{1/4}(0, T; L_2(\Gamma)) \to H$$

$$= [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega) = \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega) \tag{4.14}$$

we seek its dual $\mathscr{M}_T^*[\eta]$, defined for $[y_0, y_1] \in \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)$ by

$$\left(\mathscr{M}_T[\eta]\begin{vmatrix} u_1 \\ u_2 \end{vmatrix}, \begin{vmatrix} y_0 \\ y_1 \end{vmatrix}\right)_{\mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)} = (\zeta(T), y_0)_{\mathscr{D}(\mathscr{A}^{1/2})} + (\zeta_t(T), y_1)_{L_2(\Omega)}$$

$$= \left(\begin{vmatrix} u_1 \\ u_2 \end{vmatrix}, \mathscr{M}_T^*[\eta]\begin{vmatrix} y_0 \\ y_1 \end{vmatrix}\right)_{H_0^m(\Sigma) \times H^{1/4}(0, T; L_2(\Gamma))} \tag{4.15}$$

To find $\mathscr{M}_T^*[\eta]$ explicitly, we let φ be the solution of the corresponding homogeneous problem backward in time:

$$\begin{cases} \varphi_{tt} + \Delta^2\varphi = f'(\eta_1)\varphi & \text{in } (0, T] \times \Omega = Q, & (4.16a) \\ \varphi|_{t=T} = \varphi_0, \quad \varphi_t|_{t=T} = \varphi_1 & \text{in } \Omega, & (4.16b) \\ \varphi|_\Sigma \equiv \Delta\varphi|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma = \Sigma. & (4.16c) \end{cases}$$

Multiplying (4.13a) by φ and (4.16a) by ζ and integrating by parts, as usual, we find, after using the boundary conditions (B.C.) (4.13c, d) and (4.16c),

$$(\zeta_t(T), \varphi(T))_{L_2(\Omega)} - (\zeta(T), \varphi_t(T))_{L_2(\Omega)} = \left(u_1, \frac{\partial\Delta\varphi}{\partial\nu}\right)_{L_2(\Sigma)} + \left(u_2, \frac{\partial\varphi}{\partial\nu}\right)_{L_2(\Sigma)}. \tag{4.17}$$

We next introduce three isomorphisms:

$\mathscr{A}^{1/2}$: isomorphism $\mathscr{D}(\mathscr{A}^{1/2})$ onto $L_2(\Omega)$, self-adjoint on $L_2(\Omega)$, so that

$$(v, w)_{\mathscr{D}(\mathscr{A}^{1/2})} = (v, \mathscr{A}w)_{L_2(\Omega)}, \qquad v, w \in \mathscr{D}(\mathscr{A}^{1/2}). \tag{4.18}$$

J: isomorphism $H^m(\Sigma)$ onto $L_2(\Sigma)$, self-adjoint on $L_2(\Sigma)$, so that

$$(g_1, g_2)_{H^m(\Sigma)} = (Jg_1, Jg_2)_{L_2(\Sigma)} = (g_1, J^2g_2)_{L_2(\Sigma)}, \qquad g_i \in H^m(\Sigma). \tag{4.19}$$

$\mathscr{G}$: isomorphism $H^{1/4}(0, T; L_2(\Gamma))$ onto $L_2(0, T; L_2(\Gamma)) = L_2(\Sigma)$, self-adjoint on $L_2(\Sigma)$ so that

$$(g, h)_{H^{1/4}(0, T; L_2(\Gamma))} = (\mathscr{G}g, \mathscr{G}h)_{L_2(\Sigma)}$$
$$= (g, \mathscr{G}^2h)_{L_2(\Sigma)}, \qquad g, h \in H^{1/4}(0, T; L_2(\Gamma)). \tag{4.20}$$

Then, if we set, for $[y_0, y_1] \in \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)$,

$$\varphi_0 = \varphi(T) = y_1 \in L_2(\Omega), \qquad \varphi_1 = \varphi_t(T) = -\mathscr{A} y_0 \in [\mathscr{D}(\mathscr{A}^{1/2})]', \tag{4.21}$$

we have by (4.15)

$$\begin{aligned}
\left(\mathscr{M}_T[\eta]\begin{vmatrix} u_1 \\ u_2 \end{vmatrix}, \begin{vmatrix} y_0 \\ y_1 \end{vmatrix}\right)_{\mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)} &= (\zeta(T), y_0)_{\mathscr{D}(\mathscr{A}^{1/2})} + (\zeta_t(T), y_1)_{L_2(\Omega)} \\
&= -(\zeta(T), \varphi_t(T))_{L_2(\Omega)} + (\zeta_t(T), \varphi(T))_{L_2(\Omega)} \\
&\quad \text{(by (4.18) and (4.21))} \\
&= \left(u_1, \frac{\partial \Delta \varphi}{\partial \nu}\right)_{L_2(\Sigma)} + \left(u_2, \frac{\partial \varphi}{\partial \nu}\right)_{L_2(\Sigma)} \quad \text{(by (4.17))} \\
&= \left(u_1, J^{-2}\frac{\partial \Delta \varphi}{\partial \nu}\right)_{H^m(\Sigma)} \\
&\quad + \left(u_2, \mathscr{G}^{-2}\frac{\partial \varphi}{\partial \nu}\right)_{H^{1/4}(0,T;L_2(\Gamma))} \\
&\quad \text{(by (4.19) and (4.20))} \\
&= \left(\begin{vmatrix} u_1 \\ u_2 \end{vmatrix}, \mathscr{M}_T^*[\eta]\begin{vmatrix} y_0 \\ y_1 \end{vmatrix}\right)_{H^m(\Sigma) \times H^{1/4}(0,T;L_2(\Gamma))}
\end{aligned} \tag{4.22}$$

so that

$$\mathscr{M}_T^*[\eta]\begin{vmatrix} y_0 \\ y_1 \end{vmatrix} = \begin{vmatrix} J^{-2}\dfrac{\partial \Delta \varphi}{\partial \nu} \\ \mathscr{G}^{-2}\dfrac{\partial \varphi}{\partial \nu} \end{vmatrix}, \qquad [y_0, y_1] \in \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega). \tag{4.23}$$

Thus, to test the injectivity condition (2.35) on $\mathscr{M}_T^*[\eta]$ of assumption (C.2) we let

$$0 \equiv \mathscr{M}_T^*[\eta]\begin{vmatrix} y_0 \\ y_1 \end{vmatrix}, \qquad [y_0, y_1] \in \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega), \eta \in \mathscr{E}_T, \tag{4.24}$$

which by (4.23) implies

$$\left.\frac{\partial \Delta \varphi}{\partial \nu}\right|_{\Sigma} = \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Sigma} \equiv 0 \tag{4.25}$$

with φ solution of (4.16) with initial data as in (4.21). We then want to show that, in fact, $[y_0, y_1] = 0$. This is a consequence of the following uniqueness result.

Theorem 4.3. *Consider, for any* $T > 0$, *the problem*

$$\begin{cases}
\varphi_{tt} + \Delta^2 \varphi = p(t, x)\varphi, & \text{in } (0, T] \times \Omega = Q, \quad (4.26a) \\
\varphi|_{t=0} = \varphi_0 \in L_2(\Omega), \quad \varphi_t|_{t=0} = \varphi_1 \in [\mathscr{D}(\mathscr{A}^{1/2})]' & \text{in } \Omega, \quad (4.26b) \\
\varphi|_\Sigma = \left.\dfrac{\partial \varphi}{\partial \nu}\right|_\Sigma = \Delta\varphi|_\Sigma = \left.\dfrac{\partial \Delta \varphi}{\partial \nu}\right|_\Sigma \equiv 0, & \text{in } (0, T] \times \Gamma = \Sigma, \quad (4.26c)
\end{cases}$$

with p satisfying the following assumptions:

$$p \in L^{\infty}(Q), \qquad |\nabla_x p| \in L_2(0, T; L_2(\Omega)) = L_2(Q). \tag{4.27}$$

Then $\varphi_0 = \varphi_1 = 0$ *and so* $\varphi \equiv 0$ *in* Q.

To complete verification that assumption (C.2) on approximate controllability is satisfied we take p as follows. In the case of verifying (2.35), we take $p = f'(\eta_1)$ with $\eta_1 \in L_2(0, T; H^1(\Omega))$ (the first component of the space $\mathscr{E}_T$ in (4.3)); hence $\nabla_x p = f''(\eta_1)\nabla_x \eta_1$, so that the required assumptions (4.27) on p are guaranteed by the assumption (1.15) on f. In the case of verifying (2.36), we take $p = f_0(t, x)$ with f_0 any of the limits obtained as follows. If $\{\eta_{1n}\}$ is an arbitrary sequence in a ball of $L_2(0, T; H^1(\Omega))$, then by assumption (1.15) on f we have that

(i) $f'(\eta_{1n}(x, t))$ in a fixed ball of $L^{\infty}(Q)$ and
(ii) $\nabla_x f'(\eta_{1n}(x, t) = f''(\eta_{1n}(x, t))\nabla_x \eta_{1n}$ are in a fixed ball of $L_2(0, T; L_2(\Omega))$, uniformly in n.

Thus, there exists a subsequence η_{1n_k} such that $f'(\eta_{1n}(x, t)) \to$ some f_0 both in $L^{\infty}(Q)$ weak star and in $L_2(0, T; H^1(\Omega))$ weakly. Thus $f_0 \in L^{\infty}(Q)$ and $|\nabla_x f_0| \in L_2(Q)$, as required by (4.27).

Proof of Theorem 4.3. To begin, we have from (4.26b) the *a priori* regularity that $\varphi \in L^{\infty}(0, T; L_2(\Omega))$ for problem (4.26). Using this information, we now boost the regularity of φ in the following two-step procedure. (This is similar to the proof of Theorem 3.6 for waves.)

Step 1. **Lemma 4.4.**

(a) *For problem* (4.26) *with* $p \in L^{\infty}(Q)$ *we have, for* T *sufficiently large* $>$ *some* T_1 *and for some constant* $C_T = C(T - T_1) > 0$,

$$\infty > \|\varphi\|^2_{L^{\infty}(0, T; L_2(\Omega))} \geq C_T \|\{\varphi_0, \varphi_t\}\|^2_{\mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)}, \tag{4.28}$$

$$\mathscr{D}(\mathscr{A}^{1/2}) = \{h \in H^2(\Omega) : h|_{\Gamma} = 0\}, \qquad \|h\|^2_{\mathscr{D}(\mathscr{A}^{1/2})} = \|\mathscr{A}^{1/2} h\|^2_{L_2(\Omega)} = \int_{\Omega} (\Delta h)^2 \, d\Omega. \tag{4.29}$$

(b) *Thus, the initial data and the regularity of problem* (4.26) *are boosted to*

$$\begin{aligned} &\{\varphi_0, \varphi_1\} \in \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega), \\ &\{\varphi(t), \varphi_t(t)\} \in L^{\infty}(0, T; \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)). \end{aligned} \tag{4.30}$$

Proof of Lemma 4.4 (Sketch). We use the multipliers $h \cdot \nabla\varphi$, φ, and φ_t as applied to (4.26a), with h the radial field $h(x) = x - x_0$, $x_0 \in R^n$. By applying the first two multipliers and using the B.C. (4.26c), we obtain, with $n = \dim \Omega$ (see, e.g., [LT11]),

$$\int_0^T E_1(t)\, dt = \int_Q p\varphi h \cdot \nabla\varphi \, dQ + \left(\frac{n}{2} - 1\right) \int_Q p\varphi^2 \, dQ + \beta_{0T}, \tag{4.31}$$

$$\beta_{0T} \equiv \left[\left(-\frac{n}{2} + 1\right)(\varphi_t, \varphi)_{\Omega} + (\varphi_t, h \cdot \nabla\varphi)_{\Omega}\right]_0^T, \tag{4.32}$$

$$E_1(t) \equiv \int_{\Omega} |\Delta\varphi(t)|^2 + |\varphi_t(t)|^2 \, d\Omega. \tag{4.33}$$

Next, because of the B.C. $\varphi|_\Gamma = \Delta\varphi|_\Gamma = 0$ in (4.26c) we have

$$\int_\Omega |\nabla\varphi(t)|^2\, d\Omega = \|\mathscr{A}^{1/4}\varphi(t)\|^2_{L_2(\Omega)} \le C\|\mathscr{A}^{1/2}\varphi(t)\|^2_{L_2(\Omega)} = C\int_\Omega |\Delta\varphi(t)|^2\, d\Omega, \tag{4.34}$$

which also applies to $\varphi_0 \in \mathscr{D}(\mathscr{A})$. With such φ_0 we show (4.28) and then extend it by continuity to all $\varphi_0 \in \mathscr{D}(\mathscr{A}^{1/2})$. Using (4.34) on (4.32) (along with the Poincaré inequality) and on the first integral on the right of (4.31) we obtain, for any $\varepsilon_1 > 0$ since $p \in L^\infty(Q)$,

$$(1-\varepsilon_1)\int_0^T E_1(t)dt = \mathcal{O}(E_1(T) + E_1(0) + \|\varphi\|^2_{L_2(Q)}). \tag{4.35}$$

Next, multiplying (4.26a) by φ_t yields

$$E_1(t) = E_1(0) + 2\int_0^t\int_\Omega p\varphi\varphi_t\, d\Omega\, d\tau \tag{4.36}$$

from which, since $p \in L^\infty(Q)$,

$$E_1(t) \le E_1(0) + \varepsilon T\|\varphi_t\|^2_{L^\infty(0,T;L_2(\Omega))} + \frac{C_p}{\varepsilon}T\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}. \tag{4.37}$$

Selecting $\varepsilon = \varepsilon_2/T$, $\varepsilon_2 > 0$ preassigned, we obtain

$$(1-\varepsilon_2)\sup_{0\le t\le T} E_1(t) \le E_1(0) + \mathcal{O}_T(\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}), \tag{4.38}$$

where here and hereafter $\mathcal{O}_T$ means that the constant of upper bound may depend on T. Inserting (4.36) into the left-hand side of (4.35) and using the estimate (4.38) for the term $E_1(T)$ on the right-hand side of (4.35), we obtain

$$(1-\varepsilon_1)TE_1(0) = \mathcal{O}(E_1(0)) + \mathcal{O}_T(\|\varphi\|^2_{L^\infty(0,T:L_2(\Omega))}) - 2(1-\varepsilon_1)\alpha_T, \tag{4.39}$$

$$|\alpha_T| = \left|\int_0^T\int_0^t\int_\Omega p\varphi\varphi_t\, d\Omega\, d\tau\, dt\right|$$

$$\le \varepsilon T^2\|\varphi_t\|^2_{L^\infty(0,T;L_2(\Omega))} + \frac{C_p}{\varepsilon}T^2\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}$$

(where we select $\varepsilon = \varepsilon_3/T^2$, $\varepsilon_3 > 0$ preassigned, so that by (4.38))

$$\le \frac{\varepsilon_3}{1-\varepsilon_2}E_1(0) + \mathcal{O}_T(\|\varphi\|^2_{L^\infty(0,T;L_2(\Omega))}). \tag{4.40}$$

Then (4.40) used in (4.39) yields (4.28) for T sufficiently large, as desired, by (4.29). □

Step 2. Let now φ satisfy problem (4.26) for $0 < t \le T$, say T arbitrarily small. Define, for $0 < t \le T$, $\tilde{p} = p$ and $\tilde{\varphi} = \varphi$, and, for $t > T$, $\tilde{p} = \tilde{\varphi} \equiv 0$. Then $\tilde{p} \in L^\infty((0,\infty)\times\Omega)$ and $\tilde{\varphi}$ satisfies the same initial condition as in (4.26b) and, moreover, for almost all $t > 0$, the equation $\tilde{\varphi}_{tt} + \Delta^2\tilde{\varphi} = \tilde{p}\tilde{\varphi}$ and all four boundary conditions as in (4.26c). Then $\tilde{\varphi} \in L^\infty((0, \infty) \times \Omega)$ and Lemma 4.4 applies to $\tilde{\varphi}$ yielding the corresponding inequality (4.28) over some $[0, T_1]$, T_1 sufficiently large, where $\|\tilde{\varphi}\|_{L^\infty(0,T_1;L_2(\Omega))} = \|\varphi\|_{L^\infty(0,T;L_2(\Omega))}$.

Step 3. **Lemma 4.5.**

(a) *Consider problem* (4.26) *with p satisfying assumptions* (4.27) *for any* $T > 0$ *and with a priori regularity for* φ *given by* (4.30). *Then, if* $n \le 3$ *we have in fact that, for any* $T > 0$ *and for some constant* $C_T > 0$,

$$\infty > \| |\nabla\varphi| \|^2_{L^\infty(0,T;L_2(\Omega))} + \|\varphi_t\|^2_{L^\infty(0,T;L_2(\Omega))} + \|\varphi\|^2_{L^\infty(Q)} \| |\nabla_x p| \|^2_{L_2(Q)} \tag{4.41}$$

$$\ge C_T \|\{\varphi_0, \varphi_1\}\|^2_{\mathscr{D}(\mathscr{A}^{3/4}) \times \mathscr{D}(\mathscr{A}^{1/4})},$$

$$\mathscr{D}(\mathscr{A}^{1/4}) = H_0^1(\Omega), \qquad \mathscr{D}(\mathscr{A}^{3/4}) = \{h \in H^3(\Omega): h|_\Gamma = \Delta h|_\Gamma = 0\}$$
(with equivalent norms), (4.42)

$$\|h\|_{\mathscr{D}(\mathscr{A}^{1/4})} = \|\mathscr{A}^{1/4} h\|_{L_2(\Omega)} \qquad \textit{equivalent to} \left\{\int_\Omega |\nabla h|^2 \, d\Omega\right\}^{1/2}, \tag{4.43a}$$

$$\|h\|_{\mathscr{D}(\mathscr{A}^{3/4})} = \|\mathscr{A}^{3/4} h\|_{L_2(\Omega)} \qquad \textit{equivalent to} \left\{\int_\Omega |\nabla(\Delta h)|^2 \, d\Omega\right\}^{1/2}. \tag{4.43b}$$

(b) *Thus, the initial data and the regularity of problem* (4.26) *are further boosted to*

$$\{\varphi_0, \varphi_1\} \in \mathscr{D}(\mathscr{A}^{3/4}) \times \mathscr{D}(\mathscr{A}^{1/4}),$$
$$\{\varphi(t), \varphi_t(t) \in L^\infty(0, T; \mathscr{D}(\mathscr{A}^{3/4}) \times \mathscr{D}(\mathscr{A}^{1/4})). \tag{4.44}$$

Proof of Lemma 4.5. We now apply the multipliers $h \cdot \nabla\Delta\varphi$, $\Delta\varphi$, and $\Delta\varphi_t$ to (4.26a), with $h(x) = x - x_0$ again. Using the first two multipliers and invoking the B.C. (4.26c) we obtain (see [LT8], (2.29), (2.34), and (2.36) in the case $p \equiv 0$)

$$\int_0^T E_2(t)\, dt = -\int_Q p\varphi h \cdot \nabla(\Delta\varphi)\, dQ - \frac{n}{2}\int_Q p\varphi\Delta\varphi\, dQ + b_{0T}, \tag{4.45}$$

$$b_{0T} \equiv \left[(\varphi_t, h\cdot\nabla(\Delta\varphi))_\Omega - \frac{n}{2}\int_\Omega \nabla\varphi\cdot\nabla\varphi_t\, d\Omega\right]_0^T, \tag{4.46}$$

$$E_2(t) \equiv \int_\Omega |\nabla(\Delta\varphi(t))|^2 + |\nabla\varphi_t(t)|^2\, d\Omega$$
$$\text{equivalent to} \quad \|\varphi(t)\|^2_{\mathscr{D}(\mathscr{A}^{3/4})} + \|\varphi_t(t)\|^2_{\mathscr{D}(\mathscr{A}^{1/4})}. \tag{4.47}$$

Using the estimate (2.41) in [LT8] for b_{0T} and the Poincaré inequality on the second integral on the right of (4.45) we obtain, for any $\varepsilon_1 > 0$ since $p \in L^\infty(Q)$,

$$(1 - \varepsilon_1)\int_0^T E_2(t)\, dt \le \varepsilon_1 C_{n,h}(E_2(T) + E_2(0))$$
$$+ \mathscr{O}(\| |\nabla\varphi| \|^2_{L^\infty(0,T;L_2(\Omega))} + \|\varphi_t\|^2_{L^\infty(0,T;L_2(\Omega))} + \|\varphi\|^2_{L_2(Q)}). \tag{4.48}$$

Next, multiplying (4.26a) by $\Delta\varphi_t$ yields, e.g., [LT8]

$$E_2(t) = E_2(0) - 2\int_0^t \int_\Omega p\varphi\Delta\varphi_t \, d\Omega \, d\tau, \tag{4.49}$$

where, by Green's first theorem, using the B.C. (4.26c),

$$\int_\Omega p\varphi\Delta\varphi_t \, d\Omega = -\int_\Omega p\nabla\varphi_t \cdot \nabla\varphi \, d\Omega - \int_\Omega \varphi\nabla p \cdot \nabla\varphi_t \, d\Omega, \tag{4.50}$$

∇ being the gradient in the space variable x from which, using $p \in L^\infty(Q)$,

$$\left| \int_0^t \int_\Omega p\varphi\Delta\varphi_t \, d\Omega \, d\tau \right| \le 2\varepsilon T \| |\nabla\varphi_t| \|^2_{L^\infty(0,T;L_2(\Omega))} + \frac{C_p T}{\varepsilon} \| |\nabla\varphi| \|^2_{L^\infty(0,T;L_2(\Omega))}$$
$$+ \mathcal{O}\left(\int_0^t \int_\Omega |\varphi\nabla p|^2 \, d\Omega \, d\tau \right). \tag{4.51}$$

Selecting $2\varepsilon = \varepsilon_2/T$, $\varepsilon_2 > 0$ preassigned, and using the Sobolev embedding $\varphi \in L^\infty(0, T; H^2(\Omega))$ (from (4.30)), $\varphi \in L^\infty(Q)$ for $n \le 3$, we obtain from (4.49) via (4.51)

$$(1 - \varepsilon_2) \sup_{0 \le t \le T} E_2(t) \le E_2(0) + \mathcal{O}_T(\| |\nabla\varphi| \|^2_{L^\infty(0,T;L_2(\Omega))})$$
$$+ \mathcal{O}(\| \varphi \|^2_{L^\infty(Q)} \| |\nabla p| \|^2_{L_2(Q)}). \tag{4.52}$$

Inserting (4.49) into the left-hand side of (4.48) and using the estimate (4.52) for the term $E_2(T)$ on the right-hand side of (4.48), we obtain

$$(1 - \varepsilon_1) T E_2(0) \le \varepsilon_1 C_{n,h} \left(\frac{1}{1 - \varepsilon_2} + 1 \right) E_2(0) + \mathcal{O}(\| \varphi_t \|^2_{L^\infty(0,T;L_2(\Omega))})$$
$$+ \mathcal{O}_T(\| |\nabla\varphi| \|^2_{L^\infty(0,T;L_2(\Omega))})$$
$$+ \mathcal{O}(\| \varphi \|^2_{L^\infty(Q)} \| |\nabla p| \|^2_{L_2(Q)}) + 2(1 - \varepsilon_1)\gamma_T, \tag{4.53}$$

where from (4.51) and proceeding as in (4.52)

$$|\gamma_T| = \left| \int_0^T \int_0^t \int_\Omega p\varphi\Delta\varphi_t \, d\Omega \, d\tau \, dt \right|$$
$$\le 2\,\varepsilon\, T^2 \| |\nabla\varphi_t| \|^2_{L^\infty(0,T;L_2(\Omega))}$$
$$+ \frac{C_p T^2}{\varepsilon} \| |\nabla\varphi| \|^2_{L^\infty(0,T;L_2(\Omega))} + \mathcal{O}(T \| \varphi \|^2_{L^\infty(Q)} \| |\nabla p| \|^2_{L_2(Q)}) \tag{4.54}$$

(selecting now $2\varepsilon = \varepsilon_3/T^2$, $\varepsilon_3 > 0$ preassigned, and recalling (4.52))

$$\le \frac{\varepsilon_3}{1 - \varepsilon_2} E_2(0) + \mathcal{O}_T(\| |\nabla\varphi| \|^2_{L^\infty(0,T;L_2(\Omega))} + \| \varphi \|^2_{L^\infty(Q)} \| |\nabla p| \|^2_{L_2(Q)}). \tag{4.55}$$

Using (4.55) in (4.53) yields (4.41) for any $T > 0$, as desired, by virtue of (4.47). □

Conclusion of proof of Theorem 4.3. Having boosted the *a priori* regularity of problem (4.26) to $\{\varphi, \varphi_t\} \in L^\infty(0, T; H^3(\Omega) \times H^1(\Omega))$ from (4.44), $T > 0$ arbitrary, we can now apply the uniqueness result as in [I] twice in succession (as [I] deals with the Schrodinger equations) and conclude that, in fact, $\varphi_0 = \varphi_1 = 0$, as desired. □

Remark 4.1. The general setup of this paper may apply to other semilinear plate-like problems, in addition to the Euler–Bernoulli equation (1.14). For instance, we may replace the B.C. (1.14d), with $\Delta w + (1-\mu)Bw = u_2$ on Σ, B the boundary operator arising in the two-dimensional model, for which the corresponding exact controllability result has been recently established in [H3].

As another example, we may consider the Euler–Bernoulli equation (1.14a), this time with boundary controls

$$w|_\Sigma = u_1, \qquad \left.\frac{\partial w}{\partial \nu}\right|_\Sigma = u_2.$$

The exact controllability of the linear equation ($f = 0$) on the space

$$H = \mathscr{D}(\mathscr{A}^{1/4}) \times [\mathscr{D}(\mathscr{A}^{1/4})]' = H_0^1(\Omega) \times H^{-1}(\Omega),$$

$$\mathscr{A}h = \Delta^2 h, \qquad \mathscr{D}(\mathscr{A}) = \left\{h \in H^4(\Omega): h|_\Gamma = \left.\frac{\partial h}{\partial \nu}\right|_\Gamma = 0\right\}$$

with controls $u_1 = H_0^1(0, T; L_2(\Gamma))$ and $u_2 \equiv 0$ is given in Theorem 1.2 of [LT6], complemented by [LT12] for arbitrarily short $T > 0$. Here, we may take $\mathscr{E}_T = L_2(0, T; [\mathscr{D}(\mathscr{A}^{1/2})]')$ and $u_2 \in H_0^1(0, T; H^{-1}(\Gamma))$ for the semilinear model. All structural assumptions (A.1)–(A.5) as well as the exact controllability assumption (C.1) can then be verified. The approximate controllability assumption (C.2) leads to a uniqueness property which is apparently open at present.

Finally, it seems likely that the semilinear version of the Kirchhoff problem in [LT11] can also be covered by the present setup.

5. Proof of Theorem 1.2 and of Remark 1.2

The crux of the proof of this theorem is on the trace regularity $v|_\Sigma$ of the solution v of problem (1.6), due to the control action $\mu \in L_2(\Sigma)$. According to recent results [LT9, Main Theorem 1.3 and Remarks 1.2], we have that, in fact, $v|_\Sigma \in H^\beta(\Sigma)$, β as specified in (1.5). By assumption (1.4) on g, it then follows that $g(v|_\Sigma) \in L_2(\Sigma)$. We next define the function u by setting $u = \mu - g(v|_\Sigma) \in L_2(\Sigma)$ as in (1.7). Then (1.8) holds as desired. Note that the general question of well-posedness of problem (1.3) with a general $u \in L_2(\Sigma)$ is handled, e.g., by [L3].

The proof of the content of Remark 11.2 is similar, this time applying Main Theorem 1.2 and Remark 1.1 of [LT9].

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Appendix A. Proof of Theorem 3.1

A.1. Proof of Exact Controllability

We use a direct approach as in [LT5], [LT6], [LT8], and [T3].

Lemma A.1. *The exact controllability on* $[0, T]$ *of problem* (1.1) *with* $f \equiv 0$ *on the state space* $H_0^1(\Omega) \times L_2(\Omega)$ *and within the class of* $H_0^1(0, T; L_2(\Gamma))$*-controls as expressed in Theorem* 3.1 *is equivalent to the following property: there is a constant* $C_T > 0$ *such that*

$$\int_\Sigma \left(\frac{\partial\varphi}{\partial\nu} + K_{0T}\right)^2 d\Sigma \geq C_T \|\{\varphi_0, \varphi_1\}\|^2_{H_0^1(\Omega)\times L_2(\Omega)}, \tag{A.1}$$

where φ *solves the corresponding homogeneous problem backward in time*

$$\begin{cases} \varphi_{tt} = \Delta\varphi & \text{in } (0, T] \times \Omega \equiv Q, & \text{(A.2a)} \\ \varphi|_{t=T} = \varphi_0, \quad \varphi_t|_{t=T} = \varphi_1 & \text{in } \Omega, & \text{(A.2b)} \\ \varphi|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma, & \text{(A.2c)} \end{cases}$$

explicitly given by

$$\varphi(t) = C(t - T)\varphi_0 + S(t - T)\varphi_1, \tag{A.3}$$

$$K_{0T} = \frac{-1}{T} D^*\mathscr{A}\{[I - C(T)]\mathscr{A}^{-1}\varphi_1 - S(T)\varphi_0\}. \tag{A.4}$$

Proof. The exact controllability sought (say from the origin) means that the continuous input-solution operator $\mathscr{L}_T$ in (3.16) satisfies

$$\mathscr{L}_T: H_0^1(0, T; L_2(\Gamma)) \xrightarrow{\text{ONTO}} H_0^1(\Omega) \times L_2(\Omega) = \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega), \tag{A.5}$$

equivalently, the Hilbert space adjoint $\mathscr{L}_T^*: H_0^1(\Omega) \times L_2(\Omega) \to H_0^1(0, T; L_2(\Gamma))$ has a continuous inverse: there exists a constant $C_T > 0$ such that

$$\left\| \frac{d}{dt} \mathscr{L}_T^* \begin{vmatrix} z_0 \\ z_1 \end{vmatrix} \right\|^2_{L_2(0,T;L_2(\Gamma))} \geq C_T \|\{z_0, z_1\}\|^2_{\mathscr{D}(\mathscr{A}^{1/2})\times L_2(\Omega)} \tag{A.6}$$

since, for any $g \in H_0^1(0, T; L_2(\Gamma))$,

$$\|g\|_{H_0^1(0,T;L_2(\Gamma))} \quad \text{equivalent to} \quad \left\| \frac{d}{dt} g \right\|_{L_2(0,T;L_2(\Gamma))}. \tag{A.7}$$

We compute the adjoint $\mathscr{L}_T^*$ as usual: from (3.16) with $u \in H_0^1(0, T; L_2(\Gamma))$, $[z_0, z_1] \in \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)$ using that $\mathscr{A}$, $S(\cdot)$ (odd), $C(\cdot)$ (even) are self-adjoint, then [LT5], [T3]

$$\left(\mathscr{L}_T u, \begin{vmatrix} z_0 \\ z_1 \end{vmatrix}\right)_{\mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega)} = \left(\mathscr{A}\int_0^T S(T-t)Du(t)\,dt, \mathscr{A}z_0\right)_{L_2(\Omega)}$$

$$+ \left(\mathscr{A}\int_0^T C(T-t)Du(t)\,dt, z_1\right)_{L_2(\Omega)}$$

$$= \int_0^T (u(t), D^*\mathscr{A}[C(t-T)z_1 - S(t-T)\mathscr{A}z_0])_{L_2(\Gamma)}\,dt$$

$$= \left(u, \mathscr{L}_T^* \begin{vmatrix} z_0 \\ z_1 \end{vmatrix}\right)_{H_0^1(0, T; L_2(\Gamma))} \tag{A.8}$$

$$= \left(u_t, \frac{d}{dt}\mathscr{L}_T^* \begin{vmatrix} z_0 \\ z_1 \end{vmatrix}\right)_{L_2(0, T; L_2(\Gamma))}$$

$$= \int_0^T \left(u_t, \frac{d}{dt}\mathscr{L}_T^* \begin{vmatrix} z_0 \\ z_1 \end{vmatrix}\right)_{L_2(\Gamma)} dt$$

$$= -\int_0^T \left(u, \frac{d^2}{dt^2}\mathscr{L}_T^* \begin{vmatrix} z_0 \\ z_1 \end{vmatrix}\right)_{L_2(\Gamma)} dt \tag{A.9}$$

since $u(T) = u(0) = 0$. By comparing (A.8) with (A.9) we may take

$$-\frac{d^2}{dt^2}\left(\mathscr{L}_T^* \begin{vmatrix} z_0 \\ z_1 \end{vmatrix}\right)(t) = D^*\mathscr{A}[C(t-T)z_1 - S(t-T)\mathscr{A}z_0]. \tag{A.10}$$

Integrating (A.10) in t and requiring that

$$\mathscr{L}_T^* \begin{vmatrix} z_0 \\ z_1 \end{vmatrix} \in H_0^1(0, T; L_2(\Gamma))$$

vanishes at $t = 0$ and $t = T$, yields

$$\left(\frac{d\mathscr{L}_T^*}{dt} \begin{vmatrix} z_0 \\ z_1 \end{vmatrix}\right)(t) = -D^*\mathscr{A}[C(t-T)z_0 + S(t-T)z_1] + K_{0T}$$

$$= \frac{\partial\varphi}{\partial\nu}(t; \varphi_0 = z_0; \varphi_1 = z_1) + K_{0T} \tag{A.11}$$

since $D^*\mathscr{A} = -\partial/\partial\nu$[LT2], where φ solves (A.2) with $\varphi_0 = z_0$, $\varphi_1 = z_1$, and K_{0T} is given by (A.4). Then (A.11) used in (A.6) yields (A.1), as desired. □

Lemma A.2. *Inequality* (A.1) (*which is equivalent to exact controllability of the linear problem* (1.1) *with* $f \equiv 0$ *on the space* $H_0^1(\Omega) \times L_2(\Omega)$ *within the class of* $H_0^1(0, T; L_2(\Gamma))$*-controls) is in turn equivalent to the inequality*

$$\int_\Sigma \left(\frac{\partial\varphi}{\partial\nu}\right)^2 d\Sigma \geq C_T' \|\{\varphi_0, \varphi_1\}\|^2_{H_0^1(\Omega) \times L_2(\Omega)} \tag{A.12}$$

for some $C'_T > 0$ (which is equivalent, e.g., [L5], [H2], [T3], *to exact controllability of the linear problem* (1.1) *with $f \equiv 0$ on the space $L_2(\Omega) \times H^{-1}(\Omega)$ within the class of $L_2(0, T; L_2(\Gamma))$-controls).*

Corollary A.3.

(i) *The linear problem* (1.1) *with $f \equiv 0$ is exactly controllable in $[0, T]$ on $H_0^1(\Omega) \times L_2(\Omega)$ with $H_0^1(0, T; L_2(\Gamma))$-controls if and only if it is exactly controllable in $[0, T]$ on $L_2(\Omega) \times H^{-1}(\Omega)$ with $L_2(0, T; L_2(\Gamma))$-controls.*

(ii) *When part* (i) *holds, then problem* (1.1) *with $f \equiv 0$ is exactly controllable in $[0, T]$ on the space*

$$\mathscr{D}(\mathscr{A}^{(1-\theta)/2}) \times [\mathscr{D}(\mathscr{A}^{\theta/2})]' = H_0^{1-\theta}(\Omega) \times H^{-\theta}(\Omega), \qquad 0 \le \theta \le 1, \quad \theta \neq \tfrac{1}{2},$$

or

$$H_{00}^{1/2}(\Omega) \times [H_{00}^{1/2}(\Omega)]' \qquad \text{if} \quad \theta = \tfrac{1}{2} \tag{A.13}$$

with $H_0^{1-\theta}(0, T; L_2(\Gamma))$-controls for $\theta \neq \frac{1}{2}$, and $H_{00}^{1/2}(0, T; L_2(\Gamma))$-controls for $\theta = \frac{1}{2}$. This follows by interpolating between the statement that $(\mathscr{L}_T^*)^{-1}$ is continuous: $H_0^1(\Omega) \times L_2(\Omega) \to H_0^1(0, T; L_2(\Gamma))$ as in (A.6) *(which is equivalent to* (A.5)) *and the statement that $(\mathscr{L}_T^*)^{-1}$ is continuous: $L_2(\Omega) \times H^{-1}(\Omega) \to L_2(0, T; L_2(\Gamma))$ (which is equivalent to $\mathscr{L}_T$: $L_2(0, T; L_2(\Gamma))$ onto $L_2(\Omega) \times H^{-1}(\Omega)$), see pp.* 64–66 *of* [LM].

Proof of Lemma A.2. We adapt to present circumstances a compactness argument [L5], [L7], [LT5], [LT6], [LT8] by "absorbing" the "lower-order" term K_{0T} given by (A.4). First, we assume (A.12) and show that (A.1) holds. By contradiction, let there be a sequence $\{\varphi_{0n}, \varphi_{1n}\} \in H_0^1(\Omega) \times L_2(\Omega)$ such that with $\varphi_n(t) = C(t - T)\varphi_{0n} + S(t - T)\varphi_{1n}$, i.e.,

$$\begin{cases} \varphi_n'' = \Delta\varphi_n, \\ \varphi_n|_{t=T} = \varphi_{n0}, \ \varphi_n'|_{t=T} = \varphi_{n1}, \\ \varphi_n|_\Sigma = 0, \end{cases} \tag{A.14}$$

we have

$$\left\| \frac{\partial \varphi_n}{\partial \nu} \right\|_{L_2(\Sigma)}^2 \equiv 1, \tag{A.15}$$

$$\left\| \frac{\partial \varphi_n}{\partial \nu} + K_{0T,n} \right\|_{L_2(\Sigma)}^2 \to 0 \qquad \text{as} \quad n \to \infty. \tag{A.16}$$

But $\varphi_n(t)$ satisfies (A.12) and so there is a subsequence $\{\varphi_{0n}, \varphi_{1n}\} \to$ some $\{\tilde\varphi_0, \tilde\varphi_1\}$ in $H_0^1(\Omega) \times L_2(\Omega)$ weakly, and by compactness, for $\delta > 0$, $\delta \neq \frac{1}{2}$,

$$\{\varphi_{0n}, \varphi_{1n}\} \to \{\tilde\varphi_0, \tilde\varphi_1\}$$
$$\text{strongly in} \quad H_0^{1-\delta}(\Omega) \times H^{-\delta}(\Omega) = \mathscr{D}(\mathscr{A}^{(1-\delta)/2}) \times [\mathscr{D}(\mathscr{A}^{\delta/2})]', \tag{A.17}$$

equivalently

$$\{\mathscr{A}^{(1-\delta)/2}\varphi_{0n}, \mathscr{A}^{-\delta/2}\varphi_{1n}\} \to \{\mathscr{A}^{(1-\delta)/2}\tilde{\varphi}_0, \mathscr{A}^{-\delta/2}\varphi_1\}$$
$$\text{strongly in} \quad L_2(\Omega) \times L_2(\Omega). \tag{A.18}$$

By (A.18) with $\varepsilon = \frac{1}{4} - \delta/2$ recalling $D^*\mathscr{A}^{1/4-\varepsilon} \in L(L_2(\Omega), L_2(\Gamma))$ from (3.12), we have from (A.4) that

$$\begin{cases} K_{0T,n} = \dfrac{-1}{T} D^*\mathscr{A}^{1/4-\varepsilon}[I - C(T)]\mathscr{A}^{-1/4+\varepsilon}\varphi_{1n} \\ \qquad + \dfrac{1}{T} D^*\mathscr{A}^{1/4-\varepsilon}\mathscr{A}^{1/2}S(T)\mathscr{A}^{1/4+\varepsilon}\varphi_{0n} & \text{(A.19a)} \\ \text{converges strongly in } L_2(\Gamma) \text{ to} \\ \tilde{K}_{0T} = \dfrac{-1}{T} D^*\mathscr{A}\{[I - C(T)]\mathscr{A}^{-1}\tilde{\varphi}_0 - S(T)\tilde{\varphi}_1\}. & \text{(A.19b)} \end{cases}$$

Thus,

$$K_{0T,n} \to \tilde{K}_{0T} \qquad \text{strongly in } L_2(0, T; L_2(\Gamma)). \tag{A.20}$$

By (A.16) and (A.20)

$$\frac{\partial \varphi_n}{\partial \nu} \to -\tilde{K}_{0T} \qquad \text{strongly in } L_2(0, T; L_2(\Gamma)) \tag{A.21}$$

and by (A.15) we deduce

$$\|\tilde{K}_{0T}\|_{L_2(\Sigma)} = 1. \tag{A.22}$$

On the other hand, $\tilde{\varphi}(t) = C(t - T)\tilde{\varphi}_0 + S(t - T)\tilde{\varphi}_1$ satisfies

$$\begin{cases} \tilde{\varphi}_{tt} = \Delta\tilde{\varphi} & \text{in } Q, \\ \tilde{\varphi}|_\Sigma = 0 & \text{in } \Omega, \\ \left[\dfrac{\partial\tilde{\varphi}}{\partial\nu} + \tilde{K}_{0T}\right]_\Sigma = 0 & \text{in } \Sigma, \end{cases} \quad \text{hence} \quad \begin{cases} (\tilde{\varphi}_t)_{tt} = \Delta\tilde{\varphi} & \text{in } Q, & \text{(A.23a)} \\ \tilde{\varphi}_t|_\Sigma = 0 & \text{in } \Omega, & \text{(A.23b)} \\ \left.\dfrac{\partial\tilde{\varphi}_t}{\partial\nu}\right|_\Sigma = 0 & \text{in } \Sigma, & \text{(A.23c)} \end{cases}$$

by differentiating in t. The standard uniqueness property [H4] applies to problem (A23) right, and yields $\tilde{\varphi}_t \equiv 0$ in Q, hence $\tilde{\varphi} \equiv$ const in Q, finally $\partial\tilde{\varphi}/\partial\nu \equiv 0$ in Σ. By (A.23c), left problem, we get $\tilde{K}_{0T} = 0$ and this contradicts (A.22). Thus (A.12) implies (A.1). The proof that (A.1) implies (A.12) is identical. □

The proof of the exact controllability statement in Theorem 3.1 is now complete since (A12) is known to hold true for sufficiently smooth Ω and for $T > T(x^0) = 2R(x^0) = 2\max|x - x_0|$, for $x \in \bar{\Omega}$, by Komornik's remark [L5, Section 5]. The case where the control u in (1.1c) acts only on a prescribed portion Γ_1 of Γ can likewise be handled, see, e.g., [T3], where estimates of T are given.

A.2. Proof of Regularity Statement (3.4) in Theorem 3.1.

It is essentially contained, say, in the proof of Theorem 3.4 of [LLT]. If $u \in H_0^1(0, T; L_2(\Gamma))$ in (1.1c) of problem (1.1) with $w_0 = w_1 = 0$ and $f \equiv 0$, then integrating by parts (3.15) in t yields, since $u(0) = 0$ (see also (3.23) in [LLT]) and $\dot{u} \in L_2(\Sigma)$,

$$w(t) = Du(t) - \int_0^t C(t-\tau)D\dot{u}(\tau)\,d\tau, \tag{A.24}$$

$$w_t(t) = \mathscr{A}\int_0^t S(t-\tau)D\dot{u}(\tau)\,d\tau \in C([0, T]; L_2(\Omega)), \tag{A.25}$$

$$Du(t) \in H_0^1(0, T; H^{1/2}(\Omega)) \qquad \text{by (3.12)}, \tag{A.26}$$

$$\int_0^t C(t-\tau)D\dot{u}(\tau)\,d\tau \in C([0, T]; \mathscr{D}(\mathscr{A}^{1/2}) = H_0^1(\Omega)) \qquad \text{from (3.24) in [LLT].} \tag{A.27}$$

Thus, using (A.26), (A.27) in (A.24) and noting (A.25) yields (3.4), as desired, and Theorem 3.1 is fully proved. Moreover, for $u(T) = 0$, we get from (A.24), (A.27) that $w(T) \in H_0^1(\Omega)$, and (3.16) is also proved. □

Appendix B. Proof of (4.9) and of (4.10b)

Proof of (4.9). We actually show sharper results. Starting from (4.8) with $i = 1$, rewritten now as

$$(\mathscr{L}_1 u_1)(t) = \left[L_1 u_1(t), \frac{(dL_1 u_1)(t)}{dt}\right]$$

and integrating by parts in t time (as in Section 3 of [LLT] and in [LT7] with $u_1(t)$ satisfying

$$u_1 \in C([0, T], H^{5/2}(\Gamma)), \qquad \dot{u}_1 \in C([0, T], H^{1/2}(\Gamma)), \qquad \ddot{u}_1 \in L_2(\Sigma), \tag{B.1a}$$

$$u_1(0) = \dot{u}_1(0) = 0, \tag{B.1b}$$

we obtain

$$(L_1 u_1)(t) = G_1 u_1(t) - C(t)\not{G}_1 u_1(0) - S(t)\not{G}_1 \dot{u}_1(0) - \int_0^t S(t-\tau)G_1\ddot{u}_1(\tau)\,d\tau \in C([0, T]; H^3(\Omega)), \tag{B.2}$$

$$\frac{(dL_1 u_1)}{dt}(t) = \mathscr{A}S(t)\not{G}_1 u_1(0) + G_1\dot{u}_1(t) - C(t)\not{G}_1\dot{u}_1(0) - \int_0^t C(t-\tau)G_1\ddot{u}_1(\tau)\,d\tau \in C([0, T]; H^1(\Omega)) \tag{B.3}$$

by using (4.4) on G_1 and Theorem 1.3 of [LT7] on the integral terms. Moreover, if in addition we have

$$u_1(T) = \dot{u}_1(T) = 0, \tag{B.4}$$

then

$$(L_1u_1)(T) = -\int_0^T S(T-t)G_1\ddot{u}_1(t)\,dt \in \mathscr{D}(\mathscr{A}^{3/4}), \tag{B.5}$$

$$\frac{(dL_1u_1}{dt}(T) = -\int_0^T C(T-t)G_1\ddot{u}(t)\,dt \in \mathscr{D}(\mathscr{A}^{1/4}). \tag{B.6}$$

The regularity properties required of u_1 in (B.1a) are satisfied if u_1 belongs to the space

$$W(0,T) = \left\{h \in L_2(0,T; H^{10/3}(\Gamma)), \frac{d^2h}{dt^2} \in L_2(0,T; L_2(\Gamma))\right\} \tag{B.7}$$

as it follows via Theorem 3.1, p. 19, of [LM], while properties (B.1a), (B.1b), and (B.4) are *a fortiori* satisfied if $u_1 \in H_0^2(0,T; H^{10/3}(\Gamma))$. □

Proof of (4.10b) and (4.10c). As already observed, statement (4.10a) is proved in Theorem 1.3 of [LT7]. Next, to prove statement (4.10b), let $u_2 \in H_0^1(0,T; L_2(\Gamma))$ so that $\dot{u}_2 \in L_2(\Sigma)$. Integration by parts in t on (4.8b) with $i = 2$ yields

$$(\mathscr{L}_2u_2)(t) = \left|\begin{matrix} G_2u_2(t) - C(t)\not{G}_2u_2(0) - \int_0^t C(t-\tau)G_2\dot{u}_2(\tau)\,d\tau \\ \mathscr{A}S(t)\not{G}_2u_2(0) + \mathscr{A}\int_0^t S(t-\tau)G_2\dot{u}_2(\tau)\,d\tau \end{matrix}\right|. \tag{B.8}$$

Since, by (4.5) and, respectively, by (4.10a) with $[\mathscr{D}(\mathscr{A}^{1/4})]' = H^{-1}(\Omega)$[G], [LT7],

$$G_2u_2(t) \in H_0^1(0,T; H^{5/2}(\Omega)), \qquad \int_0^t C(t-\tau)G_2\dot{u}_2(\tau)\,d\tau \in C([0,T]; \mathscr{D}(\mathscr{A}^{3/4})), \tag{B.9}$$

we obtain via (B.8), (4.10a), and $\mathscr{D}(\mathscr{A}^{3/4}) = \{h \in H^3(\Omega)\colon h|_\Gamma = \Delta h|_\Gamma = 0\}$ [G], [LT7] that

$$(\mathscr{L}_2u_2)(t) \in C([0,T]; H^{5/2}(\Omega) \times H_0^1(\Omega)) \tag{B.10}$$

and (4.10b) is proved. Note, moreover, that since $u_2(T) = 0$, (B.8)–(B.10) yield

$$\mathscr{L}_{2T}u_2 = \left|\begin{matrix} -\int_0^T C(T-t)\dot{u}_2(t)\,dt \\ \mathscr{A}\int_0^T S(T-t)G_2\dot{u}_2(t)\,dt \end{matrix}\right| \in \left|\begin{matrix} \mathscr{D}(\mathscr{A}^{3/4}) \\ \mathscr{D}(\mathscr{A}^{1/4}) = H_0^1(\Omega) \end{matrix}\right|. \tag{B.11}$$

To prove (4.10c) note also that [G], [LT7]

$$H^{5/2}(\Omega) \subset H^{5/2-4\varepsilon}(\Omega) = \mathscr{D}(\mathscr{A}^{5/8-\varepsilon}) = \{h \in H^{5/2-4\varepsilon}(\Omega)\colon h|_\Gamma = 0\} \tag{B.12}$$

and hence interpolating between (4.10a) and (4.10b), with $\theta = \frac{3}{4}$,

$$[H^{5/2}(\Omega), H_0^1(\Omega)]_{3/4} \subset \begin{cases} [H^{5/2}(\Omega), H^1(\Omega)]_{3/4} = H^{11/8}(\Omega), & \text{(B.13)} \\ [\mathscr{D}(\mathscr{A}^{5/8-\varepsilon}), \mathscr{D}(\mathscr{A}^{1/4})]_{3/4} = \mathscr{D}(\mathscr{A}^{11/32-\varepsilon/4}), & \text{(B.14)} \end{cases}$$

$$\mathscr{D}(\mathscr{A}^{11/32-\varepsilon/4}) = \{h \in H^{11/18-\varepsilon}(\Omega)\colon h|_\Gamma = 0\} = H_0^{11/8-\varepsilon}(\Omega). \quad \square \tag{B.15}$$

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