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Shape Optimization for Dirichlet Problems: Relaxed Formulation and Optimality Conditions

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Abstract. We study an optimal design problem for the domain of an elliptic equation with Dirichlet boundary conditions. We introduce a relaxed formulation of the problem which always admits a solution, and we prove some necessary conditions for optimality both for the relaxed and for the original problem.

0. Introduction

In this paper we study a model problem in shape optimization for the domain of an elliptic equation with Dirichlet boundary conditions.

More precisely, given a bounded open subset Ω of $\mathbf{R}^N$, $N \geq 2$, and two functions $f \in L^2(\Omega)$ and $j: \Omega \times \mathbf{R} \to \mathbf{R}$, we consider the optimal design problem

$$\min_{A \in \mathscr{A}(\Omega)} \int_\Omega j(x, u_A(x))\, dx, \tag{0.1}$$

where $\mathscr{A}(\Omega)$ is the family of all open subsets of Ω and u_A is the solution of the Dirichlet problem

$$u_A \in H_0^1(A), \qquad -\Delta u_A = f \quad \text{in } A \tag{0.2}$$

extended by 0 to $\Omega \backslash A$.

It is well known that, in general, problem (0.1) has no solution (see, for instance, Example 4.3). The reason is that, although the solutions u_{A_n} of (0.2) corresponding to a minimizing sequence (A_n) of (0.1) always admit a limit point u

in the weak topology of $H_0^1(\Omega)$, we cannot find, in general, an open subset A of Ω such that $u = u_A$. On the contrary, it can be proved (see Section 3) that the limit function u is the solution of a relaxed Dirichlet problem (see [2], [9], and [10]) of the form

$$u \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \qquad -\Delta u + u\mu = f \quad \text{in } \Omega \tag{0.3}$$

for a suitable nonnegative Borel measure μ which vanishes on all sets of (harmonic) capacity 0, but may take the value $+\infty$ on some subsets of Ω.

This suggests the following relaxed formulation for the optimal design problem (0.1):

$$\min_{\mu \in \mathcal{M}_0(\Omega)} \int_\Omega j(x, u_\mu(x))\, dx, \tag{0.4}$$

where $\mathcal{M}_0(\Omega)$ is the class of all measures allowed in (0.3) and, for every $\mu \in \mathcal{M}_0(\Omega)$, u_μ denotes the corresponding solution of (0.3).

In Section 4 we prove that, under suitable hypotheses on j, the relaxed optimization problem (0.4) admits a solution, and that

$$\min_{\mu \in \mathcal{M}_0(\Omega)} \int_\Omega j(x, u_\mu(x))\, dx = \inf_{A \in \mathcal{A}(\Omega)} \int_\Omega j(x, u_A(x))\, dx.$$

Moreover, we describe the close relationship between minimum points of (0.4) and minimizing sequences of (0.1).

Similar relaxed formulations for different classes of optimal design problems have been considered by Murat and Tartar in [21], [24], [25], and [27], and by Kohn, Strang, and Vogelius in [19] and [20].

The main goal of this paper is to prove some optimality conditions for the solutions of the relaxed problem (0.4). Let $\mu \in \mathcal{M}_0(\Omega)$ be a minimum point of (0.4) and let A be the regular set of the measure μ, defined as the union of all finely open subsets B of Ω with $\mu(B) < +\infty$ (for the definition and properties of the fine topology we refer to Chapter XI of Part 1 of [14]). Let us consider the solution u of the problem

$$u \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \qquad -\Delta u + u\mu = f \quad \text{in } \Omega,$$

and the solution v of the adjoint problem

$$v \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \qquad -\Delta v + v\mu = j_s(x, u) \quad \text{in } \Omega,$$

where $j_s(x, s)$ is the partial derivative of j with respect to s.

In Sections 5 and 9 we prove the following necessary conditions for optimality:

$$uv \le 0 \qquad \text{almost everywhere (a.e.) in } A, \tag{0.5}$$

$$uv = 0 \qquad \mu\text{-a.e. in } A, \tag{0.6}$$

$$f(\cdot)j_s(\cdot, 0) \ge 0 \qquad \text{a.e. in} \quad \Omega \setminus \bar{A}, \tag{0.7}$$

$$\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} \ge 0 \qquad \text{in} \quad \Omega \cap \partial A, \tag{0.8}$$

where the normal derivatives $\partial u/\partial n$ and $\partial v/\partial n$ on ∂A are defined in a suitable weak form in Sections 6 and 7.

When the original problem (0.1) has a solution $A \in \mathscr{A}(\Omega)$, then the measure μ defined by

$$\mu(B) = \begin{cases} 0 & \text{if} \quad B\setminus A \quad \text{has capacity zero,} \\ +\infty & \text{otherwise,} \end{cases}$$

is a solution of the relaxed problem (0.4). Therefore, (0.5), (0.7), and (0.8) are optimality conditions for problem (0.1), while (0.6) is trivial in this case. If, in addition, the optimal domain A has a smooth boundary, then (0.5) and (0.8) imply

$$\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = 0 \qquad \text{on} \quad \Omega \cap \partial A.$$

This optimality condition for problem (0.1) is already known in the literature (see, for instance, [6], [22], [23], [26], and [29]), and has been obtained by using the Hadamard method of variation of domains, whereas conditions (0.5) and (0.7) seem to be new.

The results of this paper were announced without proofs in [5].

1. Notation and Preliminary Results

For every open subset A of $\mathbf{R}^N$, with $N \geq 2$, $H^1(A)$ is the usual *Sobolev space* of all (real-valued) functions of $L^2(A)$ with first-order distribution derivatives in $L^2(A)$, endowed with the scalar product

$$(u, v)_{H^1(A)} = \int_A DuDv\, dx + \int_A uv\, dx$$

and with the corresponding norm $\|\cdot\|_{H^1(A)}$. Here $H_0^1(A)$ is the closure of $C_0^\infty(A)$ in $H^1(A)$, and $H^{-1}(A)$ is the dual space of $H_0^1(A)$. The corresponding duality pairing is denoted by $\langle \cdot, \cdot \rangle_{H^{-1}(A)}$. Each function $u \in H_0^1(A)$ is extended to $\mathbf{R}^N$ by setting $u = 0$ on $\mathbf{R}^N \setminus A$. With this convention we have $H_0^1(A) \subseteq H^1(\mathbf{R}^N)$.

The lattice operations $\wedge$ and $\vee$ are defined by $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for any $a, b \in \bar{\mathbf{R}}$. For real-valued functions, the lattice operators $\wedge$ and $\vee$ are defined pointwise. It is well known (see, for instance, Theorem A.1 of Chapter II of [18]) that if $u, v \in H^1(A)$ (resp. $H_0^1(A)$), then $u \wedge v, u \vee v \in H^1(A)$ (resp. $H_0^1(A)$).

The *capacity* of a subset E of $\mathbf{R}^N$ is defined by

$$\operatorname{cap}(E) = \inf_{u \in \mathscr{U}_E} \|u\|^2_{H^1(\mathbf{R}^N)},$$

where $\mathscr{U}_E$ is the set of all functions $u \in H^1(\mathbf{R}^N)$ such that $u \geq 1$ almost everywhere in a neighborhood of E (depending on u). If a property $P(x)$ holds for all $x \in E$, except for a set $Z \subseteq E$ with $\operatorname{cap}(Z) = 0$, then we say that $P(x)$ holds *quasi-everywhere* on E (q.e. on E). The expression almost everywhere (a.e.) refers, as usual, to the Lebesgue measure.

We say that a function $u: A \to \mathbf{R}$ is *quasi-continuous* in A if for every $\varepsilon > 0$ there exists a subset E of A with $\operatorname{cap}(A \setminus E) < \varepsilon$ such that the restriction $u|_E$ of u to E is continuous on E. The notion of *quasi-lower semicontinuity* is defined in a similar way.

Functions in $H^1(A)$ can be defined q.e. in A. In fact, if $B_r(x)$ denotes the open ball centered at x with radius r, and $|B_r(x)|$ is its Lebesgue measure, then for every $u \in H^1(\Omega)$ the limit

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy$$

exists and is finite q.e. in A. We adopt the following convention concerning the pointwise values of a function $u \in H^1(A)$: for every $x \in A$ we always require that

$$\liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \le u(x) \le \limsup_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy. \tag{1.1}$$

With this convention, the pointwise value $u(x)$ is determined q.e in A and the function u is *quasi-continuous* in A. Moreover, the following property holds:

if (u_n) converges to u in $H^1(A)$, then a subsequence of (u_n) converges to u pointwise q.e. on A. (1.2)

The proof of these properties can be found in [13] and [16].

The *fine topology* is defined as the weakest topology on $\mathbf{R}^N$ making continuous every superharmonic function on $\mathbf{R}^N$. For the properties of this topology we refer to Chapter XI of Part 1 of [14]. The fine interior, the fine closure, and the fine boundary of a subset E of $\mathbf{R}^N$ are denoted by $\operatorname{int}^* E$, $\operatorname{cl}^* E$, and $\partial^* E$.

It is well known that every finely open subset A of $\mathbf{R}^N$ is *quasi-open*, i.e., for every $\varepsilon > 0$ there exists an open subset U of $\mathbf{R}^N$ such that $\operatorname{cap}(A \Delta U) < \varepsilon$, where Δ denotes the symmetric difference of sets (see Chapter IV of [3]). Moreover, a real-valued function u is quasi-continuous on an open set A if and only if u is finely continuous q.e. in A (see Proposition 3.6 of Chapter II of [13]). Therefore our convention (1.1) implies that every function $u \in H^1(A)$ is *finely continuous* q.e. in A.

If A is a finely open subset of $\mathbf{R}^N$, by $H_0^1(A)$ we denote the space of all functions $u \in H^1(\mathbf{R}^N)$ such that $u = 0$ q.e. on $\mathbf{R}^N \setminus A$, with the Hilbert space structure inherited from $H^1(\mathbf{R}^N)$. If A is open (in the Euclidean topology), the previous definition is equivalent to the usual one mentioned at the beginning of the paper (see, for instance, [17]).

Note that if A is not open, then the classical definition of $H_0^1(A)$ as the closure of a suitable space of regular functions does not work, because, in general, $C^0(\mathbf{R}^N) \cap H_0^1(A)$ is not dense in $H_0^1(A)$, as the following example shows.

Example 1.1. Given an open subset Ω of $\mathbf{R}^N$, let (x_n) be an enumeration of the points of Ω with rational coordinates and let (r_n) be a sequence of positive numbers such that $\sum_n \operatorname{cap}(B_{r_n}(x_n)) < \operatorname{cap}(\Omega)$. Then the union U of all balls $B_{r_n}(x_n)$ is dense in Ω, and, by the countable subadditivity of the capacity, we have $\operatorname{cap}(U) < \operatorname{cap}(\Omega)$. Since $\operatorname{cap}(\operatorname{cl}^* U) = \operatorname{cap}(U)$, the finely open set $A = \Omega \setminus \operatorname{cl}^* U$ has positive capacity, so that $H_0^1(A)$ contains a function which is not identically zero (see Proposition 1.2 below). On the other hand, if $u \in C^0(\mathbf{R}^N) \cap H_0^1(A)$, then $u = 0$ on U, and this implies $u = 0$ on Ω, since U is dense in Ω. Therefore the zero function is the only element in the closure of $C^0(\mathbf{R}^N) \cap H_0^1(A)$ in $H_0^1(A)$.

Given an arbitrary subset E of $\mathbf{R}^N$, its *characteristic function* 1_E is defined by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \in \mathbf{R}^N \setminus E$.

We frequently use the following proposition concerning the approximation of characteristic functions of finely open sets.

Proposition 1.2. *Let E be a finely open subset of $\mathbf{R}^N$. Then there exists an increasing sequence (u_n) in $H_0^1(E)$ converging to 1_E pointwise q.e. on $\mathbf{R}^N$, such that $0 \le u_n \le 1$ q.e. on E.*

Proof. Since each finely open set is quasi-open, the function 1_E is quasi-lower semicontinuous. Therefore the proposition follows from Lemma 1.5 of [7]. □

Let Ω be bounded open subset of $\mathbf{R}^N$. By $\mathscr{B}(\Omega)$ we denote the σ-field of all Borel subsets of Ω, and by $\mathscr{B}_c(\Omega)$ we denote the δ-ring of all Borel sets B such that $B \subset\subset \Omega$, i.e., $\bar{B}$ is compact and $\bar{B} \subseteq \Omega$. By a *Borel measure* on Ω we mean a countably additive set function $\mu\colon \mathscr{B}(\Omega) \to]-\infty, +\infty]$, not necessarily finite nor σ-finite. By a *Radon measure* on Ω we mean a countably additive set function $\mu\colon \mathscr{B}_c(\Omega) \to \mathbf{R}$. The (total) variation of a (Borel or Radon) measure μ is denoted by $|\mu|$. It is well known that any nonnegative Radon measure μ (resp. any signed Radon measure with bounded total variation) can be extended in a unique way to a nonnegative Borel measure (resp. to a real-valued Borel measure), for which we use the same symbol μ.

If μ is a (Borel or Radon) measure on Ω and if $f\colon \Omega \to \bar{\mathbf{R}}$ is a μ-measurable function, by $f\mu$ we denote the Borel (resp. Radon) measure defined by

$$(f\mu)(B) = \int_B f\, d\mu \tag{1.3}$$

for every $B \in \mathscr{B}(\Omega)$ (resp. for every $B \in \mathscr{B}_c(\Omega)$), provided that the integrals occurring in (1.3) are well defined. If E is a μ-measurable subset of Ω, by $\mu|_E$ we denote the Borel (resp. Radon) measure on Ω defined by $(\mu|_E)(B) = \mu(B \cap E)$ for every $B \in \mathscr{B}(\Omega)$ (resp. for every $B \in \mathscr{B}_c(\Omega)$).

By $\mathscr{M}_0(\Omega)$ we denote the set of all nonnegative Borel measures μ on Ω such that $\mu(B) = 0$ for every $B \in \mathscr{B}(\Omega)$ with $\operatorname{cap}(B) = 0$.

If $N - 2 < \alpha \le N$, then the α-dimensional Hausdorff measure $\mathscr{H}^\alpha$ belongs to the class $\mathscr{M}_0(\Omega)$. In particular, the N-dimensional Lebesgue measure $\mathscr{L}^N$ belongs to $\mathscr{M}_0(\Omega)$. Another example of measure of the class $\mathscr{M}_0(\Omega)$, which plays an important role in this paper, is, for every $S \subseteq \Omega$, the Borel measure ∞_S defined by

$$\infty_S(B) = \begin{cases} 0 & \text{if } \operatorname{cap}(B \cap S) = 0, \\ +\infty & \text{if } \operatorname{cap}(B \cap S) > 0 \end{cases} \tag{1.4}$$

for every $B \in \mathscr{B}(\Omega)$.

By $\mathscr{B}^*(\Omega)$ we denote the σ-field of all *finely Borel subsets* of Ω, i.e., the σ-field generated by the finely open subsets of Ω. It is well known that a subset E of Ω belongs to $\mathscr{B}^*(\Omega)$ if and only if there exists $B \in \mathscr{B}(\Omega)$ with $\mathrm{cap}(E \Delta B) = 0$, where Δ denotes the symmetric difference of sets (see Section IV of [13]). Therefore each measure $\mu \in \mathscr{M}_0(\Omega)$ can be extended in a unique way to a countably additive set function, still denoted by μ, defined on the larger σ-field $\mathscr{B}^*(\Omega)$.

The *regular set* $A(\mu)$ of a measure $\mu \in \mathscr{M}_0(\Omega)$ is defined as the union of all finely open subsets of Ω such that $\mu(A) < +\infty$. The *singular set* $S(\mu)$ is defined as the complement of $A(\mu)$ in Ω. It is easy to see that $A(\mu)$ is finely open and that if A is a finely open subset of Ω which intersects $S(\mu)$, then $\mu(A) = +\infty$. By the quasi-Lindelöf property of the fine topology (see Theorem 1.XI.11 of [14]) there exists an increasing sequence (A_n) of finely open subsets of $A(\mu)$, with $\mu(A_n) < +\infty$ for every n, such that $\mathrm{cap}(A(\mu) \setminus \bigcup_n A_n) = 0$. We refer to Section 3 of [8] for further properties of the sets $A(\mu)$ and $S(\mu)$.

We say that a Radon measure μ on Ω belongs to $H^{-1}(\Omega)$ if there exists $f \in H^{-1}(\Omega)$ such that

$$\langle f, \varphi \rangle_{H^{-1}(\Omega)} = \int_\Omega \varphi \, d\mu, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{1.5}$$

In this case we identify f and μ. It is well known (see, for instance, [4]) that if μ is a nonnegative Radon measure which belongs to $H^{-1}(\Omega)$, then $\mu \in \mathscr{M}_0(\Omega)$ and $H_0^1(\Omega) \subseteq L^1(\Omega, \mu)$. Moreover,

$$\langle \mu, v \rangle_{H^{-1}(\Omega)} = \int_\Omega v \, d\mu, \qquad \forall v \in H_0^1(\Omega). \tag{1.6}$$

If μ is a signed Radon measure on Ω, whose variation $|\mu|$ belongs to $H^{-1}(\Omega)$, then $\mu \in H^{-1}(\Omega)$ and (1.6) continues to hold.

Given $f, g \in H^{-1}(\Omega)$, we say that $f \leq g$ in $H^{-1}(\Omega)$ if

$$\langle f, v \rangle_{H^{-1}(\Omega)} \leq \langle g, v \rangle_{H^{-1}(\Omega)}, \qquad \forall v \in H_0^1(\Omega), \quad v \geq 0 \quad \text{a.e. on } \Omega.$$

By the Riesz representation theorem, if $f \in H^{-1}(\Omega)$ and $f \geq 0$ in $H^{-1}(\Omega)$, then there exists a nonnegative Radon measure μ on Ω such that (1.5) holds.

2. The Space $X_\mu(\Omega)$

For the rest of this paper we fix a bounded open subset Ω of $\mathbf{R}^N$ with $N \geq 2$. Let us fix $\mu \in \mathscr{M}_0(\Omega)$. By $X_\mu(\Omega)$ we denote the vector space of all functions $u \in H_0^1(\Omega)$ such that $\int_\Omega u^2 \, d\mu < +\infty$. This definition makes sense, because μ vanishes on all sets of capacity zero and every function $u \in H_0^1(\Omega)$ is defined up to a set of capacity zero, so that the integral $\int_\Omega u^2 \, d\mu$ is unambiguously defined. On $X_\mu(\Omega)$ we consider the scalar product

$$(u, v)_{X_\mu(\Omega)} = \int_\Omega DuDv \, dx + \int_\Omega uv \, d\mu \tag{2.1}$$

and the corresponding norm $\|\cdot\|_{X_\mu(\Omega)}$.

Proposition 2.1. *$X_\mu(\Omega)$ is a Hilbert space.*

Proof. It is enough to prove the completeness. Let (u_n) be a Cauchy sequence in $X_\mu(\Omega)$. Then (u_n) is a Cauchy sequence both in $H_0^1(\Omega)$ and in $L^2(\Omega, \mu)$. Therefore (u_n) converges to a function u in $H_0^1(\Omega)$ and to a function v in $L^2(\Omega, \mu)$. By (1.2) a subsequence (u_{n_k}) of (u_n) converges to u q.e. in Ω. Since μ vanishes on all sets with capacity zero, (u_{n_k}) converges to u μ-a.e. in Ω. On the other hand, a further subsequence of (u_{n_k}) converges to v μ-a.e. in Ω, hence $v = u$ μ-a.e. in Ω and, therefore, $u \in X_\mu(\Omega)$, and (u_n) converges to u both in $H_0^1(\Omega)$ and in $L^2(\Omega, \mu)$. This implies that (u_n) converges to u in $X_\mu(\Omega)$. □

As $X_\mu(\Omega) \subseteq H_0^1(\Omega)$, all functions in $X_\mu(\Omega)$ are defined q.e. in Ω and are finely continuous q.e. in Ω. It follows that every function in $X_\mu(\Omega)$ vanishes q.e on the singular set $S(\mu)$ of μ. In other words $X_\mu(\Omega) \subseteq H_0^1(A(\mu))$.

Let us consider now some examples which illustrate the structure of the space $X_\mu(\Omega)$ under some special assumptions on μ.

Example 2.2. Assume that $\mu = q\mathscr{L}^N$ with $q \in L^p(\Omega)$, where

$$\begin{cases} N/2 \leq p \leq +\infty & \text{if} \quad N \geq 3, \\ 1 < p \leq +\infty & \text{if} \quad N = 2. \end{cases} \tag{2.2}$$

By the Sobolev embedding theorem we have that $X_\mu(\Omega) = H_0^1(\Omega)$ with equivalent norms.

Example 2.3. Let A be a finely open subset of Ω and let $S = \Omega \backslash A$. If μ is equal to the measure ∞_S defined by (1.4), then $X_\mu(\Omega) = H_0^1(A)$, and the norms are equivalent by the Poincaré inequality. The same conclusion holds if $\mu = \infty_S + q\mathscr{L}^N$, where $q \in L^p(\Omega)$ and p satisfies the conditions of the previous example.

By $X'_\mu(\Omega)$ we denote the dual space of $X_\mu(\Omega)$, with duality pairing $\langle \cdot, \cdot \rangle_{X'_\mu(\Omega)}$; notice that the isomorphic spaces $X_\mu(\Omega)$ and $X'_\mu(\Omega)$ will not be identified. We now explain in detail how $L^2(\Omega)$, $H^{-1}(\Omega)$, $L^2(\Omega, \mu)$ can be viewed as linear subspaces of $X'_\mu(\Omega)$.

Let $i\colon X_\mu(\Omega) \to H_0^1(\Omega)$ be the natural embedding defined by $i(u) = u$ for every $u \in X_\mu(\Omega)$. The transpose map ${}^t i\colon H^{-1}(\Omega) \to X'_\mu(\Omega)$ allows us to consider $H^{-1}(\Omega)$ as a subspace of $X'_\mu(\Omega)$. With a little abuse of notation, which is discussed in a moment, we write f instead of ${}^t i(f)$ for every $f \in H^{-1}(\Omega)$. With this convention we have

$$\langle f, v \rangle_{X'_\mu(\Omega)} = \langle f, v \rangle_{H^{-1}(\Omega)}, \qquad \forall v \in X_\mu(\Omega). \tag{2.3}$$

In particular, for $f \in L^2(\Omega)$ we have

$$\langle f, v \rangle_{X'_\mu(\Omega)} = \int_\Omega fv\, dx, \qquad \forall v \in X_\mu(\Omega),$$

and this is consistent with the usual identification of $L^2(\Omega)$ with its dual.

The abuse in our notation consists in the fact that the map ${}^t i\colon H^{-1}(\Omega) \to X'_\mu(\Omega)$ is, in general, not injective, because $X_\mu(\Omega)$ is, in general, not dense in $H^1_0(\Omega)$. Therefore there may exist two elements f and g of $H^{-1}(\Omega)$ such that $f \neq g$ in $H^{-1}(\Omega)$ but $f = g$ in $X'_\mu(\Omega)$, where the last equality means ${}^t i(f) = {}^t i(g)$, according to our convention (2.3).

Example 2.4. Assume that μ is the measure ∞_Ω defined in (1.4) taking $S = \Omega$. Then $f = 0$ in $X'_\mu(\Omega)$ for every $f \in H^{-1}(\Omega)$. In fact, in this case $X_\mu(\Omega) = \{0\}$, hence ${}^t i(f) = 0$ for every $f \in H^{-1}(\Omega)$.

If the regular set $A(\mu)$ of the measure μ coincides with Ω, then ${}^t i\colon H^{-1}(\Omega) \to X'_\mu(\Omega)$ is injective, as the following proposition shows.

Proposition 2.5. *Let f and g be two elements of $H^{-1}(\Omega)$. Then $f = g$ in $X'_\mu(\Omega)$ if and only if $\langle f, v\rangle_{H^{-1}(\Omega)} = \langle g, v\rangle_{H^{-1}(\Omega)}$ for every $v \in H^1_0(A(\mu))$.*

Proof. The assertion is equivalent to the equality $\ker({}^t i) = H^1_0(A(\mu))^\perp$, where $\perp$ refers to the duality pairing $\langle\cdot,\cdot\rangle_{H^{-1}(\Omega)}$. Since $\ker({}^t i) = \mathrm{im}(i)^\perp$, the proposition is an easy consequence of the following lemma. □

Lemma 2.6. *$H^1_0(A(\mu))$ is the closure of $X_\mu(\Omega)$ in $H^1_0(\Omega)$.*

Proof. Since each element of $X_\mu(\Omega)$ vanishes q.e. on $S(\mu) = \Omega\setminus A(\mu)$, we have $X_\mu(\Omega) \subseteq H^1_0(A(\mu))$. As $H^1_0(A(\mu))$ is closed in $H^1_0(\Omega)$, it is enough to show that any function of $H^1_0(A(\mu))$ can be approximated in $H^1_0(\Omega)$ by a sequence of functions of $X_\mu(\Omega)$.

Let us fix $u \in H^1_0(A(\mu))$. By the lattice properties of $H^1_0(\Omega)$ it is not restrictive to assume that $u \geq 0$ q.e. on Ω. By the property of $A(\mu)$ mentioned in Section 1, there exists an increasing sequence (A_n) of finely open subsets of $A(\mu)$, with $\mu(A_n) < +\infty$ for every n, such that

$$\mathrm{cap}\Big(A(\mu)\setminus\bigcup_n A_n\Big) = 0.$$

For every pair of positive integers n, h we set $E^h_n = \{x \in A_n\colon u(x) > h/2^n\}$ and

$$\psi_n = 2^{-n}\sum_{h=1}^{n2^n} 1_{E^h_n}.$$

Since u is quasi-continuous, the function ψ_n is quasi-lower semicontinuous. Since $0 \leq \psi_n \leq n 1_{A_n}$ and $\mu(A_n) < +\infty$, we have $\psi_n \in L^2(\Omega, \mu)$. By construction, the sequence (ψ_n) is increasing and converges to u pointwise q.e. on Ω. By Lemma 1.6 of [7] there exists an increasing sequence (u_n) in $H^1_0(\Omega)$ such that $0 \leq u_n \leq \psi_n$ q.e. in Ω. As $\psi_n \in L^2(\Omega, \mu)$, this implies that $u_n \in X_\mu(\Omega)$ and concludes the proof of the lemma. □

Let $j\colon X_\mu(\Omega) \to L^2(\Omega, \mu)$ be the natural embedding defined by $j(u) = u$ for every $u \in X_\mu(\Omega)$. The transpose map ${}^t j\colon L^2(\Omega, \mu) \to X'_\mu(\Omega)$ allows us to consider $L^2(\Omega, \mu)$ as a subspace of $X'_\mu(\Omega)$. For every $g \in L^2(\Omega, \mu)$ the image ${}^t\!j(g)$ is denoted by $g\mu$. With this convention we have

$$\langle g\mu, v\rangle_{X'_\mu(\Omega)} = \int_\Omega vg\, d\mu, \qquad \forall v \in X_\mu(\Omega). \tag{2.4}$$

This notation is consistent with (1.3), provided $g \in L^2(\Omega, \mu) \cap L^1_{\mathrm{loc}}(\Omega, \mu)$. In this case, if the Radon measure $|g|\mu$ belongs to $H^{-1}(\Omega)$, then (2.4) is consistent with (1.6) and (2.3).

Since $X_\mu(\Omega)$ is, in general, not dense in $L^2(\Omega, \mu)$, the map ${}^t\!j\colon L^2(\Omega, \mu) \to X'_\mu(\Omega)$ is, in general, not injective. Therefore there may exist two elements f and g of $L^2(\Omega, \mu)$ such that $f \neq g$ in $L^2(\Omega, \mu)$, i.e., $\mu(\{f \neq g\}) > 0$, but $f\mu = g\mu$ in $X'_\mu(\Omega)$.

Example 2.7. Let E be the set of all points $x = (x_1, \dots, x_N)$ in Ω whose first coordinate x_1 is rational. If $\mu = \infty_E + \mathscr{L}^N$, then $X_\mu(\Omega) = \{0\}$. Therefore, taking $g = 1_{\Omega\setminus E}$, we have $g \in L^2(\Omega, \mu)$ and $g \neq 0$ in $L^2(\Omega, \mu)$, whereas $g\mu = {}^t\!j(g) = 0$ in $X'_\mu(\Omega)$.

The following proposition gives a necessary and sufficient condition for the equality $f\mu = g\mu$.

Proposition 2.8. *Let f and g be two elements of $L^2(\Omega, \mu)$. Then $f\mu = g\mu$ if and only if $f = g$ μ-a.e. on the regular set $A(\mu)$ of the measure μ.*

Proof. The assertion is equivalent to the equality

$$\ker({}^t\!j) = \{g \in L^2(\Omega, \mu)\colon g = 0 \ \mu\text{-a.e. on } A(\mu)\}.$$

Since $\ker({}^t\!j) = \mathrm{im}(j)^\perp$ and as the orthogonal complement of the set $\{g \in L^2(\Omega, \mu)\colon g = 0\ \mu\text{-a.e. on } A(\mu)\}$ is the set $\{g \in L^2(\Omega, \mu)\colon g = 0\ \mu\text{-a.e. on } S(\mu)\}$, the proposition is a consequence of the following lemma. $\square$

Lemma 2.9. *The set $Y = \{u \in L^2(\Omega, \mu)\colon u = 0\ \mu\text{-a.e. on } S(\mu)\}$ is the closure of $X_\mu(\Omega)$ in $L^2(\Omega, \mu)$.*

Proof. Since each element of $X_\mu(\Omega)$ vanishes q.e. on $S(\mu)$ and as μ vanishes on all sets of capacity zero, we have $X_\mu(\Omega) \subseteq Y$. Since Y is closed in $L^2(\Omega, \mu)$, it is enough to prove that $X_\mu(\Omega)$ is dense in Y. Let Z be the linear space generated by the functions 1_A with A finely open set with $\mu(A) < +\infty$. From the properties of $A(\mu)$ we easily obtain that Z is dense in Y. Therefore, to conclude the proof it is enough to approximate in $L^2(\Omega, \mu)$ each function 1_A by a sequence of functions of $X_\mu(\Omega)$. Since A is finely open and $\mu(A) < +\infty$, Proposition 1.2 provides the required approximation. $\square$

3. Relaxed Dirichlet Problems

In this section we recall some properties of the relaxed Dirichlet problems introduced in [9] and [10]. Let us fix $\mu \in \mathcal{M}_0(\Omega)$. By the Riesz–Fréchet representation theorem, for every $F \in X'_\mu(\Omega)$ there exists a unique $u \in X_\mu(\Omega)$ such that

$$(u, v)_{X_\mu(\Omega)} = \langle F, v\rangle_{X'_\mu(\Omega)}, \qquad \forall v \in X_\mu(\Omega). \tag{3.1}$$

By definition (2.1) of the scalar product in $X_\mu(\Omega)$, (3.1) is equivalent to

$$\int_\Omega DuDv\, dx + \int_\Omega uv\, d\mu = \langle F, v\rangle_{X'_\mu(\Omega)}, \qquad \forall v \in X_\mu(\Omega). \tag{3.2}$$

According to our conventions (2.3) and (2.4), we can write (3.2) in the form

$$\langle -\Delta u, v\rangle_{X'_\mu(\Omega)} + \langle u\mu, v\rangle_{X'_\mu(\Omega)} = \langle F, v\rangle_{X'_\mu(\Omega)}, \qquad \forall v \in X_\mu(\Omega). \tag{3.3}$$

This shows that each element F of $X'_\mu(\Omega)$ can be represented as $F = f + g\mu$ with $f \in H^{-1}(\Omega)$ and $g \in L^2(\Omega, \mu)$. Because of (3.3), we refer to the solution of (3.1) as the solution of the problem

$$u \in X_\mu(\Omega), \qquad -\Delta u + u\mu = F \quad \text{in } X'_\mu(\Omega), \tag{3.4}$$

which is called the *relaxed Dirichlet problem.* The reason for this name is explained at the end of this section.

Example 3.1. Assume that $\mu = q\mathscr{L}^N$ with $q \in L^p(\Omega)$, where p satisfies (2.2), and let $f \in H^{-1}(\Omega)$. As $X_\mu(\Omega) = H^1_0(\Omega)$ with equivalent norms (see Example 2.2), it turns out that u is a solution of the problem

$$u \in X_\mu(\Omega), \qquad -\Delta u + u\mu = f \quad \text{in } X'_\mu(\Omega) \tag{3.5}$$

if and only if

$$u \in H^1_0(\Omega), \qquad -\Delta u + qu = f \quad \text{in } H^{-1}(\Omega).$$

Note that, under our assumptions on p, the function qu belongs to $H^{-1}(\Omega)$.

Example 3.2. Let A be an open subset of Ω and let $S = \Omega\setminus A$. If μ is the measure ∞_S defined by (1.4) and $f \in H^{-1}(\Omega)$, then u is a solution of (3.5) if and only if

$$u \in H^1_0(A), \qquad -\Delta u = f|_A \quad \text{in } H^{-1}(A),$$

where $f|_A$ is defined by $\langle f|_A, v\rangle_{H^{-1}(A)} = \langle f, v\rangle_{H^{-1}(\Omega)}$ for every $v \in H^1_0(A)$.

Example 3.3. Let A be an open subset of Ω, let $S = \Omega\setminus A$, and let $f \in H^{-1}(\Omega)$. If $\mu = \infty_S + q\mathscr{L}^N$, where $q \in L^p(\Omega)$ and p satisfies (2.2), then u is a solution of (3.5) if and only if

$$u \in H^1_0(A), \qquad -\Delta u + qu = f|_A \quad \text{in } H^{-1}(A).$$

The *resolvent operator* R_μ: $X'_\mu(\Omega) \to X_\mu(\Omega)$ is defined by $R_\mu(F) = u$, where u is the unique solution of (3.4). Using (3.1) it is easy to see that R_μ is a continuous linear operator from $X'_\mu(\Omega)$ onto $X_\mu(\Omega)$ and that R_μ is *symmetric*, i.e.,

$$\langle G, R_\mu(F)\rangle_{X'_\mu(\Omega)} = \langle F, R_\mu(G)\rangle_{X'_\mu(\Omega)} \tag{3.6}$$

for every $F, G \in X'_\mu(\Omega)$. Moreover, there exists a constant $c = c(\Omega)$, independent of

μ, such that

$$\|R_\mu(f)\|_{H^1(\Omega)} \le c\|f\|_{H^{-1}(\Omega)} \tag{3.7}$$

for every $f \in H^{-1}(\Omega)$.

We often use the following result proved in Proposition 2.6 of [9].

Proposition 3.4. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $f \in H^{-1}(\Omega)$ with $f \ge 0$ in $H^{-1}(\Omega)$. If u is the solution of the problem*

$$u \in X_\mu(\Omega), \qquad -\Delta u + u\mu = f \quad \text{in } X'_\mu(\Omega),$$

then $u \ge 0$ q.e. in Ω and $-\Delta u \le f$ in $H^{-1}(\Omega)$.

The notion of γ-convergence in $\mathcal{M}_0(\Omega)$ was introduced in [10] in order to study the dependence of the solution u of (3.4) on the measure μ.

Definition 3.5. We say that a sequence (μ_n) of measures of the class $\mathcal{M}_0(\Omega)$ γ-converges to a measure $\mu \in \mathcal{M}_0(\Omega)$ if and only if

$$R_{\mu_n}(f) \to R_\mu(f) \qquad \text{strongly in } L^2(\Omega)$$

for every $f \in L^2(\Omega)$.

By (3.7) we easily obtain that (μ_n) γ-converges to μ if and only if

$$R_{\mu_n}(f) \to \text{weakly in } H_0^1(\Omega)$$

for every $f \in H^{-1}(\Omega)$. The name γ-convergence comes from the fact that, in [10], this notion is defined in an equivalent way in terms of the Γ-convergence of the functionals

$$\int_\Omega |Du|^2\,dx + \int_\Omega u^2\,d\mu_n.$$

We refer to [1], [11], and [12] for the general notion of Γ-convergence and for its applications to the study of perturbation problems in the calculus of variations. The equivalence between our definition of γ-convergence and the definition given in [10] can be proved as in Theorem 2.1 of [2], replacing $\mathbf{R}^d$ by Ω.

The main properties of the γ-convergence are the following compactness and density theorems.

Theorem 3.6. *For every sequence (μ_n) in $\mathcal{M}_0(\Omega)$ there exists a subsequence (μ_{n_k}) which γ-converges to a measure $\mu \in \mathcal{M}_0(\Omega)$.*

Proof. It is enough to replace $\mathbf{R}^n$ by Ω in the proof of Theorem 4.14 of [10]. □

Theorem 3.7. *For every $\mu \in \mathcal{M}_0(\Omega)$ there exists a sequence (S_n) of compact subsets of Ω such that the sequence (∞_{S_n}) γ-converges to μ.*

Proof. It is enough to replace $\mathbf{R}^n$ by Ω in the proof of Theorem 4.16 of [10]. □

Let $\mu \in \mathscr{M}_0(\Omega)$ and let (S_n) be the sequence given by Theorem 3.7. Let $f \in H^{-1}(\Omega)$ and let $A_n = \Omega \backslash S_n$. By Example 3.2 the solution u_n of the Dirichlet problem

$$u_n \in H_0^1(A_n), \qquad -\Delta u_n = f|_{A_n} \quad \text{in } H^{-1}(A_n) \tag{3.8}$$

can be written as $u_n = R_{\mu_n}(f)$, where $\mu_n = \infty_{S_n}$. Since (μ_n) γ-converges to μ, we have that (u_n) converges weakly in $H_0^1(\Omega)$ to the solution u of the problem

$$u \in X_\mu(\Omega), \qquad -\Delta u + u\mu = f \quad \text{in } X'_\mu(\Omega). \tag{3.9}$$

Therefore Theorem 3.7 states that, for every $f \in H^{-1}(\Omega)$, the solution of (3.9) can be approximated by the solutions of the Dirichlet problems (3.8). This is the reason for the name "relaxed Dirichlet problems" given to problems of the form (3.9). From the compactness and density properties stated in Theorems 3.6 and 3.7 we obtain that the family of relaxed Dirichlet problems (3.9) is the smallest family of equations, stable under $L^2(\Omega)$-convergence of solutions, which contains Dirichlet problems of the form (3.8).

4. The Optimization Problem

Let us fix a function $f \in L^2(\Omega)$ and a function $j: \Omega \times \mathbf{R} \to \mathbf{R}$ satisfying the following conditions:

$$j(\cdot, s) \text{ is } \mathscr{L}^N\text{-measurable in } \Omega \text{ for every } s \in \mathbf{R}; \tag{4.1}$$

$$j(x, \cdot) \text{ is continuous in } \mathbf{R} \text{ for a.e. } x \in \Omega; \tag{4.2}$$

$$\text{there exist } a_0 \in L^1(\Omega) \text{ and } c_0 \in \mathbf{R} \text{ such that } |j(x,s)| \le a_0(x) + c_0|s|^2 \text{ for a.e. } x \in \Omega \text{ and for every } s \in \mathbf{R}. \tag{4.3}$$

For every $u \in L^2(\Omega)$ we define

$$J(u) = \int_\Omega j(x, u(x))\, dx. \tag{4.4}$$

The aim of this section is to study the optimization problem

$$\min_{A \in \mathscr{A}(\Omega)} J(u_A), \tag{4.5}$$

where $\mathscr{A}(\Omega)$ is the family of all open subsets of Ω and u_A is the solution of the Dirichlet problem

$$u \in H_0^1(A), \qquad -\Delta u = f|_A \quad \text{in } H^{-1}(A).$$

It is well known that, in general, problem (4.5) has no solution (see, for instance, Example 4.3 below). In order to study the behavior of the minimizing sequences, we introduce the relaxed optimization problem

$$\min_{\mu \in \mathscr{M}_0(\Omega)} J(u_\mu), \tag{4.6}$$

where u_μ denotes the solution of the relaxed Dirichlet problem

$$u \in X_\mu(\Omega), \qquad -\Delta u + u\mu = f \quad \text{in } X'_\mu(\Omega).$$

The close connection between problems (4.5) and (4.6) is given by the following theorem.

Theorem 4.1. *Let $f \in L^2(\Omega)$ and let $j\colon \Omega \times \mathbf{R} \to \mathbf{R}$ be a function satisfying* (4.1), (4.2), *and* (4.3). *Then problem* (4.6) *admits a solution and*

$$\min_{\mu \in \mathcal{M}_0(\Omega)} J(u_\mu) = \min_{A \in \mathscr{A}(\Omega)} J(u_A). \tag{4.7}$$

Moreover, for a function $u \in H_0^1(\Omega)$ the following conditions are equivalent:

(a) *there exists a minimizing sequence (A_n) of* (4.5) *such that (u_{A_n}) converges to u weakly in $H_0^1(\Omega)$;*
(b) *there exists a minimum point μ of* (4.6) *such that $u = u_\mu$.*

Finally, if the original problem (4.5) *admits a solution $A \in \mathscr{A}(\Omega)$, then the measure ∞_S corresponding to $S = \Omega \backslash A$ is a solution of the relaxed problem* (4.6).

Proof. We first observe that the functional J is continuous in the strong topology of $L^2(\Omega)$ by the classical continuity theorems for Nemyckii operators (see, for instance, Theorem 9.1 of [28]). This implies that the function $\mu \to J(u_\mu)$ is continuous on $\mathcal{M}_0(\Omega)$ with respect to the γ-convergence. As remarked in Example 3.2, if $A \in \mathscr{A}(\Omega)$ and $S = \Omega \backslash A$, then $u_A = u_\mu$ for $\mu = \infty_S$. Therefore, all assertions of the theorem follow easily from the compactness and density properties of the γ-convergence stated in Theorems 3.6 and 3.7. ☐

Remark 4.2. By using the Sobolev embedding theorem, in the hypotheses of Theorem 4.1 it is possible to replace the inequality in (4.3) by the weaker condition

$$|j(x, s)| \leq a_0(x) + c_0 |s|^p,$$

with $0 \leq p < 2N/(N-2)$.

We conclude this section by exhibiting a simple example where problem (4.5) has no solution.

Example 4.3. Assume that $f(x) > 0$ for a.e. $x \in \Omega$ and let w be the solution of the problem

$$w \in H_0^1(\Omega), \qquad -\Delta w = f \quad \text{in } \Omega.$$

If $j(x, s) = (2s - w(x))^2$, then it is immediately seen that the relaxed problem (4.6) attains its minimum value 0 at the measure

$$\mu = \frac{f}{w} \mathscr{L}^N,$$

which corresponds to $u_\mu = w/2$. On the other hand, it is clear that there are no domains A such that $j(x, u_A(x)) = 0$ for a.e. $x \in \Omega$. By (4.7) this implies that the original problem (4.5) has no solution.

5. Two Optimality Conditions

In this section we obtain two optimality conditions for a solution $\mu \in \mathcal{M}_0(\Omega)$ of the relaxed optimization problem (4.6). The general method to prove these results consists in computing the limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [J(u_{\mu_\varepsilon}) - J(u_\mu)]$$

for suitable families $(\mu_\varepsilon)_{\varepsilon > 0}$ in $\mathcal{M}_0(\Omega)$: the optimality conditions are obtained from the fact that the limit above is nonnegative.

Let us fix a function $f \in L^2(\Omega)$ and a function $j: \Omega \times \mathbf{R} \to \mathbf{R}$ satisfying conditions (4.1), (4.2), and (4.3). Let us assume, in addition, that

$$j(x, \cdot) \text{ is differentiable and its derivative } j_s(x, \cdot) \text{ is continuous on } \mathbf{R} \text{ for a.e. } x \in \Omega; \tag{5.1}$$

$$j_s(\cdot, s) \text{ is } \mathcal{L}^N\text{-measurable on } \Omega \text{ for every } s \in \mathbf{R}; \tag{5.2}$$

$$\text{there exist } a_1 \in L^2(\Omega) \text{ and } c_1 \in \mathbf{R} \text{ such that } |j_s(x, s)| \le a_1(x) + c_1 |s| \text{ for a.e. } x \in \Omega \text{ and for every } s \in \mathbf{R}. \tag{5.3}$$

It is immediately seen that, under these hypotheses, the map J defined by (4.4) is differentiable on $L^2(\Omega)$, and that its differential J' is given by

$$\langle J'(u), v \rangle = \int_\Omega j_s(x, u) v \, dx, \qquad \forall u, v \in L^2(\Omega), \tag{5.4}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $L^2(\Omega)$ and its dual.

Given $\mu \in \mathcal{M}_0(\Omega)$, let u be the solution of the problem

$$u \in X_\mu(\Omega), \qquad -\Delta u + u\mu = f \quad \text{in } X'_\mu(\Omega), \tag{5.5}$$

and let v be the solution of the adjoint problem

$$v \in X_\mu(\Omega), \qquad -\Delta v + v\mu = j_s(x, u) \quad \text{in } X'_\mu(\Omega). \tag{5.6}$$

Theorem 5.1. *Let $\mu \in \mathcal{M}_0(\Omega)$ be a minimum point of the relaxed optimization problem* (4.6), *and let u and v be the solutions of* (5.5) *and* (5.6). *Then $uv \le 0$ q.e in Ω.*

Proof. Let us fix $\varphi \in L^\infty(\Omega)$ with $\varphi \ge 0$ a.e. in Ω. For every $\varepsilon \ge 0$, let μ_ε be the measure of the class $\mathcal{M}_0(\Omega)$ defined by $\mu_\varepsilon = \mu + \varepsilon\varphi\mathcal{L}^N$, and let u_ε be the corresponding solution of the problem

$$u_\varepsilon \in X_{\mu_\varepsilon}(\Omega), \qquad -\Delta u_\varepsilon + u_\varepsilon \mu_\varepsilon = f \quad \text{in } X'_{\mu_\varepsilon}(\Omega). \tag{5.7}$$

Note that $\mu_0 = \mu$ and $u_0 = u$. Since φ is bounded, we have $X_{\mu_\varepsilon}(\Omega) = X_\mu(\Omega)$ with equivalent norms. Therefore, u_ε satisfies

$$u_\varepsilon \in X_\mu(\Omega), \qquad -\Delta u_\varepsilon + u_\varepsilon \mu = f - \varepsilon\varphi u_\varepsilon \quad \text{in } X'_\mu(\Omega)$$

or, equivalently, $u_\varepsilon = R_\mu(f - \varepsilon\varphi u_\varepsilon)$, where R_μ is the resolvent operator introduced in Section 3. Let $\Phi\colon \mathbf{R} \times X_\mu(\Omega) \to X_\mu(\Omega)$ be the function defined by

$$\Phi(\varepsilon, w) = w - R_\mu(f - \varepsilon\varphi w).$$

Then Φ is continuously differentiable and its partial derivative $\partial\Phi/\partial w$ at $(\varepsilon, w) = (0, u)$ is the identity map on $X_\mu(\Omega)$. Therefore, by the implicit function theorem, the map $\varepsilon \to u_\varepsilon$ from $[0, +\infty[$ into $X_\mu(\Omega)$ is differentiable (on the right) at $\varepsilon = 0$, and

$$\left.\frac{du_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = -R_\mu(\varphi u).$$

Therefore, from (5.4) we obtain

$$\left.\frac{dJ(u_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = -\int_\Omega j_s(x, u)R_\mu(\varphi u)\, dx.$$

By (5.7) and by the minimum property of μ we have $J(u_\varepsilon) \geq J(u_0)$ for every $\varepsilon > 0$, hence

$$\int_\Omega j_s(x, u)R_\mu(\varphi u)\, dx \leq 0.$$

Since, by (5.6), we have $v = R_\mu(j_s(x, u))$, the symmetry condition (3.6) implies

$$\int_\Omega uv\varphi\, dx \leq 0.$$

As φ is an arbitrary nonnegative function of $L^\infty(\Omega)$, we obtain $uv \leq 0$ a.e. in Ω. Finally, since uv is finely continuous q.e. in Ω, and every nonempty finely open set has positive Lebesgue measure, we obtain that $uv \leq 0$ q.e. in Ω. □

Theorem 5.2. *Let $\mu \in \mathcal{M}_0(\Omega)$ be a minimum point of the relaxed optimization problem* (4.6), *and let u and v be the solutions of* (5.5) *and* (5.6). *Then $uv = 0$ μ-a.e. on Ω.*

Proof. For every $\varepsilon \in [0, 1[$ the measure $\mu_\varepsilon = (1 - \varepsilon)\mu$ belongs to the class $\mathcal{M}_0(\Omega)$. Let us denote by u_ε the corresponding solution of the problem

$$u_\varepsilon \in X_{\mu_\varepsilon}(\Omega), \qquad -\Delta u_\varepsilon + u_\varepsilon \mu_\varepsilon = f \quad \text{in } X'_{\mu_\varepsilon}(\Omega)$$

or, equivalently, of the problem

$$u_\varepsilon \in X_\mu(\Omega), \qquad -\Delta u_\varepsilon + u_\varepsilon \mu = f + \varepsilon u_\varepsilon \mu \quad \text{in } X'_\mu(\Omega).$$

Using the resolvent operator, we can write $u_\varepsilon = R_\mu(f + \varepsilon u_\varepsilon \mu)$. Let $\Phi\colon \mathbf{R} \times X_\mu(\Omega) \to X_\mu(\Omega)$ be the function defined by

$$\Phi(\varepsilon, w) = w - R_\mu(f + \varepsilon w \mu).$$

Then Φ is continuously differentiable and its partial derivative $\partial\Phi/\partial w$ at $(\varepsilon, w) = (0, u)$ is the identity map on $X_\mu(\Omega)$. By the implicit function theorem, the map $\varepsilon \to u_\varepsilon$ from $[0, 1[$ into $X_\mu(\Omega)$ is differentiable (on the right) at $\varepsilon = 0$, and

$$\left.\frac{du_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = R_\mu(u\mu).$$

Therefore, from (5.4) we obtain

$$\left.\frac{dJ(u_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \int_\Omega j_s(x, u)R_\mu(u\mu)\,dx.$$

By (5.8) and by the minimum property of μ we have $J(u_\varepsilon) \geq J(u_0)$ for every $\varepsilon > 0$, hence

$$\int_\Omega j_s(x, u)R_\mu(u\mu)\,dx \geq 0.$$

Since, by (5.6), we have $v = R_\mu(j_s(x, u))$, the symmetry condition (3.6) and the definition (2.4) of $u\mu$ imply

$$\int_\Omega uv\,d\mu = \langle u\mu, R_\mu(j_s(x, u))\rangle_{X'_\mu(\Omega)} \geq 0.$$

As $uv \leq 0$ μ-a.e. on Ω (Theorem 5.1), we conclude that $uv = 0$ μ-a.e. on Ω. □

6. A Boundary Measure

The following theorem associates a Radon measure ν_A with every bounded finely open subset A of $\mathbf{R}^N$. This measure, carried by the fine boundary ∂^*A, plays an important role in the weak definition of the normal derivative we introduce in Section 7.

Theorem 6.1. *Let A be a bounded finely open subset of $\mathbf{R}^N$ and let w_A be the unique solution of the problem*

$$w_A \in H_0^1(A), \qquad \int_A Dw_A Dv\,dx = \int_A v\,dx, \quad \forall v \in H_0^1(A). \tag{6.1}$$

Then there exists a unique nonnegative Radon measure ν_A belonging to $H^{-1}(\mathbf{R}^N)$ such that

$$-\Delta w_A + \nu_A = 1_{\mathrm{cl}^*A} \qquad \text{in } H^{-1}(\mathbf{R}^N). \tag{6.2}$$

Moreover, we have $\nu_A(\partial^*A) = \mathscr{L}^N(\mathrm{cl}^*A)$ and $\nu_A(\mathbf{R}^N \setminus \partial^*A) = 0$, i.e., ν_A is carried by ∂^*A.

The existence and uniqueness of a solution of (6.1) follows from the Riesz–Fréchet representation theorem. Before proving Theorem 6.1, in the following examples we show the connection between ν_A and the normal derivative $\partial w_A/\partial n$, when A

is open and smooth, and the connection between ν_A and the harmonic measure of A, when A is open.

Example 6.2. If A is a bounded open set with boundary ∂A of class $C^{1,\alpha}$, $0 < \alpha \leq 1$, then $\partial^* A = \partial A$ and the solution w_A of (6.1) belongs to $C^{1,\alpha}(\bar{A})$. An easy integration by parts shows that

$$\int_{\mathbf{R}^N} v \, d\nu_A = -\int_{\partial A} v \frac{\partial w_A}{\partial n} \, d\mathscr{H}^{N-1}, \qquad \forall v \in H^1(\mathbf{R}^N),$$

where n denotes the outer unit normal to A. Therefore

$$\nu_A = -\frac{\partial w_A}{\partial n} \mathscr{H}^{N-1}|_{\partial A}. \tag{6.3}$$

Note that $\partial w_A / \partial n > 0$ on ∂A by the Hopf maximum principle, hence ν_A and $\mathscr{H}^{N-1}|_{\partial A}$ have the same null sets.

Example 6.3. Assume that A is a bounded open subset of $\mathbf{R}^N$ with $\mathscr{L}^N(\partial^* A) = 0$, and let $h(x, B)$ be the harmonic measure of A, defined for every $x \in A$ and for every Borel subset B of ∂A. We shall prove that

$$\nu_A(B) = \int_A h(x, B \cap \partial A) \, dx \tag{6.4}$$

for every $B \in \mathscr{B}(\mathbf{R}^N)$. To this aim we introduce the bounded nonnegative Borel measure μ on $\mathbf{R}^N$ defined by $\mu(B) = \int_A h(x, B \cap \partial A) \, dx$ for every $B \in \mathscr{B}(\mathbf{R}^N)$. Given $v \in H^1(\mathbf{R}^N)$, we can consider the function $\psi \colon A \to \mathbf{R}$ defined by $\psi(x) = \int_{\partial A} v(y) h(x, dy)$. Then $\psi \in H^1(A)$, $\psi - v \in H_0^1(A)$, and $-\Delta\psi = 0$ on A. Therefore

$$\begin{aligned} \int_{\mathbf{R}^N} Dw_A Dv \, dx + \int_{\mathbf{R}^N} v \, d\mu &= \int_A Dw_A Dv \, dx + \int_{\partial A} v \, d\mu \\ &= \int_A Dw_A D\psi \, dx + \int_A Dw_A D(v - \psi) \, dx + \int_{\partial A} v \, d\mu. \end{aligned}$$

The first term in the last line is zero because ψ is harmonic and $w_A \in H_0^1(A)$, while the second term equals $\int_A (v - \psi) \, dx$ by (6.1). Therefore

$$\int_{\mathbf{R}^N} Dw_A Dv \, dx + \int_{\mathbf{R}^N} v d\mu = \int_A v \, dx - \int_A \psi \, dx + \int_{\partial A} v \, d\mu.$$

By the definition of μ and ψ we have

$$\int_A \psi \, dx = \int_A dx \int_{\partial A} v(y) h(x, dy) = \int_{\partial A} v \, d\mu,$$

hence

$$\int_{\mathbf{R}^N} Dw_A Dv \, dx + \int_{\mathbf{R}^N} v \, d\mu = \int_A v \, dx = \int_{\mathrm{cl}^* A} v \, dx, \qquad \forall v \in H^1(\mathbf{R}^N),$$

where, in the last equality, we use the fact that $\mathscr{L}^N(\partial^* A) = 0$. By definition (6.2) of ν_A we have $\mu = \nu_A$, which is equivalent to (6.4).

Proof of Theorem 6.1. Let A_0 be a bounded open subset of $\mathbf{R}^N$ such that $A \subset\subset A_0$, let $E = A_0 \setminus A$, and let $\mu = \infty_E$ be the measure of the class $\mathscr{M}_0(A_0)$ defined by (1.4) for every $B \in \mathscr{B}(A_0)$. Taking Example 2.3 into account, it is easy to see that w_A is the solution of the relaxed Dirichlet problem

$$w_A \in X_\mu(A_0), \qquad -\Delta w_A + w_A \mu = 1_{\mathrm{cl}^* A} \quad \text{in } X'_\mu(A_0).$$

By Proposition 3.4 we have $-\Delta w_A \leq 1_{\mathrm{cl}^* A}$ in $H^{-1}(A_0)$. Since $-\Delta w_A = 0 = 1_{\mathrm{cl}^* A}$ in $H^{-1}(\mathbf{R}^N \setminus \bar{A})$, we conclude that $-\Delta w_A \leq 1_{\mathrm{cl}^* A}$ in $H^{-1}(\mathbf{R}^N)$. Therefore there exists a nonnegative Radon measure ν_A, belonging to $H^{-1}(\mathbf{R}^N)$, such that

$$-\Delta w_A + \nu_A = 1_{\mathrm{cl}^* A} \qquad \text{in } H^{-1}(\mathbf{R}^N).$$

This proves (6.2). Let us prove that $\nu_A(A) = 0$. By (6.1) and (6.2) we have

$$\int_{\mathbf{R}^N} v \, d\nu_A = 0, \qquad \forall v \in H_0^1(A).$$

Since ν_A belongs to $H^{-1}(\mathbf{R}^N)$, it vanishes on all sets of capacity zero. Therefore Proposition 1.2 and the monotone convergence theorem allow us to conclude that $\nu_A(A) = 0$.

Let us prove that $\nu_A(\mathbf{R}^N \setminus \mathrm{cl}^* A) = 0$. Since $Dw_A = 0$ a.e. on $\mathbf{R}^N \setminus A$, it follows from (6.2) that

$$\int_{\mathbf{R}^N} v \, d\nu_A = 0, \qquad \forall v \in H_0^1(\mathbf{R}^N \setminus \mathrm{cl}^* A),$$

and we conclude as before that $\nu_A(\mathbf{R}^N \setminus \mathrm{cl}^* A) = 0$. This equality, together with $\nu_A(A) = 0$, implies that $\nu_A(\mathbf{R}^N \setminus \partial^* A) = 0$. Therefore, in order to prove that $\nu_A(\partial^* A) = \mathscr{L}^N(\mathrm{cl}^* A)$, it is enough to multiply equation (6.2) by a test function $\varphi \in C_0^\infty(\mathbf{R}^N)$ with $\varphi = 1$ in a neighborhood of $\bar{A}$. □

The following property of ν_A is crucial in Section 8.

Proposition 6.4. *Let $u \in H_0^1(\Omega)$, let A be a finely open subset of Ω, let $S = \Omega \setminus A$, and let ν_A be the measure defined in Theorem 6.1. Then the following conditions are equivalent*:

(a) $u = 0$ *q.e. on* S;

(b) $u = 0$ *a.e. on* $\mathrm{int}^* S$ *and* ν_A*-a.e. on* Ω.

Proof. Since the measure ν_A belongs to $H^{-1}(\mathbf{R}^N)$ it vanishes on all sets of capacity zero. As ν_A is carried by S, we immediately get that (a) implies (b).

In order to prove the converse, we introduce the space K of all functions $v \in H_0^1(\Omega)$ such that $v = 0$ a.e. on $\mathrm{int}^* S$ and $v = 0$ ν_A-a.e. on Ω. It is clear that K is a closed linear subspace of $H_0^1(\Omega)$. Since $H_0^1(A) \subseteq K$, from (6.2) we obtain that w_A is the unique solution of the problem

$$w_A \in K, \qquad \int_\Omega Dw_A Dv\, dx = \int_{A^*} v\, dx, \qquad \forall v \in K, \tag{6.5}$$

where $A^* = \Omega \cap \mathrm{cl}^* A$. We have to prove that $K \subseteq H_0^1(A)$. By the lattice properties of $H_0^1(\Omega)$, it is enough to prove that every nonnegative function of K belongs to $H_0^1(A)$. Let us fix $u \in K$ with $u \geq 0$ q.e. on Ω and let

$$K(u) = \{v \in K \colon v \leq u \text{ q.e. in } \Omega\}.$$

For every $k \in \mathbf{N}$ we consider the solutions u_k and w_k of the variational inequalities

$$u_k \in K(u), \qquad \int_\Omega Du_k D(v - u_k)\, dx \geq k \int_{A^*} (v - u_k)\, dx, \qquad \forall v \in K(u), \tag{6.6}$$

$$w_k \in K, \qquad \int_\Omega Dw_k D(v - w_k)\, dx \geq k \int_{A^*} (v - w_k)\, dx, \qquad \forall v \in K. \tag{6.7}$$

Taking the test functions $v = u_k \wedge w_k$ in (6.6) and $v = u_k \vee w_k$ in (6.7), and adding the inequalities term by term, we easily obtain that $u_k \leq w_k$ q.e. in Ω. Taking $v = u_k \vee 0$ in (6.6) we obtain that $u_k \geq 0$ q.e. in Ω. As K is a linear subspace of $H_0^1(\Omega)$, from (6.5) and (6.7) we get $w_k = kw_A$, hence

$$0 \leq u_k \leq kw_A \qquad \text{q.e. in } \Omega. \tag{6.8}$$

This implies that $u_k \in H_0^1(A)$ for every $k \in \mathbf{N}$. Taking $v = u$ as a test function in (6.6), we obtain the estimate

$$\int_\Omega |Du_k|^2\, dx + k \int_{A^*} (u - u_k)\, dx \leq \left(\int_\Omega |Du_k|^2\, dx\right)^{1/2} \left(\int_\Omega |Du|^2\, dx\right)^{1/2}.$$

Since $u_k \leq u$ a.e. on Ω by the definition of $K(u)$, the previous estimate implies that

$$\int_\Omega |Du_k|^2\, dx \leq \int_\Omega |Du|^2\, dx, \tag{6.9}$$

$$u_k \to u \qquad \text{in } L^1(A^*). \tag{6.10}$$

As $u_k = u = 0$ a.e. on $\mathrm{int}^* S$, from (6.10) we obtain that (u_k) converges to u in $L^1(\Omega)$. This fact, together with (6.9), yields that (u_k) converges to u in $H_0^1(\Omega)$. Since $H_0^1(A)$ is closed in $H_0^1(\Omega)$ and as $u_k \in H_0^1(A)$ for every $k \in \mathbf{N}$ by (6.8), we obtain that $u \in H_0^1(A)$. This proves that $K \subseteq H_0^1(A)$ and concludes the proof of the proposition. $\square$

7. Weak Definition of the Normal Derivative

In this section, given a measure $\mu \in \mathcal{M}_0(\Omega)$, we consider the solution u of the relaxed Dirichlet problem

$$u \in X_\mu(\Omega), \qquad -\Delta u + u\mu = f \quad \text{in } X'_\mu(\Omega), \tag{7.1}$$

with $f \in L^2(\Omega)$. If A denotes the regular set $A(\mu)$ of μ and ν_A is the corresponding boundary measure, the following theorem associates to the solution u a function $\alpha \in L^2(\Omega, \nu_A)$ in such a way that the measure $\alpha\nu_A$ plays the role of the normal derivative of u on the fine boundary of A.

Theorem 7.1. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $A = A(\mu)$, let $f \in L^2(\Omega)$, and let u be the solution of (7.1). Then the measures $|u|\mu$ and $u\mu$, defined by (1.3), belong to $H^{-1}(\Omega)$, and there exists a unique $\alpha \in L^2(\Omega, \nu_A)$ such that*

$$-\Delta u + u\mu + \alpha\nu_A = f 1_{\mathrm{cl}^* A} \qquad \text{in } H^{-1}(\Omega), \tag{7.2}$$

where ν_A is the boundary measure corresponding to $A = A(\mu)$, introduced in Theorem 6.1. Moreover, we have

$$\|\alpha\|_{L^2(\Omega, \nu_A)} \leq \|f\|_{L^2(\Omega)}. \tag{7.3}$$

If, in addition, $f \geq 0$ a.e. in Ω, then $\alpha \geq 0$ ν_A-a.e. in Ω.

The following examples show why the measure $\alpha\nu_A$ can be considered as a weak definition of the normal derivative $\partial u/\partial n$ on the fine boundary $\partial^* A$.

Example 7.2. Assume that A is an open set with boundary ∂A of class C^2 and that $\mu|_A = q\mathcal{L}^N|_A$ with $q \in L^\infty(A)$. Then u is the solution of the problem (see Example 3.3)

$$u \in H_0^1(A), \qquad -\Delta u + qu = f \text{ in } H^{-1}(A).$$

By the classical regularity results for elliptic equations we have $u \in H^2(A)$, hence $\partial u/\partial n \in L^2(\partial A, \mathcal{H}^{N-1})$ by the trace theorem. Thus an easy integration by parts shows that

$$\int_\Omega v\alpha \, d\nu_A = -\int_{\Omega \cap \partial A} v \frac{\partial u}{\partial n} \, d\mathcal{H}^{N-1}, \qquad \forall v \in H^1(\mathbf{R}^N).$$

Therefore

$$\alpha\nu_A = -\frac{\partial u}{\partial n} \mathcal{H}^{N-1}|_{\Omega \cap \partial A}.$$

Taking (6.3) into account we obtain that

$$\alpha = \frac{\partial u}{\partial n}\left[\frac{\partial w_A}{\partial n}\right]^{-1} \qquad \mathcal{H}^{N-1}\text{-a.e. on } \Omega \cap \partial A. \tag{7.4}$$

Example 7.3. Assume that A is an open set with a Lipschitz boundary ∂A and that $\mu|A = q\mathcal{L}^N|_A$ with $q \in L^\infty(A)$. Then $\Delta u \in L^2(A)$ and $\partial u/\partial n$ is defined as an element of $H^{-1/2}(\partial A)$. An easy integration by parts yields

$$\left\langle \frac{\partial u}{\partial n}, v \right\rangle_{H^{-1/2}(\partial A)} = \int_{\Omega \cap \partial A} v\alpha \, d\nu_A$$

for every $v \in H_0^1(\Omega)$. Therefore $\partial u/\partial n = \alpha\nu_A|_{\Omega \cap \partial A}$ in $H^{-1/2}(\Omega \cap \partial A)$.

Proof of Theorem 7.1. Let Ω_0 be a bounded open subset of $\mathbf{R}^N$ with $\Omega \subset\subset \Omega_0$ and let μ_0 be the measure of the class $\mathcal{M}_0(\Omega_0)$ defined by

$$\mu_0(B) = \begin{cases} \mu(B \cap \Omega) & \text{if} \quad \operatorname{cap}(B \setminus \Omega) = 0, \\ +\infty, & \text{if} \quad \operatorname{cap}(B \setminus \Omega) > 0. \end{cases}$$

The functions u and f are extended to Ω_0 putting $u = f = 0$ on $\Omega_0 \setminus \Omega$. Since each function of $X_{\mu_0}(\Omega_0)$ vanishes q.e. on $\Omega_0 \setminus A$, we have

$$u \in X_{\mu_0}(\Omega_0), \qquad -\Delta u + u\mu_0 = f 1_{\mathrm{cl}^* A} \quad \text{in } X'_{\mu_0}(\Omega_0). \tag{7.5}$$

If $f \geq 0$ a.e. in Ω, by Proposition 3.4 we have $-\Delta u \leq f 1_{\mathrm{cl}^* A}$ in $H^{-1}(\Omega_0)$. Therefore there exists a nonnegative Radon measure λ on Ω_0, belonging to $H^{-1}(\Omega_0)$, such that

$$-\Delta u + \lambda = f 1_{\mathrm{cl}^* A} \qquad \text{in } H^{-1}(\Omega_0). \tag{7.6}$$

In the general case $f \in L^2(\Omega)$, by considering the positive and the negative part of f we obtain that there exists a signed Radon measure $\lambda \in H^{-1}(\Omega_0)$, with $|\lambda| \in H^{-1}(\Omega_0)$, such that (7.6) holds. Let us prove that

$$\lambda(B \cap A) = \int_B u \, d\mu_0, \qquad \forall B \in \mathscr{B}(\Omega_0). \tag{7.7}$$

As $u \in L^2(\Omega_0, \mu_0)$ we have $u = 0$ μ_0-a.e. in $\Omega_0 \setminus A$. Since A is the union of a set of capacity zero and of an increasing sequence (A_n) of finely open sets with $\mu_0(A_n) < +\infty$, in order to prove (7.7) it is enough to show that

$$\lambda(B) = \int_B u \, d\mu_0 \tag{7.8}$$

for every finely open subset B of A with $\mu(B) < +\infty$. Let us fix a set B with these properties. By Proposition 1.2 there exists an increasing sequence (v_n) in $H_0^1(B)$ converging to 1_B pointwise q.e. in Ω_0, such that $0 \leq v_n \leq 1_B$ q.e. in Ω_0. Since $v_n \in X_{\mu_0}(\Omega_0)$, by (7.5) we have

$$\int_{\Omega_0} v_n u \, d\mu_0 = \int_{\mathrm{cl}^* A} f v_n \, dx - \int_{\Omega_0} Du Dv_n \, dx,$$

and from (7.6) we obtain

$$\int_{\Omega_0} v_n \, d\lambda = \int_{\mathrm{cl}^* A} f v_n \, dx - \int_{\Omega_0} Du Dv_n \, dx.$$

These equalities yield $\int_{\Omega_0} v_n u \, d\mu_0 = \int_{\Omega_0} v_n \, d\lambda$. Since μ_0 and λ vanish on all sets of capacity zero, by the monotone convergence theorem we obtain (7.8), which concludes the proof of (7.7). As $|\lambda|$ belongs to $H^{-1}(\Omega_0)$, (7.7) implies that the measures $|u|\mu_0$ and $u\mu_0$ belong to $H^{-1}(\Omega_0)$, hence the measures $|u|\mu$ and $u\mu$ belong to $H^{-1}(\Omega)$.

Let us prove that

$$\lambda(B \setminus \mathrm{cl}^* A) = 0, \qquad \forall B \in \mathscr{B}(\Omega_0). \tag{7.9}$$

It is enough to show that $\lambda(B) = 0$ for every finely open subset B of $\Omega_0 \setminus \mathrm{cl}^* A$. By Proposition 1.2 there exists an increasing sequence (v_n) in $H_0^1(B)$ converging to 1_B pointwise q.e. in Ω_0, such that $0 \le v_n \le 1_B$ q.e. in Ω_0. By (7.6) we have

$$\int_{\Omega_0} DuDv_n\,dx + \int_{\Omega_0} v_n\,d\lambda = \int_{\mathrm{cl}^*A} fv_n\,dx = 0. \tag{7.10}$$

Since $u = 0$ a.e. on B, we also have $Du = 0$ a.e. on B (see, for instance, Lemma A.4 of [18]). On the other hand, as $v_n = 0$ a.e. on $\Omega_0 \setminus B$, we also have $Dv_n = 0$ a.e. on $\Omega_0 \setminus B$. This implies that the first integral in (7.10) vanishes, hence $\int_{\Omega_0} v_n\,d\lambda = 0$. As λ vanishes on all sets of capacity zero, by the monotone convergence theorem we obtain $\lambda(B) = 0$, which concludes the proof of (7.9).

Let us denote by ν_f the Radon measure on Ω_0 defined by

$$\nu_f(B) = \lambda(B \cap \partial^* A), \qquad \forall B \in \mathscr{B}(\Omega_0).$$

Since $|\lambda| \in H^{-1}(\Omega_0)$, the measures $|\nu_f|$ and ν_f belong to $H^{-1}(\Omega_0)$. As $\partial^* A \subset\subset \Omega_0$, we have $|\nu_f|(\Omega_0) < +\infty$. By (7.7) and (7.9) we have

$$\nu_f(B) = \lambda(B) - \int_B u\,d\mu_0, \qquad \forall B \in \mathscr{B}(\Omega_0).$$

Therefore (7.6) yields

$$-\Delta u + u\mu_0 + \nu_f = f 1_{\mathrm{cl}^*A} \qquad \text{in } H^{-1}(\Omega_0). \tag{7.11}$$

The map $f \to \nu_f$ from $L^2(\Omega)$ into $H^{-1}(\Omega_0)$ is clearly linear. Since $\lambda \ge 0$ for $f \ge 0$, we have

$$\nu_f(B) \le \nu_g(B), \qquad \forall B \in \mathscr{B}(\Omega_0), \tag{7.12}$$

whenever $f, g \in L^2(\Omega)$ and $f \le g$ a.e. in Ω.

Let us prove that if $f \le 1$ a.e. in Ω, then

$$\nu_f(B) \le \nu_A(B), \qquad \forall B \in \mathscr{B}(\Omega_0), \tag{7.13}$$

where ν_A is the boundary measure introduced in Section 6. By (7.12) we may assume that $0 \le f \le 1$ a.e. in Ω, thus $u \ge 0$ q.e. in Ω by Proposition 3.4. Subtracting (7.11) from (6.2) we obtain

$$-\Delta(w_A - u) + \nu_A - \nu_f = u\mu_0 + (1 - f)1_{\mathrm{cl}^*A} \qquad \text{in } H^{-1}(\Omega_0). \tag{7.14}$$

Let $E = \Omega_0 \setminus A$ and let $\mu_1 = \infty_E$ be the measure of the class $\mathscr{M}_0(\Omega_0)$ defined by (1.4), with Ω replaced by Ω_0. As observed in Example 2.3, we have $X_{\mu_1}(\Omega_0) = H_0^1(A)$. Since w_A and u belong to $H_0^1(A)$ and as

$$\int_{\Omega_0} v\,d\nu_A = \int_{\Omega_0} v\,d\nu_f = 0, \qquad \forall v \in H_0^1(A),$$

from (7.14) we obtain

$$-\Delta(w_A - u) + (w_A - u)\mu_1 = u\mu_0 + (1 - f)1_{\mathrm{cl}^*A} \qquad \text{in } X'_{\mu_1}(\Omega_0).$$

As $u \ge 0$ and $1 - f \ge 0$, by Proposition 3.4 we have

$$-\Delta(w_A - u) \le u\mu_0 + (1 - f)1_{\mathrm{cl}^*A} \qquad \text{in } H^{-1}(\Omega_0).$$

This fact, together with (7.14), yields $\nu_A - \nu_f \geq 0$ in $H^{-1}(\Omega_0)$, which is equivalent to (7.13).

If we apply (7.13) to f and $-f$, we obtain

$$|\nu_f|(B) \leq \nu_A(B), \qquad \forall B \in \mathscr{B}(\Omega_0), \tag{7.15}$$

for every $f \in L^2(\Omega)$ with $|f| \leq 1$ a.e. on Ω. By linearity, if $f \in L^\infty(\Omega)$ we obtain from (7.15)

$$|\nu_f|(B) \leq \|f\|_{L^\infty(\Omega)} \nu_A(B), \qquad \forall B \in \mathscr{B}(\Omega_0), \tag{7.16}$$

therefore ν_f is absolutely continuous with respect to ν_A for every $f \in L^\infty(\Omega)$.

If $f \in L^2(\Omega)$ and $f \geq 0$ a.e. in Ω, then there exists an increasing sequence (f_n) in $L^\infty(\Omega)$ converging to f in $L^2(\Omega)$, with $f_n \geq 0$ a.e. in Ω. Let (u_n) be the sequence of the solutions of the problems

$$u_n \in X_{\mu_0}(\Omega_0), \qquad -\Delta u_n + u_n \mu_0 = f_n \quad \text{in } X'_{\mu_0}(\Omega_0),$$

where f_n is extended to Ω_0 setting $f_n = 0$ on $\Omega_0 \backslash \Omega$. By the continuity properties of the resolvent operator (see Section 3) the sequence (u_n) converges to u in $H_0^1(\Omega_0)$. By the comparison principle (see Theorem 2.10 of [9]) the sequence (u_n) is increasing, hence (u_n) converges to u pointwise q.e. in Ω_0 by (1.2). Writing ν_n for ν_{f_n}, by (7.11) we have

$$-\Delta u_n + u_n \mu_0 + \nu_n = f_n 1_A \qquad \text{in } H^{-1}(\Omega_0), \tag{7.17}$$

and by (7.12) the sequence of measures (ν_n) is increasing. Let ν be the Borel measure on Ω_0 defined by

$$\nu(B) = \sup_n \nu_n(B), \qquad \forall B \in \mathscr{B}(\Omega_0).$$

Then ν is absolutely continuous with respect to ν_A, since each measure ν_n has this property. From (7.17) we easily obtain that

$$-\Delta u + u\mu_0 + \nu = f 1_{\mathrm{cl}^* A} \qquad \text{in } H^{-1}(\Omega_0),$$

hence $\nu = \nu_f$ by (7.11). This proves that ν_f is absolutely continuous with respect to ν_A, provided $f \geq 0$ a.e. in Ω. The same property can be proved for an arbitrary $f \in L^2(\Omega)$ by considering the positive and the negative parts of f.

Since ν_f and ν_A are bounded Borel measures on Ω, and as ν_f is absolutely continuous with respect to ν_A, by the Radon–Nikodym theorem for every $f \in L^2(\Omega)$ there exists $\beta_f \in L^1(\Omega_0, \nu_A)$ such that

$$\nu_f = \beta_f \nu_A. \tag{7.18}$$

The map $f \to \beta_f$ from $L^2(\Omega)$ into $L^1(\Omega_0, \nu_A)$ is clearly linear. Moreover, (7.12) implies that

$$\beta_f \geq 0 \qquad \nu_A\text{-a.e. in } \Omega_0 \tag{7.19}$$

for every $f \in L^2(\Omega)$ with $f \geq 0$ a.e. in Ω. From (7.16) we have that $\beta_f \in L^\infty(\Omega, \nu_A)$ for every $f \in L^\infty(\Omega)$ and

$$\|\beta_f\|_{L^\infty(\Omega_0, \nu_A)} \leq \|f\|_{L^\infty(\Omega)}. \tag{7.20}$$

Let us prove that

$$\| \beta_f \|_{L^1(\Omega_0, \nu_A)} \le \| f \|_{L^1(\Omega)} \tag{7.21}$$

for every $f \in L^2(\Omega)$. It is not restrictive to assume that $f \ge 0$ a.e. in Ω, hence $u \ge 0$ q.e. in Ω_0 by Proposition 3.4 and $\beta_f \ge 0$ ν_A-a.e. in Ω by (7.19). Let $\varphi \in C_0^\infty(\Omega_0)$ with $\varphi = 1$ on $\bar{\Omega}$. As ν_A is carried by $\partial^* A$ and $Du = 0$ a.e. on $\Omega_0 \backslash \Omega$, taking φ as a test function in (7.11) and using (7.18) we obtain

$$\| \beta_f \|_{L^1(\Omega_0, \nu_A)} \le \int_{\Omega_0} u \, d\mu_0 + \int_{\partial^* A} \beta_f \, d\nu_A = \int_{\mathrm{cl}^* A} f \, dx,$$

which proves (7.21).

By the Riesz–Thorin interpolation theorem (see VI.10 of [15]), from (7.20) and (7.21) we obtain that $\beta_f \in L^2(\Omega_0, \nu_A)$ for every $f \in L^2(\Omega)$ and

$$\| \beta_f \|_{L^2(\Omega_0, \nu_A)} \le \| f \|_{L^2(\Omega)}. \tag{7.22}$$

Let us fix $f \in L^2(\Omega)$ and let α be the restriction of β_f to Ω. Then $\alpha \in L^2(\Omega, \nu_A)$ and (7.3) is a consequence of (7.11) and (7.18), while the positivity of α for a positive f follows from (7.19). □

We conclude this section with a corollary of Theorem 7.1.

Corollary 7.4. *In addition to the hypotheses of Theorem 7.1, assume that the regular set $A(\mu)$ of the measure μ coincides with Ω. Then the measures $|u|\mu$ and $u\mu$ belong to $H^{-1}(\Omega)$ and*

$$-\Delta u + u\mu = f \qquad \text{in } H^{-1}(\Omega).$$

Proof. Since $A = \Omega$, the measure ν_A is carried by $\partial\Omega$. Therefore (7.2) is equivalent to (7.23). □

8. A Singular Perturbation

In this section we consider a singular perturbation $(\mu_\varepsilon)_{\varepsilon > 0}$ of a measure μ of the class $\mathcal{M}_0(\Omega)$ and study the weak $L^2(\Omega)$ limit, as $\varepsilon \to 0$, of the difference quotient

$$\frac{1}{\varepsilon} [R_{\mu_\varepsilon}(f) - R_\mu(f)]$$

for every $f \in L^2(\Omega)$. Let us fix $\mu \in \mathcal{M}_0(\Omega)$ and $f \in L^2(\Omega)$. By $A = A(\mu)$ and $S = S(\mu)$ we denote the regular and the singular set of the measure μ. Let u be the solution of the problem

$$u \in X_\mu(\Omega), \qquad -\Delta u + u\mu = f \quad \text{in } X'_\mu(\Omega). \tag{8.1}$$

By Theorem 7.1 the measures $|u|\mu$ and $u\mu$ belong to $H^{-1}(\Omega)$ and there exists $\alpha \in L^2(\Omega, \nu_A)$ such that

$$-\Delta u + u\mu + \alpha\nu_A = f 1_{\mathrm{cl}^* A} \qquad \text{in } H^{-1}(\Omega), \tag{8.2}$$

where ν_A is the boundary measure introduced in Section 6. Let $\varphi, \psi \in C^0(\bar{\Omega})$ with

$$\inf_\Omega \varphi > 0, \qquad \inf_\Omega \psi > 0. \tag{8.3}$$

For every $\varepsilon > 0$ we consider the measure μ_ε of the class $\mathcal{M}_0(\Omega)$ defined by

$$\mu_\varepsilon = \mu|_A + \varepsilon^{-1}(\varphi^{-1}\mathcal{L}^N|_{\mathrm{int}^* S} + \psi^{-1}\nu_A). \tag{8.4}$$

Let u_ε be the corresponding solution of the problem

$$u_\varepsilon \in X_{\mu_\varepsilon}(\Omega), \qquad -\Delta u_\varepsilon + u_\varepsilon \mu_\varepsilon = f \quad \text{in } X'_{\mu_\varepsilon}(\Omega). \tag{8.5}$$

Since $A(\mu_\varepsilon) = \Omega$, by Corollary 7.4 the measures $|u_\varepsilon|\mu_\varepsilon$ and $u_\varepsilon\mu_\varepsilon$ belong to $H^{-1}(\Omega)$ and

$$-\Delta u_\varepsilon + u_\varepsilon \mu_\varepsilon = f \qquad \text{in } H^{-1}(\Omega). \tag{8.6}$$

Our aim is to prove that $(u_\varepsilon - u)/\varepsilon$ converges weakly in $L^2(\Omega)$ as $\varepsilon \to 0$. In order to compute the limit of the scalar products

$$\int_\Omega \frac{u_\varepsilon - u}{\varepsilon} g \, dx,$$

for every $g \in L^2(\Omega)$ we consider the solution v of the problem

$$v \in X_\mu(\Omega), \qquad -\Delta v + v\mu = g \quad \text{in } X'_\mu(\Omega). \tag{8.7}$$

As in the case of problem (8.1), the measures $|v|\mu$ and $v\mu$ belong to $H^{-1}(\Omega)$ and there exists $\beta \in L^2(\Omega, \nu_A)$ such that

$$-\Delta v + v\mu + \beta\nu_A = g1_{\mathrm{cl}^* A} \qquad \text{in } H^{-1}(\Omega). \tag{8.8}$$

The following theorem is the main result of the present section.

Theorem 8.1. *Let u be the solution of* (8.1) *and, for every $\varepsilon > 0$, let u_ε be the solution of* (8.5) *corresponding to a given pair φ, ψ of functions of $C^0(\bar{\Omega})$ satisfying* (8.3). *Then (u_ε) converges to u strongly in $H_0^1(\Omega)$ as $\varepsilon \to 0$ and $(u_\varepsilon - u)/\varepsilon$ converges weakly in $L^2(\Omega)$. More precisely, for every $g \in L^2(\Omega)$ we have*

$$\int_\Omega \frac{u_\varepsilon - u}{\varepsilon} g \, dx \to \int_{\mathrm{int}^* S} fg\varphi \, dx + \int_\Omega \alpha\beta\psi \, d\nu_A, \tag{8.9}$$

where α and β are defined by (8.2) *and* (8.8).

In order to prove the theorem, we fix $g \in L^2(\Omega)$ and, for every $\varepsilon > 0$, we consider the solution v_ε of the problem

$$v_\varepsilon \in X_{\mu_\varepsilon}(\Omega), \qquad -\Delta v_\varepsilon + v_\varepsilon \mu_\varepsilon = g \qquad \text{in } X'_{\mu_\varepsilon}(\Omega). \tag{8.10}$$

As in the case of problem (8.5), the measures $|v_\varepsilon|\mu_\varepsilon$ and $v_\varepsilon\mu_\varepsilon$ belong to $H^{-1}(\Omega)$ and

$$-\Delta v_\varepsilon + v_\varepsilon \mu_\varepsilon = g \qquad \text{in } H^{-1}(\Omega). \tag{8.11}$$

The following lemma proves the first assertion of Theorem 8.1.

Lemma 8.2. *As $\varepsilon \to 0$, the sequences (u_ε) and (v_ε) converge to u and v strongly in $H_0^1(\Omega)$.*

Proof. It is enough to prove the convergence of (u_ε). Taking u_ε as a test function in (8.5) and using definition (8.4) of μ_ε we obtain

$$\int_\Omega |Du_\varepsilon|^2\,dx + \varepsilon^{-1}\int_{\mathrm{int}^* S} \varphi^{-1} u_\varepsilon^2\,dx + \varepsilon^{-1}\int_\Omega \psi^{-1} u_\varepsilon^2\,d\nu_A + \int_A u_\varepsilon^2\,d\mu$$
$$= \int_\Omega f u_\varepsilon\,dx, \tag{8.12}$$

which immediately implies that (u_ε) is bounded in $H_0^1(\Omega)$ and in $L^2(A, \mu)$. Passing, if necessary, to a subsequence, we may assume that (u_ε) converges weakly in $H_0^1(\Omega)$ to a function w_1 and weakly in $L^2(A, \mu)$ to a function w_2.

Let us prove that $w_1 = w_2$ μ-a.e. on A. There exists a sequence (z_n) converging to w_1 strongly in $H_0^1(\Omega)$ such that each function z_n belongs to the convex hull of the set $\{u_\varepsilon : 0 < \varepsilon < 1/n\}$. By (1.2) a subsequence of (z_n) converges to w_1 q.e. in Ω, hence μ-a.e. on A. Since (u_ε) converges to w_2 weakly in $L^2(A, \mu)$, the convex combinations z_n still converge to w_2 weakly in $L^2(A, \mu)$. Since (z_n) converges to w_1 μ-a.e. on A, we conclude that $w_1 = w_2$ μ-a.e. on A. Therefore (u_ε) converges to w_1 weakly in the spaces $H_0^1(\Omega)$ and $L^2(A, \mu)$. Let us prove that $w_1 = u$. Since φ and ψ are bounded in Ω, from (8.12) we obtain that

$$\frac{1}{\varepsilon}\int_{\mathrm{int}^* S} u_\varepsilon^2\,dx \quad \text{and} \quad \frac{1}{\varepsilon}\int_\Omega u_\varepsilon^2\,d\nu_A$$

are bounded as $\varepsilon \to 0$. Therefore (u_ε) converges to 0 strongly in the spaces $L^2(\mathrm{int}^* S)$ and $L^2(\Omega, \nu_A)$. This implies that $w_1 = 0$ a.e. on $\mathrm{int}^* S$ and ν_A-a.e. on Ω. By Proposition 6.4 we obtain that $w_1 = 0$ q.e. on S. Since $w_1 \in L^2(A, \mu)$, we can conclude that $w_1 \in L^2(\Omega, \mu)$, hence $w_1 \in X_\mu(\Omega)$. Let $z \in X_\mu(\Omega)$. Since $z = 0$ a.e. on $\mathrm{int}^* S$ and ν_A-a.e. on Ω, by taking z as a test function in (8.6) we obtain

$$\int_\Omega Du_\varepsilon Dz\,dx + \int_A u_\varepsilon z\,d\mu = \int_\Omega fz\,dx.$$

Since (u_ε) converges to w_1 weakly in the spaces $H_0^1(\Omega)$ and $L^2(A, \mu)$, we get

$$\int_\Omega Dw_1 Dz\,dx + \int_A w_1 z\,d\mu = \int_\Omega fz\,dx,$$

which implies that w_1 is a solution of (8.1), hence $w_1 = u$ and the whole sequence (u_ε) converges to u weakly in the spaces $H_0^1(\Omega)$ and $L^2(A, \mu)$.

Let us prove that (u_ε) converges to u strongly in $H_0^1(\Omega)$. As (u_ε) converges to u weakly in $L^2(A, \mu)$, we have

$$\int_A u^2\,d\mu \le \liminf_{\varepsilon \to 0} \int_A u_\varepsilon^2\,d\mu. \tag{8.13}$$

Since (u_ε) converges to u strongly in $L^2(\Omega)$ by Rellich's theorem, (8.12) and (8.13) yield

$$\limsup_{\varepsilon\to 0}\int_\Omega |Du_\varepsilon|^2\,dx + \int_A u^2\,d\mu \le \int_\Omega fu\,dx. \tag{8.14}$$

Taking u as a test function in (8.1) we get

$$\int_\Omega fu\,dx = \int_\Omega |Du|^2\,dx + \int_A u^2\,d\mu,$$

which, together with (8.14), gives

$$\limsup_{\varepsilon\to 0}\int_\Omega |Du_\varepsilon|^2\,dx \le \int_\Omega |Du|^2\,dx.$$

Since (u_ε) converges to u weakly in $H_0^1(\Omega)$, this implies that (u_ε) converges to u strongly in $H_0^1(\Omega)$. □

Lemma 8.3. *As $\varepsilon \to 0$ we have*

(a) *$v_\varepsilon/\varepsilon \to g\varphi$ weakly in $L^2(\mathrm{int}^* S)$,*
(b) *$v_\varepsilon/\varepsilon \to \beta\psi$ weakly in $L^2(\Omega, \nu_A)$,*
(c) *$v_\varepsilon\mu|_A \to v\mu|_A$ weakly in $H^{-1}(\Omega)$.*

Proof. By linearity it is not restrictive to assume that $g \ge 0$ a.e. in Ω. Then we also have $v_\varepsilon \ge 0$ q.e. in Ω, $v \ge 0$ q.e. in Ω, and $\beta \ge 0$ ν_A-a.e. in Ω (see Proposition 3.4 and Theorem 7.1). Since $\int_\Omega z\,d\mu_\varepsilon \le \int_\Omega z\,d\mu$ for every quasi-continuous function $z\colon \Omega \to \mathbf{R}$ with $z \ge 0$ q.e. in Ω, by the comparison principle (Theorem 2.10 of [9]) we have

$$v \le v_\varepsilon \qquad \text{q.e. in } \Omega. \tag{8.15}$$

As $\mu_\eta \le \mu_\varepsilon$ for $\eta \ge \varepsilon > 0$, for the same reason we have

$$v_\varepsilon \le v_\eta \qquad \text{q.e. in } \Omega \text{ for } \eta \ge \varepsilon > 0. \tag{8.16}$$

By (8.16) and by Lemma 8.2 the sequence (v_ε) is decreasing and converges to v strongly in $H_0^1(\Omega)$ and pointwise q.e. in Ω (see (1.2)).

Since $v_\varepsilon\mu_\varepsilon$ belongs to $H^{-1}(\Omega)$, for every $z \in H_0^1(\Omega)$ the function $|z|$ belongs to $L^1(A, v_\varepsilon\mu)$. Therefore, by the dominated convergence theorem we have

$$\int_A z v_\varepsilon\,d\mu \to \int_A z v\,d\mu, \qquad \forall z \in H_0^1(\Omega),$$

which proves (c). From (8.8) and (8.11) we obtain

$$\begin{aligned}\int_\Omega D(v_\varepsilon - v)Dz\,dx &+ \varepsilon^{-1}\int_{\mathrm{int}^* S} v_\varepsilon z\varphi^{-1}\,dx + \varepsilon^{-1}\int_\Omega v_\varepsilon z\psi^{-1}\,d\nu_A \\ &+ \int_A (v_\varepsilon - v)z\,d\mu = \int_{\mathrm{int}^* S} gz\,dx + \int_\Omega \beta z\,d\nu_A \end{aligned} \tag{8.17}$$

for every $z \in H_0^1(\Omega)$. Taking $z = (v_\varepsilon - v)/\varepsilon$ we obtain

$$\varepsilon^{-1} \int_\Omega |D(v_\varepsilon - v)|^2 \, dx + \varepsilon^{-2} \int_{\mathrm{int}^* S} v_\varepsilon^2 \varphi^{-1} \, dx + \varepsilon^{-2} \int_\Omega v_\varepsilon^2 \psi^{-1} \, d\nu_A$$
$$+ \varepsilon^{-1} \int_A (v_\varepsilon - v)^2 \, d\mu = \varepsilon^{-1} \int_{\mathrm{int}^* S} g v_\varepsilon \, dx + \varepsilon^{-1} \int_\Omega \beta v_\varepsilon \, d\nu_A,$$

where we have used the fact that $v = 0$ q.e. on S and, therefore, $v = 0$ ν_A-a.e. on Ω, since ν_A is carried by $\partial^* A \subseteq S$. As φ and ψ are bounded in Ω, we have

$$\int_{\mathrm{int}^* S} (\varepsilon^{-1} v_\varepsilon)^2 \, dx + \int_\Omega (\varepsilon^{-1} v_\varepsilon)^2 \, d\nu_A \le c\left[\int_{\mathrm{int}^* S} g \varepsilon^{-1} v_\varepsilon \, dx + \int_\Omega \beta \varepsilon^{-1} v_\varepsilon \, d\nu_A\right]$$

for a suitable constant $c > 0$, which immediately implies that the sequence $(v_\varepsilon/\varepsilon)$ is bounded in $L^2(\mathrm{int}^* S)$ and in $L^2(\Omega, \nu_A)$. Therefore, up to a subsequence, $(v_\varepsilon/\varepsilon)$ converges weakly in $L^2(\mathrm{int}^* S)$ to a function h and weakly in $L^2(\Omega, \nu_A)$ to a function γ. Since (v_ε) converges to v weakly in $H_0^1(\Omega)$ by Lemma 8.2, from (c) and from (8.17) we obtain

$$\varepsilon^{-1} \int_{\mathrm{int}^* S} v_\varepsilon z \varphi^{-1} \, dx \to \int_{\mathrm{int}^* S} g z \, dx, \qquad \forall z \in H_0^1(\mathrm{int}^* S)$$

(recall that every function of $H_0^1(\mathrm{int}^* S)$ vanishes ν_A-a.e. in Ω, because ν_A is carried by $\partial^* A$). By (8.3) this implies

$$\int_{\mathrm{int}^* S} h z \varphi^{-1} \, dx = \int_{\mathrm{int}^* S} g z \, dx, \qquad \forall z \in H_0^1(\mathrm{int}^* S),$$

and by Proposition 1.2 we get

$$\int_B h \varphi^{-1} \, dx = \int_B g \, dx$$

for every finely open subset B of $\mathrm{int}^* S$, hence $h = g\varphi$ a.e. on $\mathrm{int}^* S$. As the limit h does not depend on the subsequence, this proves that $(v_\varepsilon/\varepsilon)$ converges to $g\varphi$ weakly in $L^2(\mathrm{int}^* S)$.

Taking now an arbitrary $z \in H_0^1(\Omega)$ and using (a) and (c) we obtain from (8.17), as $\varepsilon \to 0$, that

$$\int_{\mathrm{int}^* S} g z \, dx + \int_\Omega \gamma z \psi^{-1} \, d\nu_A = \int_{\mathrm{int}^* S} g z \, dx + \int_\Omega \beta z \, d\nu_A,$$

hence

$$\int_\Omega \gamma z \psi^{-1} \, d\nu_A = \int_\Omega \beta z \, d\nu_A, \qquad \forall z \in H_0^1(\Omega),$$

which yields $\gamma = \beta\psi$ ν_A-a.e. in Ω and concludes the proof of (b). $\square$

Proof of Theorem 8.1. Let us fix $g \in L^2(\Omega)$. Taking $(u_\varepsilon - u)/\varepsilon$ as a test function in (8.11) we obtain

$$\int_\Omega \frac{u_\varepsilon - u}{\varepsilon} g\,dx = \varepsilon^{-1}\int_\Omega Dv_\varepsilon D(u_\varepsilon - u)\,dx + \varepsilon^{-2}\int_{\mathrm{int}^* S} v_\varepsilon u_\varepsilon \varphi^{-1}\,dx$$
$$+ \varepsilon^{-2}\int_\Omega v_\varepsilon u_\varepsilon \psi^{-1}\,d\nu_A + \varepsilon^{-1}\int_A v_\varepsilon(u_\varepsilon - u)\,d\mu, \tag{8.18}$$

where we have used the fact that $u = 0$ q.e. on S and, therefore, $u = 0$ ν_A-a.e. on Ω, since ν_A is carried by $\partial^* A \subseteq S$. Taking $v_\varepsilon/\varepsilon$ as a test function in (8.2) and (8.6) we obtain that the right-hand side of (8.18) equals

$$\int_\Omega f\varepsilon^{-1}v_\varepsilon\,dx - \int_{\Omega\cap \mathrm{cl}^* A} f\varepsilon^{-1}v_\varepsilon\,dx + \int_\Omega \alpha\varepsilon^{-1}v_\varepsilon\,d\nu_A,$$

therefore

$$\int_\Omega \frac{u_\varepsilon - u}{\varepsilon} g\,dx = \int_{\mathrm{int}^* S} f\varepsilon^{-1}v_\varepsilon\,dx + \int_\Omega \alpha\varepsilon^{-1}v_\varepsilon\,d\nu_A. \tag{8.19}$$

By Lemma 8.3 the sequence $(v_\varepsilon/\varepsilon)$ converges to $g\varphi$ weakly in $L^2(\mathrm{int}^* S)$ and to $\beta\psi$ weakly in $L^2(\Omega, \nu_A)$. Therefore (8.19) implies (8.9). The weak convergence of $(u_\varepsilon - u)/\varepsilon$ in $L^2(\Omega)$ now follows from the Banach–Steinhaus uniform boundedness principle. □

9. Further Optimality Conditions

In this section we prove two necessary conditions for the solutions of the relaxed optimization problem introduced in Section 4. These optimality conditions are obtained by means of the singular perturbation studied in Section 8.

Let us fix a function $f \in L^2(\Omega)$ and a function $j: \Omega \times \mathbf{R} \to \mathbf{R}$ satisfying conditions (4.1), (4.2), (4.3), (5.1), (5.2), and (5.3). Given $\mu \in \mathcal{M}_0(\Omega)$, by $A = A(\mu)$ and $S = S(\mu)$ we denote the regular and the singular set of the measure μ. Let u be the solution of the problem

$$u \in X_\mu(\Omega), \qquad -\Delta u + u\mu = f \quad \text{in } X'_\mu(\Omega), \tag{9.1}$$

and let v be the solution of the adjoint problem

$$v \in X_\mu(\Omega), \qquad -\Delta v + v\mu = j_s(x, u) \quad \text{in } X'_\mu(\Omega). \tag{9.2}$$

By Theorem 7.1 the measures $|u|\mu$, $|v|\mu$, $u\mu$, and $v\mu$ belong to $H^{-1}(\Omega)$ and there exist α, $\mathrm{w} \in L^2(\Omega, \nu_A)$ such that

$$-\Delta u + u\mu + \alpha\nu_A = f 1_{\mathrm{cl}^* A} \qquad \text{in } H^{-1}(\Omega), \tag{9.3}$$
$$-\Delta v + v\mu + \beta\nu_A = j_s(x, u) 1_{\mathrm{cl}^* A} \qquad \text{in } H^{-1}(\Omega), \tag{9.4}$$

where ν_A is the boundary measure introduced in Section 6.

Theorem 9.1. *Let $\mu \in \mathcal{M}_0(\Omega)$ be a minimum point for the relaxed optimization problem* (4.6) *and let u and v be the solutions of* (9.1) *and* (9.2). *Then*

$$f(\cdot)j_s(\cdot, 0) \geq 0 \qquad \text{a.e. on } \operatorname{int}^* S, \tag{9.5}$$

$$\alpha\beta \geq 0 \qquad \nu_A - \text{a.e. on } \Omega, \tag{9.6}$$

where α and β are defined by (9.3) *and* (9.4).

In order to prove the theorem, we need the following lemma.

Lemma 9.2. *Let X be a Banach space with dual X' and let $F: X \to \mathbf{R}$ be a continuously differentiable function. If (u_ε) converges to u strongly in X and $(u_\varepsilon - u)/\varepsilon$ converges to w weakly in X, then*

$$\frac{F(u_\varepsilon) - F(u)}{\varepsilon} \to \langle F'(u), w \rangle, \tag{9.7}$$

where F' is the differential of F and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X' and X.

Proof. By the mean value theorem we have

$$\left| \frac{F(u_\varepsilon) - F(u)}{\varepsilon} - \left\langle F'(u), \frac{u_\varepsilon - u}{\varepsilon} \right\rangle \right| \leq \sup_{v \in I_\varepsilon} \| F'(v) - F'(u) \|_{X'} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_X,$$

where I_ε is the line segment joining u_ε to u. Since F' is continuous and $(u_\varepsilon - u)/\varepsilon$ is bounded, the right-hand side of the previous inequality tends to 0, hence (9.7) follows. □

Proof of Theorem 9.1. Given $\varphi, \psi \in C^0(\bar{\Omega})$ satisfying (8.3), for every $\varepsilon > 0$ we consider the measure $\mu_\varepsilon \in \mathcal{M}_0(\Omega)$ defined by (8.4) and the corresponding solution u_ε of (8.5). By Theorem 8.1, applied with $g(x) = j_s(x, u(x))$, we obtain that (u_ε) converges weakly in $L^2(\Omega)$ to a function w such that

$$\int_\Omega j_s(x, u) w \, dx = \int_{\operatorname{int}^* S} j_s(x, u) f \varphi \, dx + \int_\Omega \alpha\beta\psi \, d\nu_A.$$

Recalling that $u = 0$ a.e. on S, from (5.4) and from Lemma 9.2 we obtain

$$\frac{J(u_\varepsilon) - J(u)}{\varepsilon} \to \int_{\operatorname{int}^* S} j_s(x, 0) f \varphi \, dx + \int_\Omega \alpha\beta\psi \, d\nu_A.$$

Since μ is a solution of the relaxed optimization problem (4.6), we have $J(u_\varepsilon) - J(u) \geq 0$, hence

$$\int_{\operatorname{int}^* S} j_s(x, 0) f \varphi \, dx + \int_\Omega \alpha\beta\psi \, d\nu_A \geq 0.$$

As this inequality holds for any $\varphi, \psi \in C^0(\bar{\Omega})$ satisfying (8.3), we obtain (9.5) and (9.6). □

The following theorem summarizes the necessary conditions obtained in this paper for the relaxed optimization problem (4.6).

Theorem 9.3. *Let $\mu \in \mathcal{M}_0(\Omega)$ be a minimum point of* (4.6) *and let u and v be the solutions of* (9.1) *and* (9.2). *Then $u = v = 0$ q.e. on S and*

(a) $uv \leq 0$ *q.e. on* A,
(b) $uv = 0$ μ*-a.e. on* A,
(c) $f(\cdot)j_s(\cdot, 0) \geq 0$ *a.e. on* int* S,
(d) $\alpha\beta \geq 0$ ν_A*-a.e. on* $\Omega \cap \partial^* A$,

where α and β are defined by (9.3) *and* (9.4).

Proof. Property (a) is proved in Theorem 5.1, (b) in Theorem 5.2, and (c) and (d) in Theorem 9.1. □

Example 9.4. Assume that a minimum point μ of (4.6) has the form

$$\mu = \infty_S + q\mathcal{L}^N|_A,$$

where $q \in L^\infty(\Omega)$, $q \geq 0$ a.e. on Ω, $S = \Omega \backslash A$, and A is an open set with boundary ∂A of class C^2. By Example 3.3 the functions u and v are the solutions of the problems

$$u \in H_0^1(A), \qquad -\Delta u + qu = f \quad \text{in } H^{-1}(A),$$

$$v \in H_0^1(A), \qquad -\Delta v + qv = j_s(x, u) \quad \text{in } H^{-1}(A),$$

hence $u, v \in H^2(A)$. By Example 6.2 the measures ν_A and $H^{N-1}|_A$ have the same null sets, and by Example 7.2 we have

$$\alpha = \frac{\partial u}{\partial n}\left[\frac{\partial w_A}{\partial n}\right]^{-1}, \qquad \beta = \frac{\partial v}{\partial n}\left[\frac{\partial w_A}{\partial n}\right]^{-1}$$

$\mathcal{H}^{N-1}$-a.e. on $\Omega \cap \partial A$, where w_A is defined in (6.1). Therefore conditions (a)–(d) of Theorem 9.3 take the form

(a′) $uv \leq 0$ q.e. on A,
(b′) $uv = 0$ a.e. on $\{x \in A: q(x) > 0\}$,
(c′) $f(\cdot)j_s(\cdot, 0) \geq 0$ a.e. on S,
(d′) $(\partial u/\partial n)(\partial v/\partial n) \geq 0$ $\mathcal{H}^{N-1}$-a.e. on $\Omega \cap \partial A$.

Since ∂A is of class C^2 and u, $v \in H^2(A)$, conditions (a′) and (d′) imply that

(e′) $(\partial u/\partial n)(\partial v/\partial n) = 0$ $\mathcal{H}^{N-1}$-a.e. on $\Omega \cap \partial A$,

as we can easily check by considering the one-dimensional functions $t \to u(x + tn(x))$ and $t \to v(x + tn(x))$, which are continuously differentiable in a neighborhood of $t = 0$ for $\mathcal{H}^{N-1}$-a.e. point $x \in \Omega \cap \partial A$.

Example 9.5. Assume that the shape optimization problem (4.5) admits a solution A with boundary ∂A of class C^2. By Theorem 4.1 the measure ∞_S, with $S = \Omega \backslash A$, is a solution of the relaxed problem (4.6). By specializing Example 9.4 to the case

$q = 0$, we obtain that u and v are the solutions of the problems

$$u \in H_0^1(A), \qquad -\Delta u = f \quad \text{in } H^{-1}(A),$$

$$v \in H_0^1(A), \qquad -\Delta v = j_s(x, u) \quad \text{in } H^{-1}(A),$$

and conditions (a'), (c'), and (e') take the form

$$uv \leq 0 \qquad \text{q.e. on } A,$$

$$f(\cdot) j_s(\cdot, 0) \geq 0 \qquad \text{a.e. on } S,$$

$$\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = 0 \qquad \mathscr{H}^{N-1}\text{-a.e. on} \quad \Omega \cap \partial A,$$

while condition (b') is trivial in this case.

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