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Remarks on Elliptic Singular Perturbation Problems*

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Abstract. We show the effectiveness of viscosity-solution methods in asymptotic problems for second-order elliptic partial differential equations (PDEs) with a small parameter. Our stress here is on the point that the methods, based on stability results [3], [16], apply without hard PDE calculations. We treat two examples from [11] and [23]. Moreover, we generalize the results to those for Hamilton–Jacobi–Bellman equations with a small parameter.

1. Introduction

The effectiveness of viscosity-solution methods has been demonstrated in the study of asymptotic problems for second-order partial differential equations (PDEs) with small parameters. The basic scheme of applying viscosity-solution methods to such perturbation problems consists of obtaining the estimates of solutions, independent of the small parameters, which allow us to pass to the limit and of identifying the limit of the solutions, as parameters tend to zero, with the viscosity solution of the limiting equation. Such estimations of solutions usually involve hard technical calculations. We refer to [1], [9], [12], [17], and [18] for viscosity-solution approaches to singular perturbation problems, and also to [19] for various aspects of applications of viscosity solutions.

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The purpose here is to point out that there are cases where we can handle the problem without hard PDE calculations. As typical examples, we treat two asymptotic problems for linear uniformly elliptic equations with a small parameter from [11] and [23]. These problems have already been studied in [9] in light of viscosity-solution methods. However, the treatment of such problems seems to make clear the difference of our new approach from the classical ones [9]. Moreover, we generalize the results to those from Hamilton–Jacobi–Bellman (HJB in short) equations with a small parameter.

In section 2 we deal with a problem treated in [11] using a recent theory of viscosity solutions developed for a boundary problem of the Dirichlet type for Hamilton–Jacobi equations. We also discuss a generalization of the problem to that for HJB equations. In Section 3 we study a problem treated in [23] using a modification of the techniques in [6] together with an idea from [15]. We apply this method to the problem treated in [18].

After the completion of this work, the authors learned that Bardi [2] independently established an approach to Theorem 2.1 similar to theirs based on the theory of viscosity solutions of state constraint problems.

2. Application of Viscosity Solutions satisfying a Boundary Condition of the Dirichlet Type

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. We denote the space of $n \times n$ real symmetric matrices by S^n. Let $a = (a_{ij}): \bar{\Omega} \to S^n$ and $b = (b_i): \bar{\Omega} \to \mathbb{R}^n$ be given functions. We assume

$$a_{ij}, b_i \in C^2(\bar{\Omega}) \qquad \text{for} \quad i, j = 1, \dots, n, \tag{2.1}$$

and that there is positive number θ such that

$$a_{ij}\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for} \quad x \in \Omega \quad \text{and} \quad \xi \in \mathbb{R}^n. \tag{2.2}$$

Here and later we use the usual summation convention. Let Γ be a given, nonempty, relatively open subset of $\partial\Omega$. We are concerned with the boundary-value problem

$$\begin{cases} -\dfrac{\varepsilon^2}{2} a_{ij}u^\varepsilon_{x_ix_j} - b_i u^\varepsilon_{x_i} = 0 & \text{in } \Omega, \\ u^\varepsilon(x) = 1 & \text{on } \Gamma, \\ u^\varepsilon(x) = 0 & \text{on } \partial\Omega\backslash\bar{\Gamma}, \end{cases} \tag{2.3}$$

where ε is a positive parameter. Note that (2.3) has a solution belonging to $C^2(\bar{\Omega}\backslash\partial\Gamma)$ and satisfying $0 \leq u^\varepsilon \leq 1$ on Ω. See Remark (ii) below for an argument related to the existence of a solution of (2.3).

Equation (2.3) was studied by Fleming [11] in connection with the asymptotic problem for the exit probability, from Γ, of solutions of stochastic differential equations with a small parameter. Following [11], we introduce the condition:

$$\text{if}\quad \xi \in H^1_{\mathrm{loc}}([0,\infty);\mathbb{R}^n)\quad\text{and}\quad \xi(t)\in\bar{\Omega}\quad\text{for } t\geq 0,\quad\text{then}$$

$$\int_0^\infty |\dot{\xi}(t) - b(\xi(t))|^2\,dt = \infty. \tag{2.4}$$

We define $L\colon \Omega\times\mathbb{R}^n \to [0,\infty)$ by $L(x,p) = a^{ij}(x)p_ip_j$ where the $a^{ij}(x)$ denote the (i,j) components of the inverse matrix $a(x)^{-1}$, and, under the above hypothesis, $I\colon\Omega\to[0,\infty)$ by

$$I(x) = \inf \tfrac{1}{2}\int_0^T L(\xi(t), \dot{\xi}(t) - b(\xi(t)))\,dt, \tag{2.5}$$

where the infimum is taken for all $T > 0$ and $\xi \in H^1([0,T];R^n)$ satisfying $\xi(0) = x$, $\xi(t)\in\Omega$ for $t\in[0,T)$, and $\xi(T)\in\Gamma$. It is easy to check that I is Lipschitz continuous on Ω, $I\geq 0$ on Ω, and $I(x)\to 0$ as $x\to\Gamma$. We denote the continuous extension of I to $\bar{\Omega}$ again by I. Obviously, $I = 0$ on Γ. We intend to prove the following theorem due to Fleming [11] by our new method.

Theorem 2.1. *Assume* (2.1), (2.2), *and* (2.4). *For each* $\varepsilon > 0$ *let* $u^\varepsilon \in C^2(\bar{\Omega}\backslash\partial\Gamma)$ *be a solution of* (2.3) *satisfying* $0\leq u^\varepsilon \leq 1$ *on* Ω. *Then*

$$-\varepsilon^2 \log u^\varepsilon(x) \to I(x) \tag{2.6}$$

uniformly on compact subsets of $\Omega\cup\Gamma$ *as* $\varepsilon\downarrow 0$.

We begin with some preliminaries concerning condition (2.4) and viscosity solutions of boundary-value problems.

Lemma 2.2. *Condition* (2.4) *is equivalent to the condition that*

$$\begin{cases} \text{there is a } C^1 \text{ function } \psi \text{ on } \bar{\Omega} \text{ such that} \\ \quad b_i\psi_{x_i} \leq -1 \qquad \text{on } \bar{\Omega}. \end{cases} \tag{2.7}$$

Proof. The proof that (2.4) implies (2.7) can be found in [1]. We shall show that (2.7) implies (2.4). Assume (2.7), and let ψ be a C^1 function on $\bar{\Omega}$ satisfying $b_i\psi_{x_i}\leq -1$ on $\bar{\Omega}$. Let $\xi\in H^1_{\mathrm{loc}}([0,\infty);\mathbb{R}^n)$ satisfy $\xi(t)\in\bar{\Omega}$ for $t\geq 0$. Then

$$\begin{aligned}\psi(\xi(T)) - \psi(\xi(0)) &= \int_0^T D\psi(\xi(t))\cdot\dot{\xi}(t)\,dt \\ &= \int_0^T \{(\dot{\xi}(t) - b(\xi(t)))\cdot D\psi(\xi(t)) + b(\xi(t))\cdot D\psi(\xi(t))\}\,dt \\ &\leq \left\{\int_0^T |\dot{\xi}(t) - b(\xi(t))|^2\,dt\right\}^{1/2} T^{1/2}\max_{\bar{\Omega}}|D\psi| - T\end{aligned}$$

for $T > 0$, where $D\psi$ denotes the gradient of ψ. Hence

$$T \leq 2\max_{\bar{\Omega}}|\psi| + T^{1/2}\max_{\bar{\Omega}}|D\psi|\left\{\int_0^T |\dot{\xi}(t) - b(\xi(t))|^2\,dt\right\}^{1/2}$$

for $T > 0$. This implies

$$\int_0^\infty |\dot{\xi}(t) - b(\xi(t))|^2 \, dt = \infty,$$

proving our assertion. □

Following [5], [20], [3], and [16], we now recall the definition of viscosity solutions of the problem

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u = h \quad \text{or} \quad F(x, u, Du, D^2u) = 0 & \text{on } \Sigma. \end{cases} \tag{2.8}$$

Here Σ is an open subset of $\partial\Omega$, $h: \Sigma \to \mathbb{R}$ is a given function, $F: \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ is a given function, $u: \Omega \cup \Sigma \to \mathbb{R}$ is the unknown function, and D^2u denotes the Hessian matrix of u.

For function $u: \Omega \cup \Sigma \to \mathbb{R} \cup \{-\infty, \infty\}$ we define

$$u^*, u_*: \Omega \cup \Sigma \to \mathbb{R} \cup \{-\infty, \infty\}$$

by

$$u^*(x) = \lim_{r \downarrow 0} \sup\{u(y): y \in \Omega \cup \Sigma, |y - x| \le r\}$$

and

$$u_*(x) = \lim_{r \downarrow 0} \inf\{u(y): y \in \Omega \cup \Sigma, |y - x| \le r\}$$

For function $F: \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ we define

$$F^*(x, r, p, A) = \lim_{\delta \downarrow 0} \sup\{F(y, s, q, B): (y, s, q, B) \in \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^n \times S^n, \\ |y - x| < \delta, |s - r| < \delta, |q - p| < \delta, \|B - A\| < \delta\}$$

and

$$F_*(x, r, p, A) = \lim_{\delta \downarrow 0} \inf\{F(y, s, q, B): (y, s, q, B) \in \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^n \times S^n, \\ |y - x| < \delta, |s - r| < \delta, |q - p| < \delta, \|B - A\| < \delta\}$$

We call a function $u: \Omega \cup \Sigma \to \mathbb{R} \cup \{-\infty\}$ a *viscosity subsolution* of (2.8) provided $u^*(x) < \infty$ for $x \in \Omega \cup \Sigma$ and whenever $\varphi \in C^2(\Omega \cup \Sigma)$ and $u^* - \varphi$ attains its local maximum at a point $y \in \Omega \cup \Sigma$, then

$$F_*(y, u^*(y), D\varphi(y), D^2\varphi(y)) \le 0 \qquad \text{if} \quad y \in \Omega$$

and

$$u^*(y) \le h^*(y) \quad \text{or} \quad F_*(y, u^*(y), D\varphi(y), D^2\varphi(y)) \le 0 \qquad \text{if} \quad y \in \Sigma.$$

Similarly, we call a function $u: \Omega \cup \Sigma \to \mathbb{R} \cup \{\infty\}$ a *viscosity supersolution* of (2.8) provided $u_*(x) > -\infty$ for $x \in \Omega \cup \Sigma$ and whenever $\varphi \in C^2(\Omega \cup \Sigma)$ and $u_* - \varphi$ attains its local minimum at a point $y \in \Omega \cup \Sigma$, then

$$F^*(y, u_*(y), D\varphi(y), D^2\varphi(y)) \geq 0 \quad \text{if} \quad y \in \Omega$$

and

$$u_*(y) \geq h_*(y) \quad \text{or} \quad F^*(y, u_*(y), D\varphi(y), D^2\varphi(y)) \geq 0 \quad \text{if} \quad y \in \Sigma.$$

A *viscosity solution* of (2.8) is defined to be a function on $\Omega \cup \Sigma$ which is both a viscosity sub- and supersolution of (2.8). When $\Sigma = \emptyset$, a viscosity solution (resp. subsolution or supersolution) of (2.8) is also called a viscosity solution (resp. subsolution or supersolution) of $F(x, u, Du, D^2u) = 0$ in Ω.

The main tools in our proof of Theorem 2.1 are the following two propositions.

Proposition 2.3. *For $\varepsilon > 0$ let $F_\varepsilon: \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ be given and let u_ε be a viscosity subsolution (resp. supersolution) of (2.8) with F_ε in place of F. Set*

$$u(x) = \lim_{\delta \downarrow 0} \sup\{u_\varepsilon(y); 0 < \varepsilon < \delta, y \in \Omega \cup \Sigma, |y - x| < \delta\}$$

for $x \in \Omega \cup \Sigma$ and

$$\begin{aligned} F(x, r, p, A) = \lim_{\delta \downarrow 0} \inf\{F_\varepsilon(y, s, q, B)&: 0 < \varepsilon < \delta, (y, s, q, B) \in \Omega \cup \Sigma \\ &\times \mathbb{R} \times \mathbb{R}^n \times S^n, |y - x| < \delta, |s - r| < \delta, |q - p| < \delta, \\ &\|B - A\| < \delta\} \end{aligned}$$

for $(x, r, p, A) \in \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^n \times S^n$ (resp.

$$u(x) = \lim_{\delta \downarrow 0} \inf\{u_\varepsilon(y): 0 < \varepsilon < \delta, y \in \Omega \cup \Sigma, |y - x| < \delta\}$$

and

$$\begin{aligned} F(x, r, p, A) = \lim_{\delta \downarrow 0} \sup\{F_\varepsilon(y, s, q, B)&: 0 < \varepsilon < \delta, (y, s, q, B) \in \Omega \cup \Sigma \\ &\times \mathbb{R} \times \mathbb{R}^n \times S^n, |y - x| < \delta, |s - r| < \delta, |q - p| < \delta, \\ &\|B - A\| < \delta\} \end{aligned}$$

for $(x, r, p, A) \in \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^n \times S^n$). Assume u is locally bounded on $\Omega \cup \Sigma$. Then u is a viscosity subsolution (resp. supersolution) of (2.8).

Stability results for viscosity solutions were first obtained by Crandall and Lions [5]. This general result is due to Barles and Perthame [3] in the case of first-order Hamilton–Jacobi equations and is observed by Ishii [16] in the general case. We refer to [3] and [16] for the proof.

Proposition 2.4. *Let H be a real-valued continuous function on $\bar{\Omega} \times \mathbb{R}^n$. Let u and v be, respectively, viscosity sub- and supersolutions of*

$$\begin{cases} H(x, Du) = 0 & \text{in } \Omega, \\ u = h \quad \text{or} \quad H(x, Du) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Assume that h is continuous on $\partial\Omega$, $u \le h$ on $\partial\Omega$, u is Lipschitz continuous on $\bar{\Omega}$, and $p \to H(x, p)$ is convex on $\mathbb{R}^n$ for $x \in \bar{\Omega}$. Assume in addition that there is a C^1 function ψ on $\bar{\Omega}$ such that $H(x, D\psi(x)) < 0$ in $\bar{\Omega}$. Then $u \le v$ on $\bar{\Omega}$.

This assertion is proved in [16].

Let u^ε be a solution of (2.3) satisfying $0 \le u^\varepsilon \le 1$ on $\bar{\Omega}$. We observe, by setting

$$v^\varepsilon(x) = -\varepsilon^2 \log u^\varepsilon(x) \qquad \text{for} \quad x \in \Omega \cup \Gamma,$$

that v^ε is positive on Ω and solves

$$\begin{cases} -\dfrac{\varepsilon^2}{2} a_{ij} v^\varepsilon_{x_i x_j} + \frac{1}{2} a_{ij} v^\varepsilon_{x_i} v^\varepsilon_{x_j} - b_i v^\varepsilon_{x_i} = 0 & \text{in } \Omega, \\ v^\varepsilon = 0 & \text{on } \Gamma, \\ v^\varepsilon(x) \to \infty & \text{as} \quad x \to \partial\Omega \backslash \bar{\Gamma}. \end{cases} \tag{2.10}$$

Lemma 2.5. *There is a continuous function C on $\Omega \cup \Gamma$ satisfying $C = 0$ on Γ for which*

$$v^\varepsilon(x) \le C(x) \qquad \textit{for} \quad x \in \Omega \cup \Gamma \quad \textit{and} \quad 0 < \varepsilon < 1.$$

Proof. Define

$$v(x) = \sup\{v^\varepsilon(x): 0 < \varepsilon < 1\} \qquad \text{for} \quad x \in \Omega \cup \Gamma.$$

By a simple calculation, we see that, for each $r > 0$ and $\delta \in (0, r)$, there is a constant $A = A(r, \delta) > 0$ having the following properties: if $y \in \mathbb{R}^n$ and $B \in \mathbb{R}$ and if we set

$$w(x) = \frac{A}{r - |x - y|} + B \qquad \text{for} \quad x \in \bar{\Omega},$$

then we have

$$-\frac{\varepsilon^2}{2} a_{ij} w_{x_i x_j} + \tfrac{1}{2} a_{ij} w_{x_i} w_{x_j} - b_i w_{x_i} > 0$$

on $\bar{\Omega} \cap B(y; r) \backslash B(y; \delta)$ for $0 < \varepsilon < 1$. Using such functions w with appropriate r, δ, B, and y, we find that v is continuous at points of Γ and that if v^* is finite at a point $y \in \Omega$, then v^* is bounded above on any closed ball contained in Ω with center at y. Therefore the closure, relative to Ω, of the set $\{x \in \Omega: v^*(x) < \infty\}$ is nonempty and open, and hence $v^*(x) < \infty$ for $x \in \Omega$ by the connectedness of Ω. Thus, we see that the conclusion of Lemma 2.5 holds. □

Lemma 2.6. *Let h be a continuous function on $\partial\Omega$ satisfying $h \ge I$ on $\partial\Omega$ and $h = 0$ on Γ. Then I is a viscosity solution of*

$$\begin{cases} \frac{1}{2} a_{ij} u_{x_i} u_{x_j} - b_i u_{x_i} = 0 & \textit{in } \Omega, \\ u = h \quad \textit{or} \quad \frac{1}{2} a_{ij} u_{x_i} u_{x_j} - b_i u_{x_i} = 0 & \textit{on } \partial\Omega. \end{cases} \tag{2.11}$$

This lemma can be proved as in the same manner as the proof of Theorem 3.1 of [16]. See also [10]. We leave the details to the reader.

Proof of Theorem 2.1. For $\varepsilon > 0$ let v^ε be a solution of (2.10) satisfying $v^\varepsilon \geq 0$ on $\Omega \cup \Gamma$. For $\varepsilon > 0$ we define $w_\varepsilon : \Omega \cup \Gamma \to \mathbb{R} \cup \{\infty\}$ and $z_\varepsilon : \bar{\Omega} \to \mathbb{R}$ by

$$w_\varepsilon(x) = \sup\{v^\delta(y) : 0 < \delta < \varepsilon, y \in \Omega \cup \Gamma, |y - x| < \varepsilon\}$$

for $x \in \Omega \cup \Gamma$ and

$$z_\varepsilon(x) = \inf\{v^\delta(y) : 0 < \delta < \varepsilon, y \in \Omega \cup \Gamma, |y - x| < \varepsilon\}$$

for $x \in \bar{\Omega}$. Finally set

$$w(x) = \lim_{\varepsilon \downarrow 0} w_\varepsilon(x) \qquad \text{for} \quad x \in \Omega \cup \Gamma$$

and

$$z(x) = \lim_{\varepsilon \downarrow 0} z_\varepsilon(x) \qquad \text{for} \quad x \in \bar{\Omega}.$$

By definition we have $z_\varepsilon \leq z \leq w \leq w_\varepsilon$ on $\Omega \cup \Gamma$. It is easy to check that w is upper semicontinuous on $\Omega \cup \Gamma$ and that z is continuous at points of Γ and $w = z = 0$ on Γ.

It is clear that v^ε is a viscosity solution of

$$-\frac{\varepsilon^2}{2} a_{ij} u_{x_i x_j} + \tfrac{1}{2} a_{ij} u_{x_i} u_{x_j} - b_i u_{x_i} = 0 \qquad \text{in } \Omega.$$

In view of Proposition 2.3 we find that w is a viscosity subsolution of

$$\tfrac{1}{2} a_{ij} u_{x_i} u_{x_j} - b_i u_{x_i} = 0 \qquad \text{in } \Omega.$$

This implies that w is Lipschitz continuous on $\Omega \cup \Gamma$. We denote the continuous extension of w to $\bar{\Omega}$ again by w. We select a continuous function h on $\partial\Omega$ so that $h = 0$ on Γ and $w \leq h$ and $I \leq h$ on $\partial\Omega$. It is obvious that w is a viscosity subsolution of (2.11). From Lemma 2.6, I is a viscosity solution of (2.11).

Next we extend v^ε to $\bar{\Omega}$ by setting $v^\varepsilon(x) = \infty$ for $x \in \partial\Omega \backslash \Gamma$. The resulting function is a viscosity supersolution of

$$\begin{cases} -\dfrac{\varepsilon^2}{2} a_{ij} u_{x_i x_j} + \tfrac{1}{2} a_{ij} u_{x_i} u_{x_j} - b_i u_{x_i} = 0 & \text{in } \Omega, \\ u = h \quad \text{or} \quad -\dfrac{\varepsilon^2}{2} a_{ij} u_{x_i x_j} + \tfrac{1}{2} a_{ij} u_{x_i} u_{xj} - b_i u_{x_i} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by Proposition 2.3, we see that z is a viscosity supersolution of (2.11).

Let ψ be a function on $\bar{\Omega}$ satisfying (2.7) in Lemma 2.2. Note that if $\delta > 0$ is sufficiently small and we set $u(x) = \delta\psi(x)$ for $x \in \Omega$, then

$$\tfrac{1}{2} a_{ij} u_{x_i} u_{x_j} - b_i u_{x_i} \leq \delta\left(\frac{\delta}{2} a_{ij} \psi_{x_i} \psi_{x_j} - 1\right) < 0 \qquad \text{on } \bar{\Omega}.$$

Thus, applying Proposition 2.4, we have $w \le z$, $I \le z$, and $w \le I$ on $\bar{\Omega}$ and so $w = z = I$ on $\Omega \cup \Gamma$ as $w \ge z$ on $\Omega \cup \Gamma$.

Because of the monotone convergence of w_ε and z_ε, we see as in the proof of Dini's lemma that $w_\varepsilon \to w$ and $z_\varepsilon \to z$ uniformly on compact subsets of $\Omega \cup \Gamma$ as $\varepsilon \downarrow 0$. This obviously guarantees that $v^\varepsilon \to I$ uniformly on compact subsets of $\Omega \cup \Gamma$ as $\varepsilon \downarrow 0$. □

The above arguments allow us to deal with the following situation: we consider the Dirichlet problem for the HJB equation

$$\begin{cases} \max\limits_{1 \le k \le m} \left\{ -\dfrac{\varepsilon^2}{2} a_{ij}^k u_{x_i x_j}^\varepsilon - b_i^k u_{x_i}^\varepsilon \right\} = 0 & \text{in } \Omega, \\ u^\varepsilon(x) = 1 \quad \text{on } \Gamma \quad \text{and} \quad u^\varepsilon(x) = 0 & \text{on } \partial\Omega \backslash \bar{\Gamma}, \end{cases} \tag{2.12}$$

for $\varepsilon > 0$, where m is a positive integer and the matrices $(a_{ij}^k(x))_{1 \le i,j \le n}$ are real, symmetric, and positive definite for $k = 1, \ldots, m$ and $x \in \bar{\Omega}$; and we raise the same asymptotic question for the solution u^ε of (2.12) as before.

We write

$$K = \{1, \ldots, m\}, \quad a(k, x) = (a_{ij}^k(x))_{1 \le i,j \le n}, \text{ and } b(k, x) = (b_1^k(x), \ldots, b_n^k(x))$$

for $k = 1, \ldots, m$ and $x \in \bar{\Omega}$. In place of (2.4) we now assume:

$$\begin{cases} \text{if} \quad \xi \in H_{\text{loc}}^1([0, \infty); \mathbb{R}^n), \quad \xi(t) \in \bar{\Omega} \quad \text{for } t \ge 0, \quad \text{and} \\ k\colon [0, \infty) \to K \quad \text{is measurable, then} \\ \displaystyle\int_0^\infty |\dot{\xi}(t) - b(k(t), \xi(t))^2 \, dt = \infty.) \end{cases} \tag{2.13}$$

We define $L\colon K \times \Omega \times \mathbb{R}^n \to [0, \infty)$ *by* $L(k, x, p) = a^{ij}(k, x) p_i p_j$, where the $a^{ij}(k, x)$ denote the (i, j) components of the matrix $a(k, x)^{-1}$. We denote by $\mathscr{K}$ the set of measurable functions on $[0, \infty)$ taking values in K. We set

$$T_\xi = \inf\{t \ge 0\colon \xi(t) \notin \Omega\}$$

and

$$J(k, \xi) = \tfrac{1}{2} \int_0^{T_\xi} L(k(t), \xi(t), \dot{\xi}(t) - b(k(t), \xi(t))) \, dt$$

for $\xi \in H_{\text{loc}}^1([0, \infty); \mathbb{R}^n)$ and $k \in \mathscr{K}$. For $x \in \Omega$ we denote by Ξ_x the set of those $\xi \in H_{\text{loc}}^1([0, \infty); \mathbb{R}^n)$ which satisfy $\xi(0) = x$, $T_\xi < \infty$, and $\xi(T_\xi) \in \Gamma$, and by $\mathscr{A}_x$ we denote the set of those mappings $\alpha\colon \mathscr{K} \to \Xi_x$ which satisfy $\alpha(k) = \alpha(\tilde{k})$ on $[0, t]$ whenever $t > 0$, $k, \tilde{k} \in \mathscr{K}$, and $k = \tilde{k}$ a.e. on $[0, t]$. Finally, we define

$$I(x) = \inf_{\alpha \in \mathscr{A}_x} \sup_{k \in \mathscr{K}} J(k, \alpha(k)) \qquad \text{for} \quad x \in \Omega. \tag{2.14}$$

We extend this function to $\bar{\Omega}$ by continuity, and we denote the extension again by I.

Theorem 2.7. *Assume that the* a_{ij}^k *and* b_i^k *satisfy* (2.1) *and* (2.2). *Also assume* (2.13). *Let* $u^\varepsilon \in C^2(\Omega) \cap C(\bar{\Omega}\backslash\partial\Gamma)$ *be a solution of* (2.12) *satisfying* $0 \le u^\varepsilon \le 1$ *on* $\bar{\Omega}$. *Then*

$$-\varepsilon^2 \log u^\varepsilon(x) \to I(x) \tag{2.15}$$

uniformly on compact subsets of $\Omega \cup \Gamma$ *as* $\varepsilon \downarrow 0$.

Remarks. (i) Equation (2.12) is the dynamic programming equation of the following optimal control problem: for $x \in \Omega$ we consider the stochastic differential equation

$$\begin{cases} dX_t^\varepsilon = b(k_t, X_t^\varepsilon)\,dt + c(k_t, X_t^\varepsilon)\,dW_t & \text{for} \quad r > 0, \\ X_0^\varepsilon = x \quad \text{a.s.,} \end{cases} \tag{2.16}$$

where W_t is an n-dimensional Wiener process, k_t is a control (i.e., a progressively measurable process with values in K), and $c(k, x) = a(k, x)^{1/2}$. We take $\mathrm{P}(X_\tau^\varepsilon \in \Gamma) = \mathrm{E}(1_\Gamma(X_\tau^\varepsilon))$ as the cost associated with the state equation (2.16), where P and E denote the probability and the expectation, respectively, τ is the first exit time of X_t^ε from Ω, i.e.,

$$\tau = \inf\{t > 0 \colon X_t^\varepsilon \in \partial\Omega\} \quad \text{and} \quad 1_\Gamma(y) = \begin{cases} 1 & \text{for} \quad y \in \Gamma, \\ 0 & \text{for} \quad y \notin \Gamma, \end{cases}$$

and define the optimal cost by

$$u^\varepsilon(x) = \inf \mathrm{E}(1_\Gamma(X_\tau^\varepsilon)),$$

where the infimum is taken over all controls k_t. Then the function u^ε on Ω solves, at least formally, (2.12).

(ii) Equation (2.12) has a solution $u^\varepsilon \in C^2(\Omega) \cap C(\bar{\Omega}\backslash\partial\Gamma)$ satisfying $0 \le u^\varepsilon \le 1$ on $\bar{\Omega}$. This can be proved by the following argument: we choose a sequence $\{\chi_l\}$ of smooth functions on $\partial\Omega$ such that $0 \le \chi_l \le 1$ on $\partial\Omega$, $\chi_l = 1$ on Γ, and $\chi_l = 0$ on $\partial\Omega\backslash\Gamma_l$, where $\Gamma_l = (x \in \mathbb{R}^n\colon \operatorname{dist}(x, \Gamma) < 1/l\}$; solve the problem (see [8] and [21])

$$\begin{cases} \max\limits_{k \in K}\left\{ -\dfrac{\varepsilon^2}{2} a_{ij}^k u_{x_i x_j}^l - b_i^k u_{x_i}^l \right\} = 0 & \text{in } \Omega \\ u^l = \chi_l & \text{on } \partial\Omega; \end{cases}$$

and obtain a solution of (2.12) as the limit of u^l by sending $l \to \infty$. This limiting argument is justified by the interior Hölder estimate for the second derivatives of solutions of HJB equations (see [8] and [21]) and the standard barrier argument.

We refer to [14], [24], [22], [13], [12], and [7] for some asymptotic results on controlled diffusion processes related to the above theorem.

We do not give here the details of the proof of Theorem 2.7 but instead indicate just how to modify the proof of Theorem 2.1 in order to prove Theorem 2.7.

First, we define Δ to be the set of those mappings $\beta\colon \mathscr{K} \to L_{\mathrm{loc}}^2([0, \infty); \mathbb{R}^n)$ which satisfy $\beta(k) = \beta(\tilde{k})$ a.e. on $[0, t]$ whenever $t > 0$, $k, \tilde{k} \in \mathscr{K}$, and $k = \tilde{k}$ a.e. on $[0, t]$. For $x \in \Omega$, $k \in \mathscr{K}$, and $\eta \in L_{\mathrm{loc}}^2([0, \infty); \mathbb{R}^n)$ we solve

$$\begin{cases} \dot{\xi}(t) = a(k(t), \xi(t))\eta(t) + b(k(t), \xi(t)) & \text{for} \quad t > 0, \\ \xi(0) = x, \end{cases} \tag{2.17}$$

and define

$$\tilde{J}(x, k, \eta) = \tfrac{1}{2} \int_0^\tau L(k(t), \xi(t), \dot{\xi}(t) - b(k(t), \xi(t)))\, dt + \chi_\Gamma(\xi(\tau))$$
$$\equiv \tfrac{1}{2} \int_0^\tau a(k(t), \xi(t))\eta(t) \cdot \eta(t)\, dt + \chi_\Gamma(\xi(\tau)), \tag{2.18}$$

where $\tau = \inf\{t > 0: \xi(t) \in \partial\Omega\}$ and

$$\chi_\Gamma(y) = \begin{cases} 0 & \text{for} \quad y \in \Gamma, \\ \infty & \text{for} \quad y \notin \Gamma. \end{cases}$$

Then we easily see that

$$I(x) = \inf_{\beta \in \Delta} \sup_{k \in \mathscr{K}} \tilde{J}(x, k, \beta(k)) \qquad \text{for} \quad x \in \Omega.$$

The right-hand side of this identity reads as the lower value at x of the differential game with (2.17) and (2.18), respectively, as its state equation and cost. This interpretation leads to the conclusion (see [10]) that I is a viscosity solution of

$$\min_{k \in K} \left\{ \tfrac{1}{2} a_{ij}^k u_{x_i} u_{x_j} - b_i^k u_{x_i} \right\} = 0 \qquad \text{in } \Omega. \tag{2.19}$$

Instead of Lemma 2.2 we now use

Lemma 2.8. *Condition* (2.13) *is equivalent to the condition that*

$$\begin{cases} \textit{there is a } C^1 \textit{ function } \psi \textit{ on } \bar{\Omega} \textit{ such that} \\ \max_{k \in K} b_i^k \psi_{x_i} \le -1 \qquad \textit{on } \bar{\Omega}. \end{cases} \tag{2.20}$$

Also, in place of Proposition 2.4 we use

Proposition 2.9. *Let $h \in C(\partial\Omega)$ and, for $k \in K$, let H_k be a real-valued continuous function on $\bar{\Omega} \times \mathbb{R}^n$. Let u and v be, respectively, viscosity sub- and supersolutions of*

$$\begin{cases} \min_{k \in K} H_k(x, Du) = 0 & \textit{in } \Omega. \\ u = h \quad \textit{or} \quad \min_{k \in K} H_k(x, Du) = 0 & \textit{on } \partial\Omega. \end{cases} \tag{2.21}$$

Assume that u is Lipschitz continuous on $\bar{\Omega}$, $u \le h$ on $\partial\Omega$, and $p \to H_k(x, p)$ is convex on $\mathbb{R}^n$ for $x \in \Omega$ and $k \in K$, and that there is a C^1 function ψ on $\bar{\Omega}$ such that $\max_{k \in K} H_k(x, D\psi(x)) < 0$ *on $\bar{\Omega}$. Then $u \le v$ on $\bar{\Omega}$.*

The proofs of Lemma 2.2 and Proposition 2.4 are easily adapted to yield the above two assertions, and we leave the details to the reader.

The arguments of the proof of Theorem 2.1 together with the above obvservations apply in proving Theorem 2.7.

3. Estimate of the Rate of Convergence

Let Ω and $a = (a_{ij})$ be as in Section 2. Let λ be a positive number. For $\varepsilon > 0$ we shall deal with the boundary-value problem

$$\begin{cases} -\dfrac{\varepsilon^2}{2} a_{ij} u^\varepsilon_{x_i x_j} + \lambda u^\varepsilon = 0 & \text{in } \Omega, \\ u^\varepsilon = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

This problem has a unique solution belonging to $C^2(\bar{\Omega})$.

Let $L: \Omega \times \mathbb{R}^n \to \mathbb{R}$ be as in the first part of Section 2, and define

$$\operatorname{dist}(x, \partial\Omega) = \inf \int_0^1 L(\xi(t), \dot{\xi}(t))^{1/2}\, dt,$$

where the infimum is taken for all $\xi \in H^1([0, 1]; \mathbb{R}^n)$ satisfying $\xi(0) = x$, $\xi(t) \in \Omega$ for $0 \le t < 1$, and $\xi(1) \in \partial\Omega$. We set $\operatorname{dist}(x, \partial\Omega) = 0$ for $x \in \partial\Omega$, and finally

$$\begin{aligned} I(x) &= (2\lambda)^{1/2} \operatorname{dist}(x, \partial\Omega) \\ &\equiv \inf \int_0^\tau \left(\tfrac{1}{2} L(\xi(t), \dot{\xi}(t)) + \lambda\right) dt \qquad \text{for} \quad x \in \bar{\Omega}, \end{aligned} \tag{3.2}$$

where the infimum is taken over all $\xi \in H^1_{\mathrm{loc}}([0, \infty); \mathbb{R}^n)$ satisfying $\xi(0) = x$, and $\tau = \inf\{t \ge 0: \xi(t) \in \partial\Omega\}$.

The following theorem partly refines a result by Varadhan [23].

Theorem 3.1. *Assume* (2.1) *and* (2.2). *For* $\varepsilon > 0$ *let* u^ε *be the solution of* (3.1). *Then there is a constant* $C > 0$ *such that*

$$|\varepsilon \log u^\varepsilon(x) + I(x)| \le C\varepsilon^{1/2} \qquad \textit{for} \quad x \in \Omega \quad \textit{and} \quad \varepsilon > 0. \tag{3.3}$$

First we set

$$v^\varepsilon(x) = -\varepsilon \log u^\varepsilon(x) \qquad \text{for} \quad x \in \bar{\Omega},$$

and observe that $v^\varepsilon \ge 0$ on $\bar{\Omega}$ and that v^ε satisfies

$$\begin{cases} -\dfrac{\varepsilon}{2} a_{ij} v^\varepsilon_{x_i x_j} + \tfrac{1}{2} a_{ij} v^\varepsilon_{x_i} v^\varepsilon_{x_j} = \lambda & \text{in } \Omega. \\ v^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Using barrier functions $x \to r^{-p} - |x - y|^{-p}$ with appropriate $r > 0$, $p > 0$, and $y \in \Omega^c$, we easily deduce this:

Lemma 3.2. *There is a Lipschitz continuous function* C *on* $\bar{\Omega}$ *satisfying* $C = 0$ *on* $\partial\Omega$ *for which*

$$v^\varepsilon(x) \le C(x) \qquad \textit{for} \quad x \in \bar{\Omega} \quad \textit{and} \quad \varepsilon > 0. \tag{3.5}$$

Lemma 3.3. *The function I is Lipschitz continuous on $\bar{\Omega}$ and a viscosity solution of*

$$\tfrac{1}{2} a_{ij} u_{x_i} u_{x_j} = \lambda \qquad \text{in } \Omega. \tag{3.6}$$

A proof of this lemma is found in [9].

If we follow the scheme explained in Section 2, then we can conclude rather easily that

$$-\varepsilon \log u^{\varepsilon}(x) \to I(x)$$

uniformly on $\bar{\Omega}$ as $\varepsilon \downarrow 0$. This convergence result is due to Varadhan [23].

Proof of Theorem 3.1. Let I be the function defined by (3.2) and let v^{ε} be a solution of (3.4). Let $\alpha > 0$ and $0 < \nu < 1$ be numbers to be fixed later. Define $\Phi: \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ by

$$\Phi(x, y) = \nu I(x) - v^{\varepsilon}(y) - \frac{1}{\alpha}|x - y|^2 \qquad \text{for} \quad x, y \in \bar{\Omega}.$$

Suppose that Φ attains its maximum over $\bar{\Omega} \times \bar{\Omega}$ at a point $(\bar{x}, \bar{y}) \in \Omega \times \Omega$. Then

$$-a_{ij}(\bar{y})\Phi_{y_i y_j}(\bar{y}) \geq 0 \qquad \text{and} \qquad \Phi_{y_i}(\bar{y}) = 0 \qquad \text{for} \quad i = 1, \ldots, n.$$

Using these we get

$$\frac{\varepsilon}{\alpha} \operatorname{Tr} a(\bar{y}) + \frac{2}{\alpha^2} a_{ij}(\bar{y})(\bar{x}_i - \bar{y}_i)(\bar{x}_j - \bar{y}_j) \geq \lambda, \tag{3.7}$$

since v^{ε} is a solution of (3.4). Also, using that I is a viscosity subsolution of (3.6) (see Lemma 3.3), we have

$$\frac{2}{\alpha^2} a_{ij}(\bar{x})(\bar{x}_i - \bar{y}_i)(\bar{x}_j - \bar{y}_j) \leq \nu^2 \lambda. \tag{3.8}$$

This together with (2.2) implies that $|\bar{x} - \bar{y}| \leq C\alpha$ for some constant $C > 0$, independent of α and ν. In view of (2.1) we may assume that $\|a(x) - a(y)\| \leq C|x - y|$ and $|\operatorname{Tr} a(x)| \leq C$ for $x, y \in \Omega$. Now subtracting (3.8) from (3.7) yields

$$\frac{2C}{\alpha^2} |\bar{x} - \bar{y}|^3 + C\frac{\varepsilon}{\alpha} \geq (1 - \nu^2)\lambda,$$

and hence

$$2C^3\alpha + C\frac{\varepsilon}{\alpha} \geq (1 - \nu^2)\lambda. \tag{3.9}$$

Hereafter we set $C_1 = 2(2C^3 + C)/\lambda$ and assume that ε is sufficiently small; namely $\varepsilon < C_1^{-2}$. We now fix $\alpha = \varepsilon^{1/2}$ and $\nu = (1 - C_1\varepsilon^{1/2})^{1/2}$. This means that

$$2C^3\alpha + C\frac{\varepsilon}{\alpha} = \frac{\lambda}{2}(1 - \nu^2),$$

and hence we have

$$2C^3\alpha + C\frac{\varepsilon}{\alpha} < (1 - v^2)\lambda;$$

a contradiction to (3.9). That is, Φ with α and v being fixed in this way does not attain its maximum at any point of $\Omega \times \Omega$.

In view of Lemmas 3.2 and 3.3 we may assume that

$$|I(x) - I(y)| \le C|x - y| \quad \text{and} \quad |v^\varepsilon(x) - v^\varepsilon(y)| \le C|x - y|$$

for $(x, y) \in \partial(\Omega \times \Omega)$ and $\varepsilon > 0$. This yields

$$\Phi(x, y) \le C|x - y| - \frac{1}{\alpha}|x - y|^2 \qquad \text{for} \quad (x, y) \in \partial(\Omega \times \Omega).$$

Thus we conclude

$$0 \le \sup_{\Omega} (vI - v^\varepsilon) \le \sup_{\Omega \times \Omega} \Phi \le \max_{(x,y) \in \partial(\Omega \times \Omega)} \left\{ C|x - y| - \frac{1}{\alpha}|x - y|^2 \right\}.$$

That is, we have

$$0 \le \sup_{\Omega} (vI - v^\varepsilon) \le C|x - y| - \frac{1}{\alpha}|x - y|^2$$

for some $(x, y) \in \partial(\Omega \times \Omega)$, which in particular yields $|x - y| \le C\alpha$. Therefore

$$\sup(vI - v^\varepsilon) \le C\alpha,$$

and so

$$I(x) - v^\varepsilon(x) \le C\alpha + (1 - v)I(x) \le C_2 \varepsilon^{1/2}$$

for $x \in \bar{\Omega}$ and $0 < \varepsilon < C_1^{-2}$ and for some constant C_2.

If we proceed as above with

$$\Phi(x, y) = v^\varepsilon(x) - vI(y) - \frac{1}{\alpha}|x - y|^2,$$

where $v > 1$ and $\alpha > 0$, then we have

$$v^\varepsilon(x) - I(x) \le C_3 \varepsilon^{1/2}$$

for $x \in \bar{\Omega}$, for $\varepsilon > 0$ sufficiently small, and for some constant C_3. Since I is bounded on $\bar{\Omega}$ and $\{v^\varepsilon\}$ is uniformly bounded on $\bar{\Omega}$, we see that (3.3) holds for some constant C. □

The above arguments are easily adapted to the proof of the following generalization of Theorem 3.1. We consider the Dirichlet problem

$$\begin{cases} \max_{k \in K} \left\{ -\frac{\varepsilon^2}{2} a_{ij}^k u_{x_i x_j}^\varepsilon + \lambda u^\varepsilon \right\} = 0 & \text{in } \Omega, \\ u^\varepsilon = 1 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where $K = \{1, \dots, m\}$ for some positive integer m. Let $L: K \times \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $\mathscr{K}$ be as in the second part of Section 2. Let $x \in \Omega$. We denote by Ξ_x the set of those functions $\xi \in H^1_{\text{loc}}([0, \infty); \mathbb{R}^n)$ which satisfy $\xi(0) = x$ and by $\mathscr{A}_x$ we denote the set of those mappings $\alpha: \mathscr{K} \to \Xi_x$ which satisfy $\alpha(k) = \alpha(\tilde{k})$ on $[0, t]$ whenever $t > 0$, $k, \tilde{k} \in \mathscr{K}$, and $k = \tilde{k}$ a.e. on $[0, t]$. We set

$$\mathscr{A}^1_x = \{\alpha \in \mathscr{A}_x: \alpha(k)(1) \in \partial\Omega \text{ and } \alpha(k)(t) \in \Omega \text{ for } k \in \mathscr{K} \text{ and } 0 \le t < 1\}$$

and define

$$I(x) = (2\lambda)^{1/2} \inf_{\alpha \in \mathscr{A}^1_x} \sup_{k \in \mathscr{K}} \int_0^1 L(k(t), \xi(t), \dot{\xi}(t))^{1/2}\, dt, \qquad \text{where} \quad \xi = \alpha(k). \tag{3.11}$$

It is not hard to see that

$$I(x) = \inf_{\alpha \in \mathscr{A}_x} \sup_{k \in \mathscr{K}} \int_0^\tau (\tfrac{1}{2} L(k(t), \xi(t), \dot{\xi}(t)) + \lambda)\, dt,$$

where $\xi = \alpha(k)$ and $\tau = \inf\{t > 0: \xi(t) \in \partial\Omega\}$.

Our generalization of Theorem 3.1 is:

Theorem 3.4. *Assume that the a^k_{ij} satisfy* (2.1) *and* (2.2). *Then there is a constant $C > 0$ such that*

$$|\varepsilon \log u^\varepsilon(x) + I(x)| \le C\varepsilon^{1/2} \qquad \textit{for} \quad x \in \Omega \quad \textit{and} \quad 0 < \varepsilon < 1. \tag{3.12}$$

The uniform convergence of $-\varepsilon \log u^\varepsilon(x)$ to $I(x)$ as $\varepsilon \downarrow 0$ is proved by Koike [18]. We refer to [18] for the motivation to this asymptotic problem and some properties of the function I. We leave the proof of the above theorem to the reader as it is similar to the proof of Theorem 3.1.

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