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Convex Programming, Variational Inequalities, and Applications to the Traffic Equilibrium Problem*

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Abstract. The problem of determining the equilibrium distribution of the traffic flow in a city network is studied when the traffic demands on a set of given routes are known. The problem is formulated in terms of a nonlinear variational inequality over a polyhedron and a solving procedure, different from those shown in [1], [3], [4], is exhibited. This procedure is based on a very simple, necessary, and sufficient condition for a solution of the variational inequality to lie on a face of the polyhedron. Moreover, it is also compared, by means of numerical examples, with the procedures formulated in [1], [3], and [4] (see expressions (1.2) and (3.5) for a significant valuation).

1. Introduction

It is well known that variational inequalities in R^m generalize convex programming; indeed, if $S(F)$ is a continuously differentiable real-convex function defined on a nonempty, closed, convex subset K of R^m, then the problem of finding $H \in K$ such that

$$S(H) = \min_{F \in K} S(F)$$

is equivalent to that of finding H in K such that (see [3])

$$\operatorname{Grad} S(H)(F-H) = \sum_{r=1}^{m} \frac{\partial S(H)}{\partial H_r}(F_r - H_r) \geq 0, \qquad \forall F \in K.$$

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By the way, if $S(F)$ is strictly convex, there cannot exist more than one point in K minimizing $S(F)$ because grad $S(F)$ is strictly monotone (see [8]), i.e.,

$$(\operatorname{grad} S(F_1)-\operatorname{grad} S(F_2))(F_1-F_2)>0, \qquad F_1, F_2\in K, \quad F_1\neq F_2.$$

Nevertheless, variational inequalities of the type

$$H\in K, \qquad C(H)(F-H)\geq 0, \qquad \forall F\in K,$$

where $C(F)$ is a function from K to R^m, can express equilibrium conditions without $C(F)$ being the gradient of a function $S(F)$; this is the case in the traffic equilibrium problem. For the reader's convenience we recall briefly the statement of this problem. Given a transportation network (N, L), where N is a set of n nodes $P_1, \ldots, P_n$ and L a subset of $N\times N$, let us denote by $\rho(P_i, P_j)$ a function from L to R^+, which we shall think of as the steady demand for traveling from P_i to P_j, and let us introduce the following notations:

(i) *$\mathcal{R}(P_i, P_j)$ is the set of those routes(paths) R_r from P_i to P_j which traverse no link twice*;

(ii) *m denotes the number of elements of $\mathcal{R}=\bigcup_{(P_i,P_j)\in L}\mathcal{R}(P_i, P_j)$*;[1]

(iii) *$F_r\in R^+$ is the flow along the route R_r and $F=(F_1, \ldots, F_m)$ the route flow distribution*;

(iv) *$C_r(F)\in R^+$ denotes the cost along the route R_r and $C(F)=(C_1(F), \ldots, C_m(F))$ is the cost distribution*;

(v) *K denotes the set*

$$K=\left\{F: \sum_{R_r\in\mathcal{R}(P_i,P_j)} F_r=\rho(P_i, P_j) \quad (P_i, P_j)\in L\right\}.$$

Then, using the formulation by [9], the problem of traffic equilibrium is expressed by the following variational inequality

$$\text{Find} \quad H\in K \quad \text{such that} \quad C(H)(F-H)\geq 0, \qquad \forall F\in K, \tag{1.1}$$

where

$$C(H)(F-H)=\sum_{r=1}^{m} C_r(H)(F_r-H_r).$$

If we suppose $C(F)$ continuous in K, the variational inequality (1.1) admits solutions because K is a nonempty, closed, convex, bounded subset of R_m and the results of [8] hold. Under the additional assumptions that $C(F)$ is strictly monotone the variational inequality admits a unique solution. Since problem (1.1) cannot be reduced to a convex minimization problem, without making further assumptions, we cannot use the well-known algorithms of the convex programming theory in order to compute the solution of (1.1).

Some algorithms for the construction of the solution have been established in [1], [3], and [4]; we present here a new method different from those ones.

[1] We suppose that for every pair (P_i, P_j) there exists at least one route that connects P_i to P_j. Then it results $m\geq|L|=l$.

In Sections 3, 4, and 5 we compare our method with those of [1], [3], and [4], respectively. To convey an idea about these comparisons let us consider the model of a circular highway studied in [1] and the normalized measure of convergence (73) on p. 154 of the above-mentioned paper:

$$\sum_{r=1}^{5} \frac{F_r}{r/10} \frac{\Gamma_r(\tilde{F})}{C_{5+r}(F)} \tag{1.2}$$

(we use our notation). Expression (1.2) is zero if the corresponding traffic flow distribution $\tilde{F}$ is optimal. The author of [1] says that his algorithm yields near optimal flow patterns after very few iterations; but, using our method, expression (1.2) is zero (see [3.5]) because the solution of the variational inequality is given by solving the system

$$\begin{cases} F_1 = 0, \\ F_2 = 0, \\ \Gamma_3 = 0, \\ \Gamma_4 = 0, \\ \Gamma_5 = 0. \end{cases} \tag{1.3}$$

We conclude by remarking the utility of conditions (2.11) and (2.14) which allow us to construct a system of type (1.3) whose eventual solutions are the solutions of the variational inequality, without assumptions of continuity and strict monotony on the operator; we also want to explain that our method has been presented in the preliminary paper [5] and in the partial preprints [6] and [7].

2. The Computational Method

Let us start by transforming variational inequality (1.1) into an equivalent one. First, to simplify writing, let us denote by $\rho_1, \ldots, \rho_l$ and $\mathcal{R}_1, \ldots, \mathcal{R}_l$ the values $\rho(P_i, P_j)$ and the sets $\mathcal{R}(P_i, P_j)$, respectively; let us also set:

$$\varphi_{ir} = \begin{cases} 1 & \text{if } R_r \in \mathcal{R}_i, \\ 0 & \text{if } R_r \notin \mathcal{R}_i, \end{cases} \qquad i = 1, \ldots, l, \quad r = 1, \ldots, m.$$

Then K is determined by the conditions

$$\sum_{r=1}^{m} \varphi_{ir} F_r = \rho_i, \qquad i = 1, \ldots, l, \quad F_r \geq 0, \quad r = 1, \ldots, m, \tag{2.1}$$

and the structure of the system enables us to derive the values of l variables because the matrix (φ_{ir}) is such that in each column there is a unique entry which is 1, whereas all other ones are 0; so, if we suppose that we can derive the first l variables, we have[2]

$$F_i = \rho_i - \sum_{r=l+1}^{m} \varphi_{ir} F_r, \qquad i = 1, \ldots, l, \quad F_r \geq 0, \quad r = 1, \ldots, m. \tag{2.2}$$

By using (2.2) we can transform the variational inequality (1.1) into the following:

$$\text{Find} \quad \tilde{H} \in \tilde{K} \quad \text{such that} \quad \Gamma(\tilde{H})(\tilde{F} - \tilde{H}) \geq 0, \forall \tilde{F} \in \tilde{K}, \tag{2.3}$$

where

(i) $\tilde{K} = \left\{ (F_{l+1}, \ldots, F_m) \mid F_r \geq 0, r = l+1, \ldots, m, \sum_{r=l+1}^{m} \varphi_{ir} F_r \leq \rho_i, i = 1, \ldots, l \right\};$

(ii) $\tilde{F} = (F_{l+1}, \ldots, F_m)$;

(iii) $\Gamma(\tilde{F}) = (\Gamma_{l+1}(\tilde{F}), \ldots, \Gamma_m(\tilde{F}))$

$$\Gamma_r(\tilde{F}) = \tilde{C}_r(\tilde{F}) - \sum_{i=1}^{l} \varphi_{ir} \tilde{C}_i(\tilde{F})$$

with

$$\tilde{C}_r(\tilde{F}) = C_r\left(\rho_1 - \sum_{r=l+1}^{m} \varphi_{1r} F_r, \ldots, \rho_l - \sum_{r=l+1}^{m} \varphi_{lr} F_r, F_{l+1}, \ldots, F_m\right).$$

In fact, it results that

$$\begin{aligned}
\sum_{r=1}^{m} C_r(H)(F_r - H_r) &= \sum_{r=1}^{l} C_r(H)(F_r - H_r) + \sum_{r=l+1}^{m} C_r(H)(F_r - H_r) \\
&= \sum_{r=1}^{l} \tilde{C}_r(\tilde{H})\left(\rho_r - \sum_{s=l+1}^{m} \varphi_{rs} F_s - \rho_r + \sum_{s=l+1}^{m} \varphi_{rs} H_s\right) \\
&\quad + \sum_{r=l+1} \tilde{C}_r(\tilde{H})(F_r - H_r) \\
&= - \sum_{s=l+1}^{m} \sum_{r=1}^{l} \varphi_{rs} \tilde{C}_r(\tilde{H})(F_s - H_s) + \sum_{s=l+1}^{m} \tilde{C}_s(\tilde{H})(F_s - F_s) \\
&= \sum_{s=l+1}^{m} \left[\tilde{C}_s(\tilde{H}) - \sum_{r=1}^{l} \varphi_{rs} \tilde{C}_r(\tilde{H})\right](F_s - H_s) \\
&= \sum_{r=l+1}^{m} \Gamma_r(\tilde{H})(F_r = H_r).
\end{aligned} \tag{2.4}$$

If the operator $C(F)$ is continuous and strictly montone in K, then, taking into account (2.4), the same thing can easily be established for $\Gamma(\tilde{F})$; however, these assumptions are not necessary to reach Theorems 2.1 and 2.2 that we are going to prove.

[2] We observe that for fixed r there exists a unique i such that $\varphi_{ir} = 1$. The method is also available, with a slight modification, if we replace 1 with a positive number a_r when $R_r \in \mathscr{R}_i$.

Let us start with the remark that every $\tilde{H}_0$ belonging to $\tilde{K}$ and such that

$$\Gamma(\tilde{H}_0) = 0 \tag{2.5}$$

is a solution of variational inequality (2.3), whereas any other solution $\tilde{H}$ of (2.3) must belong to the boundary $\partial\tilde{K}$ of $\tilde{K}$; instead, if $\tilde{H}$ were an interior point (we observe that the interior of $\tilde{K}$ is not empty), we should have

$$\Gamma(\tilde{H}) = 0.$$

Let us search for the eventual solutions that lie on the boundary of the $(m-l)$-dimensional polyhedron $\tilde{K}$. This boundary consists of faces and we can describe a face of dimension $m-l-(h+k)$ in the following way: let us set

$$(S^h, J^k) = ((s_1, \dots, s_h), (j_1, \dots, j_k)), \qquad l \le s_q \le m, \quad 1 \le j_i \le l,$$

$$I = \{l+1, \dots, m\} - \{s_1, \dots, s_k\}, \qquad E = \{1, \dots, l\} - \{j_1, \dots, j_k\},$$

$$\tilde{K}^{h,k} = \Big\{ (F_{l+1}, \dots, F_m) \,|\, F_{sq} = 0,\ s_q \in S^h,$$

$$\sum_{r\in I} \varphi_{j_i r} F_r = \rho_{j_i}, j_i \in J^k,\ F_r \ge 0,\ r \in I,\ \sum_{r \in I} \varphi_{ir} F_r \le \rho_i,\ i \in E \Big\},$$

and let us consider the variational inequality on the face $\tilde{K}^{(h,k)}$.

Find $\tilde{H}^{(h,k)} \in \tilde{K}^{(h,k)}$ such that

$$\Gamma(\tilde{H}^{(h,k)})(\tilde{F}^{(h,k)} - \tilde{H}^{(h,k)}) \ge 0 \tag{2.6}$$

for every $\tilde{F}^{(h,k)} \in \tilde{K}^{(h,k)}$. We can rewrite this in a more convenient equivalent form as follows : let us choose the indexes $l_1, \dots, l_k \in I$ such that

$$F_{l_i} = \rho_{j_i} - \sum_{\substack{r\in I \\ r \ne l_i}} \varphi_{j_i r} F_r, \tag{2.7}$$

and let us set

$$L = I - \{l_1, \dots, l_k\},$$

and

$$\tilde{K}_{m-l-(h+k)} = \Big\{ \tilde{F}^{(h,k)} \in R^{m-l-(h+k)} \,|\, F_r \ge 0,\ r \in L,$$

$$\sum_{r \in L} \varphi_{j_i r} F_r \le \rho_{j_i}, j_i \in J^k,\ \sum_{r\in L} \varphi_{ir} F_r \le \rho_i,\ i \in E \Big\};$$

then (2.6) is equivalent to the following:

Find $\tilde{H}^{(h,k)} \in \tilde{K}_{m-l-(n+k)}$ such that

$$\Gamma^{(h,k)}(\tilde{H}^{(h,k)})(\tilde{F}^{(h,k)} - \tilde{H}^{(h,k)}) \ge 0, \qquad \forall \tilde{F}^{(h,k)} \in \tilde{K}_{m-l-(h+k)}, \tag{2.8}$$

where $\Gamma^{(h,k)}$ is the vector of $R^{m-l-(h+k)}$ whose comonents $\Gamma_r^{(h,k)}$, $r \in L$, are given by

$$\Gamma_r^{(h,k)} = \begin{cases} \Gamma_r - \Gamma_{l_i} & \text{if there exists some } i \text{ for which } \quad \varphi_{j_i r} = 1, \\ \Gamma_r & \text{if } \varphi_{j_i r} = 0, \quad i = 1, \dots, k. \end{cases} \tag{2.9}$$

Now, if there exists $\tilde{H}_0^{(h,k)} \in \tilde{K}_{m-l-(h+k)}$ such that

$$\Gamma^{(h,k)}(\tilde{H}_0^{(h,k)}) = 0, \tag{2.10}$$

then $\tilde{H}_0^{(h,k)}$ is solution of variational inequality (2.8) and we can prove the following:

Theorem 2.1. *Let us suppose that*

$$\varphi_{j_i s_q} = 0, \qquad i = 1, \dots, k, \quad q = 1, \dots, h;$$

then $\tilde{H}_0^{(h,k)}$ *is a solution of variational inequality* (2.3) *if and only if*

$$\begin{aligned} &\Gamma_r(\tilde{H}_0^{(h,k)}) \geq 0, \qquad r \in S^h, \\ &\Gamma_{l_i}(\tilde{H}_0^{(h,k)}) \leq 0, \qquad i = 1, \dots, k. \end{aligned} \tag{2.11}$$

Proof. Taking into account (2.9), for every $\tilde{F} \in \tilde{K}$ we have

$$\begin{aligned} \Gamma(\tilde{H}_0^{(h,k)})(\tilde{F} - \tilde{H}^{(h,k)}) = &\sum_{r \in S^h} \Gamma_r(\tilde{H}_0^{(h,k)})(F_r - H_r) \\ &+ \sum_{i=1}^{k} \Gamma_{l_i}(\tilde{H}_0^{(h,k)})(F_{l_i} - H_{l_i}) \\ &+ \sum_{\substack{r \in L \\ \varphi_{j_1 r} = 1}} \Gamma_r(\tilde{H}_0^{(h,k)}(F_r - H_r) + \cdots \\ &+ \sum_{\substack{r \in L \\ \varphi_{j_k r} = 1}} \Gamma_r(\tilde{H}_0^{(h,k)})(F_r - H_r) \\ &+ \sum_{\substack{r \in L \\ \varphi_{j_i r} = 0 \\ i = 1, \dots, k}} \Gamma_r(\tilde{H}_0^{(h,k)})(F_r - H_r) \\ = &\sum_{r \in S^h} \Gamma_r(\tilde{H}_0^{(h,k)}) F_r + \Gamma_{l_1}(\tilde{H}_0^{(h,k)}) \\ &\times \left(F_{l_1} + \sum_{r \in L} \varphi_{j_1 r} F_r - H_{l_1} - \sum_{r \in L} \varphi_{j_1 r} H_r \right) + \cdots \\ &+ \Gamma_{l_k}(\tilde{H}_0^{(h,k)}) \left(F_{l_k} + \sum_{r \in L} \varphi_{j_k r} F_r - H_{l_k} - \sum_{r \in L} \varphi_{j_k r} H_r \right) \\ = &\sum_{r \in S^h} \Gamma_r(\tilde{H}_0^{(h,k)}) F_r + \Gamma_{l_1}(\tilde{H}_0^{(h,k)}) \left(\sum_{r \in L \cup \{l_i\}} \varphi_{j_1 r} F_r - \rho_{j_1} \right) \\ &+ \cdots + \Gamma_{l_k}(\tilde{H}_0^{(h,k)}) \left(\sum_{r \in L \cup \{l_k\}} \varphi_{j_k r} F_r - -\rho_{j_k} \right).^3 \end{aligned} \tag{2.12}$$

Since $F \in \tilde{K}$, it results that $F_r \geq 0$ and $\sum_{r \in L \cup \{l_i\}} \varphi_{j_i r} F_r \leq \rho_{j_i}$ then from (2.12) it follows that (2.11) are sufficient. On the other hand, letting in turn all but one constraint with indexes $r \in S^h$ and $j_i \in J^k$ be satisfied as an equality we see that (2.11) are necessary. For instance, if we set $F_r = 0$, $r = s_2, \dots, s_k$, $\sum_{r \in L \cup \{l_i\}} \varphi_{j_i r} F_r = \rho_{j_i}$, $j_i \in J^k$, then we obtain $\Gamma_{s_1}(\tilde{H}^{(h,k)}) \geq 0$, and so forth. □

[3] Note that

$$H_{l_i} + \sum_{r \in L} \varphi_{j_i r} H_r = \rho_{j_i}.$$

Now suppose that there exist a subset J_p of J^k consisting of p elements ($p \leq k$) and p nonempty subsets S_{j_i}, $j_i \in J_p$, of S^h such that

$$\varphi_{j_i s_q} = 1, \qquad S_q \in S_{j_i}, \quad j_i \in J_p; \tag{2.13}$$

then we can prove the following:

Theorem 2.2. *If conditions* (2.13) *are available, then* $\tilde{H}_0^{(h,k)}$ *is a solution of the variational inequality* (2.3) *if and only if*

$$\begin{aligned}
&\Gamma(\tilde{H}_0^{(h,k)}) \geq 0, \qquad r \in S^h - \bigcup_{j_i \in J_p} S_{j_i},\\
&\Gamma_{l_i}(\tilde{H}_0^{(h,k)}) \leq 0, \qquad i = 1, \dots, k, \\
&\Gamma_{s_q}(\tilde{H}_0^{(h,k)}) - \Gamma_{l_i}(\tilde{H}_0^{(h,k)}) \geq 0, \qquad s_q \in S_{j_i}, \quad j_i \in J_p.
\end{aligned} \tag{2.14}$$

Proof. Let us observe that, when we set in the last side of (2.12)

$$\sum_{r \in L \cup \{l_i\}} \varphi_{j_i r} F_r = \rho_{j_i}, \qquad j_i \in J_p,$$

it results

$$F_{s_q} = 0, \qquad s_1 \in S_{j_i},$$

because we have the constraints

$$\sum_{r \in L \cup \{l_i\}} \varphi_{j_i r} F_r + \sum_{s_q \in S_j^i} \varphi_{j_i s_q} F_{s_q} \leq \rho_{j_i};$$

consequently we cannot infer the conditions for $\Gamma_{s_q}(\tilde{H}_0^{(h,k)})$, $s_q \in S_{j_i}$ and we must follow a different way when $s_q \in S_{j_i}$, $j_i \in J_p$.

Let us set

$$\sum_{r \in L \cup \{l_i\}} \varphi_{j_i r} F_r - \rho_{j_i} = 0, \qquad j_i \in J^k - J_p,$$

$$F_r = 0, \qquad r \in S^h - \bigcup_{J_i \in J_p} S_{j_i},$$

and

$$\sum_{r \in L \cup \{l_i\}} \varphi_{j_i r} F_r - \rho_{j_i} = 0$$

for every value $j_i \in J_p$ except one for which we set

$$\sum_{r \in L \cup \{l_1\}} \varphi_{j_i r} F_r - \rho_{j_i} = -\sum_{s_q \in S_{j_i}} \varphi_{j_i s_q} F_r = -\sum_{s_q \in S_{j_i}} F_{sq}. \tag{2.15}$$

Then expression (2.12) becomes

$$\begin{aligned}
\Gamma(\tilde{H}_0^{(h,k)})(\tilde{F} - \tilde{H}^{(h,k)}) &= \Gamma_{l_i}(\tilde{H}_0^{(h,k)}) \left(\sum_{r \in L \cup \{l_i\}} \varphi_{j_i r} F_r - \rho_{j_i} \right) + \sum_{s_q \in S_{j_i}} \Gamma_{sq}(\tilde{H}_0^{(h,k)}) F_{sq} \\
&= \sum_{s_q \in S_{j_i}} (\Gamma_{sq}(\tilde{H}_0^{(h,k)}) - \Gamma_{l_i}(\tilde{H}_0^{(h,k)})) F_{sq}.
\end{aligned} \tag{2.16}$$

Setting in (2.16), $F_{sq}=0$ for all but one index in turn, we have that conditions (2.14) are necessary. The conditions are also sufficient because for $j_i \in J_p$ and $s_q \in S_{j_i}$ it results that

$$\Gamma_{l_i}(\tilde{H}_0^{(h,k)})\Bigg(\sum_{r\in L\cup\{l_i\}} \varphi_{j_ir}F_r-\rho_{j_i}\Bigg)+\sum_{s_q\in S_{j_i}} \Gamma_{sq}(\tilde{H}_0^{(h,k)})F_{sq}$$

$$\geq \Gamma_{l_i}(\tilde{H}_0^{(h,k)})\Bigg(\sum_{r\in L\cup\{l_i\}} \varphi_{i_ir}F_r+\sum_{s_q\in S_{j_i}} \varphi_{j_is_q}F_{s_q}-\rho_{j_i}\Bigg)\geq 0.$$

The theorem is therefore proved. □

Now if (2.12) or (2.14) are not satisfied for all solutions $\tilde{H}_0^{(h,k)}$ of system (2.10), variational inequality (2.3) cannot have solutions belonging to the interior of $\tilde{K}_{m-l-(h+k)}$; whereas, if equality (2.10) does not admit any solution in $\tilde{K}_{m-l-(h+k)}$, the eventual solutions of variational inequality (2.8) must belong to the boundary of $\tilde{K}_{m-l-(h,k)}$; namely, to a face of dimension $m-l-(h+k+1)$, for which we can repeat the same considerations; consequently, if (2.10) or (2.12) and (2.14) are not satified for all faces $\tilde{K}_{m-l-(h+k)}$ with $h+k<m-l$, we can say that the eventual solutions of the variational inequality lie on face of dimension zero, that is, they are vertexes of $\tilde{K}$; in this way, we can find the eventual solutions of variational inequality (2.3) that do not verify (2.5).

If we suppose that the operator $\Gamma(F)$ is continuous and strictly nomonotone, then variational inequality (2.3) admits a unique solution that we can determine using the procedure described above; moreover, if the solution $\tilde{H}$ of (2.3) is a vertex, the following theorem holds:

Theorem 2.3. *If $\tilde{H}_i$, $i\in P=\{1,\ldots,p\}$ are the vertexes of $\tilde{K}$, the equation*

$$\Gamma(\tilde{H}_i)\tilde{H}_i=\min_{j\in P}\{\Gamma(\tilde{H}_i)\tilde{H}_j\},\qquad i\in P \tag{2.17}$$

admits a unique solution that coincides with $\tilde{H}$.

Proof. Equation (2.17) admits at least a solution; in fact, $\tilde{H}$, which is a vertex, is such that

$$\Gamma(\tilde{H})\tilde{H}_j\geq\Gamma(\tilde{H})\tilde{H},\qquad j\in P$$

and, hence, $\tilde{H}$ satisfies (2.17); moreover, (2.17) has a unique solution, because if we had two solutions h_m and H_n, it would follow

$$(\Gamma(\tilde{H}_m)-\Gamma(\tilde{H}_n))(\tilde{H}_m-\tilde{H}_n)\geq 0$$

and, owing to the strictly monotony, $H_n=H_m$. □

3. Comparison I

We compare our procedure with that of [1] by computing the solution of the traffic equilibrium problem in the case of the example considered in [1]; the author applies his method to this example and found out that a suitable normalized measure of convergence (see p. 154) gives satisfactory results even after very few iterations.

The example is that of a network constituted by five nodes: P_1, P_2, P_3, P_4, P_5, and five pairs: (P_1, P_4), (P_2, P_5), (P_3, P_1), (P_4, P_2), (P_5, P_3), each connected by two routes; R_1 and R_6 denote the two routes that connect P_1 with P_4 and R_1 is the longer one, analogously, R_2 and R_7, R_3 and R_8, R_4 and R_{10} connect the pairs (P_2, P_5), (P_3, P_1), (P_4, P_2), (P_5, P_3), respectively, and R_2, R_3, R_4, R_5 are the longer routes. F_i, $i=1, \dots, 10$ denote the flows along the routes R_i, $i=1, \dots, 10$, respectively, and the cost functions, that represent the travel time for routes, are (we consider the case when the parameter γ is zero):

$$
\begin{aligned}
C_1(F) = {} & 34F_1^2+42F_1F_2+20F_1F_3+20F_1F_4+42F_1F_5+34F_1+21F_2^2 \\
& +20F_2F_3+20F_2F_5+21F_2+10F_3^2+10F_3+10F_4^2+20F_4F_5 \\
& +10F_4+21F_5^2+21F_5+34, \\
C_2(F) = {} & 21F_1^2+42F_1F_2+20F_1F_3+20F_1F_5+21F_1+34F_2^2+42F_2F_3 \\
& +20F_2F_4+20F_2F_5+34F_2+21F_3^2+20F_3F_4 \\
& +21F_4+21F_3+10F_4^2+10F_4+10F_5^2+10F_5+34, \\
C_3(F) = {} & 10F_1^2+20F_1F_2+20F_1F_3+10F_1+21F_2^2+42F_2F_3 \\
& +20F_2F_4+21F_2+34F_3^2+42F_3F_4+20F_3F_5+34F_3+21F_4^2 \\
& +20F_4F_5+21F_4+10F_5^2+10F_5+34, \\
C_4(F) = {} & 10F_1^2+20F_1F_4+20F_1F_5+10F_1+10F_2^2+20F_2F_3+20F_2F_4 \\
& +10F_2+21F_3^2+20F_3F_5+21F_3+42F_3F_4+34F_4^2+42F_4F_5 \\
& +34F_4+21F_5^2+21F_5+34, \\
C_5(F) = {} & 21F_1^2+20F_1F_2+20F_1F_4+42F_1F_5+21F_1+10F_2^2+20F_2F_5 \\
& +10F_2+10F_3^2+20F_3F_4+20F_3F_5+10F_3+21F_4^2 \\
& +42F_4F_5+21F_4+34F_5^2+34F_5+34, \\
C_6(F) = {} & 23F_6^2+10F_7^2+10F_{10}^2 \\
& +20F_6F_7+20F_6F_{10}+23F_6+10F_7+10F_{10}+23, \\
C_7(F) = {} & 23F_7^2+10F_8^2+10F_6^2 \\
& +20F_7F_8+20F_6F_7+23F_7+10F_8+10F_6+23, \\
C_8(F) = {} & 23F_8^2+10F_9^2+10F_7^2+20F_8F_9 \\
& +20F_7F_8+23F_8+10F_9+10F_7+23, \\
C_9(F) = {} & 23F_9^2+10F_{10}^2+10F_8^2+20F_9F_{10} \\
& +20F_8F_9+23F_9+10F_{10}+10F_8+23, \\
C_{10}(F) = {} & 23F_{10}^2+10F_6^2+10F_9^2+20F_6F_{10} \\
& +20F_9F_{10}+23F_{10}+10F_6+10F_9+23.
\end{aligned}
$$

The convex set K over which we must consider the variational inequality

$$\text{“Find} \quad H \in K \quad \text{such that} \quad C(H)(F-H) \geq 0, \qquad \forall F \in K\text{”} \tag{3.1}$$

is given by

$$K=\{F=(F_1,\ldots,F_{10})\,|\,F_r\geq 0,\ r=1,\ldots,10,\ F_i+F_{i+5}=i/10,\ i=1,\ldots,5\}.$$

The starting point of our procedure is to derive the values of variables F_{i+5}, $i=1,\ldots,5$ by means of the relations

$$F_{i+5}=\frac{i}{10}-F_i,\qquad i=1,\ldots,5$$

and to the transform variational inequality (3.1) in the following one

$$\text{“Find}\quad \tilde{H}\in\tilde{K}\quad \text{such that}\quad \Gamma(\tilde{H})(\tilde{F}-\tilde{H})\geq 0,\qquad \forall\tilde{F}\in\tilde{K}\text{”},\tag{3.2}$$

where

$$\tilde{K}=\{\tilde{F}\equiv(F_1,\ldots,F_5)\,|\,0\leq F_i\leq i/10,\ i=1,\ldots,5\}$$

and $\Gamma(\tilde{F})$ is the vector with components

$$\Gamma(\tilde{F})=C_i(\tilde{F})-C_{i+5}(\tilde{F}),\qquad i=1,\ldots,5.$$

We remark that $\Gamma_i(\tilde{F})$ represent the difference between the travel times of the longer paths and those of the shorter ones. We also emphasize the fact that the steps we are going to make, in order ot find the solution of the variational inequality, are the same ones we must take to implement the method by computer.[4] For the reader's convenience we give the expression of $\Gamma_i(\tilde{F})$.

We have

$$\begin{aligned}\Gamma_1(\tilde{F})={}&11F_1^2+22F_1F_2+20F_1F_3+20F_1F_4+22F_1F_5+75.6F_1+11F_2^2\\&+20F_2F_3+20F_2F_5+37F_2+10F_3^2+10F_3\\&+10F_4^2+20F_4F_5+10F_4+11F_5^2+43F_5-2.83,\\ \Gamma_2(\tilde{F})={}&11F_1^2+22F_1F_2+20F_1F_3+20F_1F_5+37F_1+11F_2^2\\&+22F_2F_3+20F_2F_4+20F_2F_5+74.2F_2+11F_3^2+20F_3F_4\\&+41F_3+10F_4^2+10F_4+10F_5^2+10F_5-1.12,\\ \Gamma_3(\tilde{F})={}&10F_1^2+20F_1F_2+20F_1F_3+10F_1+11F_2^2+22F_2F_3+20F_1F_4\\&+41F_2+11F_3^2+22F_3F_4+20F_3F_5+82.8F_3+11F_4^2\\&+20F_4F_5+45F_4+10F_5^2+10F_5-9.57,\\ \Gamma_4(\tilde{F})={}&10F_1^2+20F_1F_4+20F_1F_5+10F_1+10F_2^2+20F_2F_3+20F_2F_4\\&+10F_2+11F_3^2+22F_3F_4+20F_3F_5+45F_3+11F_4^2+22F_4F_5\\&+91.4F_4+11F_5^2+49F_5-19.68,\\ \Gamma_5(\tilde{F})={}&11F_1^2+20F_1F_2+20F_1F_4+22F_1F_5+43F_1+10F_2^2+20F_2F_5\\&+10F_2+10F_3^2+20F_3F_4+20F_3F_5+10F_3+11F_4^2+22F_4F_5\\&+49F_4+11F_5^2+90F_5-17.95.\end{aligned}$$

[4] A program in Fortran to compute the solution of variational inequality (3.1), even when γ is different from zero, has been given by F. Turiano in her graduation thesis "Programmi di calcolo per un modello non lineare di traffico su rete". Catania, a.a. 1983-84. The results reported here are taken from this dissertation. For the same topic we also recall the thesis of A. Carolla "Un algoritmo che applica le disequazioni variazionali a problemi di equilibrio di traffico su rete", Pisa, a.a. 1982-83.

The first step consists in seeing whether the system

$$\Gamma(\tilde{H}_0) = 0 \tag{3.3}$$

admits solutions $\tilde{H}_0$ in $\tilde{K}$. Since system (3.3) has a solution with two negative components, we must pass to the second step.

In this step we have to consider four-dimensional faces $\tilde{K}^{(r,0)}$, $r = 1, \ldots, 5$ of $\tilde{K}$ and the restriction $\Gamma^{(r,0)}$ of Γ to these faces.[5] Since the system

$$\Gamma^{(r,0)}(\tilde{H}^{(r,0)}) = 0, \qquad r = 1, \ldots, 5$$

have solutions with some negative component, we must pass to the next step.

In this step we consider the three-dimensional faces of type $\tilde{K}^{(s_1,s_2,0)}$, $s_1, s_2 = 1, \ldots, 5$, $s_1 \neq s_2$, and observe that, on face $\tilde{K}^{(1,2,0)}$, the system

$$\Gamma^{(1,2,0)}(\tilde{H}^{(1,2,0)}) = 0,$$

i.e.,

$$\begin{cases} \Gamma_3(\tilde{H}^{(1,2,0)}) = 0, \\ \Gamma_4(\tilde{H}^{(1,2,0)}) = 0, \\ \Gamma_5(\tilde{H}^{(1,2,0)}) = 0, \end{cases} \tag{3.4}$$

has the solution[6]

$$\tilde{H}^{(1,2,0)}(= 0.018807, 0.135947, 0.114582)$$

for which it results

$$\Gamma_1(\tilde{H}^{(1,2,0)}) \geq 0, \qquad \Gamma_2(\tilde{H}^{(1,2,0)}) \geq 0.$$

Then $\tilde{H} = (0, 0, 0.018807, 0.135947, 0.114582)$ is the unique solution ($C(F)$ is strictly monotone) of variational inequality (3.2) and

$$H = (0,\ 0,\ 0.018807,\ 0.135947,\ 0.114582,\ 0.1,\ 0.2,\ 0.281193,$$
$$0.264053,\ 0.385418)$$

of variational inequality (3.1).

It is remarkable that it results in

$$\sum_{r=1}^{n} \frac{F_r}{r/10} \frac{\Gamma_r(\tilde{H})}{C_{5+r}(\tilde{H})} = 0, \tag{3.5}$$

and, therefore, our method gives, apart from the calculation of the solutions of the systems, an exact solution.

---

[5] $\tilde{K}^{((s_1,\ldots,s_h),(j_1,\ldots,j_k))}$ denotes the face that we obtain by setting $F_{s_1} = \cdots = F_{s_h} = 0$ and $\sum_{r=l+1}^{m} \varphi_{j_i} F_r = \rho_{j_{ir}}$, $i = 1, \ldots, k$; if $h = 0$ or $k = 0$ we write

$$\tilde{K}^{(0,j_1,\ldots,j_k)} \quad \text{and} \quad \tilde{K}^{(s_1,\ldots,s_h,0)},$$

respectively.

The meaning of $\Gamma^{((s_1,\ldots,s_h),(j_1,\ldots,j_k))}$, $\Gamma^{(0,j_1,\ldots,j_k)}$ and $\Gamma^{(s_1,\ldots,s_k,0)}$ and of $\tilde{H}^{((s_1,\ldots,s_h),(j_1,\ldots,j_k))}$, $\tilde{H}^{(0,j_1,\ldots,j_k)}$ and $\tilde{H}^{(s_1,\ldots,s_h,0)}$ is obvious.

[6] In the thesis of F. Turiano this solution is evaluated by means of the Newton–Raphson method with an approximation of 10^{-15}.

4. Comparison II

The algorithm given by [3] is close to that of [1] but less efficient (see [1, p. 141]). We solve by our method the example presented in [3] and note that we obtain the exact solution at the second step, whereas in [3] ten interactions were needed.

The network is consitituted by two modes P_1 and P_2 and three routes R_1, R_3, R_4 from P_1 to P_2 and two routes R_2, R_5 from P_2 to P_1. The travel demands are

$$\rho(P_1, P_2) = 210, \qquad \rho(P_2, P_1) = 120$$

and the cost distribution is given by

$$C(F) = ((C_1(F), C_2(F), C_3(F), C_4(F), C_5(F)),$$

where

$$C_1(F) = 10F_1 + 5F_2 + 1000,$$

$$C_2(F) = 20F_2 + 2F_1 + 1000,$$

$$C_3(F) = 15F_3 + 3F_5 + 950,$$

$$C_4(F) = 20F_4 + 3000,$$

$$C_5(F) = 25F_5 + F_3 + 1300.$$

The set K is given by

$$\{F = (F_1, F_2, F_3, F_4, F_5), F_r \geq 0, r = 1, \ldots, 5, F_1 + F_3 + F_3 = 210,$$
$$F_2 + F5 = 120\}.$$

Then we derive the values of two variables F_1 and F_2 by means of the relations

$$F_1 = 210 - F_3 - F_4, \qquad F_2 = 120 - F_5$$

and consider the variational inequality

$$\text{“Find} \quad \tilde{H} \in \tilde{K} \quad \text{such that} \quad \Gamma(\tilde{H})(\tilde{F} - \tilde{H}) \geq 0, \qquad \forall \tilde{F} \in \tilde{K}\text{”}, \tag{4.1}$$

where

$$\tilde{K} = \{\tilde{F} \equiv (F_3, F_4, F_5) \mid F_r \geq 0, r = 3, 4, 5, F_3 + F_4 \leq 210, F_5 \leq 120\},$$

and

$$\Gamma(\tilde{F}) = (\Gamma_3(\tilde{F}), \Gamma_4(\tilde{F}), \Gamma_5(\tilde{F})),$$

with

$$\Gamma_3(\tilde{F}) = 25F_3 + 10F_4 + 10F_5 - 2750,$$

$$\Gamma_4(\tilde{F}) = 10F_3 + 30F_4 + 5F_5 - 700,$$

$$\Gamma_5(\tilde{F}) = 3F_3 + 2F_4 + 45F_5 - 2520.$$

In the first step we look at whether the system

$$\Gamma(\tilde{H}_0) = 0 \tag{4.2}$$

has a solution in $\tilde{K}$. Because system (4.2) has a solution with one negative component, we pass to the second step. In this step, we consider the two-dimensional faces and note that the system

$$\Gamma^{(3,0)}(\tilde{H}^{(3,0)})=0$$

has a solution $\tilde{H}^{(3,0)}=(14.1, 54.4)$ which belongs to $\tilde{K}^{(3,0)}$. Since it results

$$\Gamma_3(0, 14.1, 54.4)<0,$$

it follows that $\tilde{H}^{(3,0)}$ is not a solution of variational inequality (4.1).

In face $\tilde{K}^{(4,0)}$ the system

$$\Gamma^{(4,0)}(\tilde{H}^{(4,0)})=0$$

i.e.,

$$\begin{cases}25H_3+10H_5=2750,\\ 3H_3+45H_5=2520,\end{cases}$$

has the solution $\tilde{H}^{(4,0)}=(95, 50)$ which belongs to $\tilde{K}^{(4,0)}$ and is such that

$$\Gamma_5(90, 0, 50)>0.$$

Then the point

$$\tilde{H}=(90, 0, 50)$$

is a solution of variational inequaltiy (4.1). The solution is unique because $C(F)$, as an easy calculation shows, is strictly monotone.

5. Comparison III

The method of [4] is not based on a projection technique; essentially, it consists in the search of those constraints which the solution is subject to, if the system

$$\Gamma(\tilde{H})=0 \tag{5.1}$$

has no solution in $\tilde{K}$.

In this sense the method of [4] is a little closer to ours than the preceding ones; however, the starting point and the procedure of [4] are quite different from our ones.

In fact, the authors of [4] assume that system (5.1) has a solution $\tilde{H}_0$ and, if $\tilde{H}_0$ does not belong to $\tilde{K}$, they consider the constraints which $\tilde{H}_0$ does not fulfil and, taking into account a previous result, search the solution subject to one of these constraints. This search is based on an effective calculation of the solutions of variational inequality over the linear manifold that are not subsets of $\tilde{K}$ and on the inspection as to whether one of these solutions belongs to $\tilde{K}$; on the contrary, our method is based on the inspection as to whether the zeros of the restrictions of the operator over faces of $\tilde{K}$ are also solutions over all the convex $\tilde{K}$; this inspection is very simple because we have at our disposal conditions (2.11) and (2.14); however, we must say that the convexes considered by [4] are more general than those considered by us. Also, in this case we solve by our method the example proposed by [4, p. 21] and we remark again on the fact that the steps that we are running long are the same ones that we must consider if we want to use the computer; but we can easily solve this example by very simple manual computations.

We have to solve the variational inequality in R^4

$$\text{“Find} \quad \tilde{x} \in K \quad \text{such that} \quad C(\tilde{x})(x-\tilde{x}) > 0, \qquad \forall x \in K\text{”}$$

where $C(x)$ has the four components:

$$C_1(x) = 20x_1 - \cos^2 x_1 + x_3 - \sin x_3 + 2x_4 - 7,$$

$$C_2(x) = 2 \tan hx_1 + 19x_2 + 2 \exp[-2x_3^2 + 0.5] + 0.5 \left[\arctan x_4 + \frac{x_4}{1+x_4^2}\right] + 5,$$

$$C_3(x) = \sin(x_1 + x_2) - x_2 + 20x_3 + \frac{x_4}{\sqrt{1+x_4^2}} - 12,$$

$$C_4(x) = \log(x_1 + 1.5 + \sqrt{x_1^2 + 3x_1 + 5}) + \sin x_2 + x_3 + 18x_4,$$

and

$$K = \{x \in R^4\colon x_i \geq 0,\ i = 1, \ldots, n,\ x_1 + 2x_3 \leq \tfrac{2}{5},\ x_2 + \tfrac{5}{4}x_4 \leq \tfrac{3}{4}\}.$$

We observe that the operator is strictly monotone and continuous, and that the system

$$C(x) = 0$$

does not admit solution in K, because $C_3(x)$ is negative in K; hence $\tilde{x} \in \partial K$.

Following our procedure, let us consider the faces $K^{(i,0)}$, $i = 1, 2, 3, 4$ and the restriction $C^{(i,0)}(x^{(i,0)})$; the system

$$C^{(i,0)}(x^{(i,0)}) = 0, \qquad i = 1, 2, 3, 4$$

does not admit solution $K^{(i,0)}$ because we find some negative component; hence $\tilde{x} \notin K^{(i,0)}$, $i = 1, 2, 3, 4$.

Let us consider the faces of type $K^{(0,j)}$, $j = 1, 2$, it is easy to check that the systems

$$C^{(0,j)}(x^{(0,j)}) = 0, \qquad j = 1, 2$$

do not admit solution $K^{(0,j)}$; hence $\tilde{x} \notin K^{(0,j)}$, $j = 1, 2$.

Analogously it is very easy to check that:

(i) $\tilde{x}$ does not belong to $K^{(h,k,0)}$, $h < k$, h, $k = 1, 2, 3, 4$;
(ii) $\tilde{x}$ does not belong to $K^{((i),(j))}$, $i = 1, 2, 3, 4$, $j = 1, 2$;
(iii) $\tilde{x}$ does not belong to $K^{((1,h),s)}$ or to $K^{((2,3),s)}$, $h = 2, 3, 4$, $s = 1, 2$.

Now let us consider the face

$$K^{((2,4),(1))} = \{x \mid x_1 \geq 0,\ x_2 = 0,\ x_3 \geq 0,\ x_4 = 0,\ x_1 = \tfrac{2}{5} - 2x_3\}$$

and the system in $K^{((2,4),(1))}$

$$C^{((2,4),(1))}(\tilde{x}^{((2,4),(1))}) = 0,$$

i.e.,

$$C_3 = 2C_1,$$

i.e.,

$$98x_3 + \sin\left(\tfrac{2}{5} - 2x_3\right) + 2\sin x_3 = 12. \tag{5.2}$$

Equation (5.2) has the (approximate) solution in $K^{((2,4),(1))}$

$$x_3 = 0.1183$$

and, since the point $\tilde{x}$, given by

$$\tilde{x} = (0.1634, 0, 0.1183, 0)$$

verifies the conditions

$$C_1(\tilde{x}) < 0, \qquad C_2(\tilde{x}) > 0, \qquad C_4(\tilde{x}) > 0,$$

$\tilde{x}$ is the unique solution (that we can calculate with the approximation that we wish) of the variational inequality. For the sake of brevity we omit to report the other example of [4, p. 19] and the other examples with 32 nodes and many routes (see [3, p. 12]).

6. The Computational Procedure

We consider the variational inequality (2.3) and we show how our algorithm can be implemented by a computer.[7] We can run through the following steps:

(1) Look whether the system

$$\Gamma(\tilde{H}) = 0 \tag{6.1}$$

has solutions in $\tilde{K}$; every solution of (6.1) is a solution of (2.3).

(2) (i) Consider the faces $\tilde{K}^{(r,0)}$, $r = l+1, \ldots, m$ and the restrictions $\Gamma^{(r,0)}$ of Γ over the faces (see (2.9)) and look whether the systems

$$\Gamma^{(r,0)}(\tilde{H}^{(r,0)}) = 0 \tag{6.2}$$

have solutions in $\tilde{K}^{(r,0)}$.

If (6.2) has solutions $\tilde{H}^{(r,0)}$ check whether it results in

$$\Gamma_r(\tilde{H}^{(r,0)}) \geq 0. \tag{6.3}$$

When inequality (6.3) is verified, $\tilde{H}^{(r,0)}$ is a solution of (2.3).

We also consider the faces $\tilde{K}^{(0,j)}$, $1 \leq j \leq l$, and the restrictions $\Gamma^{(0,j)}$ of Γ over $\tilde{K}^{(0,j)}$ (see (2.9)), and look whether the systems

$$\Gamma^{(0,j)}(\tilde{H}^{(0,j)}) = 0 \tag{6.4}$$

[7] We do not assume that Γ is strictly monotone; hence, we can have more than one solution. If the hypothesis of strict monotonicity holds we have at most one solution and, hence, in this case, if we find a solution the procedure ends.

have solutions in $\tilde{K}^{(0,j)}$ and check whether it results for the eventual solutions $\tilde{H}^{(0,j)}$

$$\Gamma_{l_j}(\tilde{H}^{(0,j)}) \leq 0 \tag{6.5}$$

(see (2.7) for the meaning of l_j). When inequality (6.5) is verified, $\tilde{H}^{(0,j)}$ is solution of (2.3).

(ii) In this step we consider faces of the type $\tilde{K}^{(s_1,s_3,0)}$, $\tilde{K}^{((r),(j))}$, $\tilde{K}^{(0,j_1,j_3)}$ (the values assumed by s_1, s_2, r, j_1, j_2 are evident) and the restrictions $\Gamma^{(s_1,s_2,0)}$, $\Gamma^{((r),(j))}$, $\Gamma^{(0,j_1,j_2)}$ and verify whether the systems

$$\Gamma^{(s_1,s_2,0)}(\tilde{H}^{(s_1,s_2,0)}) = 0, \qquad \Gamma^{((r),(j))}(\tilde{H}^{((r),(j))}) = 0, \qquad \Gamma(\tilde{H}^{(0,j_1,j_2)}) = 0$$

have solutions and whether it results for the eventual solutions

$$\Gamma_{s_1}(\tilde{H}^{(s_1,s_2,0)}) \geq 0, \qquad \Gamma_{s_2}(\tilde{H}^{(s_1,s_2,0)}) \geq 0 \tag{6.6}$$

for the first system,

$$\Gamma_r(\tilde{H}^{((r),(j))}) \geq 0, \qquad \Gamma_{l_j}(\tilde{H}^{((r),(j))}) \leq 0 \tag{6.7}$$

if $\varphi_{r,j} = 0$, or

$$\Gamma_{l_j}(\tilde{H}^{((r),(j))}) \leq 0, \qquad \Gamma_r(\tilde{H}^{((r),(j))}) - \Gamma_{l_j}(\tilde{H}^{((r),(j))}) \geq 0 \tag{6.8}$$

if $\varphi_{r,j} = 1$ for the second system,

$$\Gamma_{l_i}(\tilde{H}^{(0,j_1,j_2)}) \leq 0, \qquad \Gamma_{l_2}(\tilde{H}^{(0,j_1,j_2)}) \leq 0 \tag{6.9}$$

for the third system. In the affirmative case we have obtained solutions of (2.3).

The successive steps are evident and we so reach them $(m-l)$th step in which the faces are vertexes and the procedure has ended.

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