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Quasiconvex Quadratic Forms in Two Dimensions

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Abstract. Let f and g be two quadratic forms in $\mathbb{R}^n$. If $f(\xi)$ is positive where $g(\xi)=0$, $\xi\neq 0$, then we show that there exists a real λ such that $f-\lambda g$ is positive definite. As a consequence we obtain a new description of the old characterization by Terpstra [19] of quasiconvex quadratic forms in two dimensions.

Let u be a vector-valued function defined on an open set $\Omega\subset\mathbb{R}^2$ with values in $\mathbb{R}^2$. We denote by $u\equiv(u^1(x_1,x_2),\ u^2(x_1,x_2))$ the components of u, and by $\det Du$ the determinant of the 2×2 matrix of the gradient Du of u, i.e.:

$$\det Du \;=\; u^1_{x_1}u^2_{x_2}-u^1_{x_2}u^2_{x_1}. \tag{1}$$

If u has continuous second derivatives, then $\det Du$ is a divergence:

$$\det Du \;=\; \left(u^1u^2_{x_2}\right)_{x_1}-\left(u^1u^2_{x_1}\right)_{x_2}. \tag{2}$$

Thus, although $\det Du$ is not linear with respect to u, it has similar continuity properties as Du. In particular one can see that $\det Du$ is sequentially weakly continuous in $H^{1,2+\varepsilon}(\Omega,\mathbb{R}^2)$ for every $\varepsilon>0$ [2,15]. Thus the integral

$$\int_\Omega a(x)\det Du\,dx, \tag{3}$$

with $a\in L^\infty(\Omega)$ is an example of *nonlinear* weakly continuous functional in $H^{1,2+\varepsilon}(\Omega,\mathbb{R}^2)$. This contrasts sharply with the scalar case, i.e., the case where u is defined in $\Omega\subset\mathbb{R}^2$ (or $\mathbb{R}^n$) with values in $\mathbb{R}$ (instead of $\mathbb{R}^2$); in fact, for some integrals of the calculus of variations in the scalar case, convexity is a necessary condition for w-semicontinuity (see, i.e., [8] or [11]), and thus linearity is necessary for w-continuity.

In the calculus of variations for vector-valued functions, convexity must be replaced by a condition introduced by Morrey in 1952 [10]: the so-called quasiconvexity. A continuous real function $f(\xi)$ is *quasiconvex* if

$$\int_\Omega f(\xi + Du(x))\,dx \geq f(\xi)\,\mathrm{mis}\,\Omega, \tag{4}$$

for every $\xi \in \mathbb{R}^{nN}$ and every C^1-function $u: \mathbb{R}^n \to \mathbb{R}^N$ with support contained in Ω, i.e., $u \in C_0^1(\Omega, \mathbb{R}^N)$. We recall that quasiconvexity is necessary and sufficient for $w^* - H^{1,\infty}$ semicontinuity [10]; if $f(Du) = \det Du$ $(n = N)$, then equality holds in the above formula (4), while if f is convex then it is quasiconvex, by Jensen's inequality.

An important problem, not yet solved, is how to see if a given function is quasiconvex. A pointwise inequality, necessary for quasiconvexity, is the following *Legendre-Hadamard condition* [7] (we consider for simplicity $f \in C^2$; we denote $f = f(\xi)$, $\xi \equiv (\xi_\alpha^i)$, $\alpha = 1,\dots,n$; $i = 1,\dots,N$):

$$\sum_{i,j,\alpha,\beta} f_{\xi_\alpha^i \xi_\beta^j}(\xi)\lambda_\alpha\lambda_\beta\eta^i\eta^j \geq 0, \tag{5}$$

for every $\lambda \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^N$. If the strict inequality holds for every $\lambda, \eta \neq 0$, then (5) is also called strong ellipticity condition (see, i.e., Nirenberg [14]).

A pointwise condition, sufficient for quasiconvexity and interesting for applications to nonlinear elasticity, has been introduced by Ball in 1977 [2]. Ball says that a function f is *polyconvex* if there exists a convex function g such that

$$f(\xi) = g(\xi, D_1(\xi), D_2(\xi), \dots), \tag{6}$$

where each $D_h(\xi)$ is a subdeterminant of the matrix (ξ_α^i).

Before going on let us mention that, besides by Morrey [10] and Ball [2] (see also [3] and [11, Sect. 4.4]), this subject has been studied by many authors; for example by Murat [12,13] and Tartar [18] in the more general setting of compensated compactness (see also Bensoussan-Lions-Papanicolaou [4, Chap. 1, Sect. 11]); by Dacorogna [5,6], and also in [1,9] in the setting of calculus of variations.

We already said that polyconvexity implies quasiconvexity, and this implies the Legendre-Hadamard condition. It has been shown [19] that for $n \geq 3$ and $N \geq 3$ quasiconvexity does not imply polyconvexity; while it is not known if quasiconvexity is equivalent to the *L-H* condition (5).

If f is quadratic with respect to ξ, and n (or N) is equal to 2, then (4), (5), and (6) are equivalent to each other. This was discovered by Terpstra in 1938 [19], and proved again in 1981 by Serre [16,17]. We emphasize that this is the only known case in which (4), (5), and (6) are equivalent. Since this matter is still not well understood (for $n = N = 2$ there are not contraexamples to equivalence of (4), (5), and (6) for general f), and since the old geometric-algebraic-analytic proof by Terpstra and the recent algebraic proof by Serre are not elementary, we think it is of interest to present a new description of quasiconvex quadratic forms in two dimensions, based on simple arguments of calculus.

We assume $n = N = 2$, and we use for the 2×2 matrix ξ the vectorial notation $\xi \equiv (\xi_1, \xi_2, \xi_3, \xi_4)$. Thus, determinant of ξ means $\det\xi = \xi_1\xi_4 - \xi_2\xi_3$. We consider the quadratic form associated to a 4×4 real matrix (a_{ij}):

$$f(\xi) = \sum_{i,j=1}^{4} a_{ij}\xi_i\xi_j. \tag{7}$$

A vector ξ can be represented in the form $(\lambda_1\eta_1, \lambda_1\eta_2, \lambda_2\eta_1, \lambda_2\eta_2)$ if and only if $\det\xi = 0$; therefore, the *L-H* condition (5) is equivalent to

$$f(\xi) \geq 0 \quad \textit{for every } \xi \in \mathbb{R}^4 \textit{ such that } \det\xi = 0. \tag{8}$$

Condition (6) means that there exists a convex function $g = g(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ such that $f(\xi) = g(\xi, \det\xi)$. A simple computation gives (we assume $g \in C^2$; for a general continuous g we can use a mollifier argument):

$$\sum_{i,j=1}^{4} f_{\xi_i\xi_j}(\xi)\lambda_i\lambda_j = \sum_{i,j=1}^{5} g_{\xi_i\xi_j}(\xi)\lambda_i\lambda_j + 2g_{\xi_5}(\xi)(\lambda_1\lambda_4 - \lambda_2\lambda_3), \tag{9}$$

where $\xi, \lambda \in \mathbb{R}^4$ and $\xi_5 = \det\xi$, $\lambda_5 = \xi_1\lambda_4 - \xi_2\lambda_3 - \xi_3\lambda_2 + \xi_4\lambda_1$. Thus, since $\lambda_5 = 0$ for $\xi = 0$, we have

$$\begin{aligned} g(\xi, \det\xi) = f(\xi) &= \frac{1}{2} \sum_{i,j=1}^{4} f_{\xi_i\xi_j}(0)\xi_i\xi_j \\ &= \frac{1}{2} \sum_{i,j=1}^{4} g_{\xi_i\xi_j}(0)\xi_i\xi_j + g_{\xi_5}(0)\det\xi. \end{aligned} \tag{10}$$

Therefore, g is a sum of a convex quadratic form in ξ and a linear term in $\det\xi$. Thus, the polyconvexity condition (6) is equivalent to

$$\exists\lambda \in \mathbb{R} \quad \textit{such that } f(\xi) - \lambda\det\xi \geq 0 \textit{ for every } \xi \in \mathbb{R}^4. \tag{11}$$

Now we prove that, for quadratic forms (7), the Legendre-Hadamard condition (8) implies the polyconvexity condition (11); and thus these conditions are both equivalent to quasiconvexity. We prove also the equivalence for the corresponding *strict* inequalities. This is a consequence of the two theorems that follow.

Theorem 1. *Let f and g be two real quadratic forms in $\mathbb{R}^n$. The two conditions are equivalent*:

(i) $f(\xi) > 0$ *for every* $\xi \in \mathbb{R}^n$, $\xi \neq 0$, *such that* $g(\xi) = 0$;
(ii) *there exists* $\lambda \in \mathbb{R}$ *such that* $f(\xi) - \lambda g(\xi)$ *is positive definite.*

Proof. Of course (ii) implies (i). On changing g with $-g$ if necessary, we can assume that g is positive somewhere (the case $g \equiv 0$ is trivial). We can define

$$\lambda_1 = \inf\left\{\frac{f(\xi)}{g(\xi)} : \xi \in \mathbb{R}^n, g(\xi) > 0\right\}. \tag{12}$$

λ_1 is a minimum; in fact, let us pick up a (normalized) minimizing sequence:

$$\xi_k \in \mathbb{R}^n: g(\xi_k) > 0, \qquad |\xi_k| = 1, \quad \lim_{k+\infty} \frac{f(\xi_k)}{g(\xi_k)} = \lambda_1. \tag{13}$$

Let ξ_1 be limit of a convergent subsequence of ξ_k. If $g(\xi_1) = 0$ then by (i) $f(\xi_1) > 0$; thus, we should have $f(\xi_k)/g(\xi_k) \to +\infty$, in contradiction with the definition of λ_1. Therefore, $g(\xi_1) > 0$ and ξ_1 is a minimum point for λ_1.

Let us define

$$\lambda_2 = \begin{cases} -\infty & \textit{if } g \textit{ is positive semidefinite};\\ \sup\left\{\dfrac{f(\xi)}{g(\xi)} : \xi \in \mathbb{R}^n, g(\xi) < 0\right\} & \textit{otherwise}. \end{cases} \tag{14}$$

We want to prove that $\lambda_1 > \lambda_2$. To this aim we can assume that g is negative somewhere. Since the definition of λ_2 is analogous to that of λ_1, λ_2 is a maximum and there exists ξ_2 such that $g(\xi_2) < 0$ and $\lambda_2 = f(\xi_2)/g(\xi_2)$. Let us define

$$\begin{aligned} &\xi(t) = \xi_1 + t(\xi_2 - \xi_1);\\ &\phi(t) = g(\xi(t)); \qquad \psi(t) = f(\xi(t)) - \lambda_1 g(\xi(t)). \end{aligned} \tag{15}$$

We emphasize the coefficients of t^2 of the two polynomials ϕ and ψ:

$$\begin{aligned} \phi(t) &= g(\xi_2 - \xi_1)t^2 + \cdots\\ \psi(t) &= \{f(\xi_2 - \xi_1) - \lambda_1 g(\xi_2 - \xi_1)\}t^2 + \cdots \end{aligned} \tag{16}$$

Let us consider first the case when $g(\xi_2 - \xi_1) = 0$. In this case ϕ is a line, $\phi(0) = g(\xi_1) > 0$, $\phi(1) = g(\xi_2) < 0$, and thus there exists $t_1 \in (0,1)$ such that $\phi(t_1) = g(\xi(t_1)) = 0$. The coefficient of t^2 of ψ is $f(\xi_2 - \xi_1)$, and is positive by (i). Thus, ψ is a convex parabola with $\psi(0) = f(\xi_1) - \lambda_1 g(\xi_1) = 0$, and $\psi(t_1) = f(\xi(t_1)) > 0$ again since $g(\xi(t_1)) = 0$. It follows that $\psi(t) > 0$ for every $t \geq t_1$; in particular, $\psi(1) = f(\xi_2) - \lambda_1 g(\xi_2) > 0$, i.e., $\lambda_1 > f(\xi_2)/g(\xi_2) = \lambda_2$.

Secondly let us consider the case $g(\xi_2 - \xi_1) \neq 0$. Then ϕ is a parabola with $\phi(0) > 0$ and $\phi(1) < 0$. Then there exist t_1, t_2, t_1 inside the interval $(0,1)$ and either $t_2 < 0$ or $t_2 > 1$, such that $\phi(t_1) = \phi(t_2) = 0$. In correspondence we have $\psi(0) = 0$, $\psi(t_1) > 0$, $\psi(t_2) > 0$. This forces the (at most) second-degree polynomial $\psi(t)$ to be positive for $t = 1$. Thus, again $\lambda_1 > \lambda_2$.

Now every λ in between λ_1 and λ_2 solves our problem. In fact, for $\xi \neq 0$ and $\lambda_2 < \lambda < \lambda_1$ we have

$$f(\xi) - \lambda g(\xi) > \begin{cases} f(\xi) - \lambda_1 g(\xi) \geq 0 & \text{if } g(\xi) > 0;\\ f(\xi) - \lambda_2 g(\xi) \geq 0 & \text{if } g(\xi) < 0;\\ 0 & \text{if } g(\xi) = 0. \end{cases} \tag{17}$$

□

In the next theorem we study the case when equality may hold in (i) and (ii). Let us first show that (i) and (ii), with equality, are not always equivalent each other. In fact, if f and g are defined in $\mathbb{R}^2$ by

$$f(\xi) = \xi_1^2 + \xi_1\xi_2, \qquad g(\xi) = \xi_1^2, \tag{18}$$

then $f = 0$ when $g = 0$; but there does not exist a real λ such that $f - \lambda g = (1 - \lambda)\xi_1^2 + \xi_1\xi_2$ is positive semidefinite. In the above example g is positive semidefinite. On the contrary, the quadratic form of our application $g(\xi) = \xi_1\xi_4 - \xi_2\xi_3$ is not semidefinite. In this case, i.e., if g is indefinite, we have:

Theorem 2. *Let f and g be two real quadratic forms in $\mathbb{R}^n$. If g assumes both positive and negative values, then the two conditions are equivalent:*
(i) *$f(\xi) \geq 0$ for every $\xi \in \mathbb{R}^n$ such that $g(\xi) = 0$;*
(ii) *there exists $\lambda \in \mathbb{R}$ such that $f(\xi) - \lambda g(\xi) \geq 0$, $\forall \xi \in \mathbb{R}^n$.*

Proof. By assumption there exist in $\mathbb{R}^n$ η_1 and η_2 such that $g(\eta_1) = 1$, $g(\eta_2) = -1$. If (i) holds, for every $\varepsilon \in (0,1]$ the quadratic form $f_\varepsilon(\xi) = f(\xi) + \varepsilon|\xi|^2$ is strictly positive if $\xi \neq 0$, $g(\xi) = 0$. Thus, by Thm. 1, we deduce that there exists λ_ε such that

$$f(\xi) + \varepsilon|\xi|^2 - \lambda_\varepsilon g(\xi) > 0, \qquad \forall \xi \neq 0. \tag{19}$$

In particular, for $\xi = \eta_1$ and $\xi = \eta_2$ we have

$$-f(\eta_2) - |\eta_2|^2 < \lambda_\varepsilon < f(\eta_1) + |\eta_1|^2. \tag{20}$$

Thus, λ_ε is bounded and we can find a sequence $\varepsilon_k \to 0$ such that λ_{ε_k} converges to some $\lambda \in \mathbb{R}$. We go to the limit in (19) and we obtain (ii). □

Equivalence of polyconvexity, quasiconvexity, and *L-H* condition for quadratic forms in two dimensions clearly follows from formulations (8), (11), and Thms. 1 and 2. Now we give two other applications of Thm. 2:

Corollary 1. *Let f and g be two quadratic forms in $\mathbb{R}^n$, with g indefinite. If $f(\xi) = 0$ for every ξ such that $g(\xi) = 0$, then there exists $\lambda \in \mathbb{R}$ such that $f = \lambda g$.*

Proof. By Thm. 2 applied to f, g and $-f$, $-g$, there exist λ_1, λ_2 such that

$$\lambda_1 g(\xi) \leq f(\xi) \leq \lambda_2 g(\xi), \qquad \forall \xi \in \mathbb{R}^n. \tag{21}$$

In particular $(\lambda_2 - \lambda_1)g(\xi) \geq 0$. Since g is indefinite we must have $\lambda_1 = \lambda_2$. □

Corollary 2 (Morrey [10], [11, Sect. 4.4]). *Let $f(\xi, \eta)$, $g(\xi, \eta)$ be two bilinear forms on $\mathbb{R}^n \times \mathbb{R}^m$. If $f(\xi, \eta) = 0$ for every (ξ, η) such that $g(\xi, \eta) = 0$, then there exists $\lambda \in \mathbb{R}$ such that $f(\xi, \eta) = \lambda g(\xi, \eta)$ for every (ξ, η) $\mathbb{R}^n \times \mathbb{R}^m$.*

Proof. f and g can be considered quadratic forms with respect to the vector $(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)\in\mathbb{R}^{n+m}$. If $g\equiv 0$ the result is trivial; otherwise g is indefinite on $\mathbb{R}^{n+m}$. Thus, Cor. 2 follows from Cor. 1. □

Remark 1. I think it is of interest to report the following remarks due to Alain Bensoussan: Thms. 1 and 2 can be interpreted as results on a singular problem of Lagrange multipliers. For example, we can state Thm. 1 as follows: if $f(\xi)$ has unique minimum at $\xi=0$ under the constraint $g(\xi)=0$, then there exists a Lagrange multiplier λ such that $f(\xi)-\lambda g(\xi)$ has unique minimum on $\mathbb{R}^n$ at $\xi=0$. The problem is singular in the sense that we cannot obtain λ from the necessary condition for existence of Lagrange multipliers: $Df(0)=\lambda Dg(0)$, since both the gradients are equal to zero. Note also that the Lagrange multiplier λ is not unique! Finally, note that a Kuhn-Tucker type result holds: if f has unique minimum at $\xi=0$ under the constraint $g(\xi)\geq 0$, then there exists a *positive* λ such that $f-\lambda g$ is positive definite. In fact, from its definition (12), λ_1 is positive in this case, and we can choose $\lambda\in(\lambda_2,\lambda_1)$ positive too.

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