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Approximate Solutions of the Bellman Equation of Deterministic Control Theory*

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Abstract. We consider an infinite horizon discounted optimal control problem and its time discretized approximation, and study the rate of convergence of the approximate solutions to the value function of the original problem. In particular we prove the rate is of order 1 as the discretization step tends to zero, provided a semiconcavity assumption is satisfied. We also characterize the limit of the optimal controls for the approximate problems within the framework of the theory of relaxed controls.

1. Introduction

The dynamic programming method shows that the value function of an optimal control problem for ordinary differential equations satisfies, provided it is smooth, a nonlinear first order partial differential equation of Hamilton-Jacobi type, the Bellman equation (see, e.g., Fleming-Rishel [14]). And, on the other hand, the existence of a smooth solution of the Bellman equation often enables us to find an optimal feedback control (see [14] or Lee-Markus [18]).

However, this procedure can seldom be implemented in practice. Indeed, simple problems are known whose value functions have discontinuities in their partial derivatives, and examples show also that the Bellman equation may not have a C^1 solution, due to its fully nonlinear character. Moreover, the synthesis procedure requires regularity of the feedback control, too.

This discussion reveals that, in order to make the dynamic programming method rigorous, two main questions should be answered:

(i) In which weak sense does the value function of an optimal control problem satisfy the corresponding Bellman equation?

(ii) In what way can we construct an optimal control or a minimizing sequence based upon the information of such a weak solution?

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A major achievement concerning the first question is the observation, due to P. L. Lions [19], that under quite general conditions the value function is characterized as the (unique) viscosity solution of the associated Bellman equation. See Crandall-Lions [8], Crandall-Evans-Lions [7] for the definition of and uniqueness results for viscosity solutions.

In a previous paper [2] one of the authors proposed an approximation method of the viscosity solution of the Bellman equation associated with the infinite horizon problem with discounting. The interpretation of the solutions of the approximate problems as value functions of some discrete time control problems allows us to construct a minimizing sequence of piecewise constant controls for the original problem. See Hrustalev [16] for other results in this direction, and see also Capuzzo Dolcetta-Matzeu [4, 5], Capuzzo Dolcetta-Matzeu-Menaldi [6] for a similar approach to the stopping time and the switching problems.

The purpose of this paper is to study the rate at which the approximate solutions considered in [2] converge to the exact solution as the discretization step h tends to zero and to characterize the limit of the optimal controls for the approximate problems within the framework of the theory of relaxed control problems (see Berkovitz [1], Warga [24], and [18]).

The next section contains the precise statement of the infinite horizon control problem with discounting and its Bellman equation, with relevant matters from [2, 7, 19] about viscosity solutions and their approximations. In Sect. 3 we show that the rate of convergence of approximate solutions to the exact solution is of order $\frac{\gamma}{2}$, assuming the exact solution is Hölder continuous with exponent $0 < \gamma \leq 1$. The proof is a modification of the method introduced for difference approximations of Hamilton-Jacobi equations by [9]. The result in the case $\gamma = 1$ is due to Souganidis [21], where general approximation theorems for Hamilton-Jacobi equations are proved. In Sect. 4 the particular structure of the Bellman equations is used to prove, using both PDE and control theory methods, that the convergence rate is of order 1, provided a semiconcavity assumption is satisfied. These results can be seen as a natural development of earlier work of Cullum [10, 11] and Malanowski [20], where similar approximations have been considered under rather restrictive convexity assumptions. See [9, 19, 21, 22], for related topics. Finally, in Sect. 5, we show that the optimal controls and the corresponding states of the approximate problems converge, in a suitable sense, to an optimal relaxed control and the corresponding relaxed response.

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2. Optimal Control Problems

We will be concerned here with the infinite horizon discounted optimal control problem (see [14]); this is the problem of finding

$$V(x) \equiv \inf_{\alpha \in \mathscr{A}} \int_0^\infty f(y(x,s), \alpha(s)) e^{-\lambda s} \, ds \quad \text{for } x \in \mathbb{R}^n. \tag{2.1}$$

Here $\mathscr{A}$ denotes the set of all measurable functions of $[0, +\infty[$ to a given compact subset A of $\mathbb{R}^m$, f: $\mathbb{R}^n \times A \to \mathbb{R}$ is a given function and λ is a given positive constant. The vector $y(s) = y(x, s)$ and the control α in (2.1) are related by the *state equation*

$$\left.\begin{aligned} \dot{y}(s) &= g(y(s), \alpha(s)) \quad s > 0, \\ y(0) &= x, \end{aligned}\right\} \tag{2.2}$$

where g: $\mathbb{R}^n \times A \to \mathbb{R}^n$ is continuous and satisfies

$$|g(x,a) - g(x',a)| \le L|x - x'|, \qquad |g(x,a)| \le M \tag{2.3}$$

for all $x, x' \in \mathbb{R}^n$, $a \in A$ and for some constants L, M. The mapping y: $\mathbb{R}^n \times [0, +\infty[\to \mathbb{R}^n$ is called the *response* or the *state* corresponding to α. The constant λ represents the *discount factor* and the function f determines the *running cost*. We assume f is continuous on $\mathbb{R}^n \times A$ and satisfies

$$|f(x,a) - f(x',a)| \le M|x - x'|, \qquad |f(x,a)| \le M \quad \text{for all } x, \quad x' \in \mathbb{R}^n, \quad a \in A. \tag{2.4}$$

The function V: $\mathbb{R}^n \to \mathbb{R}$ defined by (2.1) is called the *value function* of the control problem. For $x \in \mathbb{R}^n$ and $\alpha \in \mathscr{A}$, we set

$$J(x,\alpha) = \int_0^\infty f(y(x,s), \alpha(s)) e^{-\lambda s} \, ds,$$

where y is the response to α.

It is known that under the assumptions made above, the value function V satisfies

$$|V(x)| \le \frac{M}{\lambda}, \qquad |V(x) - V(x')| \le C|x - x'|^\gamma \tag{2.5}$$

for all $x, x' \in \mathbb{R}^n$, where C is a constant depending on γ and $\gamma = 1$ if $\lambda > L$, $\gamma = \dfrac{\lambda}{L}$ if $\lambda < L$ and γ is an arbitrary number less than 1 if $\lambda = L$. Moreover, $u = V$ is the unique bounded uniformly continuous viscosity solution of the Bellman equation

$$\max_{a \in A} \{\lambda u(x) - g(x,a) \cdot Du(x) - f(x,a)\} = 0 \qquad \text{for } x \in \mathbb{R}^n. \tag{2.6}$$

Here $D = \left(\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}\right)$ and "$\cdot$" denotes the inner product in $\mathbb{R}^n$. Uniqueness is a consequence of Crandall-Lions [8, Thm. II.2]. For a proof of these facts, see [18].

For convenience, we recall here the definition of a viscosity solution of (2.6) following [7] (see also [8]). A continuous function u on $\mathbb{R}^n$ is called a *viscosity solution* of (2.6) provided for every $\varphi \in C^1(\mathbb{R}^n)$ the following holds: if x_0 is a local maximum point of $u - \varphi$,

$$\max_{a \in A} \{ \lambda u(x_0) - g(x_0, a) \cdot D\varphi(x_0) - f(x_0, a) \} \le 0,$$

and if x_1 is a local minimum point of $u - \varphi$,

$$\max_{a \in A} \{ \lambda u(x_1) - g(x_1, a) \cdot D\varphi(x_1) - f(x_1, a) \} \ge 0.$$

Let h be a positive number, and consider this approximate problem of (2.6):

$$\max_{a \in A} \{ u_h(x) - (1 - \lambda h) u_h(x + hg(x, a)) - hf(x, a) \} = 0 \qquad (2.6)_h$$

for $x \in \mathbb{R}^n$. It has been proved in [2] that if $h < 1/\lambda$ then $(2.6)_h$ has a unique bounded continuous solution u_h and that $\{u_h\}$ converges locally uniformly in $\mathbb{R}^n$ as $h \to 0^+$ to the unique bounded uniformly continuous viscosity solution of (2.6). The following representation formula has been also demonstrated in [2];

$$u_h(x) = \inf_{\alpha \in \mathscr{A}_h} J_h(x, \alpha) \quad \text{for every } x \in \mathbb{R}^n. \qquad (2.7)$$

In this formula, $\mathscr{A}_h$ denotes the subset of $\mathscr{A}$ consisting of all controls which take constant values on each interval $[kh, (k+1)h[$, $k = 0, 1, \ldots,$ and

$$J_h(x, \alpha) = h \sum_{k=0}^{\infty} f(y_h(x, k), \alpha(kh))(1 - \lambda h)^k,$$

where the sequence $\{y_h(x, k)\}$ is determined by the recursion

$$y_h(x, 0) = x, \; y_h(x, k+1) = y_h(x, k) + hg(y_h(x, k), \alpha(kh)),$$
$$k = 0, 1, \ldots. \qquad (2.8)$$

Moreover, the solution u_h of $(2.6)_h$ satisfies

$$|u_h(x)| \le \frac{M}{\lambda}, \qquad |u_h(x) - u_h(x')| \le C|x - x'|^\gamma \qquad (2.9)$$

for all $x, x' \in \mathbb{R}^n$, $h \in]0, 1/\lambda[$, where C is a constant depending on γ, but not on h, and $\gamma = 1$ if $\lambda > L$, $\gamma = \lambda/L$ if $\lambda < L$ and γ is an arbitrary number in $]0,1[$ if $\lambda = L$. One can find also in [2] a proof of the estimates (2.9) except the Hölder estimate of u_h in the case $\lambda < L$: this is proved by using the formula (2.7) and Lemma 4.1 below.

It will be useful in what follows to consider the piecewise constant extension $\tilde{y}_h(x, \cdot)$ to $[0, +\infty[$ of the mapping: $s \mapsto y_h(x, s/h)$ defined on $\{kh \mid k = 0, 1, 2, \ldots\}$. It is defined by

$$\tilde{y}_h(x, s) = y_h(x, [s/h]),$$

where $[s/h]$ denotes the largest integer which is less than or equal to s/h. As a simple consequence of the Gronwall inequality we have

$$|y(x,s) - \tilde{y}_h(x,s)| \le Mhe^{Ls} \tag{2.10}$$

for all $s \ge 0$, $x \in \mathbb{R}^n$ and $\alpha \in \mathscr{A}_h$, where y and y_h are the solutions of (2.2) and (2.8), respectively.

3. Rate of Convergence (I)

The following theorem is proved by using a simple modification of the method in [9].

Theorem 3.1. *Assume* (2.3), (2.4), *and let* $u, u_h \in \mathrm{BUC}(\mathbb{R}^n)$ *be the viscosity solution of* (2.6) *and the solution of* $(2.6)_h$, *respectively. Let* $\gamma \in]0,1]$ *be a Hölder exponent of* u. *Then*

$$\sup_{\mathbb{R}^n} |u - u_h| \le Ch^{\gamma/2} \tag{3.1}$$

for each $h \in]0, 1/\lambda[$, *where* $C > 0$ *is a constant.*

Recalling (2.5), we have immediately

Corollary 3.1. *Under the assumptions of Thm.* 3.1, *the following estimates hold for some constants* $C > 0$ (*which may depend on* γ *in the case* (3.3)).

$$\sup_{\mathbb{R}^n} |u - u_h| \le Ch^{1/2} \quad \text{if } \lambda > L; \tag{3.2}$$

$$\sup_{\mathbb{R}^n} |u - u_h| \le Ch^{\gamma/2} \quad \text{for any } \gamma < 1 \quad \text{if } \lambda = L; \tag{3.3}$$

$$\sup_{\mathbb{R}^n} |u - u_h| \le Ch^{\lambda/2L} \quad \text{if } \lambda < L. \tag{3.4}$$

Remark 3.1. The above estimate (3.2) is due to Souganidis [21], where the same estimate is obtained for more general Hamilton-Jacobi equations. Our contribution here is the estimates (3.3) and (3.4).

The basic idea of the proof of Thm. 3.1 is same as [21]; and so we only sketch the argument here.

Outline of proof. For $0 < \varepsilon < 1$, define function

$$\beta_\varepsilon(x) = -\left|\frac{x}{\varepsilon}\right|^2 \quad \text{for } x \in \mathbb{R}^n. \tag{3.5}$$

If $0 < h < 1/\lambda$, we set

$$\varphi(x,y) = u_h(x) - u(y) + \beta_\varepsilon(x-y) \quad \text{for } (x,y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Since u and u_h are bounded on $\mathbb{R}^n$, for each $\delta > 0$ there exists a point (x_1, y_1) in $\mathbb{R}^{2n}$ such that

$$\varphi(x_1, y_1) > \sup_{\mathbb{R}^{2n}} \varphi - \delta.$$

Choose $\xi \in C_0^\infty(\mathbb{R}^{2n})$ so that

$$\xi(x_1, y_1) = 1, \qquad 0 \le \xi \le 1, \quad |D\xi| \le 1,$$

and for $0 < \delta < 1$, set

$$\psi(x, y) = \varphi(x, y) + \delta\xi(x, y) \quad \text{for } (x, y) \in \mathbb{R}^{2n}.$$

Clearly, ψ takes its maximum at a point (x_0, y_0) in supp ξ. That is

$$\psi(x_0, y_0) \ge \psi(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^{2n}. \tag{3.6}$$

Note that $y \mapsto -\psi(x_0, y)$ attains its minimum at y_0. Hence, by the definition of a viscosity solution of (2.6),

$$\lambda u(y_0) + g(y_0, a^*) \cdot \big(D\beta_\varepsilon(x_0 - y_0) - \delta D_y\xi(x_0, y_0)\big) - f(y_0, a^*) \ge 0 \tag{3.7}$$

for some $a^* \in A$.

By $(2.6)_h$, we have

$$u_h(x_0) - (1 - \lambda h) u_h\big(x_0 + hg(x_0, a^*)\big) - hf(x_0, a^*) \le 0.$$

Use the inequality (3.6), with $x = x_0 + hg(x_0, a^*)$ and $y = y_0$, to cancel the term $u_h(x_0 + hg(x_0, a^*))$ and obtain

$$\begin{aligned} &\lambda u_h(x_0) + (1 - \lambda h) g(x_0, a^*) \cdot D\beta_\varepsilon(x_0 - y_0) - \frac{2h}{\varepsilon^2}|g(x_0, a^*)|^2 \\ &\quad - \delta|g(x_0, a^*)| - f(x_0, a^*) \le 0. \end{aligned} \tag{3.8}$$

Subtracting (3.7) from (3.8) and taking (2.3), (2.4), and that $D\beta_\varepsilon(x) = -\dfrac{2x}{\varepsilon^2}$ into account, we get

$$u_h(x_0) - u(y_0) \le C\left[|x_0 - y_0| + \frac{|x_0 - y_0|^2}{\varepsilon^2} + \frac{h|x_0 - y_0|}{\varepsilon^2} + \frac{h}{\varepsilon^2}\right] + 2\,\delta M. \tag{3.9}$$

(C denotes various positive constants here and in the remaining part of the proof.) Now we observe that if we choose $x = y = x_0$ in (3.6) we obtain

$$\frac{1}{\varepsilon^2}|x_0 - y_0|^2 \leq C|x_0 - y_0|^\gamma + \delta|x_0 - y_0|.$$

Since $\varepsilon, \delta < 1$, from this we have

$$|x_0 - y_0| \leq C\varepsilon^{\frac{2}{2-\gamma}}, \tag{3.10}$$

where C is independent of ε, δ, h. Thus, from (3.9) and (3.10)

$$u_h(x_0) - u(y_0) \leq C\left[\varepsilon^{\frac{2}{2-\gamma}} + \varepsilon^{\frac{2\gamma}{2-\gamma}} + h\varepsilon^{\frac{2\gamma-2}{2-\gamma}} + \frac{h}{\varepsilon^2} + \delta\right].$$

Assuming $h < 1$ and taking $\varepsilon = h^{(2-\gamma)/4}$ in the above, we have

$$u_h(x_0) - u(y_0) \leq C(h^{\gamma/2} + \delta). \tag{3.11}$$

From (3.6) with $y = x$, we have

$$u_h(x) - u(x) \leq u_h(x_0) - u(y_0) + \delta,$$

and so, from (3.11),

$$u_h(x) - u(x) \leq C(h^{\gamma/2} + \delta) \quad \text{for all } x \in \mathbb{R}^n.$$

Since $\delta \in]0,1[$ is arbitrary, we thus have

$$u_h(x) - u(x) \leq Ch^{\gamma/2} \quad \text{for all } x \in \mathbb{R}^n.$$

To prove the inequality $u(x) - u_h(x) \leq Ch^{\gamma/2}$, i.e., to complete the proof, it is enough to set $\varphi(x, y) = u(x) - u_h(y) + \beta_\varepsilon(x - y)$ and to proceed as above. □

4. Rate of Convergence (II)

Let us begin this section by showing that the interpretation of the viscosity solution of (2.6) as the value function of (2.1) and the representation formula (2.7) for u_h allow us to improve the estimate (3.1) from one side. We need the following

Lemma 4.1. *Let φ be a measurable function on $[0, +\infty[$ such that*

$$0 \leq \varphi(t) \leq \min\{Ae^{Bt}, C\}, \qquad t \geq 0 \tag{4.1}$$

for some positive constants $A < C$ and B. Let λ be a positive constant. Then

$$\int_0^{+\infty} \varphi(t)e^{-\lambda t}\,dt \leq KA^\sigma \tag{4.2}$$

for some constant K depending on σ, *where*

$$\left.\begin{array}{ll} \sigma = 1 & \text{if } \lambda > B, \\ \sigma \text{ is an arbitrary number in }]0,1[& \text{if } \lambda = B, \\ \sigma = \lambda/B & \text{if } \lambda < B. \end{array}\right\} \tag{4.3}$$

Proof. From the assumption (4.1),

$$\int_0^{+\infty} \varphi(t) e^{-\lambda t}\,dt \le A\int_0^T e^{(B-\lambda)t}\,dt + C\int_T^{+\infty} e^{-\lambda t}\,dt$$

holds for any $0 < T \le +\infty$. A direct computation of the integrals on the right hand side with the choices $T = +\infty$ if $\lambda > B$, $T = 1/\lambda \log C/A$ if $\lambda = B$, and $T = 1/B \log C/A$ if $\lambda < B$ gives

$$\int_0^{\infty} \varphi(t) e^{-\lambda t}\,dt \le \begin{cases} \dfrac{A}{\lambda - B} & \text{if } \lambda > B, \\ A\left(\dfrac{1}{\lambda} + \dfrac{1}{\lambda}\log\dfrac{C}{A}\right) & \text{if } \lambda = B, \\ A^{\lambda/B}\left(\dfrac{1}{B-\lambda} + \dfrac{1}{\lambda}\right)C^{1-\lambda/B} & \text{if } \lambda < B. \end{cases}$$

The assertion follows immediately from these inequalities. □

Theorem 4.1. *Assume* (2.3), (2.4), *and let* u_h, $u \in \mathrm{BUC}(\mathbb{R}^n)$ *be, respectively, the solution of* $(2.6)_h$ *and the viscosity solution of* (2.6). *Then the following estimate holds for some constant* $C > 0$ *depending on* σ:

$$\sup_{\mathbb{R}^n} (u - u_h) \le Ch^{\sigma}, \tag{4.4}$$

where $\sigma = 1$ *if* $\lambda > L$, σ *is an arbitrary number in* $]0,1[$ *if* $\lambda = L$, *and* $\sigma = \lambda/L$ *if* $\lambda < L$.

Proof. Using the representations

$$u_h(x) = \inf_{\alpha \in \mathscr{A}_h} J_h(x,\alpha), \qquad u(x) = \inf_{\alpha \in \mathscr{A}} J(x,\alpha),$$

we see that

$$\begin{aligned} u(x) - u_h(x) &\le \inf_{\alpha \in \mathscr{A}_h} J(x,\alpha) - \inf_{\alpha \in \mathscr{A}_h} J_h(x,\alpha) \\ &\le \sup_{\alpha \in \mathscr{A}_h} |J(x,\alpha) - J_h(x,\alpha)|. \end{aligned} \tag{4.5}$$

We fix $\alpha \in \mathscr{A}_h$, and compute that

$$|J(x,\alpha) - J_h(x,\alpha)| \leq \int_0^\infty |f(y(x,s),\alpha(s)) - f(\tilde{y}_h(x,s),\alpha(s))| e^{-\lambda s}\, ds + \int_0^\infty |f(\tilde{y}_h(x,s),\alpha(s))|\, |e^{-\lambda s} - e^{-\theta\lambda[s/h]h}|\, ds \tag{4.6}$$

where $\theta = \theta(\lambda, h)$ is given by

$$\theta = -\frac{1}{\lambda h}\log(1-\lambda h).$$

Then assumption (2.4) and (2.10) yield

$$|f(y(x,s),\alpha(s)) - f(\tilde{y}_h(x,s),\alpha(s))| \leq \min\{M^2 h e^{Ls}, 2M\},$$

and therefore, by Lemma 4.1,

$$\int_0^\infty |f(y(x,s),\alpha(s)) - f(\tilde{y}_h(x,s),\alpha(s))| e^{-\lambda s}\, ds \leq K h^\sigma \tag{4.7}$$

for some $K > 0$. On the other hand, we have

$$\begin{aligned}
&\int_0^\infty |f(\tilde{y}_h(x,s),\alpha(s))|\, |e^{-\lambda s} - e^{-\theta\lambda[s/h]h}|\, ds \\
&\quad \leq M \int_0^\infty |\lambda s - \theta\lambda[s/h]h| \max\{e^{-\lambda s}, e^{-\theta\lambda s}\}\, ds \\
&\quad \leq M\lambda(|1-\theta| + h)\int_0^\infty (s+1)\max\{e^{-\lambda s}, e^{-\theta\lambda s}\}\, ds.
\end{aligned} \tag{4.8}$$

Combining (4.5), (4.6), (4.7), and (4.8), and using the fact that

$$\lim_{h\to 0^+} \frac{\theta - 1}{h} = \frac{\lambda}{2},$$

we conclude (4.4). □

Our aim is now to obtain similar upper bounds for $\sup_{\mathbb{R}^n}(u_h - u)$. As we shall see in Thm. 4.2 below, this is possible provided the approximate solutions u_h satisfy the following condition:

$$u_h(x+z) - 2u_h(x) + u_h(x-z) \leq C|z|^{1+\tau} \tag{4.9}$$

for some $\tau \in]0,1]$ and all $x, z \in \mathbb{R}^n$, where C is a constant independent of h. For $\tau = 1$ this amounts to the (uniform) semiconcavity of u_h. Such a condition has been widely used in the study of nonlinear first order partial differential equations (see [12, 16, 17, 19]).

Theorem 4.2. *Assume* (2.3), (2.4), (4.9), *and* $\lambda > L$. *Then, for some positive constant* C, *the following estimate holds*:

$$\sup_{\mathbb{R}^n} (u_h - u) \leq Ch^{\tau} \quad \text{for all } h \in]0, \frac{1}{\lambda}[, \tag{4.10}$$

where $u_h, u \in \mathrm{BUC}(\mathbb{R}^n)$ *are the solution of* $(2.6)_h$ *and the viscosity solution of* (2.6), *respectively*.

The proof of the theorem requires a technical lemma which we state below.

Lemma 4.2. *Let* v: $\mathbb{R}^n \to \mathbb{R}$ *satisfy*

$$v(x+z) - 2v(x) + v(x-z) \leq C|z|^{1+\tau} \tag{4.11}$$

for all $x, z \in \mathbb{R}^n$ *and some* $\tau \in]0,1]$, *and*

$$v(0) = 0, \qquad \limsup_{x \to 0} \frac{v(x)}{|x|} \leq 0. \tag{4.12}$$

Then

$$v(x) \leq \frac{C}{2(2^{\tau}-1)} |x|^{1+\tau} \quad \text{for } x \in \mathbb{R}^n. \tag{4.13}$$

Proof. Let $\bar{x} \neq 0$ be an arbitrary point in $\mathbb{R}^n$ and k a nonnegative integer. Apply (4.11) with $x = \bar{x}/2^{k+1}$, $z = -\bar{x}/2^{k+1}$ to obtain that

$$v\left(\frac{\bar{x}}{2^{k+1}}\right) \geq \frac{1}{2} v\left(\frac{\bar{x}}{2^k}\right) - \frac{C}{2} \left|\frac{\bar{x}}{2^{k+1}}\right|^{1+\tau}.$$

We have used here that $v(0) = 0$. From this, by induction, one can see that for all $k = 0, 1, 2, \ldots$,

$$v\left(\frac{\bar{x}}{2^k}\right) \geq \frac{v(\bar{x})}{2^k} - \frac{C}{2^k} \left|\frac{\bar{x}}{2}\right|^{1+\tau} \frac{2^{\tau} - 2^{-\tau(k-1)}}{2^{\tau}-1}.$$

Hence,

$$\limsup_{x \to 0} \frac{v(x)}{|x|} \geq \limsup_{k \to +\infty} \frac{v\left(\frac{\bar{x}}{2^k}\right)}{\frac{|\bar{x}|}{2^k}}$$

$$\geq \lim_{k \to +\infty} \frac{2^k}{|\bar{x}|} \left[\frac{v(\bar{x})}{2^k} - \frac{C}{2^k} \left|\frac{\bar{x}}{2}\right|^{1+\tau} \frac{2^{\tau} - 2^{-\tau(k-1)}}{2^{\tau}-1} \right]$$

and therefore, taking (4.12) into account,

$$0 \geq \frac{1}{|\bar{x}|}\left(v(\bar{x}) - C\frac{|\bar{x}|^{1+\tau}}{2^{1+\tau}}\frac{2^{\tau}}{2^{\tau}-1}\right).$$

This proves (4.13). □

Proof of Thm. 4.2. Let β_ε be the function defined by (3.6), and choose $\xi \in C_0^\infty(\mathbb{R}^{2n})$ as in the proof of Thm. 3.1 so that $0 \leq \xi \leq 1$, $|D\xi| \leq 1$ and the function ψ on $\mathbb{R}^{2n}$ defined by

$$\psi(x, y) = u_h(x) - u(y) + \beta_\varepsilon(x - y) + \delta\xi(x, y)$$

attains its maximum at some point $(x_0, y_0) \in \mathbb{R}^{2n}$, i.e.,

$$\psi(x_0, y_0) \geq \psi(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^{2n}. \tag{4.14}$$

Since $y \mapsto -\psi(x_0, y)$ attains its minimum at y_0 and u is the viscosity solution of (2.6), we have

$$\lambda u(y_0) + g(y_0, a^*)\cdot\big(D\beta_\varepsilon(x_0 - y_0) - \delta D_y\xi(x_0, y_0)\big) - f(y_0, a^*) \geq 0 \tag{4.15}$$

for some $a^* \in A$.

Now we consider the function

$$v(x) = u_h(x_0 + x) - u_h(x_0) + \big(D\beta_\varepsilon(x_0 - y_0) + \delta D_x\xi(x_0, y_0)\big)\cdot x$$

on $\mathbb{R}^n$. It is easy to see that

$$v(x+z) - 2v(x) + v(x-z) = u_h(x_0 + x + z) - 2u_h(x_0 + x) + u_h(x_0 + x - z), \qquad v(0) = 0.$$

It is easily checked also that

$$v(x) = \psi(x_0 + x, y_0) - \psi(x_0, y_0) + \beta_\varepsilon(x_0 - y_0) - \beta_\varepsilon(x_0 + x - y_0) + D\beta_\varepsilon(x_0 - y_0)\cdot x + \delta\big(\xi(x_0, y_0) - \xi(x_0 + x, y_0) + D_x\xi(x_0, y_0)\cdot x\big),$$

and hence, by (4.14),

$$\limsup_{|x|\to 0} \frac{v(x)}{|x|} \leq 0.$$

By virtue of (4.9), we thus find that v satisfies the assumptions of Lemma 4.2 and so,

$$u_h(x_0 + x) - u_h(x_0) + \big(D\beta_\varepsilon(x_0 - y_0) - \delta D_x\xi(x_0, y_0)\big)\cdot x \leq \frac{C}{2(2^\tau - 1)}|x|^{1+\tau}$$

for all $x \in \mathbb{R}^n$. Choose now $x = hg(x_0, a^*)$ in the above to obtain

$$u_h(x_0 + hg(x_0, a^*)) \le u_h(x_0) - hg(x_0, a^*) \cdot (D\beta_\varepsilon(x_0 - y_0) + \delta D_x \xi(x_0, y_0)) + C|g(x_0, a^*)|^{1+\tau} h^{1+\tau}.$$

From this and the equation $(2.6)_h$ with $x = x_0$ it follows that

$$u_h(x_0) - (1 - \lambda h)\big[u_h(x_0) - hg(x_0, a^*) \cdot (D\beta_\varepsilon(x_0 - y_0) + \delta D_x \xi(x_0, y_0)) + C|g(x_0, a^*)|^{1+\tau} h^{1+\tau}\big] - hf(x_0, a^*) \le 0. \tag{4.16}$$

This inequality combined with (4.15) yields

$$\begin{aligned}\lambda(u_h(x_0) - u(y_0)) \le\ & (g(y_0, a^*) - g(x_0, a^*)) \cdot D\beta_\varepsilon(x_0 - y_0) \\ & + \delta\big(-D_x \xi(x_0, y_0) \cdot g(x_0, a^*) \\ & \quad - D_y \xi(x_0, y_0) \cdot g(y_0, a^*)\big) + f(x_0, a^*) \\ & \quad - f(y_0, a^*) + \lambda h g(x_0, a^*) \\ & \quad \cdot (D\beta_\varepsilon(x_0, y_0) + \delta D_x \xi(x_0, y_0)) + C|g(x_0, a^*)|^{1+\tau} h^\tau.\end{aligned}$$

Taking (2.3) and (2.4) into account, the above inequality gives

$$u_h(x_0) - u(y_0) \le \frac{1}{\lambda}\big[|D\beta_\varepsilon(x_0 - y_0)|(L|x_0 - y_0| + \lambda h M) + \delta M(2 + \lambda h) + M|x_0 - y_0| + CM^{1+\tau} h^\tau\big],$$

or, recalling that $D\beta_\varepsilon(x) = -\dfrac{2x}{\varepsilon^2}$,

$$u_h(x_0) - u(y_0) \le C\left[\frac{|x_0 - y_0|^2}{\varepsilon^2} + h\frac{|x_0 - y_0|}{\varepsilon^2} + |x_0 - y_0| + h^\tau + \delta\right]. \tag{4.17}$$

As in the proof of Thm. 3.1 (especially, the proof of (3.11)), we have

$$|x_0 - y_0| \le C\varepsilon^2.$$

We have used here that u is Lipschitz continuous on $\mathbb{R}^n$ under the assumption $\lambda > L$. Hence, from (4.17),

$$u_h(x_0) - u(y_0) \le C(\varepsilon^2 + h + h^\tau + \delta).$$

Choosing $\varepsilon = h^{1/2}$, we get

$$u_h(x_0) - u(y_0) \le C(h^\tau + \delta). \tag{4.18}$$

As in the proof of Thm. 3.1 we have

$$u_h(x) - u(x) \le u_h(x_0) - u(y_0) + \delta \quad \text{for all } x \in \mathbb{R}^n.$$

This combined with (4.18) proves the theorem since δ is arbitrary. □

The next lemma provides a sufficient condition for (4.9) to be satisfied.

Lemma 4.3. *Assume* (2.3), (2.4), *and*

$$|g(x+z,a) - 2g(x,a) + g(x-z,a)| \le M|z|^{1+\tau} \tag{4.19}$$

$$f(x+z,a) - 2f(x,a) + f(x-z,a) \le M|z|^{1+\tau} \tag{4.20}$$

for all $x, z \in \mathbb{R}^n$, $a \in A$, *and some* $\tau \in]0,1]$. *Then, for some* $C > 0$ *independent of* $h \in]0, 1/\lambda[$,

$$u_h(x+z) - 2u_h(x) + u_h(x-z) \le \begin{cases} C|z|^{1+\tau} & \text{if } \lambda > (1+\tau)L, \\ C|z|^{\lambda/L} & \text{if } \lambda < (1+\tau)L, \\ C|z|^{(1+\tau)\sigma} & \text{if } \lambda = (1+\tau)L, \end{cases} \tag{4.21}$$

for all $x, z \in \mathbb{R}^n$, *where* σ *is an arbitrary number in* $]0,1[$.

Proof. From (2.7) it follows that

$$u_h(x+z) - 2u_h(x) + u_h(x-z) \le \sup_{\alpha \in \mathscr{A}_h} \left[J_h(x+z,\alpha) - 2J_h(x,\alpha) + J_h(x-z,\alpha) \right]. \tag{4.22}$$

Fix $\alpha \in \mathscr{A}_h$. We observe that

$$J_h(x+z,\alpha) - 2J_h(x,\alpha) + J_h(x-z,\alpha) = h \sum_{k=0}^{\infty} A_k (1-\lambda h)^k, \tag{4.23}$$

where, denoting $a_k = \alpha(kh)$,

$$\begin{aligned} A_k = {} & f(y_h(x,k) + (y_h(x+z,k) - y_h(x,k)), a_k) - 2f(y_h(x,k), a_k) \\ & + f(y_h(x,k) - (y_h(x+z,k) - y_h(x,k)), a_k) \\ & + f(y_h(x-z,k), a_k) - f(y_h(x,k) - (y_h(x+z,k) - y_h(x,k)), a_k). \end{aligned}$$

The assumptions (4.19) and (4.20) yield

$$\begin{aligned} A_k \le {} & M|y_h(x+z,k) - y_h(x,k)|^{1+\tau} \\ & + M|y_h(x+z,k) - 2y_h(x,k) + y_h(x-z,k)|. \end{aligned} \tag{4.24}$$

Now it is easy to see that

$$|y_h(x+z,k)-y_h(x,k)| \le (1+Lh)^k|z| \quad \text{for } k = 0,1,2,\dots. \tag{4.25}$$

On the other hand we have

$$\begin{aligned}&|y_h(x+z,k)-2y_h(x,k)+y_h(x-z,k)|\\ &\quad\le Mh(1+Lh)^{k-1}\frac{(1+Lh)^{\tau k}-1}{(1+Lh)^{\tau}-1}|z|^{1+\tau} \qquad \text{for } k = 0,1,2,\dots. \end{aligned}\tag{4.26}$$

Indeed, using (2.3), (4.19), and (4.25), we have

$$\begin{aligned}&|y_h(x+z,k+1)-2y_h(x,k+1)+y_h(x-z,k+1)|\\ &\quad\le (1+Lh)|y_h(x+z,k)-2y_h(x,k)+y_h(x-z,k)|\\ &\qquad + Mh(1+Lh)^{(1+\tau)k}|z|^{1+\tau} \quad \text{for } k = 0,1,2,\dots.\end{aligned}$$

By induction, it follows from these inequalities that (4.26) holds.

Let us define now a step function $\varphi\colon \mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}\varphi(t) = \min\Bigg\{4M, M|z|^{1+\tau}\Bigg[&(1+Lh)^{(1+\tau)[t/h]}+Mh(1+Lh)^{[t/h]-1}\\ &\times\frac{(1+Lh)^{\tau[t/h]}-1}{(1+Lh)^{\tau}-1}\Bigg]\Bigg\},\end{aligned}$$

and constants $\theta=\theta(h)$, $\nu=\nu(h)$ by

$$e^{-\theta\lambda h} = 1-\lambda h, \qquad e^{\nu Lh} = 1+Lh.$$

Note that

$$\theta>1, \qquad 0<\nu<1, \quad \lim_{h\to 0^+}\theta = \lim_{h\to 0^+}\nu = 1.$$

From (4.23)–(4.26) it follows that

$$\begin{aligned}&J_h(x+z,\alpha)-2J_h(x,a)-J_h(x-z,\alpha)\\ &\quad\le h\sum_{k=0}^{\infty}\varphi(kh)e^{-\theta\lambda kh} \le e^{\theta\lambda h}\int_0^{\infty}\varphi(t)e^{-\lambda t}\,dt.\end{aligned}\tag{4.27}$$

A simple computation shows that

$$\varphi(t) \le \min\left\{4M, M|z|^{1+\tau}\left(1+\frac{M}{\tau\nu L}\right)e^{L(1+\tau)t}\right\}.$$

Therefore, applying Lemma 4.1 to the integral $\int_0^\infty \varphi(t)e^{-\lambda t}\,dt$ and recalling (4.22), we obtain the estimate (4.21). □

Let us state a theorem which summarizes the principal results of this section.

Theorem 4.3. *Assume* (2.3), (2.4), (4.19), *and* (4.20). *Let* $u_h, u \in \mathrm{BUC}(\mathbb{R}^n)$ *be the solution of* $(2.6)_h$ *and the viscosity solution of* (2.6), *respectively. Then the following estimates hold for all* $h \in]0, 1/\lambda[$:

$$\sup_{\mathbb{R}^n} |u - u_h| \leq \begin{cases} Ch^\tau & \text{if } \lambda > (1+\tau)L, \\ Ch^{\lambda/L-1} & \text{if } L < \lambda < (1+\tau)L, \\ Ch^\sigma & \text{if } \lambda = (1+\tau)L, \end{cases} \tag{4.28}$$

where σ *is an arbitrary number in* $]0,1[$ *and the letter* C *denotes constants depending on the choice of* σ.

Proof. The estimates (4.28) are straightforward consequences of Thm. 4.2, Lemma 4.3, and Thm. 4.1. □

As a consequence of (4.28), the convergence of u_h to u is of order 1 if g and f satisfy (4.19), (4.20) with $\tau = 1$, respectively, and $\lambda > 2L$. The next example shows that this result is optimal.

Example 4.1. Let $n = 1$ and A consist of a one point, i.e., $A = \{a\}$. Let $g(\cdot, a)$ and $f(\cdot, a)$ be C_0^∞ functions on $\mathbb{R}$ such that $g(x, a) = -x$ and $f(x, a) = x$ for $0 \leq x \leq 1$. It is clear that g and f satisfy (4.19) and (4.20), respectively. It is also easy to check by the representations of solutions of $(2.6)_h$ and (2.6) that their solutions are given, respectively, by

$$u_h(x) = \frac{x}{1 + \lambda - \lambda h} \quad \text{and} \quad u(x) = \frac{x}{1+\lambda} \quad \text{on } [0,1].$$

Therefore,

$$\lim_{h \to 0^+} \frac{|u_h(x) - u(x)|}{h} = \frac{x}{(1+\lambda)^2} \quad \text{for } x \in [0,1].$$

5. Convergence of Optimal Controls

We show first how the approximate equation $(2.6)_h$ allows us to synthetize, by standard dynamic programming, an optimal control $\alpha_h^* \in \mathscr{A}_h$ for the discrete time control problem

$$\inf_{\alpha \in \mathscr{A}_h} J_h(x, \alpha). \tag{5.1}$$

Let u_h be the unique bounded continuous solution of $(2.6)_h$. There exists a function a_h^*: $\mathbb{R}^n \to A$ such that for all $x \in \mathbb{R}^n$,

$$u_h(x) - (1-\lambda h)u_h\big(x + hg\big(x, a_h^*(x)\big)\big) - hf\big(x, a_h^*(x)\big) = 0. \tag{5.2}$$

Define then $y_h^*(x,k)$, $k = 0,1,2,\ldots,$ by

$$\left.\begin{aligned} &y_h^*(x,0) = x, \\ &y_h^*(x,k+1) = y_h^*(x,k) + hg\big(y_h^*(x,k), a_h^*\big(y_h^*(x,k)\big)\big) \quad \text{for } k \geq 0, \end{aligned}\right\} \tag{5.3}$$

and α_h^*: $[0,+\infty[\to A$ by

$$\alpha_h^*(t) = a_h^*\big(y_h^*(x,[t/h])\big) \quad \text{for } t \geq 0. \tag{5.4}$$

It is clear that $\alpha_h^* \in \mathscr{A}_h$ and by (5.2) that the identities

$$\begin{aligned} u_h(x) = {} & (1-\lambda h)^k u_h\big(y_h^*(x,k)\big) \\ & + h\sum_{j=0}^{k-1} f\big(y_h^*(x,j), a_h^*\big(y_h^*(x,j)\big)\big)(1-\lambda h)^j \end{aligned}$$

hold for $k = 1,2,\ldots$. Since u_h is bounded, this yields

$$u_h(x) = J_h\big(x, \alpha_h^*\big), \tag{5.5}$$

and hence, by (2.7),

$$J_h\big(x,\alpha_h^*\big) = \inf_{\alpha \in \mathscr{A}_h} J_h(x,\alpha). \tag{5.6}$$

As a consequence of the results in the previous section, we have

$$\lim_{h \to 0^+} J_h\big(x,\alpha_h^*\big) = \inf_{\alpha \in \mathscr{A}} J(x,\alpha). \tag{5.7}$$

It can be proved also that $\{\alpha_h^*\}$ forms a minimizing sequence for the problem (2.1), that is,

$$\lim_{h \to 0^+} J\big(x,\alpha_h^*\big) = \inf_{\alpha \in \mathscr{A}} J(x,\alpha) \tag{5.8}$$

(see [2] and also the estimate of (4.6) in the proof of Thm. 4.1).

We now turn our attention to the behavior of the controls α_h^* and the corresponding responses as $h \to 0^+$. It is well-known that a minimizing sequence may have no limit in any classical sense due to a highly oscillatory behavior (see [18, p. 265], for example).

It is therefore natural to set the problem in the general framework of relaxed controls. We denote by $\mathscr{M}(A)$ the space of Radon measures on A. Identifying

$\mathscr{M}(A)$ and $C(A)^*$, the dual space of the space $C(A)$ of all continuous functions on A, we may endow $\mathscr{M}(A)$ with the weak star topology of $C(A)^*$.

Following Warga [24] (see also Lee-Markus [18]), we call a *relaxed control* for the problem (2.1) any measurable mapping μ: $[0, +\infty[\to \mathscr{M}(A)$ such that μ_s, the value of μ at s, is a probability measure for almost every $s \in [0, +\infty[$. We denote by $\mathscr{A}^r$ the class of all relaxed controls for the problem (2.1). Note that any classical control $\alpha \in \mathscr{A}$ can be identified with the relaxed control μ^α: $s \mapsto \delta_{\alpha(s)}$, where δ_a denotes the Dirac measure concentrated at a. A mapping y: $\mathbb{R}^n \times [0, +\infty[\to \mathbb{R}^n$ is a *relaxed response* to $\mu \in \mathscr{A}^r$ if for all $x \in \mathbb{R}^n$ and $t \geq 0$,

$$y(x,t) = x + \int_0^t \int_A g(y(x,s), a)\, d\mu_s(a)\, ds \tag{5.9}$$

Let us introduce now the relaxation of the problem (2.1), i.e., the problem of finding

$$V^r(x) \equiv \inf_{\mu \in \mathscr{A}^r} \int_0^\infty \int_A f(y(x,s), a) e^{-\lambda s}\, d\mu_s(a)\, ds. \tag{5.10}$$

We write

$$J^r(x,\mu) = \int_0^\infty \int_A f(y(x,s), a) e^{-\lambda s}\, d\mu_s(a)\, ds \quad \text{for } \mu \in \mathscr{A}^r.$$

The main tool in what follows is the next lemma, where this observation is crucial: $L^\infty(0, T; \mathscr{M}(A))$, with $T > 0$, is the dual space of $L^1(0, T; C(A))$, under the duality

$$\langle \mu, \varphi \rangle = \int_0^T \int_A \varphi(t,a)\, d\mu_t(a)\, dt$$

for $\varphi \in L^1(0, T; C(A))$, $\mu \in L^\infty(0, T; \mathscr{M}(A))$. This is a special case of the general theorem [13, Thm. 8.18.2].

Lemma 5.1. *The convex set*

$$\mathscr{A}_T^r = \{\mu|_{[0,T]} | \mu \in \mathscr{A}^r\}$$

is sequentially compact in $L^\infty(0, T; \mathscr{M}(A))$ *with the weak star topology.*

A proof of this fact may be found in [24]. However, we give here a proof of Lemma 5.1 for the reader's convenience.

Proof. Let $\{\mu^{(k)}\}_{k=1}^\infty$ be any sequence in $\mathscr{A}_T^r$. Since

$$\int_A d\mu_t^{(k)}(a) = 1 \quad \text{for a.e. } t \in [0,T] \quad \text{and} \quad k = 1,2,\ldots,$$

$\{\mu^{(k)}\}$ is bounded in $L^\infty(0, T; \mathscr{M}(A))$. The standard compactness theorem thus asserts that there exist a subsequence of $\{\mu^{(k)}\}$ (which we denote again by $\{\mu^{(k)}\}$) and $\mu^* \in L^\infty(0, T; \mathscr{M}(A))$ such that $\mu^{(k)} \to \mu^*$ weakly star as $k \to \infty$. In view of the duality between $L^1(0, T; C(A))$ and $L^\infty(0, T; \mathscr{M}(A))$, this amounts to

$$\lim_{k \to \infty} \int_0^T \int_A \varphi(t, a)\, d\mu_t^{(k)}(a)\, dt = \int_0^T \int_A \varphi(t, a)\, d\mu_t^*(a)\, dt \tag{5.11}$$

for all $\varphi \in L^1(0, T; C(A))$.

To prove that μ_t^* is a probability measure on A for a.a. $t \in [0, T]$, let $\varphi_1 \in L^1(0, T)$ and $\varphi_2 \in C(A)$ satisfy $\varphi_i \geq 0$ for $i = 1, 2$ and plug $\varphi(t, a) = \varphi_1(t)\varphi_2(a)$ into (5.11). We then find

$$\int_0^T \varphi_1(t) \int_A \varphi_2(a)\, d\mu_t^*(a)\, dt \geq 0,$$

which implies that $\mu_t^* \geq 0$ a.e. Next take $\varphi(t, a) = \varphi_1(t) \cdot 1$, with any $\varphi_1 \in L^1(0, T)$ in (5.11), to see that

$$\int_0^T \varphi_1(t) \int_A d\mu_t^*(a)\, dt = \int_0^T \varphi_1(t)\, dt.$$

This shows that $\int_A d\mu_t^*(a) = 1$ a.e. and thus completes the proof. □

Proposition 5.1. *Assume* (2.3) *and* (2.4). *Then*

$$V^r(x) = V(x) \quad \textit{for all } x \in \mathbb{R}^n. \tag{5.12}$$

Proof. We will show that V^r satisfies the Bellman equation (2.6) in the viscosity sense. To do so, we let

$$\hat{A} = \{\mu \mid \mu \text{ is a Radon probability measure on } A\}$$

and set

$$\hat{f}(x, \mu) = \int_A f(x, a)\, d\mu(a), \qquad \hat{g}(x, \mu) = \int_A g(x, a)\, d\mu(a)$$

for $x \in \mathbb{R}^n$, $\mu \in \hat{A}$. Then the relaxed response y to $\mu \in \mathscr{A}^r$ satisfies

$$y(x, t) = x + \int_0^t \hat{g}(y(x, s), \mu_s)\, ds \quad \text{for } x \in \mathbb{R}^n \quad \text{and} \quad t \geq 0,$$

and (5.10) can be written as

$$V^r(x) = \inf_{\mu \in \mathscr{A}^r} \int_0^\infty \hat{f}(y(x, s), \mu_s) e^{-\lambda s}\, ds.$$

Moreover, by (2.3) and (2.4), we have

$$|\hat{g}(x,\mu)-\hat{g}(x',\mu)| \le L|x-x'|, \qquad |\hat{g}(x,\mu)| \le M,$$
$$|\hat{f}(x,\mu)-\hat{f}(x',\mu)| \le M|x-x'|, \qquad |\hat{f}(x,\mu)| \le M$$

for all $x, x' \in \mathbb{R}^n$, $\mu \in \hat{A}$. Note here that, regarding $\hat{A}$ as a subset of $C(A)^*$, $\hat{A}$ is metrizable, convex, and compact with the weak star topology of $C(A)^*$ (see [24, Thm. IV.1.4]). Thus, by [19, Prop. 1.1 and Thm. 1.10], we find that V^r is bounded, Hölder continuous on $\mathbb{R}^n$ and $u = V^r$ solves the Bellman equation

$$\max_{\mu\in\hat{A}} \{\lambda u(x) - \hat{g}(x,\mu)\cdot Du(x) - \hat{f}(x\cdot\mu)\} = 0 \qquad \text{in } \mathbb{R}^n \tag{5.13}$$

in the viscosity sense.

For $x, p \in \mathbb{R}^n$, we have

$$\begin{aligned}\max_{\mu\in\hat{A}} \{-\hat{g}(x,\mu)\cdot p - \hat{f}(x,\mu)\} &\ge \max_{a\in A} \{-\hat{g}(x,\delta_a)\cdot p - \hat{f}(x,\delta_a)\}\\ &= \max_{a\in A}\{-g(x,a)\cdot p - f(x,a)\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}-g(x,\mu)\cdot p - f(x,\mu) &= \int_A (-g(x,a)\cdot p - f(x,a))\, d\mu(a)\\ &\le \int_A \max_{b\in A}\{-g(x,b)\cdot p - f(x,b)\}\, d\mu(a)\\ &= \max_{a\in A}\{-g(x,a)\cdot p - f(x,a)\}\end{aligned}$$

for $x, p \in \mathbb{R}^n$, $\mu \in A$. Therefore, we have

$$\max_{\mu\in\hat{A}} \{-\hat{g}(x,\mu)\cdot p - \hat{f}(x,\mu)\} = \max_{a\in A}\{-g(x,a)\cdot p - f(x,a)\}$$

for all $x, p \in \mathbb{R}^n$ and so, (5.13) is identical to (2.6). The identity (5.12) is a direct consequence of the uniqueness of the bounded, uniformly continuous viscosity solution of (2.6) (see [8, Thm. II.2]). □

We are in a position to state the main result of this section.

Theorem 5.1. *Assume* (2.3), (2.4), *and let* α_h^* *be the function defined by* (5.4). *Then, for any* $x \in \mathbb{R}^n$, *there exist a sequence* $\{h(p)\}_{p=1}^{+\infty}$ *of positive numbers converging to zero,* $\mu^* \in \mathscr{A}^r$, *and* y^*: $[0, +\infty[\to \mathbb{R}^n$ *such that*

$$\mu^{\alpha_h^*(p)} \to \mu^* \text{ in } L^\infty(0,T;\mathscr{M}(A)) \text{ weakly star, i.e.,}$$
$$\int_0^T \varphi\big(t, \alpha_{h(p)}^*(t)\big)\, dt \to \int_0^T\int_A \varphi(t,a)\, d\mu_t^*(a)\, dt \tag{5.14}$$

for all $\varphi \in L^1(0,T;\ C(A))$ *and* $T > 0$,

$$y^*_{h(p)}(x,t) \rightarrow y^*(t) \text{ uniformly on any compact} \tag{5.15}$$

subset of $[0, +\infty[$, *and*

$$J\left(x, \alpha^*_{h(p)}\right) \rightarrow J^r(x, \mu^*) \tag{5.16}$$

as $p \rightarrow +\infty$. *In addition*,

$$y^*(t) = x + \int_0^t \int_A g(y^*(s), a)\, d\mu^*_s(a)\, ds \quad \text{for } t \geq 0, \tag{5.17}$$

$$J^r(x, \mu^*) = V^r(x). \tag{5.18}$$

Proof. In view of Lemma 5.1, we can select a sequence $\{h(p)\}_{p=1}^{+\infty}$ with $h(p) > 0$ so that $h(p) \rightarrow 0$ as $p \rightarrow +\infty$ and (5.14) holds. Let y_p: $[0, +\infty[\rightarrow \mathbb{R}^n$ be the response to $\alpha^*_{h(p)}$, i.e., the solution of (2.2) with $\alpha = \alpha^*_{h(p)}$. Since g is bounded, $\{y_p\}$ is a uniformly bounded and equicontinuous family of functions on $[0, T]$ for each $T > 0$. Therefore, choosing a subsequence if necessary, we may assume that $\{y_p\}$ converges to some continuous function y^* uniformly on any compact subset of $[0, +\infty[$. By the weak star convergence of $\mu^{\alpha^*_{h(p)}}$, we have

$$\int_0^t g\left(y^*(s), \alpha^*_{h(p)}(s)\right) ds \rightarrow \int_0^t \int_A g(y^*(s), a)\, d\mu^*_s(a)\, ds \text{ for } t \geq 0.$$

Hence, using the first assumption of (2.3) we can send $p \rightarrow +\infty$ in

$$y_p(x,t) = x + \int_0^t \int_A g\left(y_p(x,s), \alpha^*_{h(p)}(s)\right) ds \quad \text{for } t \geq 0,$$

to obtain (5.17), i.e.,

$$y^*(t) = x + \int_0^t \int_A g(y^*(s), a)\, d\mu^*_s(a)\, ds \quad \text{for } t \geq 0.$$

Similarly, we see that (5.16) holds. The identity (5.18) follows from (5.16), (5.8) and (5.12). □

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